Unit 2

FUNCTIONS AND LIMITS

By the end of this unit, the students will be able to:

2.1 Function

i. Identify through graph the domain and range of a function.

ii. Draw the graph of modulus function (i.e. $y = |x|$) and identify its domain and range.

2.2 Composition of functions

i. Recognize the composition of functions. ii. Find the composition of two given functions.

2.3 Inverse of composition of functions

i. Describe the inverse of composition of two given functions.

ıscendental functions

Recognize algebraic, trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic (and their identities), explicit and implicit functions, and parametric representation of functions.

ical representations

Display graphically:

- the explicitly defined functions like $y = f(x)$, where $f(x) = e^x,\ a^x,\ \log_a x,\ \log_e x$.
- the implicitly defined functions such as $x^2 + y^2 = a^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and distinguish between graph of a function and of an equation.
- the parametric equations of functions such as $x = at^2, y = 2at; x = a\sec\theta,\quad y = b\tan\theta$.
- the discontinuous functions of the type $y = \begin{cases} x & when\ 0 \le x \le 1, \\ x-1 & when\ 1 \le x \le 2 \end{cases}$

ii. Use MAPLE graphic commands for two-dimensional plot of:
- an expression (or a function),
- parameterized form of a function,
- implicit function by restricting domain and range.

iii. Use MAPLE package plots for plotting different types of functions.

2.6 Limit of a function

i. Identify a real number by a point on the number line.

ii. Define and represent
- open interval,
- closed interval,
- half open and half closed intervals, on the number line.

iii. Explain the meaning of phrase:
- x tends to zero $(x \to 0)$,
- x tends to a $(x \to a)$,
- x tends to infinity $(x \to \infty)$

iv. Define limit of a sequence. v. Find the limit of a sequence whose nth term is given.

vi. Define limit of a function.

vii. State the theorems on limits of sum, difference, product and quotient of functions and demonstrate through examples.

2.7 Important limits

i. Evaluate the limits of functions of the following types:
- $\frac{x^n - a^n}{x-a}, \frac{x-a}{\sqrt{x}-\sqrt{a}}$ when $x \to a$,
- $\left(1+\frac{1}{x}\right)^x$ when $x \to \infty$,
- $(1+x)^{\frac{1}{x}}, \frac{\sqrt{x+a}-\sqrt{a}}{x}, \frac{a^x-1}{x}$,
- $\frac{(1+x)^n-1}{x}$, and $\frac{\sin x}{x}$ when $x \to 0$

ii. Evaluate limits of different algebraic, exponential and trigonometric functions.

iii. Use MAPLE command limit to evaluate limit of a function.

2.8 Continuous and discontinuous functions

i. Recognize left hand and right hand limits and demonstrate through examples.

ii. Define continuity of a function at a point and in an interval.

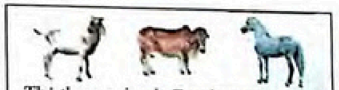
iii. Test continuity and discontinuity of a function at a point and in an interval.

iv. Use MAPLE command iscont to test continuity of a function at a point and in a given interval.

Introduction

The concept of function and its limit is fundamental idea to us, in the study of mathematics that distinguishes calculus from algebra and trigonometry. In this unit, we will revise the concept of function from unit-8 of grade-XI mathematics and then develop the concept of limit which is the fundamental building block on which all the calculus concepts are based.

2.1 Functions



The three animals Zurain owns before he started his collection.

Function are constantly encountered in mathematics and are essential for formulating physical relationships in science and technology. In our routine life function is very useful e.g. Zurain like all kind of animals. He started collecting them recently and already owns 3 animals. He plans on buying every month accordingly, of each type of animals.

Let 'x' be the number of months have past since Zurain started collecting animals. Let 'y' be the number of animals Zurain owns. How can we write a function in terms of x and y?

To write the function, at very beginning, when $x = 0$, Zurain has not bought any new animal, he owns 3 animals so, $y = 3$ animals

After the first month, when $x = 1$, Zurain owns 3 animals, plus the 1 animal he just bought. He now owns $y = 3 + 1$ animals

Similarly, after the second month, when $x = 2$, Zurain owns 3 animals, plus the 2 new animals he bought after he started animals. He now owns
$$y = 3 + 2 \text{ animals}$$

Therefore for x animals the function will be $y = 3 + x$ where, 'x' is *independent variable* and 'y' is *dependent variable.*

In mathematics, the word function is used in much the same way, but more restrictively. It is defined as:

History

Peter Gustav Lejeune Dirichlet (1805-1859)

Peter Dirichlet was a German Mathematician who made valuable contributions in the study of mathematics such as number theory, mechanics and analysis.
He was the first person who gave the modern definition of function in 1837.

"If a variable 'y' depends on a variable 'x' in such a way that each value of 'x' determines exactly one value of 'y' then we call it 'y' is a function of 'x'. e.g $y = f(x)$ "

2.1.1 Domain and range of a function through graph

The domain and range can be identified by the graph. Because domain refers to the set of all input values, so, "all the values shown on the *x-axis*. The range refers to the set of all output values. Which are shown on the *y-axis*. Consider the graph given in Figure 2.1.

$f(x) = \frac{1}{x+2}$

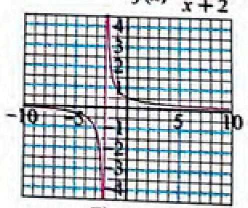


Figure 2.1

This is a graph of the function $f(x) = \frac{1}{x+2}$

Its domain is $(-\infty,-2)\cup(-2,\infty)$ and range is $(-\infty,0)\cup(0,\infty)$

Example 1 Find the domain and range of the function whose graph is shown in the Figure 2.2.

Solution Here the horizontal extent is -6 to 2. So, the domain of the function is $x \in [-6,2]$ vertical extent of the graph is 0 to 4. So the range of the function is $y \in [0,4]$. Shown in the Figure 2.3.

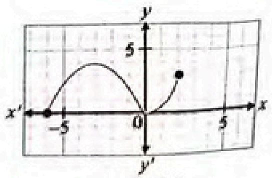


Figure 2.2

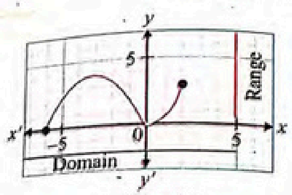


Figure 2.3

Remember

If the graph continuous beyond the given portion of the graph the domain and range can be greater than the given values.

2.1.2 Graph of modulus function (i.e. $y=|x|$) and its domain and range

"A function that contains an algebraic expression with in the absolute value symbols is called an absolute value function."

In our previous classes we have studied that the absolute value of a number is its distance from 0 on the number line.

The parent absolute value function can be written as $f(x)=|x|$ which is defined as $f(x)=\begin{cases} x & if \quad x\geq 0 \\ -x & if \quad x<0 \end{cases}$

To graph the above absolute value functions simply choose some values of x and get the values of y then draw them accordingly.

x	−4	−3	−2	−1	0	1	2	3	4	Domain values
y	4	3	2	1	0	1	2	3	4	Range values

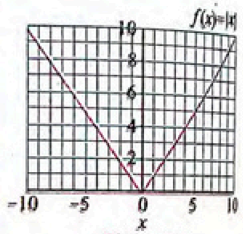


Figure 2.4

Now, plot the points

In general, domain of $f(x)=|x|$ is $(-\infty,\infty)$ and range is $[0,\infty)$ but the graph is v-shaped graph. Figure 2.4 shows the graph of $f(x)=|x|$.

Remember

In any absolute value function for vertical translation of $f(x)=|x|$ you can use the function $g(x)=f(x)+h$

(i) When $h>0$ the graph of $f(x)$ translate 'h' unit up to get $g(x)$.

(ii) When $h<0$ the graph $f(x)$ translate 'h' units down to get $g(x)$.

This is called vertical translation. For horizontal translation of $f(x)=|x|$ you can use the function $g(x)=f(x-k)$

(i) When $k>0$ the graph of $f(x)$ will translate 'k' units to the right to get $g(x)$.

(ii) When $k<0$ the graph of $f(x)$ will translate 'k' units to the left to get $g(x)$.

This is called horizontal translation.

Example 2 Graph the following absolute value functions and identify its domain and range: $f(x)=|3x-4|$

Solution From the definition of absolute value function, the given function is

$$f(x)=\begin{cases} 3x+4 & if \quad 3x+4\geq 0 \\ -(3x+4) & if \quad 3x+4<0 \end{cases}$$

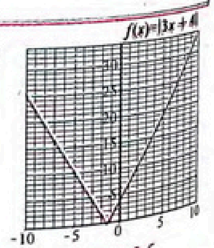


Figure 2.5

The inequality $3x+4\geq 0$ is satisfied whenever $x\geq-\frac{4}{3}$, and $3x+4<0$

is satisfied whenever $x<-\frac{4}{3}$. If $x=-\frac{4}{3}$, $y=0$, so the graph will consist of two lines that meet at $\left(-\frac{4}{3},0\right)$. Use the tabular form to obtain the graph of a function:

$f(x)=3x+4$, $3x+4\geq 0$

x	$-\frac{4}{3}$	0
y	0	4

$f(x)=-(3x+4)$, $3x+4<0$

x	$-\frac{4}{3}$	−3
y	0	5

This function has a domain set $(-\infty,\infty)$ and range set is $[0,\infty)$.

2.2 Composition of Functions

Composition of function can be described as a progression of "getting" and "dropping" "off". A function gets 'x' does something for it and drops it off. Then another function goes along and gets the drop off, does something for it, and drop it off once more. This pattern may proceed more than a few functions. Suppose a composition as a progression of car rides. 'x' boy is picked up by the first car function transported to a required location and dropped off. Then another car function come and pick up the x boy at this new location transports x boy to another location and drops x boy off.

2.2.1 Recognition of composite functions

Consider the function $h(x)$ whose rule is $h(x)=\sqrt{x^3}$. To compute h(4), you first need to find $x^3=4^3=64$ and then take the square root to obtain $\sqrt{x^3}=\sqrt{64}=8$. So the rule of $h(x)$ may be rephrased as:

$$h(x)=f(g(x)) \qquad \text{(i)}$$

Here $g(x)=x^3$, and $f(x)=\sqrt{x^3}$. We may think of the functions $f(x)$ and $g(x)$ as being "composed" to create the function h.

In other words, when the output from one function is used as the input to another function, we form what is known as a composite function.

"If f(x) and g(x) are the two functions, then, a composite function or composition of g and f is the function whose values are given by g(f(x)) for all x in the domain of f(x) such that f(x) is in the domain of g(x)."

Example 3 Let $f(x)=2x-1$ and $g(x)=\sqrt{3x+5}$. Find each of the following:

(a). $g(f(4))$ *(b).* $f(g(4))$ *(c).* $f(g(-2))$

Solution

a. The function $f(x)$ for $x=4$ is used to obtain $f(4)=2(4)-1=7$. Use $f(4)$ to obtain:
$g(f(4))=g(7)=\sqrt{3(7)+5}=\sqrt{26}$

b. The function g(x) for $x=4$ is used to obtain $g(4)=\sqrt{3(4)+5}=\sqrt{17}$. Use g (4) to obtain:
$f(g(4))=f(\sqrt{17})=2\sqrt{17}-1$

c. $f(g(-2))$ does not exist, since −2 is not in the domain of g.

2.2.2 Composition of two given functions

Example 4 Let $f(x) = 4x+1$ and $g(x) = 2x^2 + 5x$. Find each of the following:

(a). $g(f(x))$ *(b).* $f(g(x))$

Solution *a.* Using the given functions to obtain:

$$g(f(x)) = g(4x+1) = 2(4x+1)^2 + 5(4x+1) = 2(16x^2 + 8x + 1) + 20x + 5$$

$$= 32x^2 + 16x + 2 + 20x + 5 = 32x^2 + 36x + 7$$

b. Using the given functions to obtain:

$$f(g(x)) = f(2x^2 + 5x) = 4(2x^2 + 5x) + 1 = 8x^2 + 20x + 1.$$

This example shows that $f(g(x))$ is not usually equal to $g(f(x))$.

Example 5 Air pollution is a problem for many metropolitan areas. Suppose that carbon monoxide is measured as a function of the number of people according to the following information:

Number of People	Daily Carbon Monoxide Level (in parts per million)
100,000	1.41
200,000	1.83
300,000	2.43
400,000	3.05
500,000	3.72

Further studies show that a refined formula for the average daily level of carbon monoxide in the air is $L(p) = 0.7\sqrt{p^2+3}$

Further assume that the population of a given metropolitan area is growing according to the formula $p(t) = 1 + 0.02t^3$, where t is the time from now (in years) and p is the population (in hundred thousands). Based on these assumptions, what level of air pollution should be expected in 4 years?

