Unit 3

DIFFERENTIATION

By the end of this unit, the students will be able to:

3.1 Derivative of a Function

i. Distinguish between independent and dependent variables.

ii. Estimate corresponding change in the dependent variable when independent variable is incremented (or decremented).

iii. Explain the concept of a rate of change.

iv. Define derivative of a function as an instantaneous rate of change of a variable with respect to another variable.

v. Define derivative or differential coefficient of a function.

vi. Differentiate $y = x^n$, where $n \in Z$ (the set of integers), from first principles (the derivation of power rule).

vii. Differentiate $y = (ax+b)^n$, when $n = \frac{p}{q}$ and p, q are integers such that $q \neq 0$, from first principles.

3.2 Theorems on differentiation

i. Prove the following theorems for differentiation.
- the derivative of a constant is zero.
- the derivative of any constant multiple of a function is equal to the product of that constant and the derivative of the function.
- the derivative of a sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.
- the derivative of a product of two functions is equal to (the first function) × (derivative of the second function) plus (derivative of the first function × (the second function).
- the derivative of a quotient of two functions is equal to denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

3.3 Application of Theorems on differentation

i. Differentiate:
- constant multiple of x^n,
- sum (or difference) of functions,
- polynomials,
- product of functions,
- quotient of two functions.

3.4 Chain rule

i. Prove that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ when $y = f(u)$ and $u = g(x)$

ii. Show that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

iii. Use chain rule to show that $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$.

iv. Find derivative of implicit function.

3.5 Differentiation of trigonometric and inverse trigonometric functions

i. Differentiate:
- trigonometric functions ($\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$, and $\cot x$) from first principles.
- inverse trigonometric functions (arcsin x, arccos x, arctan x, arccosec x, arcsec x and arccot x) using differentiation formulae.

3.6 Differentiation of Exponential and Logarithmic Functions

i. Find the derivative of e^x and a^x from first principles.

ii. Find the derivative of $\ln x$ and $\log_a x$ from first principles.

iii. Use logarithmic differentiation to find derivative of algebraic expressions involving product, quotient and power.

3.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

i. Differentiate:
- hyperbolic functions ($\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{cosech} x$, $\operatorname{sech} x$ and $\coth x$).
- inverse hyperbolic functions ($\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\operatorname{cosech}^{-1} x$, $\operatorname{sech}^{-1} x$ and $\coth^{-1} x$).
- Use MAPLE command diff to differentiate a function.

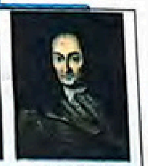
Introduction

The derivative is one of the main tools of calculus. It is instantaneous rate of change of a function at a point in the domain. It is same like the gradient or slope of the tangent line to the graph of the function at that point. Before going to the definition we need to revise the concept of limit introduced in previous unit of this book.

In this unit, we will start by defining derivative, which is the central concept of differential calculus. Then we need to develop a list of rules and formulas for finding the derivative of a variety of expressions, including polynomial functions, rational functions, exponential functions, logarithmic functions, trigonometric functions and hyperbolic functions. ***"The process of finding the derivative is known as differentiation. But the inverse process of differentiation is known as integration"*.** We will discuss in details about integration in Unit-6 of this book.

History

Isaac Newton



G.W Leibnitz

In the sense of a tangent line the concept of derivative is very old in the study of mathematics. This is familiar to Greek geometers. But the modern development of a calculus credited to Isaac Newton and G.W Leibniz. Who provided the independent and unique approaches to the derivatives and differentiation.

3.1 Derivative of a Function

The derivative of a function at some point is known as the rate of change of the function at the point. We can estimate the rate of change by calculating the ratio of change of the function Δy to the change of the independent variable Δx. In the definition of derivative, the ratio is considered in the limit as $\Delta x \to 0$.

3.1.1 Independent and dependent variable

To understand the origin of the concept of variables, some real-life situations in which one numerical quantity depends on, corresponds to, or determines another are considered. For example,

1. The amount of income tax (output/dependent variable) you pay on the amount of your income (input/independent variable). The way in which the income determines the tax is given by the tax law (rule).
2. A person in business wants to know how profit (output/dependent variable) changes with respect to advertising (input/independent variable).
3. A person in medicine wants to know how a patient's reaction to a drug (output/dependent variable) changes with respect to dose (input/independent variable).

In each case, the change in dependent variable requires the definite change in independent variable through a definite rule which is called a **function.**

3.1.2 Estimation of corresponding change in the dependent variable, when independent variable is incremented (or decremented)

A familiar situation related to change in dependent with respect to change in independent is that a driver makes the run of 120, mile trip from Peshawar to Islamabad, in 2 hours. The table shows how far the driver has traveled from Peshawar at various times:

Time	0	0.5	1.0	1.5	2.00
Distance	0	24	54	88	120

If f is the function whose rule is $f(t)$ = distance from Peshawar at time t, then, the table shows that $f(1.0) = 54$, $f(1.5) = 88$ and $f(2.0) = 120$ miles. So the distance traveled from time $t = 1.5$ to $t = 2.0$ is $f(2) - f(1.5) = 120 - 88 = 32$, the change in dependent variable (change in distance) in response of **incremented** independent variable t, while the distance traveled from time $t = 1.5$ to $t = 1.0$ is $f(1.5) - f(1.0) = 88 - 54 = 35$, the change in dependent variable (change in distance) in response of **decremented** independent variable t.

3.1.3 Concept of a rate of change

The idea of average rate of change is something we encounter every day. For example, if a car accelerates from 0 to 96 km/h in 8.0 s, then we say that it accelerates at an average rate of 12 km/h. If a spaceship climbs from 0 to 10,000 m in 2.5 s, then we say that the ship climbs at an average velocity of 4000 m/s. If corn grows a total of 28 inches in 2 weeks, then it grows an average of 2 inches per day.

In these examples, the indicated **average rate of change** is obtained by dividing the change in the **dependent variable** by the change in the **independent variable.**

Let us examine the process of finding the average rate of change of a function $y = f(x)$. If we select any value of x and increase it by an amount Δx, then a new value of the independent variable is $x+\Delta x$. As x changes from x to $x+\Delta x$, y will change to a corresponding amount of $y+\Delta y$. The ordered pairs $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ developed must satisfy the function $y = f(x)$. This is shown in the Figure 3.1

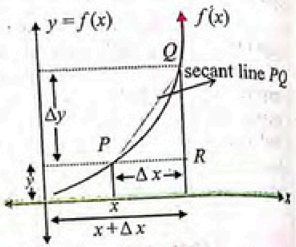


Figure 3.1

If the function value at a point $P(x, y)$ is $y = f(x)$ (i)

then, the function value at a point Q is $y+\Delta y = f(x+\Delta x)$ (ii)

The difference of equations (i) and (ii) gives the change in y;

$$(y+\Delta y)-y = f(x+\Delta x)-f(x) \Rightarrow \Delta y = f(x+\Delta x)-f(x) \quad \text{(iii)}$$

The change in x is $\Delta x = x+\Delta x - x$ (iv)

The average rate of change in y per unit change in x is the **slope of the secant line** PQ, obtained by taking the division of equation (iii) by equation (iv):

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{x+\Delta x - x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y = f(x)$$

The average rate of change y per unit change in x is given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{(v)}$$

> **Remember**
> The slope of the secant is the average rate of change which measures always "the approximate rate of change in phenomena."

Example 1 Determine the average rate of change of y per unit change in x for $y = x^2-6x+5$ as x increases from 1 to 3.

Solution According to the definition of the average rate of change:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \because y = f(x) = x^2-6x+6$$

$$= \frac{\{(x+\Delta x)^2-6(x+\Delta x)+5\}-(x^2-6x+5)}{\Delta x} = \frac{\cancel{x^2}+2x\Delta x+\Delta x^2-\cancel{6x}-6\Delta x+5-\cancel{x^2}+\cancel{6x}-5}{\Delta x}$$

$$= \frac{2x\Delta x+\Delta x^2-6\Delta x}{\Delta x} = \frac{\cancel{\Delta x}(2x+\Delta x-6)}{\cancel{\Delta x}} = 2x-6+\Delta x$$

$$\frac{\Delta y}{\Delta x} = 2x-6+\Delta x$$

As x increases from 1 to 3 then $x = 1$ and $\Delta x = 2$.

$$\frac{\Delta y}{\Delta x} = 2(1)-6+2 = -2$$

Example 2 The height h of a certain brand of corn with respect to t days $(t \geq 1)$ after the seed germinates is $h(t) = \sqrt{t}-1$.

(a). Find the average growth rate $\frac{\Delta h}{\Delta t}$. (b). Find the average growth rate between days 4 and 9.

Solution

a. The average growth rate through definition (v) is:

$$\frac{\Delta h}{\Delta t} = \frac{h(t+\Delta t)-h(t)}{\Delta t} = \frac{\sqrt{t+\Delta t}-1-(\sqrt{t}-1)}{\Delta t} = \frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t} \quad \text{(i)}$$

b. The average growth rate (i) is used for t = 4 and $\Delta t = 5$ to obtain the average growth between days 4 and 9:

$$\frac{\Delta h}{\Delta t} = \frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t} = \frac{\sqrt{4+5}-\sqrt{4}}{5} = \frac{3-2}{5} = \frac{1}{5}$$

Thus, the average rate of change of the height of the corn with respect to time (between days 4 and 9) is $\frac{1}{5}$ (1 unit change in height for each 5 units change in time). The graph is shown in Figure 3.2. The average rate of change is of course helpful in understanding the instantaneous rate of change.

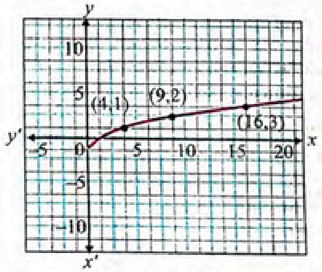


Figure 3.2

3.1.4 Derivative of a function as an instantaneous rate of change of a variable with respect to another variable

In the previous sub-section, we discussed the average rate of change, and learned that the average rate of change is the slope of the secant line joining two points on the curve $y = f(x)$. More commonly, we are asked to determine the exact or instantaneous rate of change at a particular time. For example, for an aeroplane, what is the instanta: :ous rate of change of the distance that occurs at a specific time? ⁻ :his can be dealt by the slope of a tangent line to a curve $y = f(x)$ at a specific point?

To illustrate his idea, let us examine the graph of a function $y = x^2$ at a particular point P (0.5, 0.25) with different secant lines $PQ_1, PQ_2, \ldots$ that developed from the secant line PQ:

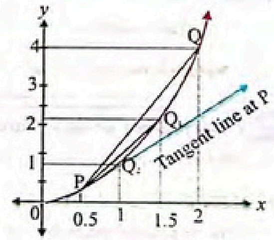


Figure 3.3

P	Q	Δx	Δy	$\frac{\Delta y}{\Delta x}$
P (0.5, 0.25)	Q(2,4)	1.5	3.75	2.5
P (0.5, 0.25)	Q_1(1.5,2.25)	1.0	2.00	2.0
P (0.5, 0.25)	Q_2(1,1)	0.5	0.75	1.5
P (0.5, 0.25)	Q_3(0.8,0.64)	0.3	0.39	1.3

The tabular form contains coordinates for the points P, Q, the change Δx in x, the change Δy in y and $\frac{\Delta y}{\Delta x}$, the slope of the secant lines $PQ, PQ_1, PQ_2, \ldots$ Notice that the slope of the secant line PQ is 2.5 $\left(\frac{\Delta y}{\Delta x} = \frac{3.75}{1.5} = 2.5\right)$. If we take values of Q closer to P (i.e, to $Q_1, Q_2, Q_3, \ldots$), then, Δx gets smaller, and smaller, and tends to zero.

The tabular form clearly shows that, as Q approaches P, Δx approaches 0, and the slope of the secant line approaches the slope of the tangent line at a particular point P (0.5, 0.25) which is 1.

Geometrically, the slope of the tangent line to a curve at a particular point P is the instantaneous (or exact) rate of change at that particular point.

This terminology develops the idea that the slope of the secant line becomes a better approximation for the slope of the tangent line to the curve at a particular point P. From our discussion on limit, it follows that the exact/actual slope of the tangent line to a curve $y = f(x)$ at a particular point P corresponds to the instantaneous rate of change at that point. That is, $\frac{\Delta y}{\Delta x}$ = slope of the secant line PQ

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ = slope of the tangent line at a particular point P.

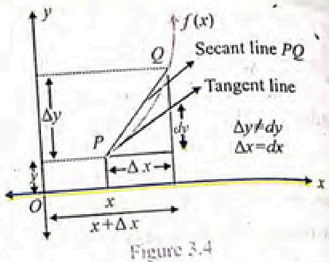


Figure 3.4

The statement $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is read "the limit as delta x approaches zero of delta y divided by delta x." If the limit exists, then the result is the slope of tangent line or the instantaneous rate of change of y with respect to x which we call the derivative of function.

Note

Different mathematicians used different notations to write derivative.

Mathematician	Newton	Leibniz	Euler	Lagrange
Notation for derivative	$\dot{y}$ or $\dot{f}$	$\frac{dy}{dx}$ or $\frac{df}{dx}$	D. $f(x)$ or Dy	$f'(x)$

3.1.5 Derivative or differential coefficient of a function

The instantaneous rate of change of a function $f(x)$ at a point P is the derivative of a function $f(x)$ at that point P, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, **if this limit exists** (i)

This is called **first principle rule** of derivative of a function $f(x)$ with respect to x.

