Unit 4

HIGHER ORDER DERIVATIVES AND APPLICATIONS

By the end of this unit, the students will be able to:

4.1 Higher Order Derivatives

i. Find higher order derivatives of algebraic, trigonometric, exponential and logarithmic functions.
ii. Find the second derivative of implicit, inverse trigonometric and parametric functions.
iii. Use MAPLE command diff repeatedly to find higher order derivative of function.

4.2 Maclaurin's and Taylor's Expansions

i. State Maclaurin's and Taylor's theorems (without remainder terms). Use these theorems to expand $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $\log_a(1+x)$ and $\ln(1+x)$.
ii. Use MAPLE command **taylor** to find Taylor's expansion for a given function.

4.3 Application of Derivatives

i. Give geometrical interpretation of derivative.
ii. Find the equation of tangent and normal to the curve at a given point.
iii. Find the angle of intersection of the two curves.
iv. Find the point on a curve where the tangent is parallel to the given line.

4.4 Maxima and Minima

i. Define increasing and decreasing functions.
ii. Prove that if $f(x)$ is a differentiable function on the open interval (a,b) then
- $f(x)$ is increasing on (a,b) if $f'(x) > 0, \forall x \in (a,b)$,
- $f(x)$ is decreasing on (a,b) if $f'(x) < 0, \forall x \in (a,b)$,

iii. Examine a given function for extreme values.
iv. State the second derivative rule to find the extreme values of a function at a point.
v. Use second derivative rule to examine a given function for extreme values.
vi. Solve real life problems related to extreme values.
vii. Use MAPLE command maximize (minimize) to compute maximum (minimum) value of a function.

Introduction

The higher order derivatives has useful physical interpretation. If $y = f(t)$ is the position of an object at time 't' then $\frac{dy}{dt} = f'(t)$ is its velocity at time 't' and $\frac{d^2y}{dt^2}$ is its acceleration at time 't'. According to the Newton's law of motion *"The acceleration of an object is proportional to the total force acting on it"*. So, the second order derivatives has importance in mechanics. The second order derivatives is also important to graph the functions. Now, in this unit we will learn in details about higher order differentiation and its applications.

4.1 Higher order derivatives

If a function $y = f(x)$ has a first derivative y', then the derivative of y', if it exists, is the **second** derivative of $y = f(x)$, written as y''. The derivative of y'', if it exists, is called the **third derivative** of $y = f(x)$, written as y'''. By continuing this process, we can find **fourth derivative** and other **higher derivatives**. For example, if $f(x) = x^4 + 2x^3 + 3x^2 - 5x + 7$, then the higher derivatives are the following:

$$y' = f'(x) = \frac{dy}{dx} = 4x^3 + 6x^2 + 6x - 5, \qquad \text{first derivative of } y$$

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 12x^2 + 12x + 6, \qquad \text{second derivative of } y$$

$$y''' = f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 24x + 12, \qquad \text{third derivative of } y$$

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Example 1 Find the second derivative of the following functions:

(a). $f(x) = 8x^3 - 9x^2 + 6x + 4$ (b). $f(x) = \frac{4x+2}{3x-1}$

> **Remember**
> **The second derivative of $y = f(x)$ can be written with any of the following notations:**
> $\frac{d^2y}{dx^2}$, y'', $f''(x)$, $D^2[f(x)]$
> **The third derivative can be written in a similar way. For derivative $n \geq 4$, the derivative holds the** notation $f^{(n)}(x)$, $n = 4, 5, ...$

Solution

a. If the given function is $f(x) = 8x^3 - 9x^2 + 6x + 4$, then, the first and second derivatives of the given function through linearity property are the following:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(8x^3 - 9x^2 + 6x + 4) = 24x^2 - 18x + 6$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(24x^2 - 18x + 6) = 48x - 18$$

b. If the given function is $f(x) = y = \frac{u}{v} = \frac{4x+2}{3x-1}$, then the first and second derivatives of the given function through quotient rule are the following:

$$\therefore \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{4x+2}{3x-1}\right) = \frac{(3x-1)\frac{d}{dx}(4x+2) - (4x+2)\frac{d}{dx}(3x-1)}{(3x-1)^2}$$

$$= \frac{(4)(3x-1) - (3)(4x+2)}{(3x-1)^2} = \frac{-10}{(3x-1)^2}$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(0)(3x-1)^2 - (-10)(2)(3x-1)(3)}{(3x-1)^4} = \frac{60(3x-1)}{(3x-1)^4} = \frac{60}{(3x-1)^3}$$

In the previous unit, we saw that the first derivative of a function represents the rate of change of the function. The second derivative, then, represents the rate of change of the first derivative. If a function describes the position of a moving object at time t, then the first derivative gives the **velocity** of the object. That is, if $y = s(t)$ describes the position of the object at time t, then $v(t) = s'(t)$ gives the velocity at a time t.

The rate of change of velocity is called **acceleration**. Since the second derivative gives the rate of change of the first derivative, the acceleration is the derivative of the velocity. Thus, if a(t) represents the acceleration at time t, then

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = s''(t)$$

Example 2 An object is moving along a straight line with its position s(t) (in feet) at time t (in seconds): $s(t) = t^3 - 2t^2 - 7t + 9$

(a). Find the velocity at any time t. (b). Find the acceleration at any time t.
(c). The object stops when velocity is zero. For $t \geq 0$, when does that occur?

Solution

a. The velocity at any time t is the first derivative of $s(t)$ w.r.t. t: $v = \frac{ds}{dt} = 3t^2 - 4t - 7$

b. The acceleration at any time t is the first derivative of $v(t)$ w.r.t. t: $a = \frac{dv}{dt} = 6t - 4$

c. Use $v(t) = 0$ to obtain the time: $3t^2 - 4t - 7 = 0$

$$(3t - 7)(t + 1) = 0, \quad t = -1, \ \frac{7}{3}$$

The object will stop at $\frac{7}{3}$ seconds, since we want time $t \geq 0$.

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4.1.1 Higher order derivatives of algebraic, trigonometric, exponential and logarithmic functions

The successive derivatives of some functions are gathered to obtain the general form of nth derivatives in the following cases:

i. The nth derivative of $f(x)=(ax+b)^m$

If $f(x)=(ax+b)^m$, m is positive integer, then the successive derivatives of the given function developed a general term for the nth derivative of a function:

$$f(x)=(ax+b)^m$$
$$f'(x)=ma(ax+b)^{m-1}$$
$$f''(x)=m(m-1)a^2(ax+b)^{m-2}$$
$$\vdots$$
$$f^n(x)=m(m-1)(m-2)\ldots(m-n+1)(a^n)(ax+b)^{m-n} \qquad \text{(i)}$$
$$=\frac{m!}{(m-n)!}a^n(ax+b)^{m-n},\ \textit{if } m \textit{ is positive integer}$$

If $m=-1$, then the nth derivative of $f(x)=\frac{1}{(ax+b)}$ is obtained by inserting $m=-1$ in equation (i):

$$f^n(x)=(-1)(-2)(-3)\ldots(-n)(a^n)(ax+b)^{-1-n}=\frac{(-1)^n\ n!a^n}{(ax+b)^{n+1}} \qquad \text{(ii)}$$

ii. The nth derivative of $f(x)=\ln(ax+b)$

If $f(x)=\ln(ax+b)$, then the successive derivatives developed a general term for the nth derivative of a function:

$$f(x)=\ln(ax+b)$$
$$f'(x)=\frac{a}{ax+b}$$
$$f''(x)=\frac{(-1)a^2}{(ax+b)^2}$$
$$f'''(x)=\frac{(-1)(-2)a^3}{(ax+b)^3}$$
$$\vdots$$
$$f^n(x)=(-1)(-2(-3)(-4)\ldots..(-(n-1))(a^n)(ax+b)^{-n}=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n} \qquad \text{(iii)}$$

iii. The nth derivative of $f(x)=a^{mx}$

If $f(x)=a^{mx}$, then the successive derivatives developed a general term for the nth derivative:

$$f(x)=a^{mx}$$
$$f'(x)=a^{mx}\log a\frac{d}{dx}(mx)=ma^{mx}\log a$$
$$f''(x)=m\log a\frac{d}{dx}(a^{mx})=m\log a(a^{mx})\log a\frac{d}{dx}(mx) \qquad \text{(iv)}$$

$$=m^2a^{mx}(\log a)^2$$
$$\vdots$$
$$f^n(x)=m^na^{mx}(\log a)^n$$

If $a=e$, then the nth derivative of $f(x)=e^{mx}$ is obtained by inserting $a=e$:

$$f(x)=e^{mx}$$
$$f'(x)=me^{mx}$$
$$f''(x)=m(m)e^{mx}=m^2e^{mx}$$
$$\vdots$$
$$f^n(x)=m^ne^{mx} \qquad \text{(v)}$$

iv. The nth derivative of $f(x)=\sin(ax+b)$:

If $f(x)=\sin(ax+b)$, then the successive derivatives developed a general term for the nth derivative of a function:

$$f(x)=\sin(ax+b)$$
$$f'(x)=a\cos(ax+b)=a\sin\left(ax+b+\frac{\pi}{2}\right)$$
$$f''(x)=a^2\cos\left(ax+b+\frac{\pi}{2}\right)=a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$
$$f'''(x)=a^3\cos\left(ax+b+\frac{2\pi}{2}\right)=a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$
$$\vdots$$
$$f^n(x)=a^n\sin\left[ax+b+\frac{n\pi}{2}\right] \qquad \text{(vi)}$$

v. The nth derivative of $f(x)=\cos(ax+b)$:

If $f(x)=\cos(ax+b)$, then the successive derivatives developed a general term for the n–th derivative:

$$f(x)=\cos(ax+b)$$
$$f'(x)=-a\sin(ax+b)=a\cos\left(ax+b+\frac{\pi}{2}\right)$$
$$f''(x)=-a^2\sin\left(ax+b+\frac{\pi}{2}\right)=a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$
$$f'''(x)=-a^3\sin\left(ax+b+\frac{2\pi}{2}\right)=a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$$
$$\vdots$$
$$f^n(x)=a^n\cos\left[ax+b+\frac{n\pi}{2}\right] \qquad \text{(vii)}$$