Solution The level of pollution at time t is given by the composite function:

$$L(p(t)) = L(1+0.02t^3) = 0.7\sqrt{(1+0.02t^3)^2 + 3}, \qquad (i) \qquad \because p(t) = 1 + 0.02t^3$$

The air pollution expected in 4 years is obtained by putting $t = 4$ in equation (:

$$L(p(t)) = L(1+0.02t^3) = 0.7\sqrt{[1+0.02(4)^3]^2 + 3} \approx 2.0 \ ppm$$

2.3 Inverse of composition of functions

"Let $y = f(x)$ be a function of x. This function takes a dependent variable y in response of independent variable x. The function that takes x as dependent variable in response of y as the independent is then called the inverse function of $f(x)$".

The inverse function is denoted by $x = f^{-1}(y)$ (i)

The symbol $f^{-1}(y)$ means the inverse of f and does not mean $\frac{1}{f}$.

For example, if $y = f(x)$ is one–to–one function, then the inverse of $y = f(x)$ is the function $x = f^{-1}(y)$ formed by interchanging the independent and dependent variables x and y for $y = f(x)$. Thus, if (a, b) is a point on the graph of $f(x)$, then (b, a) will be a point on the graph of the inverse of $f(x)$. The domain and range of $y = f(x)$ are also valid for its inverse function $x = f^{-1}(y)$.

Note

If $f(x)$ is not one–to–one, then $f(x)$ does not have an inverse function.

2.3.1 Inverse of the composition of two given functions

Example 6 Find the inverse function of $f(t) = 3t - 8$.

Solution The function $f(t)$ takes an output $3t - 8$ in response of input t. The inverse function must take an output t in response of input $3t - 8$:

$$f^{-1}(3t-8) = t \qquad (i)$$

If $z = 3t - 8$ say, then $t = \frac{z+8}{3}$. Use these values in equation (i) to obtain: $f^{-1}(z) = \frac{z+8}{3}$

Put t as its argument instead of z to obtain the inverse function of $f(t) = 3t - 8$: $f^{-1}(t) = \frac{t+8}{3}$

Example 7 Let $f(x) = 2x + 3$ and $g(x) = 3x$ and $h(x) = f(g(x))$.

Write expressions for the following functions

(a). $h(x)$ *(b).* $f^{-1}(x)$ *(c).* $g^{-1}(x)$ *(d).* $h^{-1}(x)$

Solution

a. In response of $f(x)$ and $g(x)$, the function $h(x)$ is:

$$h(x) = f(g(x))$$
$$= f(3x) \qquad \because g(x) = 3x$$
$$= 2(3x) + 3$$
$$= 6x + 3$$

b. In response of $f(x)$, the inverse of $f(x)$ is:

$$x = f^{-1}(2x+3)$$
$$x = f^{-1}(z), \qquad z = 2x + 3 \Rightarrow x = \frac{(z-3)}{2}$$
$$\frac{z-3}{2} = f^{-1}(z)$$
$$\frac{x-3}{2} = f^{-1}(x) \qquad \textit{insert x instead of z}$$

c. In response of $g(x)$, the inverse of $g(x)$ is:

$$x = g^{-1}(3x)$$
$$x = g^{-1}(z), \qquad z = 3x \Rightarrow x = \frac{z}{3}$$
$$\frac{z}{3} = g^{-1}(z)$$
$$\frac{x}{3} = g^{-1}(x) \qquad \textit{insert x instead of z}$$

d. In response of $h(x)$, the inverse of $h(x)$ is:

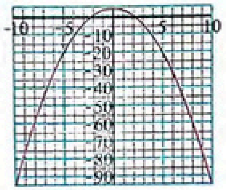
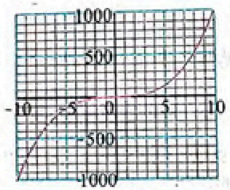
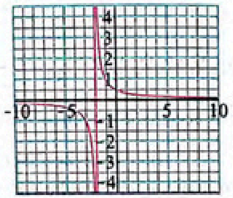
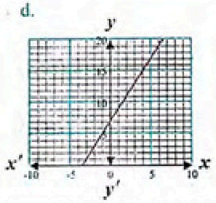
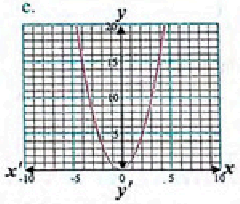
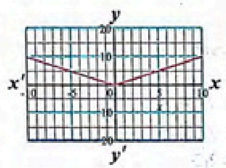
$$x = h^{-1}(6x+3)$$
$$x = h^{-1}(z), \qquad z = 6x + 3 \Rightarrow x = \frac{(z-3)}{6}$$
$$\frac{z-3}{6} = h^{-1}(z)$$
$$\frac{x-3}{6} = h^{-1}(x) \qquad \textit{insert x instead of z}$$

Facts about Calculus

Calculus was discovered as a tool of problem solving. Before the development of calculus, there were a wide range of issues that could not be addressed using the simple mathematics that was available .e.g. people did not know how to measure the speed of different objects when it was changing ever time. Another effective method was desired to calculate the area under the curve. Algebra, geometry, trigonometry and statistics were well understood, but they could not provide necessary tools to address these important issues. Some of the mathematicians of history give the credit to the ancient Greeks for discovering the calculus. But most of the scholars and mathematicians recognize Gottfried Wilhelm von Leibniz and sir. Isaac Newton developed its modern concepts in 17th century. According to the university of Laws Leibniz and Newton held different concept, while Leibniz introduced that the variables of x and y composing "sequences of infinitely close values". But Newton viewed them as variables that change with time. Leibniz considered calculus as a mathematical science for analysis but Newton took it being geometrical science.

Exercise 2.1

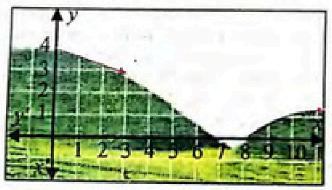
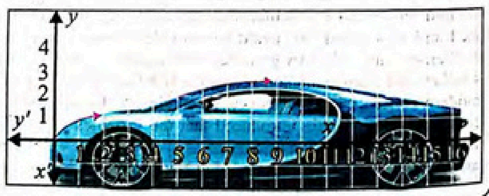
1. Read the graphs and write the function, domain and range of f.

a.

b.

c.

d.

e.

f.


2. Draw the graphs of the following functions.
   a. $2|x-3|$ b. $4|x-3|+3$
   c. $5|3t+7|-2$ d. $2|4x+3|+1$
3. Find the composite functions $f(g(x))$ and $g(f(x))$ of the following functions:
   a. $f(x)=x^2+1,\ g(x)=2x$ b. $f(x)=\sin x,\ g(x)=1-x^2$
   c. $f(x)=\dfrac{x-1}{x+1},\ g(x)=\dfrac{x+1}{1-x}$ d. $f(x)=\sin x,\ g(x)=2x+3$
4. Determine the inverse function of $f(g(x))$ and $g(f(x))$ for the following functions:
   a. $f(x)=x+5,\ g(x)=x-4$ b. $f(x)=2x+7,\ g(x)=2x$
   c. $f(x)=2(x-4),\ g(x)=\dfrac{x+5}{2}$ d. $f(x)=\dfrac{x+4}{2},\ g(x)=2x-4$

Project

Observe the following graphs and identify their domain and range.

i.

ii.


2.4 Transcendental Functions

"Functions that are not algebraic are called transcendental functions."

The functions, such as all trigonometric functions, hyperbolic functions, exponential functions and logarithmic functions are called transcendental functions.

Do You Know?

A polynomial $P_n(x)$ is a function of the form $f(x)=P_n(x)=a_n x^n+a_{n-1}x^{n-1}+....+a_2x^2+a_1x+a_0$ (i) with n is a nonnegative integer and $a_n, a_{n-1},....., a_2, a_1, a_0$ are constants. If $a_n \neq 0$, then, the integer n is called the **degree** of the polynomial.

The constant a_n is called the **leading coefficient** and the constant a_0 is called the **constant term** of the polynomial function. In particular, the polynomial (i) is going to be a

constant function by putting $n=0$: $f(x)=a_0$

linear function by putting $n=1$: $f(x)=a_1x+a_0$

quadratic function by putting $n=2$: $f(x)=a_2x^2+a_1x+a_0$

cubic function by putting $n=3$: $f(x)=a_3x^3+a_2x^2+a_1x+a_0$

2.4.1 Recognition of algebraic, trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic (and their identities), explicit and implicit functions, and parametric representation of functions

i. Algebraic functions

A function $f(x)$ is called an algebraic function if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing or taking roots) starting with polynomials. Any rational function is an algebraic function e.g.

$$f(x)=x+5,\quad g(x)=3x^2+4x-7 \quad\text{and}\quad h(t)=\frac{8t^3+5t-9}{t^4+1}$$

ii. Trigonometric functions

"Trigonometric function are the functions that describe the relationship between the sides and angles of a right triangle".

Any trigonometric function include one or more of the following 6 trigonometric ratios.

(i). sin(x) (ii). cos(x) (iii). tan(x)
(iv). csc(x) (v). sec(x) (vi). cot(x)

These function has completely discussed in grade (XI) Mathematics.

iii. Inverse trigonometric functions

"Inverse trigonometric function are simply defined as the inverse functions of the basic trigonometric function."

These functions are used to get the angle with any of the trigonometric ratios. Inverse trigonometric functions are also known as "Arc functions" particularly these are 6 functions such as:

(i) Arc sine(x) = $\sin^{-1}(x)$ where $x\in[-1, 1]$
(ii). Arc cosine(x) = $\cos^{-1}(x)$ where $x\in[-1, 1]$
(iii). Arc tangent(x) = $\tan^{-1}(x)$ where $x\in R$
(iv). Arc cosecant(x) = $\csc^{-1}(x)$ where $x\geq 1$ or $x\leq -1$
(v). Arc secant(x) = $\sec^{-1}(x)$ where $x\geq 1$ or $x\leq -1$
(vi). Arc cotangent(x) = $\cot^{-1}(x)$ where $x\in R$

Do You Know?

Inverse trigonometric functions are also termed as, cyclometric functions, arcus functions and anti trigonometric functions.

Inverse trigonometric functions are widely used in the field of physics, engineering, geometry and navigations.

iv. Exponential Functions

The exponential function has widespread application in many areas of science and engineering. Areas which utilize the exponential function include expansion of materials, laws of cooling, radioactive decay and the discharge of a capacitor.

"An equation of the form $f(x) = b^x$, $b > 0$, $b \neq 1$, b is a positive constant, defines an exponential function for each different constant b, called the base. The domain of f(x) is the set of all real numbers, and the range of f(x) is the set of all positive real numbers."

We require the base to be positive and to avoid imaginary numbers such as $(-2)^{\frac{1}{2}} = \sqrt{-2} = i\sqrt{2}$.

We conclude $b = 1$ as a base, since $f(x) = 1^x = 1$ is a constant function.

Remember

Exponent laws

If a and b are positive real numbers, $a \neq 1$ and $b \neq 1$, then,

i. $a^x a^y = a^{x+y}, \frac{a^x}{b^y} = a^{x-y}, (a^x)^y = a^{xy}, (ab)^x = a^x b^x, \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ ii. $a^x = a^y$ if and only if $x = y$

iii. For $x \neq 0$, $a^x = b^x$, if and only if $a = b$

Base e Exponential Functions

Of all possible bases b, it can use for the exponential function $y = b^x$, which ones are the most useful? If you look at the keys on a scientific calculator, you will likely see 10^x and e^x. It is clear why base 10 would be important, because our number system is a base 10 system. But what is e, and why is it included as a base? It turns out that base e is used more frequently than all other bases combined. The reason for this is that certain formulas and the results of certain processes found in calculus and more advanced mathematics take on their simplest form if this base is used. This is why you will see e used extensively in expressions and formulas that model real–world phenomena. In fact, its use is so prevalent that you will often hear people refer to $y = e^x$ as the exponential function. The base e is an irrational number (like π) it cannot be represented exactly by any finite decimal fraction. However, e can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x \qquad \text{(i)}$$

for sufficiently large x. What happens to the value of expression (i) as x increases without bound? The results are summarized in the following table:

X	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.71814...
100000	2.71827...
1000000	2.71828...

Challenge

Use binomial theorem to find the value of *e*.