If $y = f(x)$ is a function, then its **derivative or differential coefficient** is denoted by f' or y'. If x is said to be a number in the domain of $y = f(x)$ such that $y' = f'(x)$ is defined, then the function f is said to be **differentiable** at x. The process that produces the function f' from the function f is called **differentiation**.

Example 3 Determine the derivative of a function $f(x) = x^2 - 6x + 5$ by first principle rule at a point $P(4, -3)$.

Solution The derivative of a given function by first principle rule (i) is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = x^2 - 6x + 5$$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 - 6(x+\Delta x) + 5] - (x^2 - 6x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 6x - 6\Delta x + 5 - x^2 + 6x - 5}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - 6\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 6)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 6) = 2x - 6$$

The result $f'(x) = 2x - 6$ represents the slope of the tangent line at any point $P(x, y)$ on the curve $f(x) = x^2 - 6x + 5$. Thus, the slope of the tangent line at a particular point, say $P(4, -3)$ on a given curve is: $f'(x) = 2x - 6$

$f'(4) = 2(4) - 6 = 2$, at $P(4, -3)$.

From this problem, we conclude that

- **the slope of the secant line (the average rate of change) is called the approximate rate of change.**
- **the slope of the tangent line (the instantaneous rate of change) is called the exact rate of change.**

First Principle Rule

If $f(x)$ is any function, then the derivative by first principle rule is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$

The process used for finding the derivative of a function in example 3 is called the differentiation and the result 2 is the differential coefficient of a function $f(x) = x^2 - 6x + 5$ at a particular point P(4, 3).

Symbol $f'(x)$ is used to indicate the derivative of $f(x)$ with respect to x. Sometimes other symbols are used to indicate the derivative. Each of the symbols in the following box indicates the derivative of the dependent variable y with respect to the independent variable x:

"The tangent line to the graph of a function $y = f(x)$ at the point $(x, f(x))$ is the line through this point having slope

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{(ii)}$$

provided this limit exists. If this limit does not exist, then there is no tangent (no derivative) at the point.

Do You Know?

i. $f'(x)$: read "f prime of x" (derivative of $f(x)$ with respect to x)

ii. $\frac{dy}{dx}$: read " dee y, dee x" (the derivative of y with respect to x)

iii. f': read "f prime" (the derivative of the function $f(x)$ with respect to x)

iv. $D_x y$: read " D sub x, y" (the derivative of y with respect to x)

v. y': read "y prime" (the derivative of y with respect to x)

The slope of the tangent line is the instantaneous rate of change, gives "the exact rate of change in the phenomena."

Example 4 The function is $f(x) = x^2$.

(a). Find the derivative of a function at a point $P(3, 9)$.

(b). Find the tangent line on a given curve $y = x^2$ at a point $P(3, 9)$.

(c). View the slope of the tangent line on a curve $y = x^2$ at a point $P(3, 9)$ graphically.

Solution *a.* By first principle rule, the derivative of a given function is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \therefore y = f(x) = x^2$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x \quad \text{(i)}$$

b. Result (i) is used to obtain the slope of the tangent line at a $P(3, 9)$ on a curve $y = x^2$:

$f'(3) = 2(3) = 6$ (ii)

The tangent line on a curve $y = x^2$ at a point $P(3, 9)$ develops a nonhomogeneous line:

$y - y_1 = f'(x)(x - x_1)$, Point from of the line

$y - 9 = 6(x - 3)$, $\quad P(3, 9)$

$6x - y - 9 = 0$

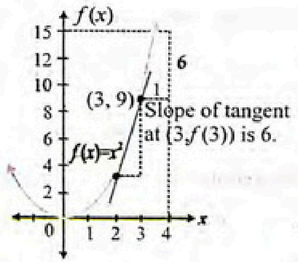


Figure 3.5

c. The graphical view of the slope of the tangent line is represented in Figure 3.5.

3.1.6 Differentiate of $y = x^n$ from first principles rule

If $f(x) = x^n$, n is any integer, then, by first principle rule, the derivative of $f(x) = x^n$ w.r.t. x is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \therefore y = f(x) = x^n$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}, \text{ by binomial expansion}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \ldots - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \ldots\right]$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots\right] = \lim_{\Delta x \to 0}[nx^{n-1}] + \lim_{\Delta x \to 0}\left[\frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots\right]$$

$$= nx^{n-1} + 0 = nx^{n-1} \qquad \text{(i)}$$

3.1.7 Differentiation of $y = (ax + b)^n$ from first principle

Proof: Let $y = (ax + b)^n$ (i). Where n is an integer

$y + \Delta y = \{a(x + \Delta x) + b\}^n$ (ii). By using the binomial theorem

$$y + \Delta y = (ax+b)^n + \binom{n}{1}(ax+b)^{n-1}(a\Delta x) + \binom{n}{2}(ax+b)^{n-2}(a\Delta x)^2 + \binom{n}{3}(ax+b)^{n-3}(a.\Delta x)^3 + \ldots + (a\Delta x)^n \quad \text{(iii)}$$

Subtracting equation (i) from equation (iii)

$$y + \Delta y - y = (ax+b)^n + \binom{n}{1}(ax+b)^{n-1}(a\cdot\Delta x) + \binom{n}{2}(ax+b)^{n-2}(a\Delta x)^2 + \binom{n}{3}(ax+b)^{n-3}(a\cdot\Delta x)^3 + \ldots + (a\cdot\Delta x)^n - (ax+b)^n$$

$$\Delta y = \binom{n}{1}(ax+b)^{n-1}(a\Delta x) + \binom{n}{2}(ax+b)^{n-2}(a\cdot\Delta x)^2 + \binom{n}{3}(ax+b)^{n-3}(a\cdot\Delta x)^3 + \ldots + (a\cdot\Delta x)^n \quad \text{(iv)}$$

Dividing equation (iv) by Δx

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x\left\{a\binom{n}{1}(ax+b)^{n-1} + \binom{n}{2}(ax+b)^{n-2}(a\cdot\Delta x)^2 + \binom{n}{3}(ax+b)^{n-3}(a\cdot\Delta x)^3 + \ldots + a^n(\Delta x)^{n-1}\right\}}{\Delta x} \quad \text{(v)}$$

Apply $\lim_{\Delta x \to 0}$ on both sides of equation (v)

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left\{a\binom{n}{1}(ax+b)^{n-1} + \binom{n}{2}(ax+b)^{n-2}(a\cdot\Delta x)^2 + \binom{n}{3}(ax+b)^{n-1}(a\Delta x)^2 + \ldots + a^n.(\Delta x)^{n-1}\right\}$$

$$= a.\binom{n}{1}(ax+b)^{n-1} + \lim_{\Delta x \to 0}\binom{n}{2}(ax+b)^{n-2}(a\cdot\Delta x)^2 + \lim_{\Delta x \to 0}\binom{n}{3}(ax+b)^{n-3} + \ldots + \lim_{\Delta x \to 0} a^n.(\Delta x)^{n-1}$$

By applying limit all terms tends to zero except first term so,

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = a\cdot\binom{n}{1}(ax+b)^{n-1}$$

Hence,

$$f'(x) = \frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}.a$$

This is generalized power rule of differentiation.

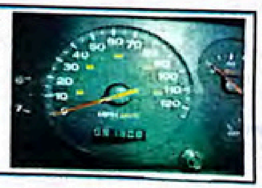
Exercise 3.1

1. Find the average rate of change of the following functions over the indicated intervals:
   a. $y = x^2 + 4$ from $x = 2$ to $x = 3$ b. $y = x^2 + \frac{1}{3}x$ from $x = -3$ to $x = 3$
   c. $s = 2t^3 - 5t + 7$ from $t = 1$ to $t = 3$ d. $h = \sqrt{2t} + 4$ from $t = 8$ to $t = 8.5$
2. Use definition for the rate of change to find out the average rate of change over the specified interval for the following functions:
   a. $s = 2t - 3$ from $t = 2$ to $t = 5$ b. $y = x^2 - 6x + 8$ from $x = 3$ to $x = 3.1$
   c. $A = \pi r^2$ from $r = 2$ to $r = 2.1$ d. $h = \sqrt{t} - 9$ from $t = 9$ to $t = 16$
3. A ball is thrown straight up. Its height after t seconds is given by the formula $h = -16t^2 + 80t$. Use definition for the rate of change to determine the average velocity $\frac{\Delta h}{\Delta t}$ for the specified intervals:
   a. From $t = 2$ to $t = 2.1$. b. From $t = 2$ to $t = 2.01$.
4. The rate of change of price is called inflation. The price p in rupees after t years is $p(t) = 3t^2 + t + 1$. Use definition for the rate of change to determine the average rate of change of inflation from $t = 3$ to $t = 5$ years. What the rate of change means? Explain.
5. A farmer plants x acres of sugar beets. The profit generated is $f(x) = 1800x - 9x^2$. Determine the average rate of change of the profit, when the planted area is in between x = 20 acres and x = 50 acres. What the rate of change means? Explain.
6. Use first principle rule to determine the derivative of the following functions:
   a. $f(x) = 3x$ b. $f(x) = (5x + 6)^{\frac{1}{2}}$ c. $f(x) = x^2 + 1$
   d. $f(x) = 12 - x^2$ e. $f(x) = 16x^2 - 7x$ f. $f($ $\frac{7}{x}$
7. Use function $f(x) = x^2 - 7x + 6$ to do the following:
   a. Find the derivative of a function at point P(5, –4).
   b. Find the tangent line on the curve $y = x^2 - 7x + 6$ at point P(5, –4).
   c. View the slope of the tangent line on the curve at P(6, 0)?
8. Use definition of derivative to determine the slope of the tangent line to the curve at a given point and then find out the tangent line equation on that curve at the same point, for the following curves:
   a. $f(x) = -x^2 + 7x, x = 3$ b. $f(x) = 6x^2 - 11x - 10, x = 1$
   c. $f(x) = 3x^2 - 6x - 10, x = 0$ d. $f(x) = 2x^2 + 3x - 4, x = 1$

Do You Know?

There is always an odometer and a speedometer in an automobile. These two things work in tandem and allow the driver to determine the speed of his/her vehicle and the distance he/she has traveled.

Electronic versions of these two gauges simply use derivatives to transform the data sent to the electronic motherboard from the tyres to miles per hour (MPH) and distance (KM).



3.2 Theorems on Differentiation

In previous section, the derivative of a function $f(x)$ is defined:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{(i)}$$

We learned that the derivative is found by applying the first principle rule. Now, after doing the exercise for the previous section, you may be wondering whether there is a shorter way of finding the derivative. In this and the next several sections, the discussion on the theorem that provides easier ways of finding derivatives.

3.2.1 Proof of differentiation theorem

Theorem –1: The derivative of a constant is zero

Proof: If $f(x)=c$, where c is any constant, then, by first principle rule, the derivative of a constant function is: $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{c-c}{\Delta x}=0,$

This calculation develops the rule that the derivative of a constant function is zero.

In general: If $f(x)=c$, where c is any constant, then: $f'(x)=0$

Example 5 Differentiate the following constant functions:

(a). $f(x)=13$ **(b).** $f(x)=3$ **(c).** $f(x)=7\pi$ **(d).** $f(x)=\sqrt{149}$

Solution The graphs of the functions are horizontal lines parallel to *x-axis*, since all function are constant. The derivative in each case is therefore going to be zero.

Theorem-2: The derivative of any constant multiple of a function is equal to the product of that const. and the derivative of the function.

Proof: If $f(x)=c.g(x)$, where c is any constant, then by the first principle rule, the derivative ... a constant multiple function is: $f(x)=y=c.g(x)$ (i)

$$y+\Delta y=c.g(x+\Delta x) \quad \text{(ii)}$$

Subtracting equation (i) from equation (ii)

$$y+\Delta y-y=c.g(x+\Delta x)-c.g(x)$$
$$\Delta y=c.g(x+\Delta x)-c.g(x) \quad \text{(iii)}$$

Dividing equation (iii) by Δy

$$\frac{\Delta y}{\Delta x}=\frac{c.g(x+\Delta x)-c.g(x)}{\Delta x}$$
$$\frac{\Delta y}{\Delta x}=c\left\{\frac{g(x+\Delta x)-g(x)}{\Delta x}\right\} \quad \text{(iv)}$$

Applying $\lim_{\Delta x\to 0}$ on both sides of equation (iv)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=c.\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Hence, $f'(x)=c.g'(x)$

This calculation develops the rule that the derivative of a constant multiple function is the product of the constant function and the derivati · of a function $f(x)$.

In general: If $g(x)=x^n$ and $f(x)=cg(x)$, c is any constant, then: $f'(x)=cg'(x)=cnx^{n-1}$

Example 6 Differentiate the following functions: **(a).** $f(x)=4x$ **(b).** ...

Solution

a. If $f(x)=4x^3$, then, the derivative of a giv... function is: $f'(x)=4(3)x^{3-1}=12..$

b. If $f(x)=0.555x^6$, then, the derivative of a given function is: $f'(x)=0.555(6)x^{6-1}=3.3..$

Theorem-3: The derivative of a sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.