Example 3 Find the 5th derivatives of the following functions:

(a). $f(x)=(6x+4)^9$ (b). $f(x)=\frac{1}{4x+3}$ (c). $f(x)=\ln(4x+7)$

(d). $f(x)=6^{4x}$ (e). $f(x)=e^{4x}$ (f). $f(x)=\sin(5x+7)$

Solution

a. If $f(x) = (6x+4)^9$ with $a = 6$, $b = 4$ and $m = 9$, then the 5^{th} derivative of the given function is obtained by inserting $n = 5$ in equation: $f^n(x) = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

$$f^5(x) = \frac{9!}{(9-5)!}6^5(6x+4)^{9-5} = \frac{6^5 9!}{4!}(6x+4)^4 = 6^5(9)(8)(7)(6)(5)(6x+4)^4 = 6^5\left(15120(6x+4)^4\right)$$

b. If $f(x) = \frac{1}{(4x+3)}$ with $a = 4$ and $b = 3$, then the 5^{th} derivative of the given function is obtained by inserting $n=5$ in equation : $f^n(x) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

$$f^5(x) = \frac{(-1)^5 5! 4^5}{(4x+3)^6} = \frac{-4^5 5!}{(4x+3)^6}$$

c. If $f(x) = \ln(4x+7)$ with $a = 4$ and $b = 7$, then the 5^{th} derivative of the given function is obtained by inserting $n = 5$ in equation: $f^n(x) = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$

$$f^5(x) = \frac{(-1)^4 4! 4^5}{(4x+7)^5} = \frac{4^5 4!}{(4x+7)^5}$$

d. If $f(x) = 6^{4x}$ with $a = 6$ and $m = 4$, then the 5^{th} derivative of the given function is obtained by inserting $n = 5$ in equation: $f^n(x) = m^n a^{mx}(\log a)^n$

$$f^5(x) = 4^5 6^{4x}(\log 6)^5$$

e. If $f(x) = e^{4x}$ with $m = 4$, then the 5^{th} derivative of the given function is obtained by inserting $n = 5$ in equation: $f^n(x) = m^n e^{mx}$

$$f^5(x) = 4^5 e^{4x}$$

f. If $f(x) = \sin(5x+7)$ with $a = 5$ and $b = 7$ then the 5th derivative of the given function is obtained by inserting $n = 5$ in equation: $f^n(x) = a^n \sin\left[ax+b+\frac{n\pi}{2}\right]$

$$f^5(x) = 5^5 \sin\left[5x+7+\frac{5\pi}{2}\right]$$

4.1.2 Second derivative of implicit, inverse trigonometric and parametric functions

Example 4 Find the second derivative of $x^2y + 2y^3 = 3x + 2y$.

Solution The equation is $x^2y + 2y^3 = 3x + 2y$ (i)

The first implicit derivative of (i) w.r.t. x is:

$$\frac{d}{dx}\left(x^2y+2y^3\right) = \frac{d}{dx}(3x+2y)$$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(2y^3) = \frac{d}{dx}(3x) + \frac{d}{dx}(2y)$$

$$2xy + x^2\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 3 + 2\frac{dy}{dx}$$

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$$(x^2+6y^2-2)\frac{dy}{dx} = 3-2xy \quad \text{(ii)}$$

The second implicit derivative of first implicit derivative (ii) w.r.t. x is: $\frac{d}{dx}\left[(x^2+6y^2-2)\frac{dy}{dx}\right] = \frac{d}{dx}(3-2xy)$

$$\frac{d}{dx}[(x^2+6y^2-2)]\frac{dy}{dx} + (x^2+6y^2-2)\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3) - \frac{d}{dx}(2xy)$$

$$(2x+12y\frac{dy}{dx}-0)\frac{dy}{dx} + (x^2+6y^2-2)\frac{d^2y}{dx^2} = 0-2y-2x\frac{dy}{dx}$$

$$(2x+12y\frac{dy}{dx})\frac{dy}{dx} + (x^2+6y^2-2)\frac{d^2y}{dx^2} = -2y-2x\frac{dy}{dx}$$

$$\left(x^2+6y^2-2\right)\frac{d^2y}{dx^2} = -2y-(2x+2x)\frac{dy}{dx} - 12y\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} = \frac{-2\left(y+2x\frac{dy}{dx}+6y\left(\frac{dy}{dx}\right)^2\right)}{\left(x^2+6y^2-2\right)} = \frac{-2\left(y+2x\left(\frac{3-2xy}{x^2+6y^2-2}\right)+6y\left(\frac{3-2xy}{x^2+6y^2-2}\right)^2\right)}{x^2+6y^2-2}$$ Putting values of $\frac{dy}{dx}$

$$= \frac{-2\left(y+\frac{6x-4x^2y}{x^2+6y^2-2}\right)+6y\left(\frac{9+4x^2y^2-12xy}{(x^2+6y^2-2)^2}\right)}{x^2+6y^2-2}$$

$$= \frac{-2\left(y+\frac{6x-4x^2y}{x^2+6y^2-2}+\frac{54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^2}\right)}{(x^2+6y^2-2)}$$

$$= \frac{\frac{-2(y(x^2+6y^2-2)^2+6x-4x^2y(x^2+6y^2-2)+54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^2}}{(x^2+6y^2-2)}$$

$$= \frac{-2(y(x^4+36y^4+4+12x^2y^2-24y^2-4x^2)+6x^3+36y^2x-12x-4x^4y-24x^2y^3+8x^2y+54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^3}$$

$$= \frac{-2(x^4y+36y^5+4y+12x^2y^3-24y^3-4x^2y+6x^3+36y^2x-12x-4x^4y-24x^2y^3+8x^2y+54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^3}$$

$$\frac{d^2y}{dx^2} = -\frac{2(36y^5+12y^3x^2-24y^3-36y^2x-3yx^4+4yx^2+58y+6x^3-12x)}{(6y^2+x^2-2)^3}$$

Example 5 Find the second derivative of $\cos^{-1}y + y = 2xy$.

Solution The given equation is $\cos^{-1}y + y = 2xy$ (i)

The first implicit derivative of (i) w.r.t. x is:

$$\frac{d}{dx}(\cos^{-1}y+y) = \frac{d}{dx}(2xy)$$

$$\frac{d}{dx}(\cos^{-1}y) + \frac{d}{dx}(y) = 2\frac{d}{dx}(xy)$$

$$\frac{-1}{\sqrt{1-y^2}}\frac{dy}{dx} + \frac{dy}{dx} = 2\left(\frac{d}{dx}(x)y + x\frac{d}{dx}(y)\right)$$

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$$\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}=2y+2x\frac{dy}{dx} \qquad \text{(ii)}$$

Remember

Equation (ii) can be written in simplified form for 1st order derivative as

$$\frac{dy}{dx}=-\frac{2y\sqrt{1-y^2}}{1-\sqrt{1-y^2}+2x\sqrt{1-y^2}}$$

The second implicit derivative of first implicit derivative (ii) w.r.t. x is:

$$\frac{d}{dx}\left[\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}\right]=\frac{d}{dx}\left(2y+2x\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}+\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{d^2y}{dx^2}=2\frac{dy}{dx}+2\frac{dy}{dx}+2x\frac{d^2y}{dx^2}$$

$$\left(\frac{-2y}{2(1-y^2)^{\frac{3}{2}}}+0\right)\frac{dy}{dx}+\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{d^2y}{dx^2}=4\frac{dy}{dx}+2x\frac{d^2y}{dx^2}$$

Equating coefficients of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to obtain: $\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)\frac{d^2y}{dx^2}=\left(\frac{y}{(1-y^2)^{\frac{3}{2}}}+4\right)\frac{dy}{dx}$

$$\frac{d^2y}{dx^2}=\frac{\left(\frac{y}{(1-y^2)^{\frac{3}{2}}}+4\right)\frac{dy}{dx}}{\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)}=\frac{\left(\frac{y}{(1-y^2)^{\frac{3}{2}}}+4\right)\left(\frac{2y\sqrt{1-y^2}}{1-\sqrt{1-y^2}+2x\sqrt{1-y^2}}\right)}{\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{2y\left(y+4(-y^2+1)^{\frac{3}{2}}\right)}{\left(2x\sqrt{1-y^2}+1-\sqrt{1-y^2}\right)\left(-1+\sqrt{1-y^2}-2x\sqrt{1-y^2}\right)\left(\sqrt{1-y^2}\right)}$$

Example 6 Find the second derivative $\frac{d^2y}{dx^2}$, when the parametric functions are:

$$x(t)=1+t^2,\ y(t)=t^3+2t^2+1$$

Solution The first derivative of the parametric functions $x=x(t)$ and $y=y(t)$ w.r.t. x is:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{(i)}$$

The second derivative of the parametric functions is obtained by taking the derivative of equation (i): $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dt}$, Multiply and divide it by dt

$$=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} \qquad \text{(ii)}$$

The quotient rule of differentiation is used to simplify the right hand side of equation (ii):

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \qquad \text{(iii)}$$

Use (ii) in (iii) to obtain the general term for second derivative of parametric functions $x(t)$ and $y(t)$:

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}\frac{dt}{dx}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \qquad \therefore \textit{replace}\ \frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}$$

In light of result (iv), the first and second derivatives of the parametric functions $x(t)=1+t^2,\ y(t)=t^3+2t^2+1$ with $\frac{dx}{dt}=2t,\ \frac{d^2x}{dt^2}=2,\ \frac{dy}{dt}=3t^2+4t,\ \frac{d^2y}{dt^2}=6t+4$ are the following:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{3t^2+4t}{2t}=\frac{t(3t+4)}{2t}=\frac{3t+4}{2}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}=\frac{(2t)(6t+4)-(3t^2+4t)(2)}{8t^3}=\frac{12t^2+8t-6t^2-8t}{8t^3}=\frac{6t^2}{8t^3}=\frac{3}{4t}$$

4.1.3 MAPLE command diff repeatedly to find higher order derivative of a function

The procedure to use of MAPLE command diff is illustrated in the following example.