Interestingly, the value of expression (i) is never close to 1, but seems to be approaching a number close to 2.7183. In fact, as x increases without bound, the value of expression (i) approaches an irrational number that we call e. The irrational number e to twelve decimal places is:

$$e = 2.718\,281\,828\,459$$

Example 8 Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as given approximately by

$$N = N_0 e^{1.386t},$$

with N is the number of bacteria present after t hours and N_0 is the number of bacteria present at the start $(t = 0)$. If we start with 25 bacteria, how many bacteria (to the nearest unit) will be present in

(a). 1 hour? (b). 3 hours? (c). 4 hours? (d). Interpret

Solution Use the amount of initial bacteria $N_0 = 25$ in the given equation to obtain: $N = 25e^{1.386t}$ (i), $N_0 = 25$

a. The bacteria at a time $t = 1$ hour is obtained by putting $t = 1$ in equation (i):

$N = 25e^{1.386(1)} = 99.97\ bacteria$

b. The bacteria at a time $t = 3$ hours is obtained by putting $t = 3$ in equation (i):

$N = 25e^{1.386(3)} = 1599\ bacteria$

c. The bacteria at a time $t = 4$ hours is obtained by putting $t = 4$ in equation (i):

$N = 25e^{1.386(4)} = 6392\ bacteria$

d. Thus, we conclude that the population of bacteria is growing when time t increases.

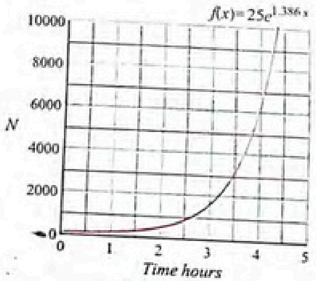


Figure 2.6

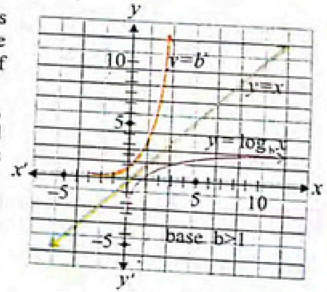
v. Logarithmic Functions

Logarithms are an alternative way of writing expressions which involve powers or indices. They are used extensively in the study of sound. The decimals used in defining the intensity of sound, is based on a logarithmic scale.

Until the development of computers and calculators, logarithms were the only effective tool for large scale numerical computations. They are no longer needed for this, but it still plays a crucial role in many applications.

For illustration, if we start with the exponential function $y = f(x)$ defined by $y = 2^x$ then the interchange of the variables is giving the inverse of $y = 2^x$:

$x = 2^y$



We call this inverse exponential function, the logarithmic function with base 2, and write this as: $y = \log_2 x$ if and only if $x = 2^y$ (i)

"The inverse of an exponential function is called a logarithmic function. For $b > 0$ and $b \neq 1$, the logarithmic function is: $y = \log_b x$ which is equivalent to $x = b^y$"

The log to the base b of x is the exponent to which b must be raised to obtain x. The domain of the logarithmic function is the set of all positive real numbers, which is also the range of the corresponding exponential function. Obviously, the range of the logarithmic function is the set of all real numbers, which is also the domain of the corresponding exponential function.

Typical graphs of an exponential function and its inverse, a logarithmic are shown in the Figure 2.

a. Common Logarithms

Common Logarithms are logarithms with base 10: $y=\log_{10}x$ means $10^y=x$

"Log x", which is read "the logarithm of x", is the answer to the question" to what exponent must 10 be raised to produce x ?

Example 9 Evaluate the following logarithmic functions:.

(*a*). $\log 10000$ (*b*). $\log .01$ (*c*). $\log\sqrt{10}=\frac{1}{2}$

Solution *a.* This is equal to: $\log 10000=\log 10^4=4\log 10=4$

b. This is equal to:

$$\log .01=\log\frac{1}{100}=\log(10)^{-2}=-2\log 10=-2$$

c. This is equal to: $\log\sqrt{10}=\log(10)^{\frac{1}{2}}=\frac{1}{2}\log 10=\frac{1}{2}$

> **Do You Know ?**
> The common logarithm is also called Briggsian logarithms and the natural logarithm is also called Napierian logarithms.

b. Natural Logarithms

Natural Logarithms are logarithms with base e: $y=\ln x$ means $e^y=x$

"ln x", which is read "the *el–en* of x", is the answer to the question" to what exponent must e be raised to produce x"?

$$y=\log_b x \text{ means } b^y=x$$

" $y=\log_b x$ ", which is read "y is the logarithm of x to the base b", is the answer to the question" to what power must b be raised to produce x?

Logarithmic Notation

Common logarithmic: $\log x=\log_{10}x$ Natural logarithmic: $\ln x=\log_e x$

Logarithmic–Exponential Relationships

$\log x=y$ is equivalent to $x=10^y$

$\ln x=y$ is equivalent to $x=e^y$

Properties of Logarithms

If b, M and N are positive real numbers $b\neq 1$, and p and x are also any positive real numbers, then:

i. $\log_b 1=0$ ii. $\log_b b=1$ iii. $\log_b b^x=x$ iv. $b^{\log_b x}=x, x>0$ v. $\log_b MN=\log_b M+\log_b N$

vi. $\log_b\frac{M}{N}=\log_b M-\log_b N$ vii. $\log_b M^p=p\log_b M$ viii. $\log_b M=\log_b N,\ M=N$

Example 10 Find x to four decimal places for the following indicated exponential functions:

(*a*). $10^x=2$ (*b*). $e^x=3$

Solution

a. $10^x=2$

$\log 10^x=\log 2,$ log of both sides

$x\log 10=\log 2,$ $\because \log 10=1$

$x=0.3010$

b. $e^x=3$

$\ln e^x=\ln 3,$ ln of both sides

$x\ln e=\ln 3,$ $\because \ln e=1$

$x=1.0986$

Example 11 Two people with the covid-19 positive visited the campus of Peshawar University. The number of days T that it took for the corona virus to infect n people is given by

$$T(n)=-1.43\ln\left(\frac{10{,}000-n}{4998n}\right)$$

How many days will it take for the virus to infect a. 500 people? *b.* 5000 people?

Solution The number of days T that will take for the flu virus to infect n people is given by

$$T(n)=-1.43\ln\left(\frac{10{,}000-n}{4998n}\right) \quad \text{(i)}$$

The number of days that will take for the virus to infect 500 people, is obtained by putting $n=500$ in equation (i):

a. $T(500)=-1.43\ln\left(\frac{10{,}000-500}{4998(500)}\right)=-1.43\ln\left(\frac{9500}{2499000}\right)=-1.43\ln(0.00380)$

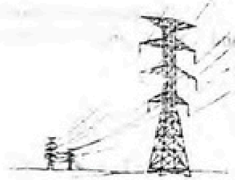
$=-1.43(-5.57275)=7.96903\approx 8$ *days*

b. $T(5000)=-1.43\ln\left(\frac{10{,}000-5000}{4998(5000)}\right)=-1.43\ln\left(\frac{5000}{24990000}\right)=-1.43\ln\left(\frac{1}{4998}\right)=12.17\approx 12$ *days*

vi. Hyperbolic functions and their identities

In physics, it is shown that a heavy, flexible cable (for example a power line) that is suspended between two points at the same height assumes the shape of a curve called a **catenary**, with an equation of the form $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ (i)

This is one of several important applications that involve combinations of exponential functions. In certain ways, the functions we shall study are analogous to be trigonometric functions, and they have essentially the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason, these functions are called hyperbolic functions. Three basic functions are the hyperbolic sine (denoted "sinh$x$" and pronounced "cinch"), the hyperbolic cosine (cosh$x$; pronounced "kosh") and the hyperbolic tangent (tanh$x$; pronounced "tansh"). They are listed as under:



Hanging cable

Figure 2.8

$$\sinh x=\frac{e^x-e^{-x}}{2},\quad \cosh x=\frac{e^x+e^{-x}}{2},\quad \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}},\quad \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

$\cosh^2 x-\sinh^2 x=1$

$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$

$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$

The name "hyperbolic functions" comes from the fact that the functions sinh$t$ and cosh$t$ play the same role in the parametric representation of the hyperbolic $x^2-y^2=1$, as the trigonometric functions sin$t$ and cos$t$, do in the parametric representation of the circle $x^2+y^2=1$.

Eliminating the parameter t from the parametric equations $x=\cos t$, $y=\sin t$ to obtain the equation of the circle: $x^2+y^2=\cos^2 t+\sin^2 t=1$

Similarly, the equations $x=\cosh t$, $y=\sinh t$ are the parametric equations of the hyperbola. Squaring these equations and subtracting the second from the first to obtain the equation of hyperbola: $x^2-y^2=\cosh^2 t-\sinh^2 t=1$

vii. Explicit and Implicit Functions

So far we have met many functions of the form $y=f(x)$: $y=x^2+3,\ y=\sin x,\ y=e^{3x}-2x$ (i)

If y is equated to an expression involving only x terms, then we say that y is expressed **explicitly** in terms of x that is in equation (i).

Sometimes we have an equation connecting x and y but it is impossible to write it in the form of $y=f(x)$:

$y=x^2-y^3+\sin x-\cos y=1,\ \sin(x+y)+e^x+e^{-y}=x^3+y^3$ (ii)

In these cases we say that y is expressed **implicitly** in terms of x.

Remember

The following curves are modeled through implicit functions:

a. The Bifolium curve: $(x^2+y^2)^2 = 4x^2y$

b. Lemniscate curve: $(x^2+y^2)^2 = \frac{25}{3}(x^2-y^2)$

c. Folium of Descartes: $x^3+y^3-\frac{9}{2}xy = 0$

d. Cissoid of Diocles: $y^2(6-x) = x^3$

e. Cardioid curve: $(x^2+y^2)^{\frac{3}{2}} = \sqrt{x^2+y^2}+x$

f. Ellipse curve: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

g. Hyperbola curve: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

viii. Parametric representation of functions

It is sometimes useful to define the variables x and y in the ordered pair (x, y), so that they are each functions of some other variable, say t:

$$x = f(t) \quad \text{and} \quad y = g(t) \qquad \text{(i)}$$

The domain of these functions $f(t)$ and $g(t)$ is some interval D. The variable t is called a parameter and $x = f(t)$ and $y = g(t)$ are called the parametric equations.

" If $f(t)$ and $g(t)$ are continuous functions of parameter t on an interval D, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called the parametric equations for the plane curve generated by the set of ordered pairs in the plane:

$$(x, y) = (x(t), y(t)) = (f(t), g(t)) \qquad \text{(ii)}$$

Example 12 Sketch the graph of the parametric functions $(x(t), y(t)) = (3-t, 2t)$ for all t.

Solution The graph is the collection of all points (x, y) with $x = 3-t, y = 2t$ for different real values of t:

$t = 0 \Rightarrow (x(t), y(t)) = (3,0)$

$t = 1 \Rightarrow (x(t), y(t)) = (2,2)$

$t = 2 \Rightarrow (x(t), y(t)) = (1,4)$

The plot of the position vectors $t_0 = (3,0)$, $t_1 = (2,2), t_2 = (1,4)$ in the Figure 2.9 developed a straight line parallel to the direction vector $u = (-1,2)$ and passing through the point p (3,0).

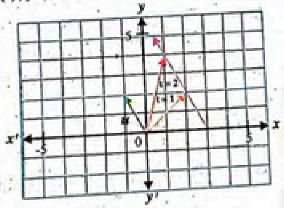


Figure 2.9

2.5 Graphical Representation

In our previous classes we have learnt that graphical representations refers to the use of intuitive charts to clearly visualize and simplify the given data sets. The data is ingested into the graphical representation of software and then represented by the different symbols. Like, curves, bars and slices on the chart.

(a) Graphical display of explicit defined functions like $y = f(x)$, where $f(x) = e^x$, a^x, $\log_a x$, $\log_e x$

i. **Graphically representation of $f(x) = e^x$**

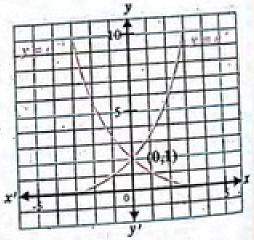


Figure 2.10

Example 13 Draw the graphs of

$$y = e^x \text{ and } y = e^{-x}.$$

Solution Use a scientific calculator to create the table of points. Plot these points and then join them to obtain the graphs of smooth curves in the Figure 2.10.

The domain set is $(-\infty,\infty)$, while the range set is $(0,\infty)$.

ii. **Graphically representation of $f(x) = a^x$**

Example 14 Sketch the graph of $y = 2^x$.

Solution To hand sketch graphs of equations such as $y = 2^x$ or $y = 2^{-x}$, simply make a tables by assigning integers to x, plot the resulting points, and then join these points with a smooth curve as shown in Figure 2.11.