Proof: To determine the derivative of a polynomial, such as the derivative of the sum or difference of two or more functions, we need to develop a rule that could be used in the determination of a derivative like $f(x)=3x^5+2x^2+3$. In this situation, if $h(x)=f(x)+g(x)$, then, our task is to determine $h'(x)$ by first principle rule of differentiation:

$$h(x)=y=f(x)+g(x) \quad \text{(i)}$$
$$y+\Delta y=f(x+\Delta x)+g(x+\Delta x) \quad \text{(ii)}$$

By subtraction equation (i) from equation (ii)

$$y+\Delta y-y=f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)$$
$$\Delta y=f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x) \quad \text{(iii)}$$

Dividing equation (iii) by Δx then we have

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}$$
$$=\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x} \quad \text{(iv)}$$

Now, apply $\lim_{\Delta x\to 0}$ on both sides of equation (iv)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Hence, $h'(x)=f'(x)+g'(x)$

We can say that the derivative of a sum of two functions is the sum of the derivatives of two functions. The difference of two functions $f(x)-g(x)$ can be written as the sum of $f(x)-g(x)=f(x)+[-g(x)]$. Thus, the derivative of the difference of two functions is the difference of their derivatives.

In general: If $u=f(x)$ and $v=g(x)$, then, the sum rule can be restated using the notations:

$$\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$$

This rule generalizes to the sum and difference of any given number of functions.

Example 7 If $f(x)=3x^2+4x$ and $g(x)=7x-2$ the differentiate $f(x)+g(x)$ and $f(x)-g(x)$

Solution Since, $f(x)=3x^2+4x$ and $g(x)=7x-2$

$$f(x)+g(x)=(3x^2+4x)+(7x-2)=3x^2+11x-2$$
$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}(3x^2+11x-2)=\frac{d}{dx}(3x^2)+\frac{d}{dx}(11x)-\frac{d}{dx}(2)=6x+11$$

Now,

$$f(x)-g(x)=(3x^2+4x)-(7x-2)=3x^2-3x+2$$
$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}(3x^2-3x+2)=\frac{d}{dx}(3x^2)-\frac{d}{dx}(3x)+\frac{d}{dx}(2)=6x-3$$

Theorem–4: The derivative of a product of two functions is equal to (the first function)×(derivative of the second function) plus (derivative of the first function)×(the second function).

Proof: If $h(x)=f(x)g(x)$, and $f(x)$ and $g(x)$ are differentiable functions of x, then by first principle rule,

$$h(x)=y=f(x)g(x) \quad \text{(i)}$$
$$y+\Delta y=f(x+\Delta x).g(x+\Delta x) \quad \text{(ii)}$$

Subtracting equation (i) from equation (ii)

$$y+\Delta y-y=f(x+\Delta x).g(x+\Delta x)-f(x).g(x)$$
$$\Delta y=f(x+\Delta x).g(x+\Delta x)-f(x).g(x) \quad \text{(iii)}$$

The addition and subtraction of $f(x+\Delta x).g(x)$ to the right side of equation

$$\Delta y = f(x+\Delta x).g(x+\Delta x) - f(x+\Delta x).g(x) + f(x+\Delta x).g(x) - f(x).g(x)$$

$$= f(x+\Delta x)\{g(x+\Delta x) - g(x)\} + g(x)\{f(x+\Delta x) - f(x)\} \quad \text{(iv)}$$

Now, dividing equation (iv) by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)\{g(x+\Delta x)-g(x)\} + g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x} \quad \text{(v)}$$

Now, apply $\lim\limits_{\Delta x \to 0}$ on both sides of equation (v)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)\{g(x+\Delta x)-g(x)\} + g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} f(x+\Delta x).\lim_{\Delta x\to 0}\frac{g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x} + g(x).\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Hence, $h'(x) = f(x).g'(x) + g(x).f'(x)$

This calculation develops the idea that the derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.

In general: If $y = f(x)\,g(x) = u\,v$ with $u = f(x)$ and $v = g(x)$, then the product rule can be restated using the notations: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Theorem-5: The derivative of a quotient of two functions is equal to denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator all divided by the square of the denominator.

Proof: If $h(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$ and $f(x)$ and $g(x)$ are differentiable functions of x, then, the derivative of $h(x)$ can be found by first principle rule:

$$h(x) = y = \frac{f(x)}{g(x)} \quad \text{(i)}$$

$$y + \Delta y = \frac{f(x+\Delta x)}{g(x+\Delta x)} \quad \text{(ii)}$$

Subtracting equation (i) from equation (ii)

$$y + \Delta y - y = \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}$$

$$\Delta y = \frac{f(x+\Delta x).g(x) - g(x+\Delta x).f(x)}{g(x+\Delta x).g(x)} \quad \text{(iii)}$$

The addition and subtraction of $f(x).g(x)$ to the numerator of equation (iii)

$$\Delta y = \frac{f(x+\Delta x).g(x) - f(x).g(x) - f(x)g(x+\Delta x) + f(x).g(x)}{g(x+\Delta x).g(x)}$$

$$\Delta y = \frac{g(x)\{f(x+\Delta x) - f(x)\} - f(x)\{g(x+\Delta x) - g(x)\}}{g(x+\Delta x).g(x)} \quad \text{(iv)}$$

Now, divide equation (iv) by Δx. Then

$$\frac{\Delta y}{\Delta x} = \frac{g(x)\left\{\frac{f(x+\Delta x)-f(x)}{\Delta x}\right\} - f(x)\left\{\frac{g(x+\Delta x)-g(x)}{\Delta x}\right\}}{g(x+\Delta x).g(x)} \quad \text{(v)}$$

Apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation (v)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{g(x)\left\{\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\right\} - f(x)\left\{\lim\limits_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\right\}}{\lim\limits_{\Delta x\to 0} g(x+\Delta x).g(x)}$$

Hence, $h'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$

In general: If $y = \frac{f(x)}{g(x)} = \frac{u}{v}$, with $u = f(x)$ and $v = g(x)$, then the quotient rule can be restated using the notations:

$$\frac{dy}{dx} = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

Example 8 Differentiate the following functions:

(a). $y = 4x^3 - 2x^2 + 5x$ (b). $y = (x^2 - 2x)(x^3 - 3)$ (c). $y = \frac{x^2+13x+9}{x^2+11x+3}$

Solution

a. If $y = u + v + w$ then $\frac{dy}{dx} = \frac{d}{dx}(u+v+w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$ so, the derivative of a given function is:

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^3) - (2x^2) + (5x)] = \frac{d}{dx}(4x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(5x) = 12x^2 - 4x + 5$$

b. If $y = u\cdot v$ then $\frac{dy}{du} = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}$ so, the derivative of the given function is:

$$\frac{dy}{dx} = (x^2 - 2x)\cdot\frac{d}{dx}(x^3 - 3) + (x^3 - 3)\cdot\frac{d}{dx}(x^2 - 2x)$$

$$= (x^2 - 2x)\cdot\left\{\frac{d}{dx}(x^3) - \frac{d}{dx}(3)\right\} + (x^3 - 3)\cdot\left\{\frac{d}{dx}(x^2) - 2\frac{d}{dx}(x)\right\}$$

$$= (x^2 - 2x)\cdot(3x^2 - 0) + (x^3 - 3)(2x - 2) = 3x^2(x^2 - 2x) + x^3(2x - 2) - 3(2x - 2)$$

$$= 3x^4 - 6x^3 + 2x^4 - 2x^3 - 6x + 6 = 5x^4 - 8x^3 - 6x + 6$$

c. If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}$ so, the derivative of a given function is:

$$\frac{dy}{dx} = \frac{(x^2+11x+3)\frac{d}{dx}(x^2+13x+9) - (x^2+13x+9)\frac{d}{dx}(x^2+11x+3)}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+11x+3)(2x+13) - (x^2+13x+9)(2x+11)}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2x^3+22x^2+6x+13x^2+143x+39) - (2x^3+26x^2+18x+11x^2+143x+99)}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{2x^3+35x^2+149x+39-2x^3-39x^2-161x-99}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{-4x^2-12x-60}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{-4(x^2+3x+15)}{(x^2+11x+3)^2}$$

3.3 Application of Theorems on Differentiation

Calculus is used in both applied mathematics and pure mathematics, the biological and medicine, physical sciences, computer science, engineering, statics, economics, artificial intelligence and many more areas of other fields.

Few simple examples of applications of differentiation are given in this section.

3.3.1 Differentiation of

- Constant multiple of x^n
- Product of functions
- Sum (or difference) of functions
- Quotient of two functions
- Polynomials

Example 9 The cost in (million) dollars to produce x units of wheat is given by $C(x)=5000+20x+10\sqrt{x}$. Find the marginal cost, when

(a). $x=9$ units *(b).* $x=16$ units

(c). $x=25$ units *(d).* As more wheat is produced; what happens to the marginal cost?

Solution

a. If $C(x)=5000+20x+10\sqrt{x}$, then the marginal cost is the derivative of $C(x)$ with respect to x:

$$C'(x)=20+10\left(\frac{1}{2}\right)(x^{-\frac{1}{2}})=20+\frac{5}{\sqrt{x}}$$

The marginal cost at $x=9$ units is obtained by inserting $x=9$ in $C'(x)$:

$$C'(9)=20+\frac{5}{\sqrt{9}}=20+\frac{5}{3}=\frac{65}{3}=\$21.67$$

b. The marginal cost at $x=16$ units is obtained by inserting $x=16$ in $C'(x)$:

$$C'(16)=20+\frac{5}{\sqrt{16}}=20+\frac{5}{4}=\frac{85}{4}=\$21.25$$

c. The marginal cost at $x=25$ units is obtained by inserting $x=25$ in $C'(x)$:

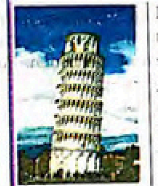
$$C'(25)=20+\frac{5}{\sqrt{25}}=20+\frac{5}{5}=\$21$$

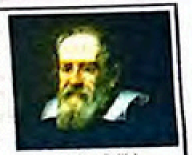
d. It decreases and approaches $20.

Marginal Analysis

In business and economics the rates of change of such variables as cost, revenue and profit are most important. Economists use the word **marginal** to refer to rates of change. For example, the marginal cost refers to the rate of change of cost. Since the derivative of a function gives the rate of change of the function, a marginal cost (or revenue of profit) function is found by taking the derivative of the cost (or revenue of profit) function. The marginal cost at some level of production x is the cost to produce the $(x+1)$st item (i.e., one more item).

Do You Know?





Galileo Galilei (1564-1642)

Does a feather fall more slowly than a rock? An Italian mathematician astronomer and physicist raise this question before 400 years ago. He theorized that the rate of falling objects depends on the air resistance, not on mass. It is believed that he tested his idea by dropping spheres of different masses but the same diameter from the top of the Leaning tower of Pisa in Italy. The result was exactly as he predicted they fell at the same rate.

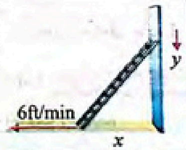
In 1971 during the Apollo 15 lunar landing, David Scott (commander) performed a demonstration on live television show. This is because of the surface of the moon is essentially a vacuum, a hammer and a feather fell at the same rate.

Exercise 3.2

1. Differentiate $f(x)+g(x)$ and $f(x)-g(x)$ if:
   a. $f(x)=3x+7$ and $g(x)=6x^2+2x-3$
   b. $f(x)=17x^2-15$ and $g(x)=\frac{1}{3}x^2+\frac{1}{2}x-5$
   c. $f(x)=x^3-\frac{3}{2}$ and $g(x)=3x^3-4x^2+2$
   d. $f(x)=4x^3-5x$ and $g(x)=\frac{3}{5}x^2-2x$
2. Use the product rule to find out the derivative of the following functions:
   a. $y=(x^2-2)(3x+1)$
   b. $y=(7x^4+2x)(x^2-4)$
   c. $y=(2x-3)(\sqrt{x}-1)$
   d. $y=(-3\sqrt{x}+6)(4\sqrt{x}-2)$
3. Use the quotient rule to find out the derivative of the following functions:
   a. $y=\frac{3x-5}{x-4}$
   b. $y=\frac{-x^2+6x}{4x^3+1}$
   c. $y=\frac{5x+6}{\sqrt{x}}$
   d. $f(p)=\frac{(2p+3)(4p-1)}{(3p+2)}$
4. Find an equation of a tangent line to the graph of the function at the particular point in the following problems:
   a. $f(x)=3x-7$ at $(3,2)$
   b. $f(x)=x^3$ at $x=\frac{-1}{2}$
   c. $f(x)=\frac{1}{x+3}$ at $x=2$
   d. $f(x)=\frac{x}{x-2}$ at $(3,3)$
5. For a thin lens of constant focal length P, the object distance x and the image distance y are related by the formula $\frac{1}{x}+\frac{1}{y}=\frac{1}{P}$
   a. Solve the above equation for y in terms of x and P.
   b. Determine the rate of change of y with respect to x.

Project

At your home. Take a ladder measure its length and put if with the wall as shown in the figure. Pull it away from the wall at the constant rate of 6ft/min. Calculate how fast is the top of the ladder moving down the wall when the bottom of the ladder is 6 feet from the wall.