Example 7 Differentiate $f(x)=x^4+x^2\sin x+x+2$ w.r.t. 'x'

Solution

Command

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> diff(x^4 + x^2·sin(x) + x + 2, x);
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$$12x^2+2\sin(x)+4x\cos(x)-x^2\sin(x)$$

For second derivative, after command, press the "Enter" key two times to obtain the second derivative of a given function above result.

Exercise 4.1

1. Find the indicated higher derivatives of the following functions:
   a. $f(x)=3x^3+4x+5,\ f''(x)$ b. $f(x)=x+\frac{1}{x},\ f'''(x)$
   c. $s(t)=\sqrt{5t+7},\ s''(t)$ d. $y=\frac{x+1}{x-1},\ y''$
2. Use implicit rule to find out the second derivative of the following functions:
   a. $b^2x^2+a^2y^2=a^2b^2$ b. $x^2+y^2=r^2$
   c. $y^2-2xy=0$ d. $e^x+x=e^y+y$
3. Use parametric differentiation to find out $\frac{d^2y}{dx^2}$ for the following parametric functions $x\,(t)$ and $y\,(t)$:
   a. $x=4t^2+1,\ y=6t^3+1$ b. $x=3at^2+2,\ y=6t^4+9$
   c. $x=a\cos 2t,\ y=b\sin 2t$ d. $x=\frac{3at}{1+t^3},\ y=\frac{3at^2}{1+t^3}$
4. Find the indicated higher order derivative of the following function.
   a. $f(x)=(x^3+4x-5)^4,\ f^{iv}(x)$ b. $f(x)=\tan^2(x),\ f'''(x)$
   c. $f(x)=\frac{1}{2}e^{9x};\ f'''(x)$ d. $f(x)=x^4.\ln|x^2|,\ f^{iv}(x)$
5. Use MAPLE command "diff" to find the indicated higher order derivative of the following functions.
   a. $f(x)=\sqrt{\sec(2x)},\ f'''(x)$ b. $f(x)=\sin(\sin x),\ f'''(x)$
6. Find the indicated derivative of the following by using rule.
   a. $y=(3x+7)^{11}$, 7th derivative
   b. $f(x)=\ln(2x-4)$, 10th derivative
   c. $g(x)=4\cos(3x+8)$, 6th derivative
   d. $h(x)=7e^{5x+4}$, 12th derivative

Project

By using the chain rule and other differential rules, some of the derivative computations can be radius to perform. For complicated derivatives mathematicians, scientists and engineers use computer softwares. Such as mathematica, maple and Matlab, use computer software to compute.

- $\frac{d}{dx}\left[\frac{(x^2+4)^{10}\sin^5(\sqrt{x})}{\sqrt{1+\cos(x)}}\right]$
- $\frac{d}{dx}\left[\frac{\sqrt{1+\csc(x)}}{(x^2+4)^{10}\sin^5(\sqrt{x})}\right]$

Although we have all mathematical tools to compute above type of problems by hand. But the computation involving computer software may be more efficient.

4.2 Maclaurin's and Taylor's Expansions

Often the value of a function and the values of its derivatives are known at a particular point and from this information it is desired to obtain values of the function around that particular point. The Taylor polynomials and Taylor series allow us to make such estimates.

4.2.1 Maclaurin's and Taylor's theorems. Using these theorems to expand sinx, cos x, tan x, a^x, e^x, loga(1+x) and ln(1+x)

A. Taylor's Theorem

If $f(x)$ and its n derivatives at $x=x_0$ are $f'(x_0), f''(x_0),\dots,f^n(x_0)$, then the nth order Taylor polynomial $p_n(x)$ may be written as:

$$p_n(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\dots\dots+\frac{(x-x_0)^n}{n!}f^n(x_0) \qquad \text{(i)}$$

This polynomial provides an approximation to $f(x)$. The polynomial and its n derivatives are very much matched with the values of $f(x)$ and its first n derivatives evaluated at $x=x_0$:

$$p_n(x_0)=f(x_0),\ p_n'(x_0)=f'(x_0),\ p_n''(x_0)=f''(x_0),\ \dots\dots,\ p_n^n(x_0)=f_n^n(x_0)$$

Example 8 The function $y=f(x)=e^x$ and its derivatives evaluated at $x_0=0$ are known by $f(0)=1$, $f'(0)=1$, $f''(0)=1$, $f'''(0)=1$, $f^{iv}(0)=1$. Use fourth order Taylor polynomial about $x_0=0$ to estimate $f(0.2)$ at $x=0.2$.

Solution The fourth order Taylor polynomial $p_4(x)$ is obtained by terminating the Taylor polynomial (i) after fourth order derivative term:

$$p_4(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\frac{(x-x_0)^4}{4!}f^{iv}(x_0) \qquad \text{(ii)}$$

Insert $x_0=0$ in (ii) to obtain:

$$p_4(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+f^{iv}(0)\frac{x^4}{4!}$$
$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!} \qquad \text{(iii)}$$

The Taylor polynomial (iii) is used to obtain approximation of a function $y=f(x)=e^x$ at $x=0.2$:

$$p_4(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$
$$p_4(0.2)=1+0.2+\frac{(0.2)^2}{2!}+\frac{(0.2)^3}{3!}+\frac{(0.2)^4}{4!}$$
$$=1+0.2+0.02+0.00133+0.00007=1.2214$$

Notice that the Taylor polynomial approximation equals the actual function value $y=f(0.2)=e^{0.2}=1.2214$ at $x=0.2$.

Remember

Taylor's and Maclaurin's theorems are also known as Taylor's and Maclaurin's series.

Remember

Taylor's Series: The Taylor polynomials have been used to estimate the values of $y=f(x)$ at various x values. It is reasonable to ask:

i. How accurate Taylor polynomials generated by $y=f(x)$ at x_0 to approximate $y=f(x)$ at values of x other than x_0?

ii. If more and more terms are used in the Taylor polynomial, then this will produce a better and better approximation to $y = f(x)$.

To answer these questions, we introduce the Taylor series. As more and more terms are included in the Taylor polynomial, we obtain an infinite series, known as a **Taylor series**:

$$p(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \ldots\ldots + \frac{(x - x_0)^n}{n!}f^n(x_0) \quad (i)$$

For some Taylor series, the value of the series equals the value of the function for every value of x. That is, the Taylor series approximations of e^x, $\sin x$ and $\cos x$ equal the values of e^x, $\sin x$ and $\cos x$ for every value of x. However, some functions have a Taylor series which equals the function only for a limited range of x values. For example, the value of a function $f(x) = \frac{1}{(1+x)}$ which equals its Taylor series only when $-1 < x < 1$.

B. Maclaurin's Series

A special case of a Taylor series occurs, when the function $y = f(x)$ is known only at the origin $x_0 = 0$. This special condition imposed on Taylor series, develops the **Maclaurin's series**:

$$p(x) = f(0) + xf'(0) + \frac{(x)^2}{2!}f''(0) + \ldots + \frac{(x)^n}{n!}f^n(0) \quad (i)$$

The Taylor and Maclaurin's series of $y = f(x)$ about a particular point x_0 are of course:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + \ldots + \frac{(x - x_0)^n}{n!}f^n(x_0) \quad (ii)$$

$$f(x) = f(0) + xf'(0) + \frac{(x)^2}{2!}f''(0) + \frac{(x)^3}{3!}f'''(0) \ldots\ldots + \frac{(x)^n}{n!}f^n(0) \quad (iii)$$

If we use $x - x_0 = h$, then equations (ii) and (iii) take the popular notation for the Taylor and Maclaurin's series of order n:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \ldots + \frac{h^n}{n!}f^n(x_0) \quad (iv)$$

$$f(x_0 + h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^3}{3!}f'''(0) + \ldots + \frac{h^n}{n!}f^n(0) \quad (v)$$

The graphical view of a function $y = f(x)$ at $x = x_0$ is shown in the Figure 4.1.

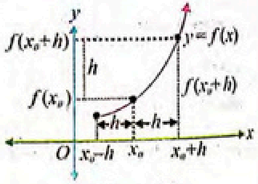

Figure 4.1

Note

The popular notation for the Taylor & Maclaurin's series of order *n* are:

i. $f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \ldots \frac{h^n}{n!}f^n(x_0)$

ii. $f(x_0 + h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^3}{3!}f'''(0) + \ldots \frac{h^n}{n!}f^n(0)$

If a function $y = f(x)$ is known at a particular point $x_0 \neq 0$, then the Taylor series (iv) at a forward or backward point $x = x_0 \pm h$ of a function $y = f(x)$ are:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \ldots \qquad x = x_0 + h$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \frac{h^3}{3!}f'''(x_0) + \ldots \qquad x = x_0 - h$$

NOT FOR SALE

Now, look at the following examples the procedure to the use of Taylor and Maclaurin's Theorem is illustrated in these examples.

Example 9 Use Taylor's series to approximate the value of a function $f(x) = e^x$ at a point $x_0 = 2$.

Solution The function and its derivatives at $x_0 = 2$

$f(x) = e^x$, $f(2) = e^2 = 7.3891$, $f'(x) = e^x$, $f'(2) = e^2 = 7.3891$, $f''(x) = e^x$, $f'''(2) = e^2 = 7.3891$

are used in Taylor series (ii) to obtain the Taylor series approximation of e^x at a point $x_0 = 2$:

$$e^x = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \ldots\ldots$$

$$= 7.3891 + 7.3891(x-2) + 7.3891\frac{(x-2)^2}{2!} + 7.3891\frac{(x-2)^3}{3!} + \ldots\ldots$$

D. Maclaurin's theorem for the functions of the type $f(x) = a^x$

Example 10 Use Maclaurin's series to approximate the value of a function $f(x) = a^x$ at a point $x_0 = 0$.

