Unit 12 INTRODUCTION TO NUMERICAL METHODS

By the end of this unit, the students will be able to:

12.1 Numerical solution of non-linear equations

i. Describe importance of numerical methods.
ii. Explain the basic principles of solving a non-linear equation in one variable.
iii. Calculate real roots of a non-linear equation in one variable by
- Bisection method.
- Regula-falsi method,
- Newton-Raphson method.

iv. Use MAPLE command *fsolve* to find numerical solution of an equation and demonstrate through examples.

12.2 Numerical quadrature

i. Define numerical quadrature. Use
- Trapezoidal rule,
- Simpson's rule,

to compute the approximate value of definite integrals without error terms.

Use MAPLE command **trapezoid** for trapezoidal rule and **simpson** for Simpson's rule and demonstrate through examples.

Introduction

Scientists, economists, engineers, and other researchers study relationships between quantities. For example, an engineer may need to know how the illumination from a light source on an object is related to the distance between the object and the source; a biologist may wish to investigate how the population of a bacterial colony varies with time in the presence of a toxin; an economist may wish to determine the relationship between demand for a certain commodity and its market price. The mathematical study of such relationships involves the concept of non-linear equations. For example, the value of the assets of a certain company at time t years is modeled by a non-linear equation $f(t)=100,000-75,000e^{-2t}$ where t is measured in years. The standing rule for solving non-linear equations algebraic is quadratic formula. In this case, it is not valid to obtain the actual number of t (years) at which the asset function $f(t)$ is going to be zero. Now we are in position to obtain the approximate number of year's t that can be found by using some numerical procedures. In this unit, we will learn the numerical procedures recommended are the bracketing methods and iterative methods.

12.1 Numerical Solution of Non-linear Equations

Numerical analysis is the theory of constructive methods in mathematical analysis. Constructive methods in their turn mean a procedure that permits us to obtain the solution of a mathematical problem with an arbitrary precision in a finite number of steps that can be prepared rationally.

12.1.1 Importance of numerical methods

Numerical analysis is both a Science and an Art. As a Science, it is concerned with the process by which a mathematical problem can be solved arithmetically. As an Art, numerical analysis is concerned with choosing that procedure which is best suited to the solution of a particular problem.

Students learning numerical solution of non-linear equations should have the following objectives in view. First, he should obtain an intuitive and working understanding of some numerical methods for the basic problems of numerical analysis. Second, he should gain some appreciation of the concept of error and of the need to analyze and predict it. Third, he should develop some experience in the implementation of numerical method by using **computer software**.

12.1.2 Basic principles of solving non-linear equations in one variable

*"If $f(x)$ is any **continuous function** of a single variable x, then any number r for which $f(r)=0$ is called a root of $f(x)=0$. Also we say that r is a zero of the function $f(x)$."*

If $f(x)$ is any algebraic function (non-linear equation), then the actual roots of $f(x)$ can be found by direct rules, such as quadratic formula, common factors procedure and synthetic division.

For example, the quadratic equation $x^2+5x+6=0$ has two **actual** (or exact) roots $r_1=-2$ and $r_2=-3$ obtained by common factors procedure/quadratic formula:

$$f(x)=x^2+5x+6=0$$
$$=x^2+2x+3x+6=0$$
$$=(x+2)(x+3)=0 \Rightarrow x+2=0,\ x+3=0 \Rightarrow x=-2=r_1,\ x=-3=r_2$$

On the other hand, the actual roots of non-algebraic equation $x^{2}+5x+6=0$ are not possible by applying quadratic formula. The only way is to find out the **approximate roots** that can be found by using some **numerical procedures.** The numerical procedures are the bracketing methods and iterative methods.

12.1.3 Real roots of non-linear equation in one variable

For approximate roots of a non-linear equations, the numerical procedures **Bisection method** and **regula-falsi method** are the bracketing methods that depend on two initial approximations that must be in the shape of a closed interval $[a, b]$.