$y = 2^x$						
x	−5	−1	0	1	2	3
y	0.03	0.5	1	2	4	8

$y = 2^{-x}$						
x	−3	−2	−1	0	1	5
y	8	4	2	1	0.5	0.03

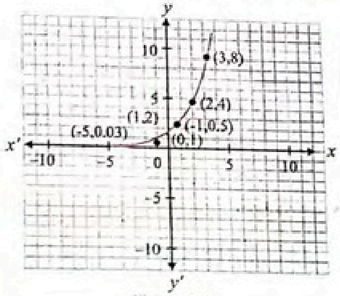


Figure 2.11

It is useful to compare the graphs $y = 2^x$ and $y = 2^{-x}$ by plotting both on the same set of coordinate axes as shown in Figures 2.12. The graph of $f(x) = b^x$, $b > 1$ shown in Figure 2.13 looks very much like the graph of $y = 2^x$, and the graph of $f(x) = b^{-x}$, $0 < b < 1$ in Figure 2.13 looks very much like the graph of $y = 2^{-x}$.

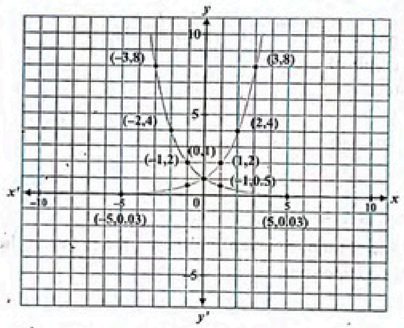


Figure 2.12

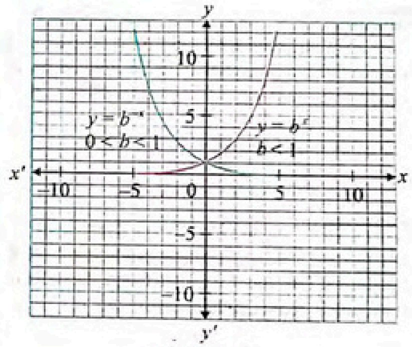


Figure 2.13

The graphs in Figures 2.12 and Figures 2.13 suggest the following important general properties of exponential functions that are summarized in the box below:

Remember

Basic properties of the graph of $f(x) = b^x, b > 0, b \neq 1$

1. All graphs will pass through the point (0,1).
2. All graphs are continuous curves, with no holes or jumps.
3. If $b > 1$, then b^x increases as x increases.
4. If $0 < b < 1$, then b^x decreases as x increases.

iii. Graphically representation of $\log_a x$ and $\log_e x$

Example 15 Sketch the graph of $y = \log_2 x$.

Solution We can graph $y = \log_2 x$ by plotting $x = 2^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components.

For example, ordered pair (3,8) satisfies $y = 2^x$ and (8,3) satisfies equation $x = 2^y$.

$y = 2^x$						
x	−5	−1	0	1	2	3
y	0.03	0.5	1	2	4	8

and

$x = 2^y$ or $y = \log_2 x$						
y	0.03	0.5	1	2	3	8
x	−5	1	0	1	2	3

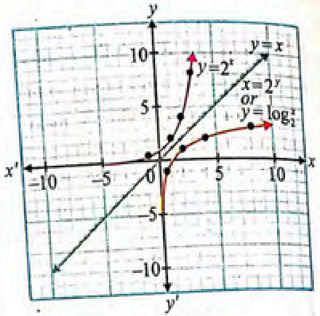


Figure 2.14

The graphs of $y = 2^x$ and $y = \log_2 x$ are shown in the Figure 2.14.

Example 16 Sketch the graph of $y = \log_2(x-2)$.

Solution We can graph $y = \log_2(x-2)$ by plotting $x - 2 = 2^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components.

The graph is shown in the Figure 2.15.

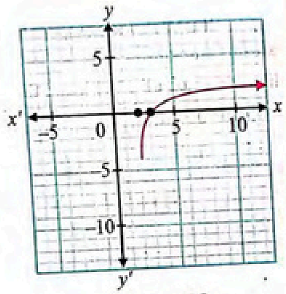


Figure 2.15

Example 17 Sketch the graph of $y = \ln x$.

Solution We can graph $y = \ln x$ by plotting $x = e^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components.

The graph is shown in the Figure 2.16.

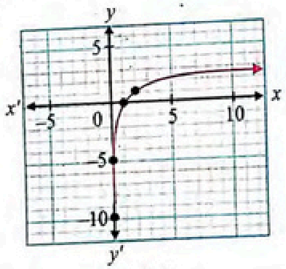


Figure 2.16

Do You Know?

If we fold the paper along the dashed line $y = x$, the two graphs match exactly.
The line $y = x$ is a line of symmetry for the two graphs.

(b) Graphical display of implicit defined function such as $x^2 + y^2 = a^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and distinguish between graph of a function and of an equation

Example 18 Sketch the graph of $x^2 + y^2 = 9$.

Solution This equation $x^2 + y^2 = 9$ is an equation of circle having radius 3 with centred at origin (0, 0).

Rewrite the equation $x^2 + y^2 = 9$ as

$$(x-0)^2 + (y-0)^2 = 3^2$$

The standard form of equation of circle is $(x-h)^2 + (y-k)^2 = r^2$

With centre $(h, k) = (0, 0)$ and radius $r = 3$

So, this is a circle of radius 3 centred at origin (0, 0).

Figure 2.17 is showing the graph of $x^2 + y^2 = 9$.

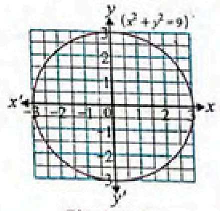


Figure 2.17

Example 19 Sketch the graph of $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution The equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is an equation of ellipse.

Compare it with the standard equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \therefore c^2 = a^2 - b^2$

where, $h = 0, k = 0$ $a = \sqrt{16} = 4$, $b = \sqrt{9} = 3$ and $c = \sqrt{16-9} = \sqrt{7}$

So,
Center: $(h, k) = (0, 0)$
Foci: $(h \pm c, k) = (\pm\sqrt{7}, 0)$
Vertices: $(h \pm a, k) = (\pm 3, 0)$
Co-vertices: $(h, k \pm b) = (0, \pm 4)$

Graph the center, vertices, foci and axes on the graph paper as shown in Figure 2.18.

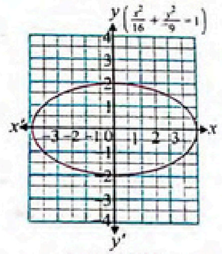


Figure 2.18

Note

The students will learn more about conic in unit 8 and 9.

(c) Graphical display of parametric equation functions such as $x = at^2, y = 2at$; $x = a\sec\theta, y = b\tan\theta$

Example 20 Sketch the graph of the parametric function $(x(t), y(t)) = (3t^2, 4t + 3)$.

Solution To sketch the graph of the parametric equation, Let's make a table to get the idea of the shape and direction of the graph.

t	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
x(t)	108	75	48	27	12	3	0	3	12	27	48	75	108
y(t)	−21	−17	−13	−9	−5	−1	3	7	11	15	19	23	27

$$y = 4t + 3 \Rightarrow t = \frac{y-3}{4} \qquad \dots \text{(i)}$$

$$x = 3t^2 \qquad \dots \text{(ii)}$$

$$x = 3\left(\frac{y-3}{4}\right)^2$$

$$x = \frac{1}{16}(y^2 - 6y + 9) \Rightarrow x = \frac{3}{8}(y^2 - 6y + 9)$$

This is a right opening parabola, its graph is shown in Figure 2.19.

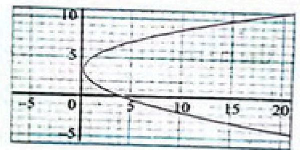


Figure 2.19

Example 21 Sketch the graph of the parametric function $(x(t), y(t)) = (3.\cos(t), 2.\sin(t))$ for $0 \le t \le 2\pi$.

Solution To sketch the graph of the parametric equation. Let's make a table to get the idea of the shape and direction of the graph.

t	0	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{2}\right)$	$\left(\frac{3\pi}{4}\right)$	π	$\left(\frac{5\pi}{4}\right)$	$\left(\frac{3\pi}{2}\right)$	$\left(\frac{7\pi}{4}\right)$	2π
x(t)	3	2.12	0	−2.12	−3	−2.12	0	2.12	3
y(t)	0	1.41	2	1.41	0	−1.41	−2	−1.41	0

Now, convert the standard form by eliminating the parameter. e.g.

$x = 3\cos(t) \Rightarrow x^2 = 9\cos^2(t)$ ... (i)

$y = 2\sin(t) \Rightarrow y^2 = 4\sin^2(t)$ ... (ii)

From (i) and (ii). $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2(t) + \sin^2(t) = 1$

So, the formula of ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ where $a = 3, b = 2$.

This is an ellipse, which will be discussed in detail in unit-9.

(d) Graphical display of discontinuous functions of the type

$$y = \begin{cases} x & when \quad 0 \le x < 1 \\ x-1 & when \quad 1 \le x \le 2 \end{cases}$$

Example 22 Graph the compound function:

$$f(x) = \begin{cases} 3-x & if \quad x < -2 \\ x+2 & if \quad -2 \le x < 2 \\ 1 & if \quad x \ge 2 \end{cases}$$

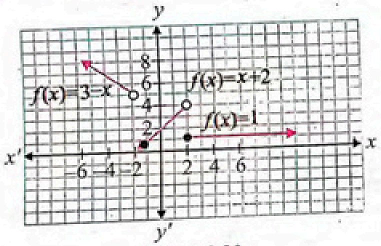


Figure 2.20

Solution

a. Use the function $f(x) = 3 - x$ for $x < -2$ to obtain a set of points:

x	−3	−2
f(x)	6	5

b. Use the function $f(x) = x + 2$ for $-2 \le x < 2$ to obtain a set of points:

x	−2	2
f(x)	0	4

Do You Know ?

A function that defined by more than one equation is called compound function.

c. Use the function $f(x) = 1$ for $x \ge 2$ to obtain a set of points:

x	2	4
f(x)	1	1

Use these tabular points to obtain the graph of a compound function in Figure 2.20.

2.5.2 MAPLE graphic commands for two-dimensional plot of

- an expression (or a function)
- parameterized form of a function
- implicit function by restricting domain and range

Look at the following example, the procedure to use the maple graphic commands is illustrated.

Example 23 Use maple commands to draw the graphs of the given function.

(a). Function $f(x) = -(x+2)^2$, with domain $[0,-4]$.

(b). Parametric function $(x(t), y(t)) = \cos(t).\sin(t)$ for $t = -3.5$ to 3.5, x from -1.5 to 1.5 and y from -1.5 to 1.5.

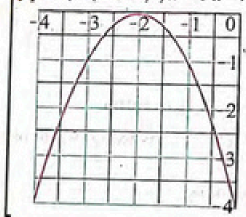
(c). An implicit function $x^2 - y^2 = 1$ x from -5 to 5 and y from -5 to 5.

Solution The command below will show you full detail of plotting expressions/ functions on line by typing:

> ?plots

a. **Command**

> $plot(-(x+2)^2, x = 0..-4);$



>

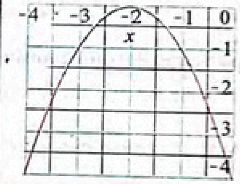
Figure 2.21

Context Menu

> $-(x+2)^2$

$-(x+2)^2$ (1)

> $plot(-(x+2)^2, x = 0..-4)$

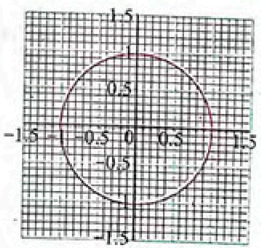


>

Figure 2.22

b. **Command**

> $plot([\cos(t), \sin(t), t = -3.5..3.5], -1.5..1.5, -1.5..1.5);$



>

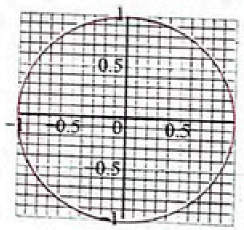
Figure 2.23

Context Menu

> $\sin(t), \cos(t)$

$\sin(t), \cos(t)$ (1)

> $plot([\sin(t), \cos(t), t = -3.5..3.5])$



>

Figure 2.24

This graph is obtained through right-click on the last end of the expression by selecting "Plots < Plot Builder < 2D Parametric Plot" on the context menu.

c. **Command**

```
> plots[implicitplot](x^2 - y^2 - 1,
  x=-5 ..5, y=-5 ..5);
```

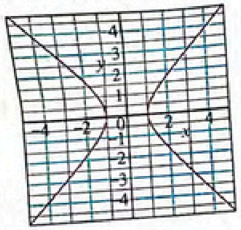


Figure 2.25

Context Menu

```
> x^2 - y^2 = 1
                x^2 - y^2 = 1          (1)
> smartplot[x, y](x^2 - y^2 = 1)
```

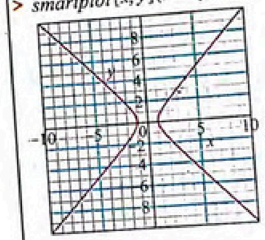


Figure 2.26

This graph is obtained through right-click on the last end of the expression by selecting " Plots < 2D–Implicit Plot < x, y " on the context menu.