Solution The function and its derivatives at $x_0 = 0$

$f(x) = a^x$, $f(0) = 1$, $f'(x) = a^x \log_e a$, $f'(0) = \log_e a$

$f''(x) = a^x(\log_e a)^2$, $f''(0) = (\log_e a)^2$

are used in Maclaurin series (iii) to obtain the Maclaurin series approximation of a^x at a point $x_0 = 0$:

$$a^x = f(0) + x f'(0) + \frac{(x)^2}{2!}f''(0) + \frac{(x)^3}{3!}f'''(0) + \ldots\ldots$$

$$= 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \ldots\ldots$$

E. Maclaurin's theorem for the functions of the type $f(x) = e^x$

Example 11 Use Maclaurin's series to approximate the value of a function $f(x) = e^x$ at a point $x_0 = 0$.

Solution The function and its derivatives at $x_0 = 0$

$f(x) = e^x$, $f(0) = 1$, $f'(x) = e^x$, $f'(0) = 1$, $f''(x) = e^x$, $f''(0) = 1$

are used in Maclaurin series (iii) to obtain the Maclaurin's series approximation of e^x at a point $x_0 = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

F. Maclaurin's theorem for the function of the type $f(x) = \sin(x)$

Example 12 Use Maclaurin's series to approximate the value of a function $f(x) = \sin(x)$ at a point $x_0 = 0$.

Solution The function and its derivatives at $x_0 = 0$

$f(x) = \sin x$, $f(0) = \sin(0) = 0$, $f'(x) = \cos x$, $f'(0) = \cos(0) = 1$

$f''(x) = -\sin x$, $f''(0) = -\sin(0) = 0$, $f'''(x) = -\cos x$, $f'''(0) = -\cos(0) = -1$

are used in Maclaurin series (iii) to obtain the Maclaurin series approximation of $\sin x$ at a point $x_0 = 0$:

$$\sin x = f(0) + (x)f'(0) + \frac{(x)^2}{2!}f''(0) + \frac{(x)^3}{3!}f'''(0) + \ldots$$

$$= 0 + x - 0 - \frac{(x)^3}{3!} + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

NOT FOR SALE

G. Maclaurin's theorem for the function of the type $f(x)=\cos(x)$

Example 13 Use Maclaurin's series to approximate the value of a function $f(x)=\cos x$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$.

$$f(x)=\cos x,\ f(0)=\cos 0=1,\quad f'(x)=-\sin x,\ f'(0)=-\sin 0=0$$

$$f''(x)=-\cos x,\ f''(0)=-\cos 0=-1,\quad f'''(x)=\sin x,\ f'''(0)=\sin 0=0$$

are used in Maclaurin series (iii) to obtain the Maclaurin series approximation of a function $\cos x$ at a point $x_0=0$:

$$\cos x=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)f'''(0)+\dots$$

$$=1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\dots=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$$

H. Maclaurin's theorem for the function of the type $f(x)=\tan(x)$

Example 14 Use Maclaurin's series to approximate the value of a function $f(x)=\tan x$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$$f(x)=\tan x,\ f(0)=\tan 0=0,\ f'(x)=\sec^2 x,\ f'(0)=\sec^2 0=1$$

$$f''(x)=2\sec^2 x\tan x,\ f''(0)=2(1)(0)=0,\ f'''(x)=2\sec^4 x+4\tan^2 x\sec^2 x$$

$$f'''(0)=2\sec^4 0+4\tan^2 0\sec^2 0=2$$

are used in Maclaurin's (iii) to obtain the Maclaurin's series approximation of a function $\tan x$ at a point $x_0=0$:

$$\tan x=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots$$

$$=0+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(2)\dots\dots=x+2\frac{x^3}{3!}+\dots$$

$$=x+\frac{x^3}{3}+\dots\dots$$

I. Maclaurin's theorem for the function of the type $f(x)=\log_a(1+x)$

Example 15 Use Maclaurin's series to approximate the value of a function $f(x)=\log_a(1+x)$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$$f(x)=\log_a(1+x),\ f(0)=\log_a(1)=0$$

$$f'(x)=\frac{1}{1+x}\log_a e,\ f'(0)=\log_a e=\log_a e$$

$$f''(x)=-\frac{1}{(1+x)^2}\log_a e,\ f''(0)=-\log_a e$$

$$f'''(x)=\frac{2}{(1+x)^3}\log_a e,\ f'''(0)=2\log_a e$$

are used in Maclaurin's series (iii) to obtain the Maclaurin's series approximation of a function $f(x)=\log_a(1+x)$ at a point $x_0=0$:

$$\log_a(1+x)=f(0)+(x)f'(0)+\frac{(x)^2}{2!}f''(0)+\frac{(x)^3}{3!}f'''(0)+\dots$$

$$=0+x\log_a e-\frac{(x)^2}{2!}\log_a e+2\frac{x^3}{3!}\log_a e-\dots$$

$$=x\log_a e-\frac{x^2}{2!}\log_a e+\frac{2x^3}{3!}\log_a e-\dots$$

$$\log_a(1+x)=x\log_a e-\frac{x^2}{2}\log_a e+\frac{x^3}{3}\log_a e-\dots$$

J. Maclaurin's theorem for the function of the type $f(x)=\ln(1+x)$

Example 16 Use Maclaurin's series to approximate the value of a function $f(x)=\ln(1+x)$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$$f(x)=\ln(1+x),\ f(0)=\ln(1)=0,\quad f'(x)=\frac{1}{(1+x)},\ f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2},\ f''(0)=-1,\quad f'''(x)=\frac{2}{(1+x)^3},\ f'''(0)=2$$

are used in Maclaurin's series (iii) to obtain the Maclaurin's series approximation of $\ln(1+x)$ at a point $x_0=0$:

$$\ln(1+x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots$$

$$=0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\dots=x-\frac{x^2}{2!}+2\frac{x^3}{3!}-\dots=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$$

Challenge!

Use Taylor's theorem to compute the series of the following functions at $x_0=3$.

i. $f(x)=\sin x$ ii. $f(x)=\cos x$ iii. $f(x)=\tan x$ iv. $f(x)=a^x$
v. $f(x)=e^x$ vi. $f(x)=\log_a(1+x)$ vii. $f(x)=\ln(1+x)$

4.2.2 MAPLE command "Taylor" to find Taylor's expansion for a given function

The use of MAPLE command 'Taylor' is illustrated in the following example.

Example 17 Use Maple command taylor for the function

(a). $f(x)=e^x$ by Taylor's series expansion to first four terms.
(b). $f(x)=\sin x$ by Taylor's series expansion to first 5 terms.

Solution

a. Command:

```
> taylor(e^x, x = 0, 4);
```

$$1+\ln(e)x+\frac{1}{2}\ln(e)^2x^2+\frac{1}{6}\ln(e)^3x^3+0(x^4)$$

Context Menu:

```
> e^x
> series( e^x,x,4)
```

$$1+\ln(e)x+\frac{1}{2}\ln(e)^2x^2+\frac{1}{6}\ln(e)^3x^3+0(x^4)$$

This result is obtained through right click on the last end of the expression by selecting "Series $< x$" on the context menu.

b. Command:

```
> taylor(sin(x), x = 0, 5);
```

$$x-\frac{1}{6}x^3+0(x^5)$$

Context Menu:

```
> sin(x)
> series( sin(x),x,5)
```

$$x-\frac{1}{6}x^3+0(x^5)$$

4.3 Application of Derivatives

In this section, we will see how to use derivatives to determine the tangent, and normal lines, the angles in between two curves, the maximum and minimum values of a function as well as the intervals where the function is increasing or decreasing.

4.3.1 Geometrical interpretation of derivative

Consider a function $y=f(x)$ as shown in the Figure 4.2.

Let $P(x_0, y_0)$ be a point on a curve $y=f(x)$.

The change Δx in x develops a change Δy in y.

The coordinates of a point Q are therefore $Q(x_0+\Delta x, y_0+\Delta y)$. Notice that the slope of the secant line PQ

$$\frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \qquad \text{(i)}$$

If we take values of Q closer to P, then Q approaches P, and Δx approaches 0 and the slope of the secant line PQ automatically approaches the slope of the tangent line at a particular point P and is denoted by:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0) \qquad \text{(ii)}$$

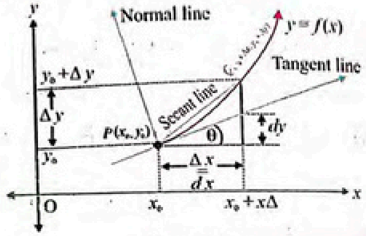

Figure 4.2

4.3.2 Equations of tangent and normal to the curve at a given point

If the slope of the tangent line on a curve $y=f(x)$ at a particular point $P(x_0, y_0)$ is $f'(x_0)$, then the tangent line on this curve at a particular point $P(x_0, y_0)$ is the nonhomogeneous line (developed from the definition of the point form of the straight line):

$$y-y_0=f'(x_0)(x-x_0)$$

$$y-y_0=m(x-x_0), \qquad m=f'(x_0) \qquad \text{(i)}$$

The normal line is the line perpendicular to the tangent line on this curve at a particular point $P(x_0, y_0)$ with slope $\frac{-1}{f'(x_0)}$:

$$(y-y_0)=\frac{-1}{f'(x_0)}(x-x_0)$$

$$(y-y_0)=-\frac{1}{m}(x-x_0), \qquad m=f'(x_0) \qquad \text{(ii)}$$

Example 18 Find the equations of the tangent and normal lines on a curve $y=x^2$ at a point $P(2,4)$.

Solution If the given curve is $y=x^2$, then, the slope of the tangent line is the first derivative of the given curve at a particular point $P(2,4)$:

$$f'(x)=2x$$

$$f(2)=2(2)=4=m, \ \text{say}, \ \text{at a point } P(2,4)$$

The tangent line (i) on the given curve at a particular point P(2,4) is:

$$y-y_0=m(x-x_0)$$

$$(y-4)=4(x-2)$$

$$-4x+y-4+8=0 \Rightarrow -4x+y+4=0 \Rightarrow 4x-y-4=0$$

NOT FOR SALE

The normal line (ii) on the given curve at a particular point P(2, 4) is:

$$(y-y_0)=\frac{-1}{m}(x-x_0)$$

$$(y-4)=\frac{-1}{4}(x-2)$$

$$4(y-4)=-(x-2)$$

$$x+4y-16-2=0 \Rightarrow x+4y-18=0$$

Example 19 Find the equations of the tangent and normal lines on the curve $y=9-x^2$ at a point, when y crosses the x- axis.