3.4 Chain Rule

"The chain rule is a rule which we use to differentiate the composite functions". We have learnt about composition of functions in unit-2 that a function is a composite functions of the two similar functions $f(x)$ and $g(x)$ if it is written as $f[g(x)]$. In other words it is a function of a function. For example $\sin(x^2)$ is a composite functions because of we consider $f(x) = \sin(x)$ and $g(x) = x^2$ then $f[g(x)] = \sin(x^2)$. Generally, we write chain rule as;

$$\frac{d}{dx}[f(g(x))] = f'[g(x)].g'(x) \qquad \text{(i)}$$

3.4.1 Prove that $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ when $y = f(u)$ and $u = g(x)$

Proof: For our convenience, we set $\quad y = F(x) = f[g(x)] \qquad$ (i)

Let $u = g(x)$ the equation (i) will be $\quad y = F(x) = f[g(x)] = f(u) \qquad$ (ii)

$$y + \Delta y = F(x + \Delta x) \qquad \text{(iii)}$$

Now, subtracting equation (i) from equation (iii)

$$y + \Delta y - y = F(x + \Delta x) - F(x)$$

$$\Delta y = F(x + \Delta x) - F(x) \qquad \text{(iv)}$$

Equation (iv) can be written as

$$\Delta y = f[g(x + \Delta x)] - f[g(x)] \qquad \text{(v)}$$

Where, $\quad \Delta u = g(x + \Delta x) - g(x)$

$$\Rightarrow g(x + \Delta x) = \Delta u + g(x) \qquad \text{(vi)}$$

Substitute the value of $g(x + \Delta x)$ from equation (vi) to equation (v)

$$\Delta y = f[\Delta u + g(x)] - f[g(x)]$$

$$F(x + \Delta x) - F(x) = f(u + \Delta u) - f(u) \qquad \because u = g(x)$$

Divide equation (iv) by Δx then we have

$$\frac{\Delta y}{\Delta x} = \frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta x} \qquad \text{(vii)}$$

Multiply and divide the right side of equation by Δu so,

$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u}.\frac{\Delta u}{\Delta x}$$

$$= \frac{f(u + \Delta u) - f(u)}{\Delta u}.\frac{g(x + \Delta x) - g(x)}{\Delta x} \qquad \text{viii)}$$

Apply $\lim\limits_{\Delta x \to 0}$ and $\lim\limits_{\Delta u \to 0}$ on equation (viii)

We have $\quad \lim\limits_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim\limits_{\Delta u \to 0}\frac{f(u + \Delta u) - f(u)}{\Delta u}.\lim\limits_{\Delta x \to 0}\frac{g(x + \Delta x) - g(x)}{\Delta x}$

Hence, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ as required.

It can also written as $\frac{d}{dx}[f(g(x))] = f'(u).g'(x) = f'[g(x)].g'(x)$

Example 10 Differentiate the following functions w.r.t. x:

(a). $f(x) = (4x - 3)^3$ $\qquad$ (b). $f(x) = \sqrt{15x^2 + 1}$

Solution

a. If $y = f(x) = (4x-3)^3 = u^3$ with $u = 4x - 3$, then, the first derivative w.r.t. x by chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(u^3)\frac{d}{dx}(4x-3) = (3u^2)(4) = 12u^2 = 12(4x-3)^2, u = (4x-3)$$

b. If $y = f(x) = \sqrt{15x^2+1} = \sqrt{u} = u^{\frac{1}{2}}$ with $u = 15x^2 + 1$, then, the first derivative by chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \therefore u = (15x^2+1)$$

$$= \frac{d}{du}(u^{\frac{1}{2}})\frac{d}{dx}(15x^2+1) = \frac{1}{2}(u^{\frac{1}{2}-1})(30x) = \frac{15x}{\sqrt{u}} = \frac{15x}{\sqrt{15x^2+1}}$$

Example 11 The revenue realized by a small city from the collection of fines from parking tickets is given by $R(x) = \frac{8000x}{x+2}$ where x is the number of work hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work hours per day. How fast is revenue from parking fines decreasing during the epidemic?

Solution We need to find $\frac{dR}{dt}$, the change in revenue with respect to time t. The chain rule is used to obtain $\frac{dR}{dt}$.

$$\frac{dR}{dt} = \frac{dR}{dx}\frac{dx}{dt} \qquad \text{(i)}$$

First find $\frac{dR}{dx}$ as follows.

$$R'(x) = \frac{dR}{dx} = \frac{(x+2)((8000) - 8000x(1)}{(x+2)^2} = \frac{16000}{(x+2)^2} \Rightarrow R'(30) = \frac{16000}{(30+2)^2} = 15.625; \quad at\ x = 30$$

$\frac{dR}{dx}$=15.625 and $\frac{dx}{dt} = -6$ are used in equation (i) to obtain: $\frac{dR}{dt} = \frac{dR}{dx}\frac{dx}{dt} = (15.625)(-6) = -93.75$

This tells us that the revenue is being lost at the rate of approximately \$94 per day.

3.4.2 Show that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

If $y = f(x)$ is any differential function of x, then it admits an inverse function $x = g(y)$.

Suppose y is changed by a small amount Δy. This will cause x to change by an amount Δx. The increment Δx in x corresponds to the increment Δy in y is determined from

$x = g(y)$, $\qquad x = g(y)$ is inverse of $y = f(x)$

That gives: $\quad 1 = \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta y}$

By letting $\Delta x \to 0$ to obtain: $\quad 1 = \lim\limits_{\Delta x \to 0}\frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta y}$

Do You Know?

Leibniz was the first person who mentioned the composite of the square root function and the function on 1676. The common notation of the chain rule is also due Leibniz.

$$1=\frac{dy}{dx}\frac{dx}{dy} \quad \Rightarrow \frac{1}{\frac{dx}{dy}}=\frac{dy}{dx}$$

Thus $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocal to each other.

12 Verify result $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$ for the following problems:

(a). $f(x)=(4x-3)^3$ *(b).* $f(x)=\sqrt{15x^2+1}$

Solution

a. The derivative of $y=(4x-3)^3$ is $\frac{dy}{dx}=12(4x-3)^2$. This result agrees to result :

The derivative of $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{12(4x-3)^2}}=(1)\left(\frac{12(4x-3)^2}{1}\right)=12(4x-3)^2$

b. The derivative of $y=\sqrt{15x^2+1}$ is $\frac{dy}{dx}=\frac{15x}{\sqrt{15x^2+1}}$.

This result agrees to result :

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{\frac{\sqrt{15x^2+1}}{15x}}}=(1)\left(\frac{15x}{\sqrt{15x^2+1}}\right)=\frac{15x}{\sqrt{15x^2+1}}$$

3.4.3 Use of chain rule to show that $\frac{d}{dx}[f(x)]^n=n[f(x)]^{n-1}f'(x)$

Proof: Let $y=[f(x)]^n$ and $u=f(x)$ then $y=u^n$ and $\frac{dy}{du}=nu^{n-1}$ (by power rule)

But $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=nu^{n-1}\cdot\frac{du}{dx}$

Or $\frac{d}{dx}[f(x)]^n=n[f(x)]^{n-1}f'(x) \qquad \because \frac{du}{dx}=f'(x)$

In words, if $f(x)$ is equal to an expression in x raised to a power of n, then $f'(x)$ is equal to the product of n times the expression to the n-1 power times the derivative of the expression with respect to the variable. The statement is known as the **general power rule**.

13 Differentiate the following functions:

(a). $f(x)=(11x^2-7)^8$ *(b).* $f(x)=\sqrt{2x^3+11}$

Solution

a. If $y=f(x)=(11x^2-7)^8$ then, the first derivative of a given function is:

$$\frac{dy}{dx}=\frac{d}{dx}(11x^2-7)^8=8(11x^2-7)^{8-1}\frac{d}{dx}(11x^2-7)=8(11x^2-7)^7(22x)=176x(11x^2-7)^7$$

b. If $y=f(x)=\sqrt{2x^3+11}$, then, the first derivative of a given function is:

$$\frac{dy}{dx}=\frac{d}{dx}(2x^3+11)^{\frac{1}{2}}=\frac{1}{2}(2x^3+11)^{\frac{1}{2}-1}\frac{d}{dx}(2x^3+11)=\frac{1}{2}(2x^3+11)^{-\frac{1}{2}}(6x^2)=\frac{3x^2}{\sqrt{2x^3+11}}$$

Note

If two differential functions $x=f(t)$ and $y=g(t)$ of parameter t. If $t=h(x)$ is an inverse function of $x=f(t)$, then $y=g[h(x)]$ is a function of x.

By chain rule, the differentiation of $y=g[h(x)]$ w.r.t. x is $\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{dy}{dt}\frac{1}{\frac{dx}{dt}}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)}$

Find $\frac{dy}{dx}$, when $x=at^2$ *and* $y=2at$.

for the assumptions $\frac{dx}{dt}=2at, \frac{dy}{dt}=2a$ is used to obtain:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=(2a)\left(\frac{1}{2at}\right)=\frac{1}{t}$$

3.4.4 Derivative of implicit function

Whether y is expressed explicitly or implicitly in terms of x, we can still differentiate to find the derivative $\frac{dy}{dx}$. If y is expressed **explicitly** in terms of x, then $\frac{dy}{dx}$ will also be expressed explicitly in terms of x. If y is expressed **implicitly** in terms of x, then $\frac{dy}{dx}$ will be expressed in terms of x and y.

Fortunately, there is a simple technique based on the chain rule that allows us to find $\frac{dy}{dx}$ without first solving the equation for y explicitly. This technique is known as **implicit differentiation.** It consists differentiation of the both sides of the equation with respect to x and then solving the resultant equation algebraically for $\frac{dy}{dx}$.

14 Differentiate the implicit equation $x^2y+2y^3=3x+2y$.

Solution The implicit equation is $x^2y+2y^3=3x+2y$.

The implicit differentiation of is obtained by differentiating both sides w.r.t. x:

$$\frac{d}{dx}\left(x^2y+2y^3\right)=\frac{d}{dx}(3x+2y)$$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(2y^3)=\frac{d}{dx}(3x)+\frac{d}{dx}(2y)$$

$$2xy+x^2\frac{dy}{dx}+6y^2\frac{dy}{dx}=3+2\frac{dy}{dx}$$

$$(x^2+6y^2-2)\frac{dy}{dx}=3-2xy$$
$$\frac{dy}{dx}=\frac{3-2xy}{x^2+6y^2-2}$$

Example 15 Find the slope of a tangent line to the circle $x^2+y^2=5x+4y$ at a particular point P(5,4).

Solution The slope of a tangent line to the given curve is $\frac{dy}{dx}$ that can be found by taking the derivative of $x^2+y^2=5x+4y$ with respect to x:

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(5x+4y)$$
$$2x+2y\frac{dy}{dx}=5+4\frac{dy}{dx}$$
$$(2y-4)\frac{dy}{dx}=5-2x$$
$$\frac{dy}{dx}=\frac{5-2x}{2y-4}$$

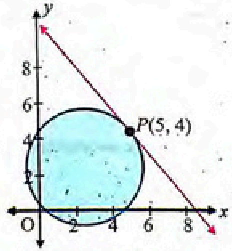


Figure 3.6

At a point P(5, 4), the slope of the tangent line is:

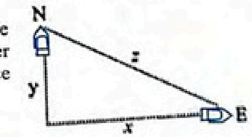
$$\frac{dy}{dx}=\frac{5-2(5)}{2(4)-4}=\frac{-5}{4}$$

Note that the expression is undefined at $y=2$. This makes sense, when you see that the tangent is vertical there.

Exercise 3.3

1. Find the derivative of the following functions w.r.t involved independent variable:
   a. $w=4(x^3-4x+2)^5$ b. $y=\frac{(4x-x^3)^{11}}{5}$
   c. $u=\sqrt[3]{1-3t^2}$ d. $s=\frac{1}{(3t+1)^7}$
2. Determine the derivative $f'(x)$ in each case:
   a. $f(x)=(2x-5)^3(5x-7)$ b. $f(x)=\frac{(x+2)^2}{x-1}$
   c. $f(x)=\left(\frac{2x-5}{x-4}\right)^4$ d. $f(x)=x\sqrt{2x^2+11}$
3. Find $\frac{dy}{dx}$ of the following function in terms of parameter t:
   a. $x=1+t^2,\ y=t^3+2t^2+1$ b. $x=3at^2+2,\ y=6t^4+9$
   c. $x=\frac{a(1-t^2)}{1+t^2},\ y=\frac{2bt}{1+t^2}$ d. $x=\frac{3at}{1+t^3},\ y=\frac{3at^2}{1+t^3}$

4. At a certain factor, the total cost of manufacturing q units during the daily production run is $C(q)=0.2q^2+q+900$ dollars. From experience, it has been determined that approximately $q(t)=t^2+100t$ units are manufactured during the first t hours of a production run. Compute the rate at which the total manufacturing cost is changing with respect to time one hour after production begins.
5. Use implicit differentiation to perform $\frac{dy}{dx}$ for the following functions:
   a. $x^2+y^2=25$ b. $xy(2x+3y)=2$ c. $(x+y)^3+3y=3$ d. $\frac{1}{y}+\frac{1}{x}=1$
6. Arrange the following functions explicitly and implicitly to perform $\frac{dy}{dx}$:
   a. $x^2y^3+y^3=1$ b. $xy+2y=x^2$ c. $x+\frac{1}{y}=5$ d. $xy-x=y+2$
7. Let $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$, where a and b are nonzero constants. Find:
   a. $\frac{du}{dv}$ b. $\frac{dv}{du}$
8. Determine the slope of the tangent line to the curve $3x^2-7y^2+14y=27$ at the point $P(-3,0)$.
9. Suppose two motor boats leave from the same point at the same time. If one boat travels north at 15 miles per hour and the other boat travels east at 20 miles per hour. How fast will the distance between them be changing after 2 hours?