2.5.3 MAPLE package plots for plotting different types of functions

Look at the following example the procedure of plotting the functions using maple package is illustrated.

Example 24 Use MAPLE commands to draw the following functions.

Solution *(a).* $f(x) = x^2 - b, x$ from –1 to 10 and y from –5 to 5.

(b). $(x(t), y(t)) = \cos(tx)\sin(ty)), t$ from –1 to 2, x from $-\pi$ to π, y from $-\pi$ to π.

The command below will show you full detail of plotting packages on line by typing

```
> ? plots[animate]
```

a. **Command**

```
> plots[animate](plot, [x^2 - b,
  x=-10 ..10], b=-5 ..5);
                b=-5.
```

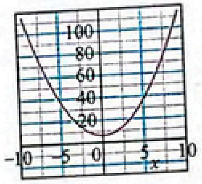


Figure 2.27

Context Menu

```
> x^2 - b
                x^2 - b                (1)
>
> plots[:-animate]('plot', [x^2-b, x=-10..10,
  labels=[x, ""]], b=-5 ..5)
                b=-5.
```

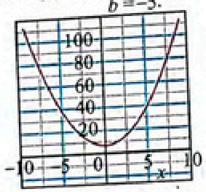


Figure 2.28

This graph is obtained through right-click on the last end of the expression by selecting "Plots < Plot Builder < Animation (choose 2D–Implicit Plot)" on the context menu.

b. **Command**

```
> plots[animate]
  ('plot3d', [cos(t*x)*sin(t*y),
  x=-π..π, y=-π..π], t=1 ..2);
                t=1.
```

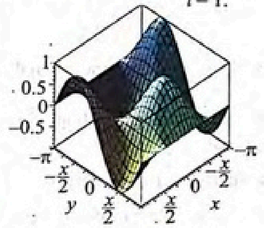


Figure 2.29

Context Menu

```
> plots[:-animate]('plot3d', [cos(t*x)
  *sin(t*y), x=-π..π, y=-π..π,
  labels=[x, y, ""]], t=1 ..2)
                t=1.
```

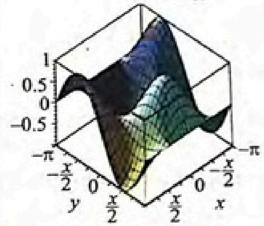


Figure 2.30

This graph is obtained through right-click on the last end of the expression by selecting "Plots < Plot Builder < Animation (choose 3D-Implicit Plot + Parameter) " on the context menu.

Exercise 2.2

1. Recognize and write the type for each of the following functions:
   a. $y = a\sin^2 x + x$ b. $y = 7x^4 + 3x^2 - 4x + 5$ c. $y = \text{arc}\tan x - 7$
   d. $y = \log_2 16 + 7\log_2(x)$ e. $x^2 + y^2 = 36$ f. $y = \dfrac{e^{3x} + e^{-3x}}{e^{3x} - e^{-3x}}$ g. $y = \sqrt{x-8}$
2. A sealed box contains radium. The number of grams present at time t is given by $Q(t) = 100e^{-0.00043t}$ where t is measured in years. Find the amount of radium in the box at the following times:
   a. $t = 0$ b. $t = 800$ c. $t = 1600$ d. $t = 5000$
   e. How did you guess from the above results?
3. Using a calculator and point–by–point to plot the following exponential functions:
   a. $h(x) = (2^x); [-5, 0]$ b. $m(x) = (3^{-x}); [0, 3]$ c. $N = e^t; [0, 5]$ d. $N = e^{-t}; [0, 5]$
4. Using a calculator and point–by–point to plot the following logarithmic functions:
   a. $y = \ln x$ b. $u = -\ln x$ c. $y = 2\ln(x+2)$
   d. $y = 4\ln(x-3)$ e. $y = 4\ln x - 2$
5. Sketch the following parametric curves:
   a. $(x(t), y(t)) = (3-t, 2t)$, t is real number. b. $(x(t), y(t)) = (4\cos t, -3\sin t)$
6. Sketch the graph of:
   a. e^{-2x} b. $\dfrac{2}{3}e^{3x}$ c. 4^x d. 4^{-x}
   e. $\log_2(x+5)$ f. $\log_2 x^2$ g. $\log_e(2x - 5)$
7. Sketch the graph of:
   a. $x^2 + y^2 = 4$ b. $x^2 + y^2 = 16$ c. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ d. $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$
8. Use maple commands to plot the graphs of the functions given in Q.1.

2.6 Limit of Function

The algebraic problems considered in earlier sections dealt with static situations:

What is the revenue when x items are sold?

How much interest is earned in 2 years?

Calculus, on the other hand, deals with dynamic situations:

At what rate is the economy growing?

How fast is a rocket going at any instant after lift–off?

The techniques of calculus will allow us to answer many questions like these that deal with rates of change.

The key idea underlying the development of calculus is the concept of limit. So we begin by studying limits after explaining the location of intervals on the real number line.

2.6.1 Identification of a real number by a point on the number line

The various types of numbers used in this book can be illustrated with a diagram called a number line. Each real number corresponds to exactly one point on the line and vice–versa. A number line with several sample numbers located on it is shown in Figure 2.31:

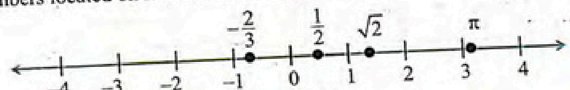


Figure 2.31

2.6.1 Representation of open interval, closed interval, half open and half closed intervals on the number line

"A set that consists of all the real numbers between two points is called an interval."

A special notation will be used to indicate an interval on the real number line.

For example, the interval including all numbers x, where $-2 < x < 3$ is written as $(-2, 3)$. The parentheses indicate that the number -2 and 3 are not included.

If -2 and 3 are to be included in the interval, square brackets are used, as in $[-2, 3]$.

The chart below shows several typical intervals, where $a < b$:

Inequality	Interval Notation	Explanation
o $a \le x \le b$	$[a,b]$: Closed	Both a and b are included.
o $a \le x < b$	$[a,b)$: Half open/Closed	a is included, b is not.
o $a < x \le b$	$(a,b]$: Half Open/Closed	b is included, a is not.
o $a < x < b$	(a,b): Open	Both a and b are not included.

Interval notation is also used to describe sets such as the set of all numbers x, with $x \ge -2$. This interval is written $[-2,\infty)$.

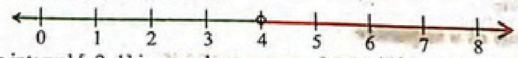
Example 24 Represent the following intervals on number line.

(a). $[-2,\infty)$ (b). $[4,\infty)$ (c). $[-2,1]$

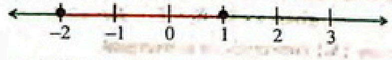
Solution

a. Start at -2 and draw a heavy line to the right, as in graph. Use a solid hole at -2 to show that -2 is itself a part of the graph.
The symbol ∞, read "infinity" does not represent a number. It simply indicates that all numbers greater than -2 are in the interval. Similarly, the notation $(-\infty,2)$ indicates the set of all real numbers with $x < 2$.

b. The graph of the interval $[4,\infty)$ is as under:



c. The graph of the interval $[-2, 1]$ is as under:

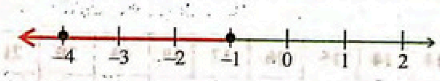


Example 25 Use number line to indicate the interval notation:

(a).

(b).

(c).



Solution The given graphs indicate the following intervals:

a. $(-5, 3)$ b. $[4, 7]$ c. $(-\infty,-1]$

2.6.3 Explanation of phrase

o x tends to zero $(x \to 0)$ o x tends to a $(x \to a)$ o x tends to infinity $(x \to \infty)$

i. x tends to zero $(x \to 0)$

The answer to the phrase x tends to "0" is easy to see that the value of a function $y = f(x) = \frac{x^2-4}{x-2}$ gets closer and closer to a single real number "2" on both left and right sides of "2", when x is a number very close to "0" on both left and right sides of "0". In this situation, we are in position to say that x approaches to "0" or x tends to "0" and is denoted by $x \to 0$, when $f(x)$ tends to a single number "say $L = 2$".

ii. x tends to a $(x \to a)$

The answer to the phrase x tends to "a" (a is any real number) is easy to see that the value of a function

$$y = f(x) = \frac{x^2-a^2}{x-a},$$

gets closer and closer to a single real number "$2a$" on both left and right sides of "$2a$", when x is a number very close to "a" on both left and right sides of "a". In this situation, we say that x approaches to "a" or x tends to "a" and is denoted by $x \to a$, when $f(x)$ tends to a single number "say $L = 2a$".

iii. x tends to infinity $(x \to \infty)$

The answer to the phrase x tends to "infinity" is easy to see that the function $f(x) = \frac{3x+2}{x+1}$

gets smaller and smaller, when x approaches "infinity" from either side of a number say 3. In this situation, we say that the function $f(x)$ gets closer and closer to a single number "say $L = 3$" when $x \to \infty$ from either side.

2.6.4 Limit of a sequence

The number L is the limit of the sequence $\{S_n\}$ if

$$\text{given } \varepsilon > 0,\ S_n \approx L \quad for\ n \geq 1 \qquad (i)$$

If such an L exist, we say $\{S_n\}$ converges, or convergent.
If 'L' does not exist, $\{S_n\}$ diverges or divergent. There are two notations which we use to show the limit of a sequence.

$$\lim_{n\to\infty}\{S_n\} = L, \quad S_n \to L \ \ as\ \ n \to \infty$$

These notations are abbreviated as $\lim S_n = L$ or $S_n \to L$

2.6.5 Limit of a sequence whose nth term is given

Let $S_n = \dfrac{n+2}{n^2}$ be a sequence. To get the first few terms of this sequence we need to plug in values of n into the general form of the sequence. We will get the sequence terms considering n as positive integer.

n	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$\frac{n+2}{n^2}$	$\frac{13}{121}$	$\frac{7}{72}$	$\frac{15}{169}$	$\frac{4}{49}$	$\frac{17}{225}$	$\frac{9}{128}$	$\frac{19}{289}$	$\frac{5}{81}$	$\frac{21}{361}$	$\frac{1}{200}$	$\frac{23}{441}$	$\frac{6}{121}$	$\frac{25}{529}$	$\frac{1}{288}$	$\frac{27}{625}$

Similarly, we can get more terms by using this process and write the above sequence in the following notation:

$$S_n = \left\{\frac{n+2}{n^2}\right\}_{n=1}^{\infty} = 3, 1, \frac{5}{9}, \frac{3}{8}, \frac{7}{25}, \frac{2}{9}, \frac{9}{49}, \ldots$$

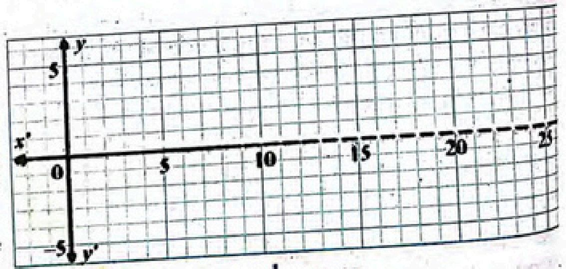


Figure 2.32

In the above sequence we treated it as a function that can only have integers plugged into them. This is an important idea which allows us to do many things with sequences that we can not compute by using other methods. To graph the sequence $\{S_n\}$ we plot the points (n, S_n) as n ranges over all possible values on the graph representing first 25 terms of the given sequence. From the graph we noticed that as n increases the terms of the sequence get closer and closer to zero, but not exactly equal to zero. We then say that zero is limiting value of the sequence and it can be written as

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n+2}{n^2} = 0$$

Example 26 Represent the sequence on one dimensional space whose nth term is $s_n = \dfrac{8n}{n+3}$.