Solution The coordinates of a particular point P at which the given curve $y=9-x^2$ crosses the x axis are $y=0$.

Put y=0 in $y=9-x^2$ to obtain a set of points: $0=y=9-x^2 \Rightarrow x^2=9 \Rightarrow x=\pm 3 \Rightarrow (3,0), (-3,0)$

If the given curve is $y=9-x^2$, then, the slope of the tangent line is the first derivative of the given curve at a particular point $P(\pm 3,0)$:

$$f'(x)=-2x$$

$$f'(3)=-2(3)=-6=m, \ \text{at a point } P(+3,0)$$

$$f'(-3)=-2(-3)=6=m, \ \text{at a point } P(-3,0)$$

The tangent lines (i) on the given curve at the particular points are:

$$(y-0)=-6(x-3), \quad m=-6, \ P(3,0)$$

$$6x+y-18=0$$

$$(y-0)=6(x+3), \quad m=6, \ P(-3,0)$$

$$6x-y+18=0$$

The normal lines (ii) on the given curve at the particular points are:

$$(y-0)=\frac{-1}{-6}(x-3), \quad m=-6, \ P(3,0)$$

$$6y=x-3$$

$$x-6y-3=0$$

$$(y-0)=\frac{-1}{6}(x+3), \quad m=6, \ P(-3,0)$$

$$6y=-x-3 \Rightarrow x+6y+3=0$$

Remember

i. **The tangent equation at a point** $P(x_0, y_0)$ is $(y-y_0)=m(x-x_0)$.

ii. **The normal equation at a point** $P(x_0, y_0)$ is $(y-y_0)=\frac{-1}{m}(x-x_0)$.

4.3.3 Angle of intersection of the two curves

If m_1 is the slope of the first curve and m_2 is the slope of the second curve, then the angle of intersection in between these two curves at a point of intersection is the angle in between their tangents at that point. This angle takes the notation:

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2} \qquad \text{(i)}$$

Example 20 Find the angle of intersection in between the curves $y=x^3-2x+1$ and $y=x^2+1$ at the point of intersection (2, 5).

Solution The required angle of intersection in between the given two curves is: $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ (i)

For point of intersection, solve the system of nonlinear equations for the unknowns x and y:

NOT FOR SALE

$y = x^3 - 2x + 1, \ y = x^2 + 1$ (ii)

Using first equation of the nonlinear system (ii) in second equation to obtain:

$$x^3 - 2x + 1 = x^2 + 1$$
$$x^3 - x^2 - 2x = 0$$
$$x(x^2 - x - 2) = 0 \Rightarrow x = 0, -1, 2$$

The set of x values is used in first equation of the nonlinear system (ii) to obtain a set of y values:

Put $x = 0$ to obtain $y = x^3 - 2x + 1 = 1$

Put $x = -1$ to obtain $y = x^3 - 2x + 1 = -1 + 2 + 1 = 2$

Put $x = 2$ to obtain $y = x^3 - 2x + 1 = 8 - 4 + 1 = 5$

This process developed a set of points of intersection: (0,1), (−1,2), (2,5).

The slope of the first curve at a point (2, 5) is: $\frac{dy}{dx} = 3x^2 - 2 \Rightarrow \left(\frac{dy}{dx}\right)_{(2,5)} = 3(2)^2 - 2 = 10 = m_1$, say

The slope of the second curve at a point (2, 5) is: $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,5)} = 2(2) = 4 = m_2$, say

The slopes m_1 and m_2 are used in (i) to obtain the angle of intersection in between the given two curves:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{10 - 4}{1 + (10)(4)} = \frac{6}{41}$$

$$\theta = \tan^{-1}\frac{6}{41} = 0.1453$$

4.3.4 Point on a curve where the tangent is parallel to the given line

Look at the following example the procedure to find the point on a curve where tangent is parallel to the given line is illustrated in this example.

Example 21 Find all the points on the curve $y = 2x^3 + 4x^2$ where tangent line is parallel to the line $y = 8x - 4$

Solution Since the given line is $y = 8x - 4$

Slope of the given line = 8

Given curve = $y = 2x^3 + 4x^2$ (i)

$$\frac{dy}{dx} = 6x^2 + 8x$$
$$8 = 6x^2 + 8x$$
$$3x^2 + 4x - 4 = 0$$
$$3x^2 + 6x - 2x - 4 = 0$$
$$3x(x+2) - 2(x+2) = 0$$
$$(x+2)(3x-2) = 0$$
$$\Rightarrow x + 2 = 0, \ 3x - 2 = 0$$
$$x = -2 \ \text{and} \ x = +\frac{2}{3}$$

$y = 2x^3 + 4x^2$ when $x = -2$

Then $y = 2(-2)^3 + 4(-2)^2 = -16 + 16 = 0$

$y = 2\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^2$ when $x = \frac{2}{3}$

$$= 2\left(\frac{8}{27}\right) + 4\left(\frac{4}{9}\right)$$
$$= \frac{16}{27} + \frac{16}{9} = \frac{16 + 48}{27} = \frac{64}{27}$$
$$y = 0, \frac{64}{27}$$

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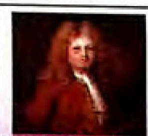
Exercise 4.2

1. In each case, find the equation of the tangent line to the curve at the indicated value of x:
   a. $y = \sqrt{x+1}, \ x = 3$ b. $y = \sin(2x + \pi), \ x = 0.$
   c. $y = x^2 e^{-x}, \ x = 1$ d. $y = \frac{x}{x^2 + 1}, \ x = 1$
2. In each case, find the equation of normal to the curve at the indicated value of x:
   a. $y = xe^x, \ x = 1$ b. $y = (2x+1)^6, \ x = 0$
   c. $y = \cos(x - \pi), \ x = \frac{\pi}{2}$ d. $y = x^3 \ln x, \ x = 1$
3. a. Find an equation of the tangent line to the curve $x^2 + y^2 = 13$ at (−2, 3).
   b. Find an equation of the tangent line to the curve $\sin(x - y) = xy$ at $(0, \pi)$.
   c. Find an equation of the normal line to the curve $x^2 + 2xy = y^3$ at (1, −1).
   d. Find an equation of the normal line to the curve $x^2\sqrt{y-2} = y^2 - 3x - 1$ at (1, 2).
4. a. Show that the first four terms in the Taylor series expansion of $f(x) = \tan x$ about $x = \frac{\pi}{4}$ are: $1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$
   b. Show that the first four terms in the Taylor series expansion of $f(x) = \sqrt{x}$ about $x = 4$ are: $2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$
   c. Show that the first four terms in the Taylor series expansion of $f(x) = x + e^x$ about $x = 1$ are: $(1+e)x + e\left[\frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \ldots\right]$
5. Find the Maclaurin series expansion for the following functions:
   a. $f(x) = \frac{1}{1+x}$ b. $f(x) = \sin^2 x$ c. $f(x) = \cosh x$ d. $f(x) = \ln(1 - 4x)$
6. a. Use the Maclaurin series for e^x to show that the sum of the infinite series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$ is e.
   b. Use part (a) to find out the value of e that must be accurate to 4 decimal places.
7. Find the angle of intersection between the following curves:
   a. $x^2 - y^2 = a^2, \ x^2 + y^2 = a^2\sqrt{2}$ b. $y^2 = ax, \ x^3 + y^3 = 3axy$
8. Find the points on the curve $y = 5x^3 - 4x^2$ where tangent line is parallel to the line $y = 5x - 3$.

History

B.Taylor was a British mathematician who is known by his invention of Taylor's theorem and the Taylor's series. In 1708 he obtained the solution of the problem of the "centre of oscillation" and published on 1714. Calculus of finite differences add to the branch of higher mathematics in 1715 with the name "Methodus Incrementorum Directa et Inversa". This word contain the well known layrange realized its importance and termed it as the main foundation of differential calculus.



Brook Taylor (1685–1731)

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4.4 Maxima and Minima

Always the maximum and minimum values of a function can be read from its graphical view. For a quadratic function (whose graph is parabola), the maximum or minimum values can be determined without graphing by finding the vertex algebraically. For functions whose graphs are not known, other techniques are needed. In this unit, we shall see how to use derivatives to determine the maximum and minimum values of a function as well as the intervals where the function is increasing or decreasing.

4.4.1 Increasing and decreasing functions

Suppose an ecologist has determined the size of a population of a certain species as a function $f(t)$ of time t (months). If it turns out that the population is increasing until the end of the first year and decreasing thereafter. It is reasonable to expect the population to be maximized at time $t = 12$ and for the population curve to have a high point at $t = 12$ as shown in the Figure 4.3.

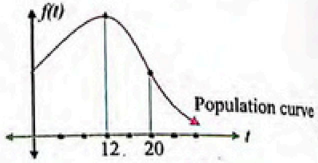

Figure 4.3

If the graph of a function $f(t)$, such as this population curve, is rising throughout the interval $0 < t < 12$, then we say that $f(t)$ is strictly increasing on that interval. Similarly, the graph of the function in Figure 4.3 is strictly decreasing on the interval $12 < t < 20$. These terms are defined more formally in the Figure 4.4.

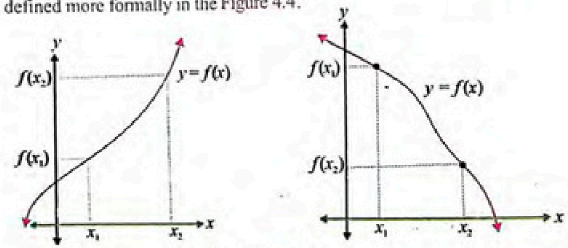

Figure 4.4

- The function $f(x)$ is **strictly increasing** on an interval (a, b), if $f(x_1) < f(x_2)$, whenever $x_1 < x_2$ for x_1 and x_2 on (a, b).
- The function $f(x)$ is **strictly decreasing** on an interval (a, b), if $f(x_1) > f(x_2)$, whenever $x_1 < x_2$ for x_1 and x_2 on (a, b).