3.5 Differentiation of Trigonometric and inverse Trigonometric Functions

To understand this section we need to know about trigonometric function. For differentiating all trigonometric functions we use the basic rule of differentiation that we have already learnt e.g. We will use product, quotient and chain rules to differentiation functions that are the combination of the trigonometric function.

3.5.1 Differentiation of trigonometric functions (sinx, cosx, tanx, cosecx, secx and cotx) from first principle

i. **Derivative of sin x:** If $y=\sin x$ then the derivative of $y=\sin x$ is $\frac{dy}{dx}=\cos x$.

Proof: By the rule of first principle.

Let $y=\sin(x)$ (i) $\quad\because y=f(x)$

$y+\Delta y=\sin(x+\Delta x)$ (ii)

Subtracting equation (i) from equation (ii).

$y+\Delta y-y=\sin(x+\Delta x)-\sin(x)$

$\Delta y=\sin(x+\Delta x)-\sin(x)$

$=2\cos\left(\frac{x+\Delta x+x}{2}\right).\sin\left(\frac{x+\Delta x-x}{2}\right)$ $\quad\because \sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right).\sin\left(\frac{\alpha-\beta}{2}\right)$

$$\Delta y = 2\cos\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)$$

Now, divide equation by the Δx.

$$\frac{\Delta y}{\Delta x} = \frac{2\cos.\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

Now, apply $\lim\limits_{\Delta x\to 0}$ on both side of equation.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right).\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}\right]$$

$$= \lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \cos(x+0)\cdot 1$$

$$= \cos(x)$$

$\because \lim\limits_{\Delta x\to 0}\dfrac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} = 1$

$\because \dfrac{\Delta x}{2}\to 0$ as $\Delta x\to 0$

Hence, $\dfrac{dy}{dx} = \cos(x)$

ii. **Derivative of cos x:** If $y = \cos x$, then the derivative of $y = \cos x$ is $\dfrac{dy}{dx} = -\sin x$.

Proof: By the rule of first principle.

Let $y = \cos x$ $\quad \because y = f(x)$

$y+\Delta y = \cos(x+\Delta x)$

Subtracting equation from equation .

$$y+\Delta y - y = \cos(x+\Delta x) - \cos(x)$$

$$\Delta y = \cos(x+\Delta x) - \cos(x)$$

$$\Delta y = -2\sin\left(\frac{x+\Delta x+x}{2}\right)\cdot\sin\left(\frac{x+\Delta x-x}{2}\right)$$

$$\Delta y = -2\sin\left(\frac{x+\Delta x}{2}\right)\cdot\sin\left(\frac{\Delta x}{2}\right) \qquad \because \cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Delta y = -2\sin\left(x+\frac{\Delta x}{2}\right)\cdot\sin\left(\frac{\Delta x}{2}\right)$$

Now, divide equation by the Δx.

$$\frac{\Delta y}{\Delta x} = -2\sin\left(x+\frac{\Delta x}{2}\right)\frac{\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = -\sin\left(x+\frac{\Delta x}{2}\right)\cdot\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation .

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} \qquad \because \lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x} = 1$$

$$= -\sin(x+0)\cdot 1$$

$$= -\sin x$$

Hence, $\dfrac{dy}{dx} = -\sin(x)$

iii. **Derivative of tan x:** If $y = \tan x$, then the derivative of $y = \tan x$ is $\dfrac{dy}{dx} = \sec^2 x$.

Proof: By the rule of first principle.

Let $y = \tan(x)$ and

$y+\Delta y = \tan(x+\Delta x)$

Subtracting equation from equation .

$$y+\Delta y - y = \tan(x+\Delta x) - \tan(x)$$

$$\Delta y = \frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin(x)}{\cos(x)}$$

Divide equation (iii) by Δx

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}\left\{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin(x)}{\cos(x)}\right\}$$

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation .

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin(x)}{\cos(x)}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x).\cos(x) - \sin(x).\cos(x+\Delta x)}{\Delta x.\cos(x).\cos(x+\Delta x)}$$

$$= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x-x)}{\Delta x.\cos(x).\cos(x+\Delta x)} = \lim_{\Delta x\to 0}\left(\frac{1}{\cos(x+\Delta x)}\cdot\frac{1}{\cos(x)}\cdot\frac{\sin\Delta x}{\Delta x}\right)$$

$$= \lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)}\cdot\lim_{\Delta x\to 0}\frac{1}{\cos(x)}\cdot\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = \frac{1}{\cos(x)}\cdot\frac{1}{\cos(x)}\cdot(1) = \frac{1}{\cos^2(x)} = \sec^2(x)$$

Hence, $\dfrac{dy}{dx} = \sec^2(x)$

iv. **Derivative of sec x:** If $y = \sec x$, then the derivative of $y = \sec(x)$ is $\dfrac{dy}{dx} = \sec(x).\tan(x)$.

Proof: By the rule of first principle.

Let $y = \sec(x)$

and $y+\Delta y = \sec(x+\Delta x)$

Subtracting equation from equation .

$$y+\Delta y - y = \sec(x+\Delta x) - \sec(x)$$

$$\Delta y = \sec(x+\Delta x) - \sec(x)$$

Now, divide equation by the Δx

$$\frac{\Delta y}{\Delta x} = \frac{\sec(x+\Delta x) - \sec(x)}{\Delta x}$$

Do You Know?

The traffic police officers uses radar guns to take the advantage of the easy use of derivatives. When a radar gun is pointed and fired at a car on the motorway. The gun is able to determine the time and distance at which the radar was able to hit a certain section of the car with the use of derivative it is able to calculate the speed at which the car was going and also report the distance that the car was from the radar gun.

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{\cos(x+\Delta x)}-\frac{1}{\cos(x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cos(x)-\cos(x+\Delta x)}{\Delta x.\cos(x).\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{-2\sin\left(\frac{x+x+\Delta x}{2}\right).\sin\left(\frac{x-x-\Delta x}{2}\right)}{\Delta x.\cos(x).\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{-2\sin\left(\frac{2x+\Delta x}{2}\right).\sin\left(-\frac{\Delta x}{2}\right)}{\Delta x.\cos(x).\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{2\sin\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x.\cos(x).\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)}.\lim_{\Delta x\to 0}\frac{1}{\cos(x)}.\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$=\sin(x+0)\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos(x)}\cdot(1)=\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos x}=\sec(x)\cdot\tan(x)$$

Hence, $\frac{dy}{dx}=\sec(x)\cdot\tan(x)$

v. **Derivative of cosec x:** If $y=\text{cosec } x$, then the derivative of $y=\text{cosec}(x)$ is $\frac{dy}{dx}=-\cot(x).\text{cosec}(x)$.

Proof: By the rule of first princir' :.

Let $y=\text{cosec}(x)$ (i)

and $y+\Delta y=\text{cosec}(x+\Delta x)$ (ii).

Subtracting equation (i) from equation (ii).

$y+\Delta y-y=\text{cosec}(x+\Delta x)-\text{cosec}(x)$

$\Delta y=\text{cosec}(x+\Delta x)-\text{cosec}(x)$ (iii)

Now, divide equation (iii) by the Δx

$\frac{\Delta y}{\Delta x}=\frac{\text{cosec}(x+\Delta x)-\text{cosec}(x)}{\Delta x}$ (iv)

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\text{cosec}(x+\Delta x)-\text{cosec}(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{\sin(x+\Delta x)}-\frac{1}{\sin(x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x)-\sin(x+\Delta x)}{\Delta x.\sin(x).\sin(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{2\cos\left(\frac{x+x+\Delta x}{2}\right).\sin\left(\frac{x-x-\Delta x}{2}\right)}{\Delta x.\sin(x).\sin(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{2.\cos\left(x+\frac{\Delta x}{2}\right).\sin\left(-\frac{\Delta x}{2}\right)}{\Delta x.\sin(x).\sin(x+\Delta x)}=-\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\left(\frac{1}{\sin(x).\sin(x+\Delta x)}\right).\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$=-\cos(x+0).\frac{1}{\sin(x).\sin(x+0)}.\frac{1}{\cos(x)}.(1)=-\frac{\cos(x)}{\sin(x)}.\frac{1}{\sin(x)}=-\cot(x).\text{cosec}(x)$$

Hence, $\frac{dy}{dx}=\cot(x).\text{cosec}(x)$

vi. **Derivative of cot x:** If $y=\cot x$, then the derivative of $y=\cot x$ is $\frac{dy}{dx}=-\text{cosec}^2(x)$.

Proof: By the rule of first principle.

Let $y=\cot(x)$ (i)

and $y+\Delta y=\cot(x+\Delta x)$ (ii)

Subtracting equation (i) from equation (ii).

$\cancel{y}+\Delta y-\cancel{y}=\cot(x+\Delta x)-\cot(x)$

$\Delta y=\cot(x+\Delta x)-\cot(x)$ (iii)

Now, divide equation (iii) by the Δx.

$\frac{\Delta y}{\Delta x}=\frac{\cot(x+\Delta x)-\cot(x)}{\Delta x}$ (iv)

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cot(x+\Delta x)-\cot x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{\cos(x+\Delta x)}{\sin(x+\Delta x)}-\frac{\cos x}{\sin x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin x\cos(x+\Delta x)-\cos x\sin(x+\Delta x)}{\Delta x\sin x\sin(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x-x-\Delta x)}{\Delta x\sin x\sin(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x)}\cdot\frac{1}{\sin x}\cdot\frac{-\sin\Delta x}{\Delta x},\qquad \sin(-\Delta x)=-\sin\Delta x$$

$$=-\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x)}\lim_{\Delta x\to 0}\frac{1}{\sin x}\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=-\frac{1}{\sin x}\frac{1}{\sin x}(1)=-\frac{1}{\sin^2 x}=-\cos ec^2 x,$$

The trigonometric formulae are listed below.

i. $\frac{d}{dx}(\sin x)=\cos x$ *ii.* $\frac{d}{dx}(\cos x)=-\sin x$ *iii.* $\frac{d}{dx}(\tan x)=\sec^2 x$

iv. $\frac{d}{dx}(\cos ecx)=-\cot x\cos ecx$ *v.* $\frac{d}{dx}(\sec x)=\tan x\sec x$ *vi.* $\frac{d}{dx}(\cot x)=-\cos ec^2 x$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the trigonometric functions, as summarized in the box:

i. $\frac{d}{dx}(\sin u)=\cos u\frac{d}{dx}(u)$ *ii.* $\frac{d}{dx}(\cos u)=-\sin u\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\tan u)=\sec^2 u\frac{d}{dx}(u)$ *iv.* $\frac{d}{dx}(\cos ecu)=-\cot u\cos ecu\frac{d}{dx}(u)$

v. $\frac{d}{dx}(\sec u)=\tan u\sec u\frac{d}{dx}(u)$ *vi.* $\frac{d}{dx}(\cot u)=-\cos ec^2 u\frac{d}{dx}(u)$

Example 16 Differentiate the following trigonometric functions:

(a). $p(t)=(t^2+t)\sin t$ *(b).* $f(x)=\frac{1+\sin x}{2-\cos(x)}$

Solution

a. If the given function is $p(t)=(t^2+t)\sin t$, then the product rule of differentiation w.r.t. t is used to obtain: $\frac{dp}{dt}=(t^2+t)\frac{d}{dt}(\sin t)+(\sin t)\frac{d}{dt}(t^2+t)=(t^2+t)\cos t+\sin t(2t+1)$

b. If the given function is $f(x)=\frac{1+\sin x}{2-\cos x}$, then the quotient rule of differentiation w.r.t. x is used to obtain:

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{1+\sin(x)}{2-\cos(x)}\right)=\frac{(2-\cos x)\frac{d}{dx}(1+\sin x)-(1+\sin x)\frac{d}{dx}(2-\cos x)}{(2-\cos x)^2}=\frac{(\cos x)(2-\cos x)-(\sin x)(1+\sin x)}{(2-\cos x)^2}$$

$$= \frac{2\cos x - \cos^2 x - \sin x - \sin^2 x}{(2-\cos x)^2}$$

$$= \frac{2\cos x - \sin x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2} = \frac{2\cos x - \sin x - 1}{(2-\cos x)^2} \qquad \because \sin^2 x + \cos^2 x = 1$$

17 Differentiate the following trigonometric functions:

(a). $f(x) = \sec x \tan x$ (b). $f(x) = \frac{x^2 + \tan x}{3x + 2\tan x}$

Solution

a. If the given function is $f(x) = \sec x \tan x$, then the product rule of differentiation w.r.t. x is used to obtain:

$$\frac{df}{dx} = \frac{d}{dx}\sec x \tan x = \sec x \frac{d}{dx}(\tan x) + (\tan x)\frac{d}{dx}(\sec x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$$

b. If the given function is $f(x) = \frac{x^2 + \tan x}{3x + 2\tan x}$, then the quotient rule of differentiation w.r.t. x is used to obtain:

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{x^2 + \tan(x)}{3x + 2\tan(x)}\right) = \frac{(3x + 2\tan x).\frac{d}{dx}(x^2 + \tan x) - (x^2 + \tan x)\frac{d}{dx}(3x + 2\tan x)}{(3x + 2\tan(x))^2}$$

$$= \frac{(2x + \sec^2 x)(3x + 2\tan x) - (3 + 2\sec^2 x)(x^2 + \tan x)}{(3x + 2\tan x)^2} = \frac{3x^2 + (4x - 3)\tan x + x(3 - 2x)\sec^2 x}{(3x + 2\tan x)^2}$$

3.5.2 Differentiation of inverse trigonometric functions

i. **Derivative of $\sin^{-1}x$:** If $y = \sin^{-1} x$, then $x = \sin y$.