Solution The sequence $\{s_n\}$ in terms of function notation is $s(n) = \dfrac{8n}{n+3}$, whose domain is the set of non-negative integers. The functional values of $s(n)$ develop

n: Integers	Function: $s(n)$
1	$s_1 = s(1) = 2.0$
2	$s_2 = s(2) = 3.2$
3	$s_3 = s(3) = 4.0$
4	$s_4 = s(4) = 4.57$

The one dimensional view on a real number line is shown in figure

Remember

Limit theorem of sequence

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then the limit exist are the following:

(i). $\lim_{n\to\infty}(ra_n + sb_n) = rL + sM$ Linearity rule

(ii). $\lim_{n\to\infty}(a_n b_n) = LM$ Product rule

(iii). $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{L}{M}, M \neq 0$ Quotient rule

(iv). $\lim_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{L}$ Root rule

Example 27 Find the limit of each of these convergent/divergent sequences.

(a). $\left\{\dfrac{8n}{n+3}\right\}$ (b). $\left\{\dfrac{n^5+n^3+2}{7n^4+n^2+3}\right\}$ (c). $\{(-1)^n\}$

Solution

a. Let $\{a_n\} = \dfrac{8n}{n+1}$

$$\lim_{n\to\infty}\{a_n\} = \lim_{n\to\infty}\frac{8n}{n+1}$$

$$= \lim_{n\to\infty}\frac{n(8)}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{8}{1+\frac{1}{n}}$$

$$= \frac{8}{1+\frac{1}{\infty}} = \frac{8}{1} = 8 \qquad \because \frac{1}{\infty} = 0$$

Hence, the sequence is converging to 8.

Remember

Some of the sample points near n = infinity are:

n	$\lim(-1)^n$
1	-1
10	1
100	1
1000	1
10000	1
100000	1

b. Let $a_n = \dfrac{n^5+n^3+2}{7n^4+n^2+3}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^5+n^3+2}{7n^4+n^2+3} = \lim_{n\to\infty}\frac{n^5\left(1+\frac{1}{n^2}+\frac{2}{n^5}\right)}{n^5\left(\frac{7}{n}+\frac{1}{n^3}+\frac{3}{n^5}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}+\frac{2}{n^5}}{\frac{7}{n}+\frac{1}{n^3}+\frac{3}{n^5}} = \frac{1+\frac{1}{(\infty)^2}+\frac{2}{(\infty)^5}}{\frac{7}{\infty}+\frac{1}{(\infty)^3}+\frac{3}{(\infty)^5}} = \frac{1}{0}$$

In the above expression the numerator tends to 1 as $n \to \infty$, but the denominator approaches to 0. So, the quotient increases without bound. Hence, the sequence is divergent.

c. Let $a_n = (-1)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n$$

The sequence does not approaches to any specific number. So it is divergent sequence by oscillation. The nth term is always either 1 or −1. It is 1 when n is even and −1 when n is odd.

2.6.6 Limit of a function

"Let f (x) be a function defined on an open interval X. Containing x = c (the value f (c) do not needs to be defined)

The specific number L is called the limit of function f (x) as x → c if and only if, for every ε > 0 there exist δ → 0 such that $|f(x)-L|<\varepsilon$ *when ever,* $0<|x-c|<\delta$"

This definition is also known as the couchy definition for limit.

Usually limit of a function is written as $\lim_{x\to c} f(x) = L$ and read as "Lim of $f(x)$ as $x \to c$ is L"

This is neither desirable nor practicable to find the limit of a function by numerical approach. You must be able to evaluate a limit in some mechanical way.

2.6.7 Theorems on limits of sum, difference, product and quotient of functions and demonstrate through examples

Let $f(x)$ and $g(x)$ be two functions, for which $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

i. The limit of the sum of two functions is equal to the sum of their limits

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L+M$$

Example 28 If $f(x)=x^2+2x+3$ and $g(x)=x-4$ then calculate $\lim_{x\to 2}[f(x)+g(x)]$

Solution Since, $f(x)=x^2+2x+3$ (i)

$g(x)=x-4$ (ii)

By adding (i) and (ii) $f(x)+g(x)=x^2+3x-1$ (iii)

Applying $\lim_{x\to 2}$ on both sides of (iii).

$$\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2}(x^2+3x-1)$$

$$= \lim_{x\to 2} x^2 + 3\lim_{x\to 2}(x) - \lim_{x\to 2}(1) = 4+6-1=9$$

Hence, $\lim_{x\to 2}[f(x)+g(x)]=9$

Remember

If $f(x)=K$, where K is any constant than

$\lim_{x\to c} f(x) = \lim_{x\to c} K = K$ (constant rule)

$\lim_{x\to c} K f(x) = K \lim_{x\to c} f(x) = K.L$ (multiple rule)

ii. The limit of the difference of two functions is equal to the difference of their limits

$$\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L-M$$

Example 29 If $f(x)=x-7$ and $g(x)=x^2+3x+2$ then calculate $\lim_{x\to 1}[f(x)-g(x)]$

Solution Since, $f(x)=x-7$ (i)

$g(x)=x^2+3x+2$ (ii)

By subtracting (ii) from (i)

$$f(x)-g(x) = (x-7)-(x^2+3x+2) = x-7-x^2-3x-2$$

$$= -x^2-2x-9 = -(x^2+2x+9) \quad \text{(iii)}$$

By applying $\lim_{x\to 1}$ on both sides of equation (iii).

$$\lim_{x\to 1}[f(x)-g(x)] = \lim_{x\to 1} -(x^2+2x+9) = -\lim_{x\to 1}(x^2+2x+9)$$

$$= -\left[\lim_{x\to 1}(x^2) + 2\lim_{x\to 1} x + \lim_{x\to 1}(9)\right] = -[1+2+9] = -12$$

Hence, $\lim_{x\to 1}[f(x)-g(x)] = -12$

iii. The limit of the product of the functions is equal to the product of their limits

$$\lim_{x\to c}[f(x).g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = LM$$

Example 30 If $f(x)=x+5$ and $g(x)=2x-4$ then calculate $\lim_{x\to 3}[f(x).g(x)]$

Solution Since, $f(x)=x+5$ (i)

$g(x)=2x-4$ (ii)

By multiplying equation (i) and equation (ii).

$$f(x).g(x) = (x+5)(2x-4) = 2x^2-4x+10x-20 = 2x^2+6x-20 \quad \text{(iii)}$$

By applying $\lim_{x\to 3}$ on both sides of equation (iii).

$$\lim_{x\to 3}[f(x).g(x)] = \lim_{x\to 3}(2x^2+6x-20) = 2\lim_{x\to 3}(x^2) + 6\lim_{x\to 3}(x) - \lim_{x\to 3}(20)$$

$$= 2(9)+6(3)-20 = 18+18-20 = 16$$

Hence, $\lim_{x\to 3}[f(x).g(x)] = 16$

iv. The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M} \text{ where } g(x)\neq 0 \text{ and } M\neq 0.$$

Example 31 If $f(x)=x-4$ and $g(x)=x^2+3$ then calculate $\lim_{x\to -2}\left[\frac{f(x)}{g(x)}\right]$

Solution Since, $f(x)=x-4$ (i)

$g(x)=x^2+3$ (ii)

By using equation (i) and equation (ii). $\frac{f(x)}{g(x)} = \frac{x-4}{x^2+3}$ (iii)

By applying $\lim_{x\to -2}$ on both sides of equation (iii).

$$\lim_{x\to -2}\frac{f(x)}{g(x)} = \lim_{x\to -2}\left[\frac{x-4}{x^2+3}\right] = \frac{\lim_{x\to -2}(x-4)}{\lim_{x\to -2}(x^2+3)} = \frac{-2-4}{4+3} = -\frac{6}{7}$$

Hence, $\lim_{x\to -2}\left[\frac{f(x)}{g(x)}\right] = -\frac{6}{7}$

Important Limits

2.7.1 Limits of the functions of the following types

i. $\dfrac{x^n - a^n}{x-a}, \dfrac{x-a}{\sqrt{x}-\sqrt{a}}$ when $x \to a$

a. $\lim\limits_{x\to a}\left(\dfrac{x^n - a^n}{x-a}\right) = na^{(n-1)}$

Proof: In this situation, we need to divide out the numerator by denominator to obtain:

$$\frac{x^n - a^n}{x-a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots\dots + a^{n-2}x + a^{n-1} \quad (i)$$

Being a polynomial, the function to the right of the above expression (i) is continuous for all values of x and as such its limit, when $x \to a$ must equal to its value at $x = a$. Thus, the limit of the expression (i), when x tends to a is:

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = \lim_{x\to a}(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots\dots + a^{n-2}x + a^{n-1})$$
$$= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \dots\dots + aa^{n-2} + a^{n-1}$$
$$= a^{n-1} + a^{n-1} + a^{n-1} + \dots\dots + a^{n-1} + a^{n-1} = na^{n-1}$$

b. $\lim\limits_{x\to a}\left(\dfrac{x-a}{\sqrt{x}-\sqrt{a}}\right) = 2\sqrt{a}$

Proof: When $x \to a$, the limit of a function is of the form $\left(\dfrac{0}{0}\right)$, which is undefined. In this situation, we need to rationalize the given function to obtain the required limit:

$$\lim_{x\to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right) = \lim_{x\to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right)\times\left(\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right) = \lim_{x\to a}\frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}$$
$$= \lim_{x\to a}(\sqrt{x}+\sqrt{a}) = \lim_{x\to a}\sqrt{x} + \lim_{x\to a}\sqrt{a} = \sqrt{a}+\sqrt{a} = 2\sqrt{a}$$

ii. $\left(1+\dfrac{1}{x}\right)^x$ when $x \to \infty$

Prove that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$

Proof: The base "e" is an irrational number (like π), it cannot be represented exactly by any finite decimal fraction. However, e can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x \quad (i)$$

for sufficiently large x. What happens to the value of the expression as x increases without bound? The results are summarized in the following table:

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.71814...
100000	2.71827...
1000000	2.71828...

Activity

Use binomial theorem to generate e.

Interestingly, the value of expression (i) is never close to 1, but seems to be approaching a number close to 2.7183. In fact, as x increases without bound, the value of expression approaches an irrational number that we call e. The irrational number e to twelve decimal places is:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 2.718281828459 = e$$

From result (ii), the new result deduced is:

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = 2.718281828459 = e$$

iii. $(1+x)^{\frac{1}{x}}, \dfrac{\sqrt{x+a}-\sqrt{a}}{x}, \dfrac{a^x-1}{x}, \dfrac{(1+x)^n-1}{x}$ and $\dfrac{\sin x}{x}$ when $x \to 0$

a. **Show that** $\lim\limits_{x\to\infty}(1+x)^{\frac{1}{x}} = e$

Proof: If we put $y = \dfrac{1}{x}$, then $y \to \infty$, when $x \to 0$, and the left-hand side of the limit thus gives the right-hand side: $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = \lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^y = e$

b. **Prove that** $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \ln a,\ a > 0.$

Proof: If we put $a^x - 1 = y$, then x is obtained by taking log of both sides:

$$a^x - 1 = y \quad \Rightarrow a^x = 1 + y$$
$$\log(a^x) = \log(1+y) \quad \Rightarrow x\log a = \log(1+y)$$
$$x = \frac{\log(1+y)}{\log a}$$

Use this x in the expression $\dfrac{a^x-1}{x}$ to obtain: $\Rightarrow \dfrac{a^x-1}{x} = \dfrac{y}{\dfrac{\log(1+y)}{\log a}}$

Taking limit $y \to 0$, when $x \to 0$:

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\frac{\log(1+y)}{\log a}}=\lim_{y\to 0}\frac{1}{\frac{1}{y}\log(1+y)\left(\frac{1}{\log a}\right)}=\lim_{y\to 0}\frac{\log a}{\log(1+y)^{\frac{1}{y}}}$$

$$=\log a\frac{1}{\log\lim_{y\to 0}(1+y)^{\frac{1}{y}}}=\log a\,\frac{1}{\log e}=\log_e a=\ln a$$

By replacing 'a' with 'e' the following result can be deduced.

$$\lim_{x\to 0}\frac{e^x-1}{x}=\ln e=1$$

c. Prove that $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$

Proof: We have to show that: $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$

Take L.H.S $\quad \lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{\sqrt{0+a}-\sqrt{a}}{0}=\frac{0}{0}$ *which is undefined.*

Now, rationalize the function $f(x)$.

$$f(x)=\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\right)\left(\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right)$$

$$=\frac{x+a-a}{x\left(\sqrt{x+a}+\sqrt{a}\right)}=\frac{x}{x\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$\Rightarrow \lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{1}{\sqrt{x+a}+\sqrt{a}}$$

$$=\frac{1}{\sqrt{0+a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}=\text{R.H.S}$$

d. Prove that $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$

Proof: If we put $(1+x)^n-1=z$, then: $(1+x)^n-1=z$

$\quad (1+x)^n=(1+z)$ Taking log of both sides to obtain:

$$\log(1+x)^n=\log(1+z)$$

$$n\log(1+x)=\log(1+z)$$

Use these expressions in the left-hand side of the limit to obtain the right-hand side:

$$\frac{(1+x)^n-1}{x}=\frac{z}{x}=\frac{z}{x}\times\frac{\log(1+z)}{\log(1+z)}=\frac{z}{\log(1+z)}\times\frac{n\log(1+x)}{x}=\frac{1}{\log(1+z)^{\frac{1}{z}}}\times n\log(1+x)^{\frac{1}{x}}$$

Taking limit $z\to 0$, when $x\to 0$:

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{z\to 0}\frac{1}{\log(1+z)^{\frac{1}{z}}}\times\lim_{x\to 0}n\log(1+x)^{\frac{1}{x}}=\frac{1}{\log e}\times n\log e=n$$

Do You Know?