Example 22 Find the intervals at which the function $f(x) = x^2$ is increasing or decreasing.

Solution The function $f(x) = x^2$ is a parabola passing through the origin. Take any two points x_1 and x_2 in the interval (a, b) for which: $f(x_2) - f(x_1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$

If $x_1, x_2 \in (0, \infty)$ with condition $x_2 > x_1$, then the function $f(x)$ is increasing in the interval $(0, \infty)$:

$f(x_2) - f(x_1) > 0$

$f(x_2) > f(x_1)$, *both* $(x_2 - x_1)$ *and* $(x_2 + x_1)$ *are +ve, when* $x_2 > x_1$

If $x_1, x_2 \in (-\infty, 0)$ with condition $x_2 > x_1$, then the function $f(x)$ is decreasing in the interval $(-\infty, 0)$:

$f(x_2) - f(x_1) < 0$

$f(x_2) < f(x_1)$, $(x_2 - x_1)$ *is +ve while* $(x_2 + x_1)$ *is −ve, when* $x_2 > x_1$

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4.4.2 Prove that if $f(x)$ is a differentiable function on the open interval (a, b) then

- $f(x)$ is increasing on (a, b) if $f'(x) > 0$, $\forall x \in (a, b)$
- $f(x)$ is decreasing on (a, b) if $f'(x) < 0$, $\forall x \in (a, b)$

Proof: Let $x_1, x_2 \in (a, b)$ such that $x_2 > x_1$ then there exist a point c between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$(x_2 - x_1)f'(c) = f(x_2) - f(x_1)$

For $f'(c) > 0$ and so, $x_2 - x_1$

Therefore, $f(x_2) - f(x_1) > 0$ if $x_2 > x_1$

Or $f(x_2) > f(x_1)$ if $x_2 > x_1$

Thus, f is an increasing function.

Remember

If a function f is continuous on [a, b] and differentiable on (a, b) then there exists a point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Similarly, the proof of part (ii) can be done which is left as an exercise for the reader.

Example 23 Determine the values of x at which the function $f(x) = x^2 + 2x - 3$ is increasing or decreasing. Also find the point at which the given function is neither increasing nor decreasing.

Solution For graphical view, the given function through completing square

$$f(x) = x^2 + 2x - 3 = x^2 + 2x + 1 - 1 - 3 = (x+1)^2 - 4$$

is compared with the general equation of parabola $f(x) = a(x-h)^2 + k$ to obtain a parabola with vertex $(-1, -4)$ that opens upward ($a = 1$ is positive). The graph of a parabola through the points $(-4, 5)$ and $(2, 5)$ is shown in the Figure 4.5.

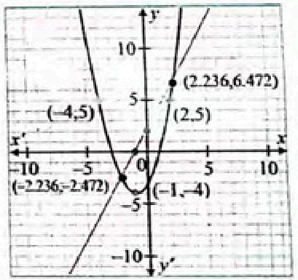

Figure 4.5

The derivative of a given function with respect to x is the slope of the parabola: $f'(x) = 2x + 2$

If the slope of parabola is $f'(x) > 0$ (positive), then it gives

$f'(x) > 0$

$2x + 2 > 0 \quad \Rightarrow \quad 2x > -2 \quad \Rightarrow \quad x > -1$

This shows that the given function $f(x)$ is increasing in the interval $(-1, \infty)$.

If the slope of parabola is $f'(x) < 0$ (negative), then it gives

$f'(x) < 0$

$2x + 2 < 0 \quad \Rightarrow \quad 2x < -2 \quad \Rightarrow \quad x < -1$

This shows that the given function f(x) is decreasing in the interval $(-\infty, -1)$.

If the slope of parabola is $f'(x) = 0$ (zero), then it gives

$f'(x) = 0 \Rightarrow 2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$

This shows that the given function $f(x)$ is neither increasing nor decreasing at a vertex $(-1, -4)$.

4.4.3 Examination of a given function for extreme values

Typically the extrema of a continuous function occur either at endpoints of the interval or at points where the graph has a "peak" or a "valley" (points where the graph is higher or lower than all nearby points). For example, the function $f(x)$ in Figure 4.6 has "peaks" at B and D and "valleys" at C and E. Peaks and valleys are what we call the **relative extrema.**

The exact location of a relative maximum or minimum rather than a graphic's approximation can normally be found by using derivatives. The concept developed is as under:

Let $f(x)$ be a function as a roller coaster track with a roller coaster car moving from left to right along the graph in the Figure 4.6. As the car moves up towards a peak, its floor tilts upward. At the

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instant the car reaches the peak, its floor is level, but then it begins to tilt downward as the car rolls down toward a valley. At any point along the graph, the floor of the car (a straight-line segment in the figure) represents the tangent line to the graph at that point. Using this analogy, we see that as the car passes through the peaks and valleys at A, B, C, the tangent line is horizontal and has slope 0. At peak D and valley E, however, a real roller coaster car would have trouble. It would fly off the track at peak D and be unable to make the 90^0 change of direction at valley E. There is no tangent line at D or E, because of the sharp corners.

Thus, the points where a peak or a valley occurs have this property: the tangent line is horizontal and has slope 0 there or no tangent line is defined there. The slope of the tangent line to the graph of the function $f(x)$ at a point $P(x, f(x))$ is the value of the derivative $f'(x)$.

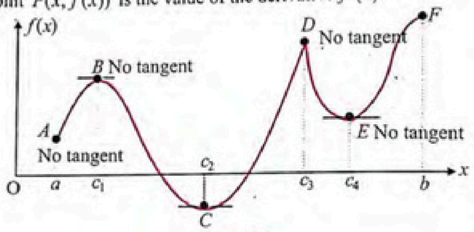

Figure 4.6

A. **Relative Maximum and Relative Minimum:** The function $f(x)$ is said to have a **relative maximum** at a number c if $f(c) \geq f(x)$ for all x in an open interval containing c. Also, $f(x)$ is said to have a **relative minimum** at a number d if $f(d) \leq f(x)$ for all x in an open interval containing d. In general, the relative maxima and relative minima are called **relative extrema.**

B. **Critical Values and Critical Point:** Suppose $f(x)$ is defined at a number c and either $f'(c)=0$ or $f'(c)$ does not exist. Then the number c is called a **critical value** of $f(x)$ and the point $P(c, f(c))$ on the graph of $f(x)$ is called a **critical point.**

Note that if $f(c)$ is not defined, then c cannot be a critical value. If there is a relative maximum at c, then the functional value $f(c)$ at that point is the maximum value. Similarly, if there is a relative minimum at c, then the functional value $f(c)$ at that point is the minimum value.

Example 24 Find the critical values for the following functions:

(a). $f(x)=4x^3-5x^2-8x+20$ *(b).* $f(x)=\dfrac{x^2}{x-2}$ *(c).* $f(x)=12x^{\frac{1}{2}}-2x^{\frac{3}{2}}$

(d). $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$ *(e).* $f(x)=6x^{\frac{2}{3}}-4x$

Solution

a. The first derivative of the given function is: $f'(x)=12x^2-10x-8$

$f'(x)=12x^2-10x-8$ is defined for all values of x. Set $f'(x)=0$ to obtain the critical values:

$f'(x)=12x^2-10x-8=0 \Rightarrow 2(3x-4)(2x+1)=0 \Rightarrow x=\frac{4}{3}, \frac{-1}{2}$

b. The first derivative of the given function is: $f'(x)=\dfrac{x(x-4)}{(x-2)^2}$

The derivative is not defined at $x=2$, also the original function $f(x)$ is not defined at $x=2$. So $x=2$ is not a critical value. Set $f'(x)=0$ to obtain the other critical values:

$f'(x)=\dfrac{x(x-4)}{(x-2)^2}=0 \Rightarrow x(x-4)=0 \Rightarrow x=0,4$

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c. The first derivative of the given function is: $f'(x)=6x^{\frac{-1}{2}}-3x^{\frac{1}{2}}$

The derivative is not defined at $x=0$, but the original function $f(x)$ at $x=0$ is $f(0)=12(0)^{\frac{1}{2}}-2(0)^{\frac{3}{2}}=0$ defined. So $x=0$ is a critical value. For other critical values, set $f'(x)=0$ to obtain: $f'(x)=6x^{\frac{-1}{2}}-3x^{\frac{1}{2}}=0 \Rightarrow 3x^{\frac{-1}{2}}(2-x)=0 \Rightarrow 2-x=0 \Rightarrow x=2$

Thus, the critical values are $x=0, 2$.

d. The derivative of a given function is: $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

$f'(x)=3\left(\frac{4}{3}\right)x^{\frac{4}{3}-1}-12\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}=4x^{\frac{1}{3}}-4x^{\frac{-2}{3}}=4x^{\frac{1}{3}}-\dfrac{4}{x^{\frac{2}{3}}}=\dfrac{4x-4}{x^{\frac{2}{3}}}$

The derivative fails to exist when $x=0$, but the original function $f(x)$ is defined when $x=0$. So $x=0$ is a critical value of $f(x)$.

If $x \neq 0$, then $f'(x)$ is going to be 0 only, when the numerator $4x-4=0$ is zero for $x=1$. So $x=1$ is also the critical value of $f(x)$. Thus, the critical values of $f(x)$ are 0 and 1.

e. The derivative of a given function is: $f(x)=6x^{\frac{2}{3}}-4x$

$$f'(x)=6\left(\frac{2}{3}\right)x^{\frac{2}{3}-1}-4=4x^{\frac{1}{3}}-4=\frac{-4-4x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

The derivative fails to exist when $x=0$, but the original function $f(x)$ is defined when $x=0$. So $x=0$ is a critical value of $f(x)$. If $x \neq 0$, then $f'(x)$ is going to be 0 only when the numerator $4-4x^{\frac{1}{3}}=0$ is zero for $x=1$. So $x=1$ is the critical value of $f(x)$. Thus, the critical values of $f(x)$ are 0 and 1.