The differentiation of $x = \sin y$ w.r.t. y is: $\frac{dx}{dy} = \cos y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \pm\frac{1}{\sqrt{1 - \sin^2 y}} = \pm\frac{1}{\sqrt{1 - x^2}} \qquad \because \sin^2 y + \cos^2 y = 1,\ \sin y = x$$

Here, the sign of the radical is the same as that of $\cos y$. By definition of $\sin^{-1} x$:

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \quad or \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2},$$

Hence, $\cos y$ is positive: $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$

ii. **Derivative of $\cos^{-1}x$:** If $y = \cos^{-1} x$, then $x = \cos y$.

The differentiation of $x = \cos y$ w.r.t. y is: $\frac{dx}{dy} = -\sin y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = -\frac{1}{\sin y} = \pm\frac{-1}{\sqrt{1 - \cos^2 y}} = \pm\frac{-1}{\sqrt{1 - x^2}} \qquad \because \sin^2 y + \cos^2 y = 1,\ \cos y = x$$

Here, the sign of the radical is the same as that of $\sin y$. By definition of $\cos^{-1} x$:

$$0 \le \cos^{-1} x \le \pi \quad or \quad 0 \le y \le \pi,$$

Also, if y lies between 0 and π, then, $\sin y$ is necessarily positive. Hence $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$

iii. **Derivative of $\tan^{-1}x$:** If $y = \tan^{-1} x$, then $x = \tan y$.

The differentiation of $x = \tan y$ w.r.t. y is: $\frac{dx}{dy} = \sec^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2} \qquad \because \sec^2 y = 1 + \tan^2 y,\ \tan y = x$$

iv. **Derivative of $\sec^{-1}x$:** If $y = \sec^{-1} x$, then $x = \sec y$.

The differentiation of $x = \sec y$ w.r.t. y is: $\frac{dx}{dy} = \sec y \tan y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \pm\frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \pm\frac{1}{x\sqrt{x^2 - 1}} \qquad \because 1 + \tan^2 y = \sec^2 y,\ \sec y = x$$

We take + sign before the radical sign to obtain: $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$

v. **Derivative of $\text{cosec}^{-1}x$** If $y = \text{cosec}^{-1} x$, then $x = \text{cosec}\, y$.

The differentiation of $x = \text{cosec}\, y$ w.r.t. y is: $\frac{dx}{dy} = -\text{cosec}(y)\cot(y)$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\text{cosec} y \cot y} = \pm\frac{-1}{\text{cosec} y\sqrt{\text{cosec}^2 y - 1}} \qquad \because 1 + \cot^2 y = \text{cosec}^2 y,\ \text{cosec}\, y = x$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{-1}{x\sqrt{x^2 - 1}}$$

We take + sign before the radical sign to obtain: $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$

vi. **Derivative of $\cot^{-1}x$:** If $y = \cot^{-1} x$, then $x = \cot y$.

The differentiation of $x = \cot y$ w.r.t. y is: $\frac{dx}{dy} = -\text{cosec}^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = -\frac{1}{\text{cosec}^2 y} = \frac{-1}{1 + \cot^2 y} = \frac{-1}{1 + x^2} \qquad \because \text{cosec}^2 y = 1 + \cot^2 y,\ \cot y = x$$

These inverse trigonometric formulas are listed in the box:

i. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ ii. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$ iii. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$

iv. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$ v. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$ vi. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the inverse trigonometric functions, as summarized in the box:

i. $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{d}{dx}(u)$ ii. $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1 - u^2}}\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2}\frac{d}{dx}(u)$ iv. $\frac{d}{dx}(\text{cosec}^{-1} u) = \frac{-1}{u\sqrt{u^2 - 1}}\frac{d}{dx}(u)$

v. $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{u\sqrt{u^2 - 1}}\frac{d}{dx}(u)$ vi. $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1 + u^2}\frac{d}{dx}(u)$

Example 18 Differentiate the following inverse trigonometric functions:

(a). $y=\tan^{-1}\sqrt{x}$ (b). $y=\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$

Solution a. $y=\tan^{-1}\sqrt{x}$

$y=f(x)=\tan^{-1}\sqrt{x}$ (i)

Let $u=\sqrt{x}\Rightarrow\dfrac{du}{dx}=\dfrac{1}{2\sqrt{x}}$ (ii)

Now, $y=\tan^{-1}.(u)\Rightarrow\dfrac{dy}{du}=\dfrac{1}{1+u^2}$ (iii)

By using the chain rule $\dfrac{dy}{dx}=\dfrac{dy}{du}.\dfrac{du}{dx}$

From equation (ii) and equation (iii) $\dfrac{du}{dx}=\left(\dfrac{1}{1+u^2}\right)\left(\dfrac{1}{2\sqrt{x}}\right)\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}(1+x)}$

b. Given $y=\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$

Let $u=\dfrac{x-x^{-1}}{x+x^{-1}}$

$u=\dfrac{x^2-1}{x^2+1}$

$$\frac{du}{dx}=\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

$$=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

Now, $y=\cos^{-1}u$ where $u=\dfrac{x-x^{-1}}{x+x^{-1}}=\dfrac{x^2-1}{x^2+1}$

$$\Rightarrow\frac{dy}{du}=\frac{d}{du}\cos^{-1}u$$

$$\frac{dy}{du}=\frac{-1}{\sqrt{1-u^2}}$$

By chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\frac{-1}{\sqrt{1-u^2}}\times\frac{4x}{(x^2+1)^2}=\frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{(x^2+1)^2}$$

$$\frac{dy}{dx}=\frac{-4x}{\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2+1)^2}\times(x^2+1)^2}}$$

$$=\frac{-4x}{\sqrt{\frac{4x^2}{(x^2+1)^2}\times(x^2+1)^2}}=\frac{-4x}{2x(x^2+1)}$$

$$\frac{dy}{dx}=\frac{-2}{(x^2+1)}$$

Exercise 3.4

1. Use of first principle rules to differentiate the following functions.
   a. $y=\sin(2x)$ b. $y=\cot(3x)$ c. $y=\cos(3x)+\tan(3x)$
   d. $y=\cot^2(x)$ e. $y=\tan\sqrt{x}$ f. $y=\sin^3(x)$
2. Differentiate the following trigonometric functions by using any suitable rule.
   a. $x^2\cot(3x)$ b. $y=(\sin 2x+\cot 3x)^2$ c. $y=4\operatorname{cosec}2x$
   d. $y=2\tan(x+3)^2$ e. $y=\dfrac{\sqrt{\tan(x)}}{\cos\sqrt{x}}$ f. $y=\dfrac{1+\tan 2x}{\operatorname{cosec}3x}$
3. Use any suitable rule of differentiation to perform $\dfrac{dy}{dx}$ for the following functions.
   a. $y=\cos^{-1}\left(\dfrac{x}{a}\right)$ b. $y=\tan^{-1}\left(\dfrac{x}{p}\right)$ c. $y=\cot^{-1}\left(\dfrac{a}{x}\right)$
   d. $y=\operatorname{cosec}^{-1}\sqrt{(1+x^2)}$ e. $y=\operatorname{cosec}^{-1}(t+3)$ f. $y=\dfrac{1}{x}.\tan^{-1}\left(\dfrac{x+1}{x-1}\right)$
4. Suppose profits on the sale of swimming suits in a departmental store are given approximately by
   $P(t)=5-5\cos\dfrac{\pi t}{26},\quad 0\le t\le 104$
   where P(t) is profit (in hundreds of dollars) for a week of sales t weeks after January first.
   a. What is the rate of change of profit t weeks after the first of the year?
   b. What is the rate of change of profit 8 weeks after the first of the year? 26weeks after the first of the year? 50 weeks after the first of the year?
5. A normal seated adult breathes in and exhales about 0.8 liter of air every 4 seconds. The volume of air $V(t)$ in the lungs t seconds after exhaling is given approximately by $V(t)=0.45-0.35\cos\dfrac{\pi t}{2}, 0\le t\le 8$.
   a. What is the rate of flow of air t seconds afterexhaling?
   b. What is the rate of flow of air 3 seconds after exhaling? 4 seconds after exhaling? 5 seconds after exhaling?

3.6 Differentiation of Exponential and Logarithmic Functions

The goal of this section is to develop the differential calculus of logarithmic and exponential functions. We shall begin by deriving differentiation formulas for $\ln x$ and e^x. The derived formulas will be applied to a number of differentiation problems and applications.

3.6.1 Derivative of e^x and a^x from first principle

i. **Derivative of e^x:** If $y=e^x$, then the derivative of $y=e^x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=e^x$$

$$=\lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)}-e^x}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^x e^{\Delta x}-e^x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{e^x(e^{\Delta x}-1)}{\Delta x}=\lim_{\Delta x\to 0}e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=e^x(1)=e^x \qquad \therefore \lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=1$$

ii. **Derivative of a^x:** If $y=a^x$, then the derivative of $y=a^x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=a^x$$

$$=\lim_{\Delta x\to 0}\frac{a^{(x+\Delta x)}-a^x}{\Delta x}=\lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{a^x(a^{\Delta x}-1)}{\Delta x}=\lim_{\Delta x\to 0}a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=a^x\log_e a \qquad \therefore \lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=\log_e a=\ln a$$

3.6.2 Derivative of $\ln x$ and $\log_a x$ from first principle

i. **Derivative of $\ln x$:** If $y=\ln x$, then the derivative of $y=\ln x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=\ln x$$

$$=\lim_{\Delta x\to 0}\frac{\ln(x+\Delta x)-\ln x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\ln\left(\frac{x+\Delta x}{x}\right), \quad \textit{Lograithmic-rule}$$

$$=\lim_{\Delta x\to 0}\frac{x}{x}\frac{1}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right) \quad \textit{multipy and divide out by } x$$

$$=\lim_{\Delta x\to 0}\frac{1}{x}\left(\frac{x}{\Delta x}\right)\ln\left(1+\frac{\Delta x}{x}\right)=\lim_{\Delta x\to 0}\frac{1}{x}\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}=\frac{1}{x}\ln e=\frac{1}{x} \qquad \because \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

ii. **Derivative of $\log_a x$:** If $y=\log_a x$, then the derivative of $y=\log_a x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=\log_a x$$

$$=\lim_{\Delta x\to 0}\frac{\log_a(x+\Delta x)-\log_a x}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\log_a\left(\frac{x+\Delta x}{x}\right)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)$$

$$=\lim_{\Delta x\to 0}\frac{x}{x}\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right), \quad \textit{multiply and divide out by } x$$

$$=\frac{1}{x}\lim_{\Delta x\to 0}\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\lim_{\Delta x\to 0}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}=\frac{1}{x}\log_a e \qquad \because \lim_{\Delta x\to 0}(1+x)^{\frac{1}{x}}=e$$

These exponential and logarithmic formulas are listed in the box:

i. $\frac{d}{dx}(e^x)=e^x$ ii. $\frac{d}{dx}(a^x)=a^x\log_e a$ iii. $\frac{d}{dx}(\ln x)=\frac{1}{x}$ iv. $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the exponential and logarithmic functions, as summarized in the box:

i. $\frac{d}{dx}(e^u)=e^u\frac{d}{dx}(u)$ ii. $\frac{d}{dx}(a^u)=a^u\log_e a\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{d}{dx}(u)$ iv. $\frac{d}{dx}(\log_a u)=\frac{1}{u}\log_a e\frac{d}{dx}(u)$

19 Differentiate the following functions:

(a). $f(x)=7^{(4-3x^5)}$ (b). $f(x)=\log_{10}\sqrt{x^2-7x}+x^3$ (c). $f(x)=\ln(e^{mx}+e^{-mx})$ (d). $f(x)=\frac{e^{2x}}{\ln x}$

Solution

a. If the given function is $f(x)=7^{(4-3x^5)}$, then the derivative of the given function w.r.t. x is

$$\frac{dy}{dx}=\frac{d}{dx}\left[7^{(4-3x^5)}\right]$$

Let $u=4-3x^5$ then $\frac{du}{dx}=-15x^4$

Now, $y=7^u \Rightarrow \frac{dy}{du}=\frac{d}{du}(7^u)=7^u.\log_e(7)$

By using chain rule $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

$$\frac{dy}{dx}=7^u.\log_e(7).(-15x^4)\Rightarrow\frac{dy}{dx}=-15x^4.7^{(4-3x^5)}.\log_e(7) \qquad \because u=4-3x^5$$

b. If the given function is $f(x)=\log_{10}\sqrt{(x^2-7x)}+x^3$, then the derivative of a given function w.r.t. x is:

$$\frac{df}{dx}=\frac{d}{dx}\left[\log_{10}\sqrt{(x^2-7x)}+x^3\right]=\frac{d}{dx}[\log_{10}\sqrt{(x^2-7x)}]+\frac{d}{dx}(x^3)=\frac{1}{u}\log_{10}e\frac{d}{dx}(u)+3x^2, \quad u=\sqrt{(x^2-7x)}$$

$$=\frac{1}{2(x^2-7x)}\log_{10}e\frac{d}{dx}(x^2-7x)+3x^2=\frac{2x-7}{2(x^2-7x)}\log_{10}e+3x^2$$

c. If the given function is $f(x)=\ln(e^{mx}+e^{-mx})$, then the derivative of a given function w.r.t. x is:

$$\frac{df}{dx}=\frac{d}{dx}[\ln(e^{mx}+e^{-mx})]=\frac{d}{du}(\ln u)\frac{d}{dx}(e^{mx}+e^{-mx}) \qquad \because u=e^{mx}+e^{-mx}$$

$$=\frac{1}{\ln u}(me^{mx}-me^{-mx})=\frac{1}{\ln(e^{mx}+e^{-mx})}(m)(e^{mx}-e^{-mx})=\frac{m(e^{mx}-e^{-mx})}{\ln(e^{mx}+e^{-mx})}$$

d. If the given function is $f(x)=\frac{e^{2x}}{\ln x}$, then the derivative of a given function w.r.t. x is:

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{e^{2x}}{\ln x}\right)=\frac{\frac{d}{dx}[e^{2x}].\ln(x)-e^{2x}.\frac{d}{dx}[\ln(x)]}{(\ln x)^2}=\frac{e^{2x}.\frac{d}{dx}(2x).\ln(x)-\frac{e^{2x}}{x}}{(\ln x)^2}$$

$$=\frac{2e^{2x}\ln x-\frac{1}{x}e^{2x}}{(\ln x)^2}=\frac{2xe^{2x}\ln x-e^{2x}}{x(\ln x)^2}=\frac{e^{2x}(2x\ln x-1)}{x(\ln x)^2}$$

3.6.3 Use of logarithmic differentiation to algebraic expressions involving product, quotient and power

Logarithmic differentiation is a procedure in which logarithms are used to trade the task of differentiating products and quotients for that of differentiating sums and differences. It is especially valuable as a means for handling complicated product or quotient functions and power functions where variables appear in both the base and the exponent.