The sandwich theorem: This is a theorem that is used in calculus to evaluate a limit of a function. It is particularly useful to evaluate limits where other techniques might be unnecessarily complicated. To define sandwich theorem:

"Let $f(x)$, $g(x)$ and $h(x)$ be functions such that $f(x)\le g(x)\le h(x)$ for all x in some open interval containing "c", except possibly at c itself. If $\lim_{x\to c}f(x)=L$ *and* $\lim_{x\to c}h(x)=L$ *then* $\lim_{x\to c}g(x)=L$

e. Prove that $\lim_{x\to 0}\frac{\sin x}{x}=1$ if angle x is measured in radian.

Proof: Take a positive acute central angle of a circle with radius $r=1$ as shown in the figure.

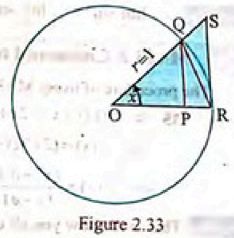


Figure 2.33

Given that ΔOPQ, $\sin x=\frac{|QP|}{|QO|}=|PQ|$ $\quad |QO|=1$ (radius of unit circle)

In ΔORS, $\tan x=\frac{|SR|}{|RO|}=|RS|$

In term of x, the areas are expressed as produce QO to S so, that $SR\perp RO$ join QR.

(i) Area of $\Delta ORQ=\frac{1}{2}|OR||PQ|=\frac{1}{2}(1)(\sin x)=\frac{1}{2}\sin x$

(ii) Area of sector $ORQ=\frac{1}{2}r^2x=\frac{1}{2}(1)(x)=\frac{x}{2}$

(iii) Area of $\Delta ORS=\frac{1}{2}|OR||RS|=\frac{1}{2}(1)(\tan x)=\frac{1}{2}\tan x$

From the figure we observed

Area of $\Delta QRO<$ Area of sector $QRO<$ Area of Δ SRO.

$$\Rightarrow\frac{1}{2}\sin x<\frac{1}{2}x<\frac{1}{2}\tan x$$

As $\sin x$ is positive, so, dividing by $\frac{1}{2}\sin x$ we get

$$0<\frac{x}{\sin x}<\frac{1}{\cos x}\left(1<x<\frac{\pi}{2}\right)\text{ i.e. } 1>\frac{\sin x}{x}>\cos x \text{ or } \cos x<\frac{\sin x}{x}<1$$

When $x\to 0$, $\cos x\to 1$

Since $\frac{\sin x}{x}$ is sandwiched between 1 and a quantity approaches 1 itself.

Therefore, by the sandwich theorem, it must also approach 1 i.e., $\lim_{x\to 0}\frac{\sin x}{x}=1$.

2.7.2 Limits of different algebraic, exponential and trigonometric functions

The idea of limits in the above situations is illustrated in the following examples:

Example 32 Evaluate $\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6}$.

Solution $\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6}=\frac{\lim_{x\to 1}(x^3-3x+7)}{\lim_{x\to 1}(5x^2+9x+6)}=\frac{1-3+7}{5+9+6}=\frac{5}{20}=\frac{1}{4}$

Example 33 Evaluate $\lim_{x\to -2} \sqrt[3]{(x^2-3x-2)}$.

Solution

$$\lim_{x\to -2} \sqrt[3]{x^2-3x-2} = \lim_{x\to -2}(x^2-3x-2)^{\frac{1}{3}} = [\lim_{x\to -2}(x^2-3x-2)]^{\frac{1}{3}} = [(-2)^2-3(-2)-2]^{\frac{1}{3}} = (8)^{\frac{1}{3}} = 2$$

Example 34 Evaluate the limits

(a). $\lim_{x\to 0}\sin^2 x$ (b). $\lim_{x\to 0}(1-\cos x)$, when $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0}\cos x = 1$.

Solution a. $\lim_{x\to 0}\sin^2 x = \left[\lim_{x\to 0}\sin x\right]^2 = 0$ b. $\lim_{x\to 0}(1-\cos x) = \lim_{x\to 0} 1 - \lim_{x\to 0}\cos x = 1-1 = 0$

2.7.3 MAPLE Command to evaluate limit of a function

The procedure of using MAPLE command 'limit' is illustrated in the following example.

Example 35 (a). $f(x) = x^2+2x+2$, when x tends to 2.

(b). $f(x) = (2x^3)(x-4)$, when x tends to 3.

(c). $f(x) = \dfrac{(x^3-a^3)}{(x-a)}$, when x tends to a.

Solution This will show you all commands about the limits.

a. Command

> limit$((x^2)+(2x+2), x=2)$; 10

Using Palettes: Use cursor button to select limit palette. Click-the required limit palette and replace a by 2. Click $(a+b)$ (for sum rule of a function), then press "Enter" key to obtain the required limit:

> $\lim_{x\to 2}((x^2)+(2\cdot x+2))$

10

Remember

If the command for the required limit of a function is not known to you, then, easily on line, call the command by typing:

> *limit.*

b. Command

> limit$((2x^3)(x-4), x=3)$;

-10

Using Palettes: Use cursor button to select limit palette. Click-the required limit, and replace a by 3. Click-$(a*b)$ (for product rule of a function), then "Enter" key to obtain the required limit:

> $\lim_{x\to 3}((2\cdot x^3)\cdot(x-4))$

-54

c. Command

> $limit\left(\dfrac{(x^3-a^3)}{(x-a)}, x=a\right)$;

$3a^2$

Using Palettes: Use cursor button to select limit palette. Click-the required limit and replace a by a. Click-$\left(\frac{a}{b}\right)$ (the quotient rule of a function), then "Enter" key to obtain the required limit:

> $\lim_{x\to a}\dfrac{x^3-a^3}{x-a}$

$3a^2$

Exercise 2.3

1. Evaluate the following limits:

a. $\lim_{x\to 4}\left(\dfrac{3}{x}+\dfrac{1}{x-5}\right)$ b. $\lim_{x\to 1}\left(\dfrac{x^2+3x+2}{x^2+x+2}\right)^2$ c. $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$

d. $\lim_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$ e. $\lim_{x\to 0}\dfrac{1-\sin x}{\cos^2 x}$ f. $\lim_{x\to 0}\dfrac{\tan x}{x}$

g. $\lim_{x\to 0}\dfrac{\sec x-1}{x\sec x}$ h. $\lim_{x\to 0}\dfrac{\sin^2(7x)}{7x}$

2. Use algebra and the rules of limits to evaluate the following limits:

a. $\lim_{x\to 4}\dfrac{-6}{(x-4)^2}$ b. $\lim_{x\to 0}\dfrac{\left(\frac{1}{(x+3)}\right)-\frac{1}{3}}{x}$ c. $\lim_{x\to 5}\dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

3. Find the limit of the convergent of the following sequences:

a. $\left\{\dfrac{5n}{n+7}\right\}$ b. $\left\{\dfrac{4-7n}{8+n}\right\}$ c. $\dfrac{(-1)^n}{n^2}$

4. Weekly sales (in rupees) at big store x weeks after the end of an advertising campaign are given by: $S(x) = 5000+\dfrac{3600}{x+2}$

Find the sale for the indicated weeks limits:

a. $S(5)$ b. $\lim_{x\to 5}S(x)$ c. $\lim_{x\to 16}S(x)$

5. Use MAPLE command "limit" to evaluate the limit of all parts of Q.1.

6. Use algebraic techniques to evaluate the following.

a. $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ b. $\lim_{x\to a}\dfrac{x^n-a^n}{x^m-a^m}$ c. $\lim_{\theta\to 0}\dfrac{1-\cos p\theta}{1-\cos q\theta}$

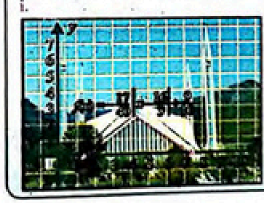
d. $\lim_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$ e. $\lim_{n\to +\infty}\left(1-\dfrac{1}{n}\right)^n$ f. $\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$

g. $\lim_{x\to +\infty}\left(\dfrac{x}{1+x}\right)^x$

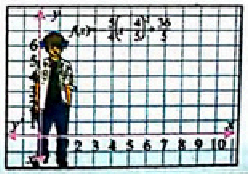
Project

What type of function are represented by the curves drawn on each image?

i.



ii.



2.8 Continuous and Discontinuous Functions

Before discussion about continuous and discontinuous function we will revise the concept of limit of a function, which we have done in previous Section 2.6.

2.8.1 Recognition of left and right hand limits

It is a value the function approaches as the x-values approach the limit from one side only i.e. the left side limit and the right side limit.

i. **The left hand limit of a function**

A given function $f(x)$ has a left hand limit if $f(x)$ can be made as close to the number. 'L' as we please for all values of $x < c$ e.g. $\lim_{x\to c^-} f(x) = L$

ii. **The right hand limit of a function**

A function $f(x)$ has a right hand limit if $f(x)$ can be made as to the number 'L' as we please for all values of $x > c$. $\lim_{x\to c^+} f(x) = L$

"In general, the function has a limit as x approaches c if both the left hand and right hand limit at c exist and are e... l. $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c} f(x) = L$ "

Example 36 Determine whether $\lim_{x\to 3} f(x)$ and $\lim_{x\to 5} f(x)$ exist, if

$$f(x) = \begin{cases} 3x+4 & if\ 0 \le x < 3 \\ 16-x & if\ 3 \le x < 12 \\ x & if\ 12 \le x < 14 \end{cases}$$

Solution

(a). $\lim_{x\to 3^-} f(x) = \lim_{x\to 3} f(x) = \lim_{x\to 3}(3x+4) = 3(3)+4 = 13$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = \lim_{x\to 3}(16-x) = 16-3 = 13$

Since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = 13$

$\Rightarrow \lim_{x\to 3} f(x)$ exists and equal to 13.

(b). $\lim_{x\to 12^-} f(x) = \lim_{x\to 12} f(x) = \lim_{x\to 12}(16-x) = 16-12 = 4$

$\lim_{x\to 12^+} f(x) = \lim_{x\to 12} f(x) = \lim_{x\to 12}(x) = 12$

Since $\lim_{x\to 12^-} f(x) \ne \lim_{x\to 12^+} f(x)$ Therefore, $\lim_{x\to 12} f(x)$ does not exist.

Here, we observed that sometimes $\lim_{x\to c} f(x) = f(x)$ and sometime it does not and also sometimes f(c) is not defined whereas $\lim_{x\to c} f(x) = f(x)$ exist.

2.8.2 Continuity of a function at a point and in an interval

i. **Continuity of a function at a point**

A function is said to be a continuous function at a point if two sided limit at that point exist and equal to the function's value e.g. Consider a function $f(x)$, it is continuous at the point $x = c$ if.

(i) $f(c)$ *is exist.*

(ii) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$ *exist or* $\lim_{x\to c} f(x)$ *exist.* ....(i)

(iii) $\lim_{x\to c} f(x) = f(c)$.

If any one of the above condition does not satisfied then the function is not continuous.

Example 37 Discuss the continuity of $f(x) = x^3 - 2x^2 - 3x + 5$ at $x = 1$

Solution

a. $f(x) = x^3 - 2x^2 - 3x + 5$ For $x = 1$

$f(1) = (1)^3 - 2(1)^2 - 3(1) + 5$

$= 1 - 2 - 3 + 5 = 1$

b. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^3 - 2x^2 - 3x + 5)$

$= (1)^3 - 2(1)^2 - 3(1) + 5 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^3 - 2x^2 - 3x + 5)$

$= (1)^3 - 2(1)^2 - 3(1) + 5$

$= 1 - 2 - 3 + 5 = 1$

c. $\Rightarrow \lim_{x\to 1} f(x) = f(1)$

$\therefore$ $f(x)$ is continuous at $x = 1$.