Theorem 4.1: If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.

Example 25 The function $f(x)$ is defined by $f(x)=x^3-3x^2-9x+1$. Determine the intervals at which the function $f(x)$ is strictly increasing or decreasing.

Solution First, we need to find out the derivative of the given function, which is: $f'(x)=3x^2-6x-9$

For critical values, set $f'(x)=0$ to obtain: $3x^2-6x-9=0 \Rightarrow 3(x+1)(x-3)=0 \Rightarrow x=-1,3$

These critical values divide the $x-$ *axis* into three parts, as shown in the Figure 4.7. Next, we select a typical number from each of these intervals. For example, we select −2, 0 and 4, evaluate the derivative at these values and mark each interval as increasing or decreasing, according to whether the derivative is positive or negative respectively. This is shown in Figure 4.7.

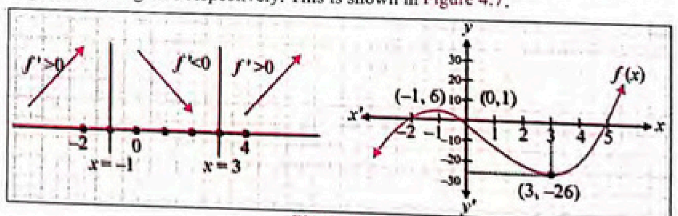

Figure 4.7

Thus, the function $f(x)$ increases in the intervals for $x<-1$ and $x>3$, but decreases in the interval $-1<x<3$.

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Example 26 Draw the function $f(x)=x^3-3x^2-9x+1$ and its derivative $f'(x)=3x^2-6x-9$. Use these graphs to tell about the following questions:

(a). When $f'(x)$ is positive, what does that mean in terms of the graph of $f(x)$?

(b). When the graph of $f(x)$ is decreasing, what does that mean in terms of the graph of $f'(x)$?

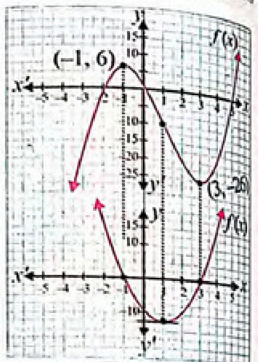

Figure 4.8

Solution The graphs of $f(x)=x^3-3x^2-9x+1$ and $f'(x)=3x^2-6x-9$ are shown in Figure 4.8.

These graphs develop the idea that the critical values of $f(x)$ are always intercepts for the graph of $f'(x)=3x^2-6x-9$:

- If $f'(x)$ is positive, then $f(x)$ is increasing.
- If $f'(x)$ is negative, then $f(x)$ is decreasing.

4.4.4 State the second derivatives rule to find the extreme values of a function at a point

The first derivative of a function can be used to determine whether the function is increasing or decreasing on a given interval. We shall use this information to develop a procedure called the first **derivative test** for classifying a given point as a relative maximum, a relative minimum, or neither.

The steps involved in first–derivative test for relative extrema are the following:

1. Find all critical values of $f(x)$. That is, find all numbers c such that $f(c)$ is defined and either $f'(c)=0$ or $f'(c)$ does not exist.
2. The point $(c, f(c))$ is a **relative maximum** if $f'(x)>0$ (rising) for all x in an open interval (a, c) to the left of c, and $f'(x)<0$ (falling) for all x in an open interval (c, b) to the right of c.
3. The point $(c, f(c))$ is a **relative minimum** if $f'(x)<0$ (falling) for all x in an en interval (a, c) to the left of c, and $f'(x)>0$ (rising) for all x in an open interval (c, b) to the right of c.
4. The point $(c, f(c))$ is not an **extremum** if the derivative $f'(x)$ has the same sign in open intervals (a, c) and (c, b) on both sides of c.

In light of first-derivative test, the function $f(x)=x^3-3x^2-9x+1$ (example 24) has the critical values −1 and 3. The function $f(x)$ is increasing when $x<-1$ and $x>3$ and decreasing when $-1<x<3$. The first derivative test tells us that there is a relative maximum of 6 at $x=-1$ and a relative minimum of −26 at $x=3$.

Example 27 Examine the function $f(x)=2x^3+3x^2-12x-5$ for the relative extrema using first-derivative test.

Solution The first derivative of $f(x)=2x^3+3x^2-12x-5$ is: -

$$f'(x)=6x^2+6x-12=6(x+2)(x-1)$$

Set $f'(x)=0$ to obtain the critical values:

$$f'(x)=6x^2+6x-12=0=6(x+2)(x-1)=0 \Rightarrow x=-2, 1$$

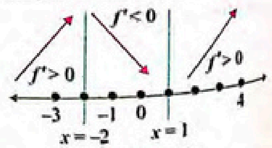

Figure 4.9

To test the critical values −2, 1, we can use the test values −3, 0 and 2. Many other choices of the test values are also possible, but we try to select numbers that will make the computations easy. This is shown in the Figure 4.9.

The test values −3 and 0 are used for the critical value $x=-2$ to obtain:

$$f'(-3)=6(-3+2)(-3-1)=24>0 \text{ (positive)}$$
$$f'(0)=6(0+2)(0-1)=-12<0 \text{ (negative)}$$

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The value of the derivative is positive (rising) to the left of −2 and negative (falling) to the right of −2. Thus, $x=-2$ leads a **relative maximum point**

$$f(-2)=2(-2)^3+3(-2)^2-12(-2)-5=-16+12+24-5=15.$$

The test values 0 and 2 are used for the critical value $x=1$ to obtain:

$$f'(0)=6(0+2)(0-1)=-12 \text{ (negative)}$$
$$f'(2)=6(2+2)(2-1)=24 \text{ (positive)}$$

The value of the derivative is negative (falling) to the left of 1 and positive (rising) to the right of 6. Thus, $x=1$ leads a **relative minimum point.** $f(1)=2(1)+3(1)-12(1)-5=-12$

Thus, the arrow pattern in the figure suggests that the graph of $f(x)$ has a relative maximum at $(-2, 15)$ and a relative minimum at $(1, -12)$.

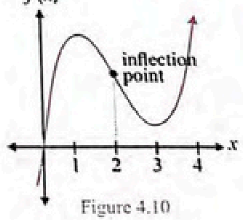
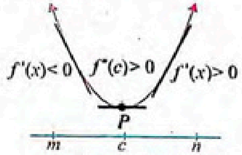


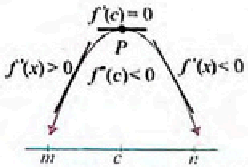
Figure 4.10

The Second Derivative Rule: It is often possible to classify a critical point $P(c, f(c))$ on the graph of $f(x)$ by examining the sign of $f''(c)$. Specifically, if $f'(c)=0$ and $f''(c)>0$, then there is a horizontal tangent line at P and the graph of $f(x)$ is concave up in the neighborhood of P. This means that the graph of $f(x)$ is cupped upward from the horizontal tangent at P and to expect P to be a relative minimum, as shown in Figure 4.11.

Similarly, we expect P to be a relative maximum, if $f'(c)=0$ and $f''(c)<0$, because the graph is cupped down beneath the critical point P, as shown in Figure 4.12.



$f'(c)=0$ and $f''(c)>0$
implies $f(c)$ is a relative minimum
Figure 4.11



$f'(c)=0$ and $f''(c)<0$
implies $f(c)$ is a relative maximum
Figure 4.12

In other words,

- The point $P(c, f(c))$ is said to be a relative maximum, if the slope $f'(c)$ of the tangent line from left to right along a curve through P, is decreasing from **positive to zero to negative** and the second derivative $f''(c)$ is negative.
- The point $P(c, f(c))$ is said to be relative minimum, if the slope $f'(c)$ of the tangent line from left to right along a curve through P, is increasing from **negative to zero to positive** and the second derivative $f''(c)$ is positive. These observations lead to the second–derivative test for relative extreme.

Remember

The Second Derivative Rule for Relative Extrema: Let $f(x)$ be a function such that $f'(c)=0$ and the second derivative exists on an open interval (a, b) containing c.

1. If $f''(c)>0$, then there is a **relative minimum** at $x=c$ and the graph of $f(x)$ is **concave up** in the neighborhood of $P(c, f(c))$.
2. If $f''(c)<0$, then there is a **relative maximum** at $x=c$ and the graph of $f(x)$ is **concave down** in the neighborhood of $P(c, f(c))$.
3. If $f''(c)=0$, then the second derivative test fails and gives no information.

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Example 28 Use the second-derivative test to determine whether each critical value of the function $f(x)=3x^5-5x^3+2$ corresponds to a relative maximum, a relative minimum, or neither.

Solution The first and second derivatives of $f(x)$ are the following:

$f'(x)=15x^4-15x^2=15x^2(x-1)(x+1)$, $f''(x)=60x^3-30x=30x(2x^2-1)$

Put $f'(x)=15x^4-15x^2=15x^2(x-1)(x+1)=0$ to obtain the critical values 0, 1 and -1.

The second derivative $f''(x)$ at a critical point $x=0$ is: $f''(0)=30(0)(0-1)=0$

The critical value $x=0$ declares the failure of second derivative test.

The second derivative $f''(x)$ at a critical point $x=1$ is: $f''(1)=30>0$

The critical value $x=1$ leads to a relative minimum of $f(1)=3(1)-5(1)+2=0$.

The second derivative $f''(x)$ at a critical point $x=-1$ is: $f''(-1)=-30<0$

The critical value $x=-1$ gives a relative maximum of $f(-1)=-3-5(-1)+2=4$

The second derivative test works only for those critical values c that make $f'(c)=0$. This test does not work for those critical values c for which $f'(c)$ does not exist or that make $f''(c)=0$. In both of these cases, use the first derivative test to proceed the process of relative extrema.