Example 20 Differentiate the following functions

(a). $y=\ln\left[\dfrac{x(x^2-3)^2}{\sqrt{(x^2-4)}}\right]$ *(b).* $y=x^{\sin x}$

Solution *a.* $y=\ln\left[\dfrac{x(x^2-3)^2}{\sqrt{(x^2-4)}}\right]$

If the given function after simplification is

$$y=\ln\left[\frac{x(x^2-3)^2}{(x^2-4)^{\frac{1}{2}}}\right]=\ln[x(x^2-3)^2]-\ln[(x^2-4)^{\frac{1}{2}}],\quad \text{logarithms rules}$$

$$=\ln x+\ln(x^2-3)^2-\ln(x^2-4)^{\frac{1}{2}}=\ln x+2\ln(x^2-3)-\frac{1}{2}\ln(x^2-4),$$

then the derivative of y w.r.t. x is

$$\frac{dy}{dx}=\frac{d}{dx}\left[\ln x+2\ln(x^2-3)-\frac{1}{2}\ln\left(x^2-4\right)\right]=\frac{d}{dx}[\ln(x)]+2\frac{d}{dx}[\ln(x^2-3)]-\frac{1}{2}\frac{d}{dx}\ln(x^2-4)$$

$$=\frac{1}{x}+2\frac{1}{x^2-3}(2x)-\frac{1}{2}\frac{1}{x^2-4}(2x)=\frac{4x^4-20x^2+12}{x(x^2-3)(x^2-4)}=\frac{4(x^4-5x^2+3)}{x(x^2-3)(x+2)(x-2)}$$

b. $y=x^{\sin x}$:

$\ln y=\ln(x^{\sin x})$, taking ln of both sides

$\ln y=\sin x\ln x$

then on differentiation w.r.t. x. It becomes;

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(\sin x\ln x)$$

$$\frac{1}{y}\frac{dy}{dx}=\sin x\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{\sin x}{x}+\ln x\cos x$$

$$\frac{dy}{dx}=y\left[\frac{\sin x}{x}+\ln x\cos x\right]=x^{\sin(x)}\left[\frac{\sin x}{x}+\ln x\cos x\right]$$

3.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

The concept of hyperbolic functions is completely discussed in **Unit-2**. The differentiation of hyperbolic functions can be found as follows:

3.7.1 Differentiation of the hyperbolic functions

i. **Derivative of sinhx:** If $y=\sin hx=\dfrac{e^x-e^{-x}}{2}$, then on differentiation w.r.t. x, it becomes

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{e^x-e^{-x}}{2}\right]=\frac{1}{2}\left[\frac{d}{dx}(e^x)-\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}(e^x+e^{-x})=\cos hx$$

ii. **Derivative of coshx:** If $y=\cos hx=\dfrac{e^x+e^{-x}}{2}$, then on differentiation w.r.t. x, it becomes

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{e^x+e^{-x}}{2}\right]=\frac{1}{2}\left[\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}(e^x-e^{-x})=\sin hx$$

iii. **Derivative of tanhx:** If $y=\tan hx=\dfrac{\sin hx}{\cos hx}$, then on differentiation w.r.t. x through quotient rule, it becomes $\dfrac{dy}{dx}=\dfrac{\cos hx\frac{d}{dx}(\sin hx)-\sin hx\frac{d}{dx}(\cos hx)}{\cos h^2x}$ $\because \cos h^2x-\sin h^2x=1$

$$=\frac{\cos hx\cos hx-\sin hx\sin hx}{\cos h^2x}=\frac{\cos h^2x-\sin h^2x}{\cos h^2x}=\frac{1}{\cos h^2x}=\sec h^2x$$

iv. **Derivative of sechx:** If $y=\text{sech}\,x=\dfrac{1}{\cosh x}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\cos hx\frac{d}{dx}(1)-(1)\frac{d}{dx}(\cos hx)}{\cos h^2x}=\frac{\cos hx(0)-\sin hx}{\cos h^2x}=\frac{-\sin hx}{\cos h^2x}=-\frac{\sin hx}{\cos hx}\frac{1}{\cos hx}=-\tan hx\sec hx$$

v. **Derivative of cosechx:** If $y=\text{cosech}x=\dfrac{1}{\sin hx}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\sin hx\frac{d}{dx}(1)-(1)\frac{d}{dx}(\sin hx)}{\sin h^2x}=\frac{\sin hx(0)-\cos hx}{\sin h^2x}=\frac{-\cos hx}{\sin h^2x}$$

$$=-\frac{\cos hx}{\sin hx}\frac{1}{\sin hx}=-\cot hx\ \ \text{cosech}x$$

vi. **Derivative of cothx:** If $y=\cot hx=\dfrac{\cos hx}{\sin hx}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\sinh x\frac{d}{dx}(\cosh x)-\cosh x\frac{d}{dx}(\sinh x)}{\sinh^2x}\qquad \because \cos h^2x-\sin h^2x=1$$

$$=\frac{\sinh x\sinh x-\cosh x\cosh x}{\sinh^2x}=\frac{-(\cosh^2x-\sinh^2x)}{\sinh^2x}=-\frac{1}{\sinh^2x}=-\text{cosech}^2x$$

> The hyperbolic formulas are listed below:
>
> *i.* $\dfrac{d}{dx}(\sinh x)=\cosh x$ *ii.* $\dfrac{d}{dx}(\cosh x)=\sinh x$ *iii.* $\dfrac{d}{dx}(\tanh x)=\sec h^2x$
>
> *iv.* $\dfrac{d}{dx}(\text{cosech}x)=-\coth x\ \text{cosech}x$ *v.* $\dfrac{d}{dx}(\sec hx)=-\tanh x\sec hx$ *vi.* $\dfrac{d}{dx}(\cot x)=-\text{cosech}^2x$
>
> The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the hyperbolic functions, as summarized in the box:

i. $\frac{d}{dx}(\sinh u) = \cosh u \frac{d}{dx}(u)$	ii. $\frac{d}{dx}(\cosh u) = \sinh u \frac{d}{dx}(u)$
iii. $\frac{d}{dx}(\tanh u) = \sec h^2 u \frac{d}{dx}(u)$	iv. $\frac{d}{dx}(\text{cosech}u) = -\coth u \,\text{cosech}u \frac{d}{dx}(u)$
v. $\frac{d}{dx}(\sec hu) = -\tanh u \sec hu \frac{d}{dx}(u)$	vi. $\frac{d}{dx}(\coth u) = -\text{cosec}^2 u \frac{d}{dx}(u)$

Example 21 Differentiate the following functions: *(a).* $y = \cos h(2x^2-1)$ *(b).* $y = \sec h\left(\frac{1-x}{1+x}\right)$

Solution *a.* If the given function is $y = \cos h(2x^2-1)$, then the derivative of y w.r.t. x is:

$$\frac{dy}{dx} = \frac{d}{dx}[\cos h(2x^2-1)] = \sin h(2x^2-1).\frac{d}{dx}(2x^2-1)$$

$$= \sin h(2x^2-1)\left(2.\frac{d}{dx}[x^2] + \frac{d}{dx}[-1]\right) = \sin h(2x^2-1)(2(2x)+0) = 4x.\sin h(2x^2-1)$$

b. If the given function is $y = \sec h\left[\frac{1-x}{1+x}\right]$, then the derivative of y w.r.t. x is:

$$y' = \frac{d}{dx}\left[\sec h\left(\frac{1-x}{1+x}\right)\right] = \frac{d}{du}(\sec hu)\frac{d}{dx}(u) \qquad \because u = \frac{1-x}{1+x}$$

$$= -\tanh u \sec hu \frac{d}{dx}\left(\frac{1-x}{1+x}\right) = -\tan hu \sec hu\left[\frac{(-1)(1+x)-(1-x)(1)}{(1+x)^2}\right]$$

$$= \frac{2}{(1+x)^2}\tan h\left(\frac{1-x}{1+x}\right)\sec h\left(\frac{1-x}{1+x}\right)$$

3.7.2 Differentiation of inverse hyperbolic functions

i. **Derivative of $\sin h^{-1}x$:** If $y = \sinh^{-1}x$, then $x = \sinh y$, the differentiation of $x = \sin h\, y$ w.r.t y is:

$\frac{dx}{dy} = \cosh y$ Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\cos hy} = \pm\frac{1}{\sqrt{1+\sin h^2 y}} = \pm\frac{1}{\sqrt{1+x^2}} \qquad \therefore \sin hy = x, \quad \cos h^2 y - \sin h^2 y = 1$$

Here, the sign of the radical is the same as that of cos hy which we know is always positive.

Hence, $\frac{d}{dx}(\sin h^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

ii. **Derivative of $\cos h^{-1}x$:** If $y = \cosh^{-1}x$, then $x = \cosh y$,

then the differentiation of $x = \cos h\, y$ w.r.t. y is: $\frac{dx}{dy} = \sin hy$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\sin hy} = \pm\frac{1}{\sqrt{\cos h^2 y - 1}} = \pm\frac{1}{\sqrt{x^2-1}} \qquad \because \cos hy = x, \ \cos h^2 y - \sin h^2 y = 1$$

Here, the sign of the radical is the same as that of cos hy which we know is always positive.

Hence, $\frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

iii. **Derivative of $\tan h^{-1}x$:** If $y = \tan h^{-1}x$, then $x = \tanh y$,

then the differentiation of $x = \tan h\, y$ w.r.t. y is: $\frac{dx}{dy} = \sec h^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\sec h^2 y} = \frac{1}{1-\tan h^2 y} = \frac{1}{1-x^2}, \quad \sec h^2 y = 1 - \tan h^2 y, |x| < 1, \tan hy = x$$

iv. **Derivative of $\sec h^{-1}x$:** If $y = \sec h^{-1}x$, then $x = \sec hy$.

The differentiation of $x = \sec hy$ w.r.t. y is: $\frac{dx}{dy} = -\sec hy \tan hy$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\sec hy \tan hy} = \pm\frac{-1}{\sec hy\sqrt{1-\sec h^2 y}} = \pm\frac{-1}{x\sqrt{1-x^2}} \qquad \because 1 - \sec h^2 y = \tan h^2, y \sec hy = x$$

Here, the sign of the radical is the same as that of tanhy but we know that $\sec h^{-1}x$ is always positive, so that tanhy is always positive. Hence, $\frac{d}{dx}(\sec h^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}$

v. **Derivative of $\text{cosec}h^{-1}x$:** If $y = \text{cosec}h^{-1}x$, then $x = \text{cosec}hy$.

The differentiation of $x = \text{cosec}\, hy$ w.r.t. y is: $\frac{dx}{dy} = -\text{cosec}\, hy \coth y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\text{cosec}hy \cot hy} = \pm\frac{-1}{\text{cosec}hy\sqrt{\text{cosec}h^2 y + 1}} \qquad \because \cot h^2 y = \text{cosec}h^2 y + 1$$

$$= \pm\frac{-1}{x\sqrt{x^2+1}}, \qquad \text{cosec}hy = x$$

Here, the sign of the radical is the same as that of cothy which. Here cothy is positive or negative according as x is positive or negative.

$$\frac{d}{dx}(\text{cosec}h^{-1}x) = \frac{-1}{x\sqrt{x^2+1}} \text{ if } x > 0 \text{ and } \frac{d}{dx}(\text{cosec}h^{-1}x) = \frac{-1}{-x\sqrt{x^2+1}} \text{ if } x < 0.$$

Thus $\frac{d}{dx}(\text{cosec}h^{-1}x) = \frac{-1}{|x|\sqrt{x^2+1}}$ for all values of x.

vi. **Derivative of $\coth^{-1}x$:** If $y = \cot h^{-1}x$, then $x = \coth y$.