The non-linear curve $f(x)$ has the function values $f(a)$ and $f(b)$ in the interval $[a, b]$ that must be opposite in signs for showing its continuity. Once the interval has been found, no matter how large, the **iteration** will be preceded until an approximate root is obtained. The fundamental principle in computer science is the **iteration.** As the name suggests, it means that a process of bisection or regula-falsi method is repeated until an answer is achieved.

(a) Bisection Method

If $y=f(x)$ is continuous function in the interval $[a, b]$, then, it will cross the x-axis at a point $(r,0)$ whose x-coordinate $x=r$ will keep as actual root that lies somewhere in the interval $[a, b]$. This is shown in the Figure 12.1.

The bisection method systematically moves the endpoints of the interval $[a,b]$ closer and closer together till it reaches an interval of small width that brackets the root r. The decision step for this process of interval halving is to choose the midpoint $c=\dfrac{(a+b)}{2}$ and then analyze the three possibilities that might arise:

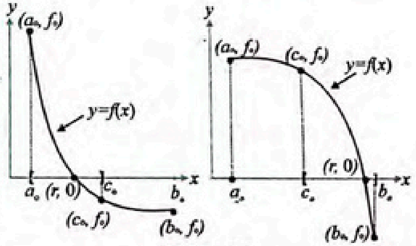


Figure 12.1

1. If the function values $f(a)$ and $f(c)$ at $x=a$ and $x=c$ have opposite signs, then the approximate root lies in interval $[a, c]$ and discard b.
2. If the function values $f(c)$ and $f(b)$ at $x=c$ and $x=b$ have opposite signs, then the approximate root lies in interval $[c, b]$ and discard a.
3. If the function value at $x=c$ is $f(c)=0$, then **c is our approximate root to actual root r.**

If either of cases 1 or 2 occurs, we have an interval half as wide as the original interval that contains the root, and we are "squeezing down on it" see Figure 12.1. To continue the process, relabel the new smaller interval and repeat the sequence of nested intervals and their midpoints.

The procedure in detail is as under:

The given interval$[a_0, b_0]$ is the initial interval at which the function $f(x)$ must be opposite in signs. At this stage, the initial interval brackets the actual root r whose midpoint is $c_0 = \frac{(a_0+b_0)}{2}$. It develops the first iterate c_o(through properties 1 or 2) to actual root r in the interval $[a_0, b_0]$.

The next interval$[a_1, b_1]$is the first interval which brackets the actual root r and c_1 is its midpoint $c_1 = \frac{(a_1+b_1)}{2}$. It develops the second iterate c_1(through properties 1 or 2) to actual root r in the interval$[a_1, b_1]$.

Similarly, the interval $[a_n, b_n]$is the nth interval which brackets the actual root r and c_n is its midpoint $c_n = \frac{(a_n+b_n)}{2}$. It develops the $(n-1)$-th iterate c_{n-1} to actual root r in the n-th interval $[a_n, b_n]$.

This completes the n times iteration of the bisection method and the midpoint c_{n-1} is taken as the desired approximation to the actual root $r \approx c_{n-1}$ of $y = f(x)$.

Remember

To find approximate root of an equation $f(x)=0$ in the interval $[a, b]$. Proceed with the method only if $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs.

Example 1 Perform two iterations of the bisection method to approximate the actual root r of the non-linear equation $f(x)=\sin x - e^{-x}$ (x is in radians) in the interval [0.5, 0.7].