4.4.6 Solve real-life problems related to extreme values

Example 29 A truck burns fuel at the rate of $G(x)=\frac{1}{200}\left(\frac{800+x^2}{x}\right)$, $x>0$ gallons per mile when traveling x miles per hour on a straight level road. If fuel costs \$2 per gallon, find the speed that will produce the minimum total cost for a 1000 mile trip. Find the maximum total cost.

Solution The total cost of the trip in dollars is the product of the (number of gallons per mile) (the number of miles) (the cost per gallon) that develops the rule: $C(x)=\frac{1}{200}\left(\frac{800+x^2}{x}\right)(1000)(2)=\frac{8000+10x^2}{x}$

The independent variable x represents speed, only positive values of x make sense here. Thus, the domain of $C(x)$ is the open interval $(0,\infty)$ and there are no endpoints to check.

The first and second derivatives of $C(x)$ are the following: $C'(x)=\frac{10x^2-8000}{x^2}$, $C''(x)=\frac{16000}{x^3}$

Put $C'(x)=0$ to obtain the critical values: $\frac{10x^2-8000}{x^2}=0 \Rightarrow 10x^2-8000=0 \Rightarrow \bar{x}=8000 \Rightarrow x=\pm 28.3mph$

The only critical number in the domain is $x=28.3$. The second derivative test at a critical value $x=28.3$ is: $C''(28.3)=\frac{16000}{(28.3)^3}=0.72>0$

The second derivative test shows that the critical value $x=28.3$ leads to a minimum value. The minimum total cost is found by inserting $x=28.3$ in the cost function: $C(28.3)=\frac{8000+10(28.3)^2}{28.3}=565.69dollars$

Example 30 The supporting cable of a pipeline suspension system forms a parabolic arc between the supports, which is described by the equation $y=0.03125x^2-1.25x$. The distances are measured in meters. The origin of the axis system is at the point where the cable attaches to the left support tower. Where the point is on the and how far is it below the attachment point?

Solution For the low point of the cable, we need to find the first and second derivatives of the given function:

$y=0.03125x^2-1.25x$, $y'=0.0625x-1.25$, $y''=0.0625$ Set $y'=0$ to obtain the critical value: $0.0625x-1.25=0 \Rightarrow x=20$

Since the second derivative is positive for all values of x, the critical value $x=20$ will produce the minimum value on the curve.

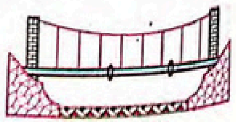

Figure 4.13

NOT FOR SALE

The low point on the cable occurs 20.0 m to the right of the left tow... Put the critical value for x in the original function to obtain the distance from the low point of the cable which is below the attachment point: $y=0.03125(20)^2-1.25(20)=-12.5\ m$

Therefore, the low point of the cable is 20.0 m to the right ... m below its point of attachment to the support.

4.4.7 MAPLE command maximize (minimize) to compute maximum (minimum) value of a function

The procedure to the use of MAPLE command maximize (minimize) to complete maximum (minimum) value of a function is illustrated in the following example.

Example 31 Use MAPLE commands to compute

(a). $f(x)=\cos x$.

(b). $f(x)=x^4-2x^2+3$ in the interval $[-1,2]$.

Solution

a. Command:

```
> maximize(cos(x));
                                1
> minimize(cos(x));
                                -1
```

Context Menu:

```
> cos(x)
> Optimization[Maximize]( cos(x) )
                [1., [x = 5.58237824894110 10^-17]]
```

This result is obtained through right click on the last end of the expression by selecting "Optimization < maximize local"on the context menu.

```
> cos(x)
> Optimization[Minimize]( cos(x) )
                [-1., [x = 3.14159265358977]]
```

b. Command:

```
> maximize(x^4 - 2x^2 + 3, x=-1..2);
                                11
> minimize(x^4 - 2x^2 + 3, x=-1..2);
                                2
```

Exercise 4.3

1. Find the critical values of the given functions in the following problems and show where the function is increasing and where it is decreasing.
   a. $f(x)=x^3+3x^2+1$
   b. $f(x)=x^3+35x^2-125x-9.375$
2. Find the critical values of the following functions:
   a. $f(x)=2x^3-3x^2-72x+15$
   b. $f(x)=\frac{1}{3}x^3-x^2-15x+6$
   c. $f(x)=6x^{\frac{2}{3}}-4x$
   d. $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

NOT FOR SALE

3. Determine whether the given function has a relative maximum, a relative minimum, or neither at the given critical values for the following problems:

a. $f(x)=(x^3-3x+1)^7=0$ at $x=1; x=-1$ b. $f(x)=(x^4-4x+2)^5$ at $x=1$

c. $f(x)=(x^2-4)^4(x^2-1)^3$ at $x=1; x=2$ d. $f(x)=\sqrt[3]{x^3-48}$ at $x=4$

4. Find all critical points of the functions in the following problems, and determine where the graph of the function is rising, falling, concave up, or concave down. Sketch the graph.

a. $f(x)=2(x+20)^2-8(x+20)+7$ b. $f(x)=\frac{1}{3}x^3-9x+2$

5. Find all relative extrema of the following functions:

a. $f(x)=x^3-3x^2+1$ b. $f(x)=x^3+6x^2+9x+2$

6. a. Suppose $f(x)$ is a differential function with derivative $f'(x)=(x-1)^2(x-2)(x-4)(x+5)^4$ Find all critical values of $f(x)$ and determine whether each corresponds to a relative maximum, a relative minimum, or neither.

b. Suppose $f(x)$ is a differential function with derivative $f'(x)=\frac{(2x-1)(x+3)}{(x-1)^2}$

Find all critical values of $f(x)$ and determine whether each corresponds to a relative maximum, a relative minimum, or neither.

7. A company has found through experience that increasing its advertising also increases its sales up to a point. The company believes that the mathematical model connecting profit in hundreds of dollars P(x) and expenditures on advertising in thousands of dollars x is: $P(x)=80+108x-x^3,\ 0\le x\le 10$

a. Find the expenditure on advertising that leads to maximum profit.

b. Find the maximum profit.

8. The total profit P(x) (in thousands of dollars) from the sale of x hundred thousands of automobile tires is approximated by $P(x)=-x^3+9x^2+120x-400,\ 3\le x\le 15$

Find the number of hundred thousands of tires that must be sold to maximize profit. Find the maximum profit.

9. The percent of concentration of a drug in the bloodstream x hours after the drug is administered is given by: $K(x)=\frac{4x}{3x^2+27}$

a. On what time intervals is the concentration of the drug increasing?

b. On what intervals is it decreasing?

c. Find the time at which the concentration is a maximum.

d. Find the maximum concentration.

10. A diesel generator burns fuel at the rate of $G(x)=\frac{1}{48}\left(\frac{300}{x}+2x\right)$ gallons per hour when producing x thousand kilowatt hours of electricity. Suppose that fuel costs \$2.25 a gallon and find the value of x that leads to minimum total cost if the generator is operated for 32 hours. Find the minimum cost.

Review Exercise 4

1. **Choose the correct option.**

i. If $f(t)=3t^2+4t-5$ then $f'(t)$ is

(a). $3t^2-4t+5$ (b). $6t+4$ (c). 12 (d). $6t-5$

ii. If $f(t)=3t^2+4t-5$ *then* $f''(t)$ is

(a). $3t^2-4t+5$ (b). $6t-4$ (c). 6 (d) $6t+5$

iii. If $y=e^{mx}$ then $\frac{d^ny}{dx^n}=$

(a). $me^{mx}\log(e^x)$ (b). m^ne^{mx} (c). me^{nx} (d) me^ne^{mx}

iv. The 5[th] derivative of $f(x)=e^x$ is

(a). e^x (b). e^{x+5} (c). $5e^x$ (d). e^{5x}

v. The 4[th] derivative of $\sin x$ is

(a). $\frac{d^4y}{dx^4}=\sin x$ (b). $\frac{d^4y}{dx^4}=\cos x$ (c). $\frac{d^4y}{dx^4}=-\sin x$ (d). $\frac{d^4y}{dx^4}=-\cos x$

vi. If $f(x)$ and its derivatives at $x=x_0$ are $f'(x_0), f''(x_0),\dots,f^n(x_0)$ then the n[th] order polynomial $f(x)$ will be equal to:

(a). $f(x)+f(x_0-x)f'(x)+\dots+\frac{(x_0-x)^n}{n!}f^n(x)$ (b). $f(x_0)+(x-x_0)f'(x)+\dots+\frac{(x-x_0)^n}{n!}f^n(x_0)$

(c). $f(x+h)+(x+h)f'(h)+\dots+\frac{(x-h)}{n!}f^n(h)$ (d). $f(x-h)+(x-h)f'(h)+\dots+\frac{(x-h)}{n!}f'(h)$

vii. To calculate the first five terms of taylor's series for $f(x)=e^{2x}$, the MAPLE command is used as

(a). taylor $(x, e^{2x}=0)$ (b). taylor (e^{2x})

(c). taylor $(e^{2x}, x=0,5)$ (d). taylor $(e^{2x}, x=5)$

viii. The equation of normal at point (x_0, y_0) is:

(a). $x-x_0=\frac{1}{m}(y-y_0)$ (b). $(y-y_0)=\frac{1}{m}(x_0-x)$

(c). $(y-y_0)=m(x-x_0)$ (d). $(y-y_0)=-\frac{1}{m}(x-x_0)$

ix. The angle of intersection of the two curves can be calculated by using the formulas.

(a). $\theta=\sin^{-1}\frac{1+m_1m_2}{1-m_1m_2}$ (b). $\theta=\tan^{-1}\frac{1-m_1m_2}{1+m_1m_2}$

(c). $\theta=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$ (d). $\theta=\sin^{-1}\frac{m_1+m_2}{1+m_1m_2}$

x. If $f(x)$ is differentiable on the open interval (a, b) the $f(x)$ is strictly increasing if:

(a). $f(x)>0$ (b). $f'(x)>0$ (c). $f''(x)>0$ (d). $f''(x)\angle 0$