The differentiation of $x = \cot hy$ w.r.t. y is: $\frac{dx}{dy} = -\text{cosec}h^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\text{cosec}h^2 y} = \frac{1}{\cot h^2 y - 1} = \frac{-1}{x^2-1} \qquad \because \text{cosec}h^2 y = \cot h^2 y - 1, |x| > 1, \ \coth y = x$$

The inverse hyperbolic formulas are listed below:

i. $\frac{d}{dx}(\sin h^{-1}x)=\frac{1}{\sqrt{1+x^2}}$ ii. $\frac{d}{dx}(\cos h^{-1}x)=\frac{1}{\sqrt{x^2-1}}$ iii. $\frac{d}{dx}(\tan h^{-1}x)=\frac{1}{1-x^2}$

iv. $\frac{d}{dx}(\cos ech^{-1}x)=\frac{-1}{|x|\sqrt{x^2+1}}$ v. $\frac{d}{dx}(\sec h^{-1}x)=\frac{-1}{x\sqrt{1-x^2}}$ vi. $\frac{d}{dx}(\cot h^{-1}x)=\frac{-1}{x^2-1}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the inverse hyperbolic functions, as summarized in the box:

i. $\frac{d}{dx}(\sin h^{-1}u)=\frac{1}{\sqrt{1+u^2}}\frac{d}{dx}(u)$ ii. $\frac{d}{dx}(\cos h^{-1}u)=\frac{1}{\sqrt{u^2-1}}\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\tan h^{-1}u)=\frac{1}{1-u^2}\frac{d}{dx}(u)$ iv. $\frac{d}{dx}(\cos ech^{-1}u)=\frac{-1}{|u|\sqrt{u^2+1}}\frac{d}{dx}(u)$

$\frac{d}{dx}(\sec h^{-1}u)=\frac{-1}{u\sqrt{1-u^2}}\frac{d}{dx}(u)$ v. $\frac{d}{dx}(\cot h^{-1}u)=\frac{-1}{u^2-1}\frac{d}{dx}(u)$

Example 22 Differentiate the following functions: ***(a).*** $y=\sin h^{-1}(x^3)$ ***(b).*** $y=\frac{\sin h^{-1}x}{\cos h^{-1}x}$

Solution

a. If the given function is $y=\sin h^{-1}(x^3)$, then the derivative of y w.r.t. x is:

$$\frac{dy}{dx}=\frac{d}{dx}[\sin h^{-1}(x^3)] \qquad \because \frac{d}{dx}[\sin h^{-1}(x)]=\frac{1}{\sqrt{1+x^2}}$$

$$=\frac{1}{\sqrt{1+(x^3)^2}}.\frac{d}{dx}(x^3)$$

$$=\frac{3x^2}{\sqrt{1+x^6}}$$

b. If the given function is $y=\frac{\sin h^{-1}(x)}{\cos h^{-1}(x)}$, then the derivative of y w.r.t. x is:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\sin h^{-1}(x)}{\cos h^{-1}(x)}\right)=\frac{\cos h^{-1}(x).\frac{d}{dx}[\sin h^{-1}(x)]-\sin h^{-1}(x).\frac{d}{dx}[\cos h^{-1}(x)]}{[\cos h^{-1}(x)]^2}$$

$$=\frac{\cos h^{-1}(x)\frac{1}{\sqrt{x^2+1}}-\sin h^{-1}(x).\frac{1}{\sqrt{x^2-1}}}{[\cos h^{-1}(x)]^2}=\frac{\frac{\cos h^{-1}(x)}{\sqrt{x^2+1}}-\frac{\sin h^{-1}(x)}{\sqrt{x^2-1}}}{[\cos h^{-1}(x)]^2}$$

$$=\frac{1}{\sqrt{x^2+1}.\cos h^{-1}(x)}-\frac{\sin h^{-1}(x)}{\sqrt{x^2-1}.[\cos h^{-1}(x)]^2}$$

3.8 MAPLE Command 'diff' to differentiate a function

The procedure to use the MAPLE command 'diff' to differentiate a function is illustrated in the following example.

Example 23 Use MAPLE command 'diff' to differentiate

(a). $f(x)=x^5+7x+2$ w.r.t. variable x. ***(b).*** $f(x)=\frac{(x^4+2x+16)}{(x^3+3x-2)}$ w.r.t. variable x.

(c). $f(x)=(x^3+\sin(x)^2+\arccos x)$ w.r.t. variable x.

(d). $f(x)=x^2\cosh x+\text{arcsinh}x$ w.r.t. variable x.

Solution

a. Command:

> $diff(x^5+7\cdot x+2,x);$

$5x^4+7$

Context Menu:

> $x^5+7\cdot x+2$

> $diff(x\text{^}5+7*x+2,x)$

$5x^4+7$

This result is obtained through right-click on the last end of the expression by selecting " Differentiate < x " on the context menu.

b. Command:

> $diff\left(\frac{(x^4+2\cdot x+16)}{(x^3+3\cdot x-2)},x\right);$

$\frac{4x^3+2}{x^3+3x-2}-\frac{(x^4+2x+16)(3x^2+3)}{(x^3+3x-2)^2}$

Context Menu:

> $\frac{(x^4+2\cdot x+16)}{(x^3+3\cdot x-2)}$

> $diff((x\text{^}4+2*x+16)/(x\text{^}3+3*x-2),x)$

$\frac{4x^3+2}{x^3+3x-2}-\frac{(x^4+2x+16)(3x^2+3)}{(x^3+3x-2)^2}$

c. Command:

> $diff(x^3+\sin(x)^2+\arccos(x),x);$

$3x^2+2\sin(x)\cos(x)-\frac{1}{\sqrt{1-x^2}}$

Context Menu:

> $x\text{^}3+\sin(x)\text{^}2+\arccos(x)$

> $diff(x\text{^}3+\sin(x)\text{^}2+\arccos(x),x)$

$3x^2+2\sin(x)\cos(x)-\frac{1}{\sqrt{1-x^2}}$

d. Command:

> $diff(x^2\cdot\cosh(x)+\text{arcsinh}(x),x);$

$2x\cosh(x)+x^2\sinh(x)+\frac{1}{\sqrt{1+x^2}}$

Context Menu:

> $x^2\cdot\cosh(x)+\text{arcsinh}(x)$

> $diff(x\text{^}2*\cosh(x)+\text{arcsinh}(x),x)$

$2x\cosh(x)+x^2\sinh(x)+\frac{1}{\sqrt{1+x^2}}$

Exercise 3.5

1. Use the rule of first principle to find the derivative of the following functions:
   a. $f(x)=e^{2x}$ b. $f(x)=\frac{1}{3}e^{3x}$ c. $f(x)=\frac{5}{8}e^{x^2}+1$ d. $f(x)=2^x$
   e. $f(x)=4^{x+4}$ f. $f(x)=\log(x+1)$ g. $f(x)=\log_a(x^2)$ h. $f(x)=\sinh 2x$
2. Find $f'(x)$ if $f(x)$ is:
   a. $11^{(3-4x^3)}$ b. $e^{\sqrt{x}-5}$ c. $x^5.e^{\frac{1}{x}}$
   d. $\frac{e^{2x}}{e^{-2x}+1}$ e. $\ln(e^{mx}-e^{-mx})$ f. $\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$
3. Find $\frac{dy}{dx}$ by using any suitable rule of differentiation.
   a. $y=x^3.\ln\sqrt{x}$ b. $y=x^2\sqrt{\ln(x)}$ c. $y=\ln\sqrt{\frac{x^2+1}{x^2-1}}$
   d. $y=\ln(x-\sqrt{x^2+1})$ e. $y=e^{-2x}.\cos(2x)$ f. $y=x^2.e^{\sin(x)}$
4. Differentiate the following functions.
   a. $y=\log(x+2)^3$ b. $y=\cos h(3x)$ c. $y=\sin h^{-1}(\cos x)$
   d. $y=\tan h^{-1}\left(\frac{x}{2}\right)$ e. $y=\ln(\cot h(x))$ f. $y=x.\cos h^{-1}(x)-\sqrt{x^2-1}$

5. A research group (used hospital records) developed the approximate mathematical model related to systolic blood pressure and age is: $p(x)=40+25\ln(x+1),\ \ 0\le x\le 65$
where $p(x)$ is the pressure measured in millimeters of mercury and x is age in years. What is the rate of change of pressure at the end of 10 years? at the end of 30 years? at the end of 60 years?

6. A single cholera bacterium divides every 0.5 hour to produce two complete cholera bacteria. If we start with a colony of 5,000 bacteria, then after t hours there will be a
$A(t)=5000\,2^{2t}$ bacteria. Find $A'(t)$, $A'(1)$ and $A'(5)$. Interpret the results.

7. Use MAPLE command "diff" to differentiate all the functions given in Q.3 and Q.4.

Review Excercise 3

1. Choose the correct option.

i. If $f(t)=2t^2+3t+2$ then $f(-3)$ is:
(a). 9 (b). 11 (c). 21 (d). 29

ii. The average rate of change for $f(x)=x^2-6x+5$ is______ if x increase at $x\in[1,3]$.
(a). 1 (b). −1 (c). 2 (d). −2

iii. If $y=f(x)$ then $f'(x)=$
(a). $\lim\limits_{\Delta x\to 0}\dfrac{f(x)-f(\Delta x)}{\Delta x}$ (b). $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$
(c). $\lim\limits_{\Delta x\to 0}\dfrac{f(x-\Delta x)+f(x)}{\Delta x}$ (d). $\lim\limits_{\Delta x\to 0}\dfrac{f(x-\Delta x)+f(x)}{f(x)}$

iv. If $f(t)=2t^3-3t^2+4$ then $f'(3)$ is
(a). 54 (b). 45 (c). 36 (d). 27

v. If $f(x)=\dfrac{h(x)}{g(x)}$ then $f'(x)=$
(a). $\dfrac{h'(x)-g'(x)}{[g(x)]^2}$ (b). $\dfrac{g(x).h'(x)-h(x)g'(x)}{[g(x)]^2}$
(c). $\dfrac{f'(x).g(x)+g(x)f'(x)}{[g(x)]^2}$ (d). $\dfrac{f'(x).g'(x)-f'(x)g'(x)}{[g(x)]^2}$

vi. If $h(t)=\sqrt{t^3}$ then $h'(t)$ is:
(a). $\dfrac{3}{2}\sqrt{t}$ (b). $\dfrac{2}{3}\sqrt{t}$ (c). $\dfrac{3t}{2\sqrt{t^2}}$ (d). $\dfrac{3t^2}{2\sqrt{t^3}}$

vii. If $f(x)=x\tan x$ then $f'(x)=$
(a). $\tan x+x\sec^2 x$ (b). $x\tan x-x\sec^2 x$ (c). $\sec^2 x$ (d). $\tan(x)+x[1+\tan(x)^2]$

viii. $\dfrac{d}{dy}\cos^{-1}y=$
(a). $\dfrac{1}{\sqrt{y^2-1}}$ (b). $\dfrac{x}{\sqrt{x^2-y^2}}$ (c). $\dfrac{-1}{\sqrt{1-y^2}}$ (d). $\dfrac{x}{\sqrt{1+y^2}}$

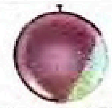
ix. $-\dfrac{1}{|x|\sqrt{x^2+1}}=\dfrac{d}{dx}($__________$)$
(a). $\sec^{-1}(x)$ (b). $\cos^{-1}(x)$ (c). $\operatorname{cosec}^{-1}(x)$ (d). $\tan^{-1}(x)$

x. If $f(x)=\ln(x)$ then $f'(x)$
(a). $\dfrac{1}{\ln(x)}$ (b). $\dfrac{2}{x}$ (c). $\dfrac{1}{x^2}$ (d). $\dfrac{1}{x}$

Project

Take a spherical balloon or a ball. It must be inflated.

- Find the general formula for instantaneous rate of change of the volume 'V' w.r.t, radius r. given that $V=\frac{4}{3}\pi r^3$
- Find its rate of change of V, w.r.t, r at the instant when $r=3$.



Summary

- The **average rate of change** $y=f(x)$ per unit change in x is given by:
$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)$$
- The slope of the secant is the average rate of change which measures **"the approximate rate of change in phenomena."**
- The **instantaneous rate of change** of a function $y\,f(x)$ at a particular point $P(x,f(x))$ is the **derivative** of a function $y=f(x)$ at that point, $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta)-f(x)}{\Delta x}$, $y=f(x)$ provided this limit exists. This is named by **first principle rule** of derivative of a function $f(x)$.
- The **tangent line to the graph of a function** $y=f(x)$ at the point $P(x,f(x))$ is the line through this point having slope $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, $y=f(x)$ if this limit exists. If this limit does not exist, then there is no tangent at the point.
- The slope of the tangent is the instantaneous rate of change which measures **"the exact rate of change in phenomena."**
- In business terminology,
  - the instantaneous rate of change of cost is the **marginal cost** which counts "as the approximate rate of change in business phenomena".
  - the average rate of change is the exact rate of change which counts "as the actual rate of change in business phenomena"
- For any real number n, if $f(x)=x^n$, then: $f'(x)=nx^{n-1}$
- The **chain rule** is a rule which we use to differentiate the composite function. It is generally written as $\dfrac{d}{dx}[f(g(x)]=f'[g(x)].g'(x)$