Now, look at the following graph of function.

It can be seen that the side limits consider with the value of the function with the point.

Remember

A function is said to be a discontinuous function at a point 'c' if one of the three conditions of continuity does not satisfy.

ii. **Continuity of a function in an interval**

"A function is said to be a continuous function in an interval when the function i... fined at every point in that interval and no jumps or breaks invol...

If some functions $f(x)$ satisfies these criteria from $x = a$ to $x = b$ and we say that $f(x)$ is continuous on the interval $[a,b]$.

Note

$f(x)$ is continuous over the closed interval $[a, b]$ if it is continuous on the (a, b) interval.

2.8.3 Test of continuity and discontinuity of a function at a point and in an interval

Example 38 Discuss the continuity of $f(x) = \frac{x^2-4}{x+2}$, at $x = -2$.

Solution In order to check the continuity of the function $f(x)$ at $x = -2$. We will have to check the function for all three conditions as we have done in Example 37.

$$f(x) = \frac{x^2-4}{x+2}$$

$$f(-2) = \frac{(-2)^2-4}{(-2)+2} = \frac{4-4}{-2+2} = \frac{0}{0} = \text{Undefined}$$

Hence, $f(-2)$ is not defined. We know that if any of the three conditions of continuity does not satisfy, the function will discontinuous.

Therefore, $f(x)$ is discontinuous function at $x = -2$.

However, if we try to find the limit of $f(x)$, we conclude that $f(x)$ is continuous on all the values other than -2.

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2}(x-2) = -4$$

This implies that $f(x)$ is continuous at all the values of x other than -2.

Example 39 First-class postage in 1995 was $0.32 for the first ounce and $0.23 for each additional ounce up to 11 ounces. If $p(x)$ is the amount of postage for a letter weighing in x ounces, then we write:

$$P(x)=\begin{cases} \$0.32, & if\ 0<x\le 1 \\ \$0.55, & if\ 1<x\le 2 \\ \$0.78, & if\ 2<x\le 3 \\ and\ so\ on \end{cases}$$

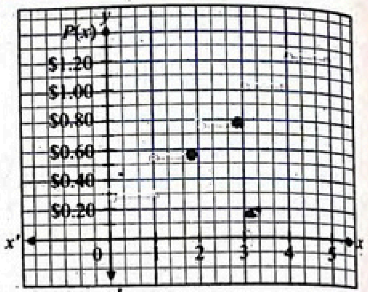


Figure 2.34

(a). Graph $p(x)$ for $0<x\le 5$

(b). Find $\lim_{x\to 1^-} p(x)$, $\lim_{x\to 1^+} p(x)$ and $p(1)$.

(c). Find $\lim_{x\to 4.5} p(x)$ and $p(4.5)$

Solution

a. The graph of $p(x)$ is shown in the Figure 2.34

b. From the graph of the function, the left, right limits and the value of the function at $x=1$ are:

$\lim_{x\to 1^-} p(x)=0.32$, $\lim_{x\to 1^+} p(x)=0.55$ and $p(1)=0.32$.

c. From the graph of the function, the limit and the value of the function are equal:

$\lim_{x\to 4.5} p(x)=1.24$, $p(4.5)=1.24$

Thus the function is continuous at $x=4.5$.

Note

Sometimes functions need to be defined in pieces, because they have a split domain. These functions require more than one formula to define the function, and therefore these types of functions are called **piecewise continuous functions**.

2.8.4 MAPLE command iscont to test continuity of a function at a point and in a given interval

Look at the following example The procedure of using the maple command for continuity of a function is illustrated in this example.

Example 40 Use maple command "iscount" to check the continuity of function

$$f(x)=x^2+4 \text{ in:}$$

(a). Internal from 0 to 1.

(b). Closed interval [0, 1].

(c). Open interval (0, 1).

Solution

```
> iscont(x^2 + 4, x = 0..1);                 true
> iscont(x^2 + 4, x = 0..1,'closed');        true
> iscont(x^2 + 4, x = 0..1,'open');          true
```

Project

Create at least five functions randomly then use MAPLE command "iscont" to check their continuity on interval (0,1)

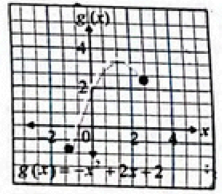
Exercise 2.4

1. Use properties of continuous function to test the continuity and discontinuity of the following functions:

   a. $f(x)=2x-3$ b. $h(x)=\dfrac{2}{x-5}$ c. $g(x)=\dfrac{x-5}{(x-3)(x+2)}$

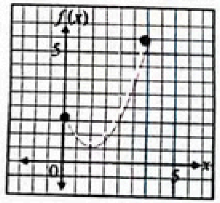
2. Show that function $f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$ is continuous at $x=0$.

3. Use the graph of the function $g(x)$ to answer the following questions:



   a. Is $g(x)$ continuous on the open interval $(-1, 2)$?
   b. Is $g(x)$ continuous from the right at $x=-1$?
      Is $\lim_{x\to -1^+} g(x)=g(-1)$?
   c. Is $g(x)$ continuous from the left at $x=2$?
      Is $\lim_{x\to 2^-} g(x)=g(2)$?
   d. Is $g(x)$ continuous on the closed interval $[-1, 2]$?

4. Use the graph of the function $f(x)$ to answer the following questions:



   a. Is $f(x)$ continuous on the open interval $(0, 3)$?
   b. Is $f(x)$ continuous from the right at $x=0$?
      Is $\lim_{x\to 0^+} f(x)=f(0)$?
   c. Is $f(x)$ continuous from the left at $x=3$?
      Is $\lim_{x\to 3^-} f(x)=g(3)$?
   d. Is $g(x)$ continuous on the closed interval $[0, 3]$?

5. Graph and locate all points of discontinuity of the following piecewise functions:

   a. $f(x)=\begin{cases}1+x, & if\ x<1\\ 5-x, & if\ x\ge 1\end{cases}$ b. $f(x)=\begin{cases}-x, & if\ x<0\\ 1, & if\ x=0\\ x, & if\ x>0\end{cases}$

6. Personal computer salesperson receives a base salary of \$1,000 per month and a commission of 5% of all sales over \$10,000 during the month. If the monthly sales are \$20,000 or more, the salesperson is given an additional \$500 bonus. Let $E(s)$ represents the person's earnings during the month as a function of the monthly sales.

   a. Graph E(s) for $0\le s\le 30{,}000$ b. Find $\lim_{s\to 10{,}000} E(s)$ and E(10,000).

   c. Find $\lim_{s\to 20{,}000} E(s)$ and E(20,000) d. Is E continuous at $s=10{,}000$? At $s=2{,}000$

7. Use MAPLE command "iscont" to test the continuity of $f(x)=\dfrac{x^2+1}{x^3+2.7}$ in closed interval $[-5, 5]$.

Review Exercise 2

1. Choose the correct option.

i. The independent variable in the function $y=\frac{x^2+4x-3}{(x+3)^3}$ is:
(a). x (b). x^2 (c). x^3 (d). y

ii. If $f(x)=\frac{3x^2-2}{3x+9}$ then $f(-3)$ is:
(a). $\frac{25}{18}$ (b). $\frac{29}{18}$ (c). $\frac{3}{1}$ (d). undefined

iii. The domain of $f(x)=\frac{3x^2-2}{3x+9}$ is:
(a). $(-\infty,-3)\cup(-3,\infty)$ (b). $(-\infty,-3)\cup(-3,\infty)$
(c). $[-\infty,-3]$ (d). $(-\infty,3)$

iv. The Domain of $f(x)=\frac{3x-2}{3x^2+9}$ is:
(a). $x\geq -3$ (b). $x\leq -3$ (c). $\infty\leq x\leq\infty$ (d). $-\infty<x<\infty$

v. If $f(x)=2x+3$ then $f^{-1}(5)$ is:
(a). 13 (b). -13 (c). 1 (d). -1

vi. If y is expressed in term of x as $y=f(x)$ then y is called:
(a). implicit function (b). explicit function
(c). linear function (d). identity function

vii. If $f(x)=3x^2+2x-1$ and $g(x)=x+1$ then $f\{g(x)\}$ is:
(a). $3x^2+2x+1$ (b). $3x^2+8x+4$ (c). $3x^2+8x-1$ (d). $3x^2-8x-4$

viii. The value of e is:
(a). 3.142 (b). $\frac{22}{7}$ (c). 2.71 (d). 3.8

ix. If $5^x=7$ then the value of x is:
(a). $\frac{\ln(7)}{\ln(5)}$ (b). $\frac{\ln(5)}{\ln(7)}$ (c). $\frac{\ln(x)}{\ln(5)}$ (d). $\frac{\ln(7)}{\ln(x)}$

x. Cot $h\,x=$ ______
(a). $\frac{e^x-e^{-x}}{2}$ (b). $\frac{e^x+e^{-x}}{2}$ (c). $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ (d). $\frac{e^x-e^{-x}}{e^x+e^{-x}}$

xi. $\ln(m)-\ln(n)=$ ______
(a). $\ln(mn)$ (b). $\ln(m+n)$ (c). $\ln(m-n)$ (d). $\ln\frac{m}{n}$

xii. The function $f(x)=\frac{2}{x^2-9}$ is discontinuous at point:
(a). $-1, 1$ (b). $-2, 2$ (c). $-3, 3$ (d). $4, 4$

xiii. $\lim_{\theta\to\infty}\frac{\sin 7\theta}{7\theta}=$
(a). 1 (b). 2 (c). 3 (d). 90

xiv. If $f(\theta)=\theta.\sec\theta$ then $f(0)=$
(a). 0 (b). 1 (c). $\sqrt{2}$ (d). -1

xv. The inverse function of $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ is:
(a). $\frac{1}{2}\log\left(\frac{2-x}{2+x}\right)$ (b). $\frac{1}{2}\log\left(\frac{2+x}{2-x}\right)$ (c). $\frac{1}{2}\log\left(-\frac{1-x}{x-1}\right)$ (d). $\frac{1}{2}\log\left(\frac{-1-x}{1-x}\right)$

Summary

- A function $y=f(x)$ is a rule that assigns for each value of the independent variable x a unique value of the dependent variable y: $y=f(x)$
- A function that defined by more than one equation is called a **compound function**.
- The graph of a function $f(x)$ consists of all points whose coordinates (x, y) satisfy a function $y=f(x)$, for all x in the domain of $f(x)$.
- Let $y=f(x)$ be a function of x. This function takes an dependent variable y in response of independent variable x. The function that takes x as dependent variable in response of y as the independent is then called the **inverse function** of $f(x)$ and is denoted by: $x=f^{-1}(y)$
- A function $f(x)$ is called **algebraic** if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing, or taking roots) starting with polynomials. Any rational function is an **algebraic function.**
- Functions that are not algebraic are called **transcendental functions.**
- If a function is defined by an equation of the form $y=f(x)$, one says that the function is defined explicitly or is explicit. The terms **"explicit function"** and **"implicit function"** do not characterize the nature of the function but merely the way it is defined. Every explicit function $y=f(x)$ may also be represented as an implicit function $y=f(x)=0$.
- If $f(t)$ and $g(t)$ are continuous functions of parameter t on an interval D, then the equations $x=f(t)$ and $y=g(t)$
  are called the **parametric equations** for the plane curve C generated by the set of ordered pairs in plane: $(x,y)=(x(t),y(t))=(f(t),g(t))$
- If $f(x)$ is a function of x, and c, L are the real numbers, then L is the limit of a function $f(x)$ as x approaches c : $\lim_{x\to c}f(x)=L$
- A function $f(x)$ is said to be a continuous at $x=c$, if all three of the following conditions are satisfied:
  - The function is defined at $x=c$; that is, $f(c)$ exists.
  - The function approaches a definite limit as x approaches c; that is $\lim_{x\to c}f(x)$ exists.
  - The limit of a function is equal to the value of a function when $x=c$; that is, $\lim_{x\to c}f(x)=f(c)$.

History

H. Steinhaus was polish mathematician and educator. He earned his ph.D degree from his mutable contribution to functional analysis through the bancach-steinhaus theorem. He is also one of the early founders of probability and game theory. He also proposed sandwich theorem in 1938 first time specifically n=3 case of bisecting 3-solids with a plane.


Hugo Steinhaus (1887-1972)