Solution Reset the given interval to obtain the initial interval$[a_0, b_0]=[0.5, 0.7]$ and compute the function $f(x)=\sin x - e^{-x}$ values at $x = a_0 = 0.5$ and $x = b_o = 0.7$ to obtain:

$$\left.\begin{aligned} f(a_o) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) &= f(0.7) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \end{aligned}\right\} \textit{opposite in signs}$$

The function values $f(a_o)$ and $f(b_o)$ are opposite in signs, so the actual root r of $f(x)$ lies in the interval [0.5, 0.7].

i. The midpoint of the initial interval $[a_0, b_0]$is $c_o = \frac{a_o+b_o}{2} = \frac{0.5+0.7}{2} = 0.6$, and the function values at $x = a_0 = 0.5$, $x = b_0 = 0.7$, $c_0 = 0.6$ are the following:

$$\left.\begin{aligned} f(a_o) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) &= f(0.7) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \\ f(c_o) &= f(0.6) = \sin 0.6 - e^{-0.6} = +0.016 > 0 \end{aligned}\right\}$$

The function values $f(a_o)$and $f(c_o)$are opposite in signs, so the approximation to the actual root r of $f(x)$ lies in the interval [0.5, 0.6] and discard $b_0 = 0.7$. The initial interval is reset to obtain the first interval $[a_1, b_1]=[0.5, 0.6]$.

ii. The midpoint of the first interval is $c_1 = \frac{a_1+b_1}{2} = \frac{0.5+0.6}{2} = 0.55$, and the function values at $x = a_1 = 0.5$, $x = b_1 = 0.6$, and $x = c_1 = 0.55$ are the following:

$$\left.\begin{aligned} f(a_1) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(c_1) &= f(0.55) = \sin 0.55 - e^{-0.55} = -0.054 < 0 \\ f(b_1) &= f(0.6) = \sin 0.6 - e^{-0.6} = +0.016 > 0 \end{aligned}\right\}$$

The function values $f(b_1)$ and $f(c_1)$ are opposite in signs, so the approximation to actual root r of $f(x)$ lies in the interval [0.55,0.6] and discard $a_1 = 0.5$. The first interval is reset to obtain the second interval

$$[a_2, b_2]=[0.55, 0.6].$$

After second iteration of the bisection method, the midpoint $c_1 = 0.55$ is declared approximate root to actual root $r \approx c_1$. The approximation value of a function $f(x)=\sin x - e^{-x}$ at approximate root $r \approx 0.55$ is $f(0.055) = -0.054$.

(b) Regula-Falsi method

The next bracketing method is the method of regula-falsi method. It was developed because the bisection method converges at a fairly slow speed. As before, we assume that $f(a)$ and $f(b)$ have opposite signs. The bisection method always used the midpoint of the interval as the next iterate, but in regula-falsi method, the next iterate is anywhere in the interval $[a, b]$ represented by the point of intersection $(c, 0)$ of the straight line formed by the points $(a, f(a))$, and $(b, f(b))$ and the x-axis

$$\left.\begin{aligned} \frac{y-f(a)}{x-a} &= \frac{f(b)-f(a)}{b-a} \\ \frac{0-f(a)}{c-a} &= \frac{f(b)-f(a)}{b-a} \\ c &= a - \frac{(a-b)f(a)}{f(a)-f(b)} \end{aligned}\right\}, \; at\ (x, y)=(c, 0) \qquad \text{(i)}$$

and then analyze the three possibilities that might arise :

i. If the function values $f(a)$ and $f(c)$ at $x = a$ and $x = c$ have opposite signs, then the root lies in the interval $[a, c]$ and discard b.

ii. If the function values $f(c)$ and $f(b)$at $x = c$ and $x = b$ have opposite signs, then the root lies in the interval $[c, b]$ and discard a.

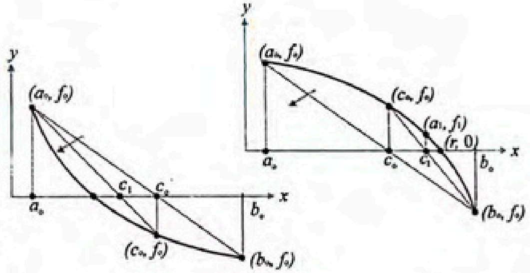


Figure 12.2

iii. If the function value at $x = c$ is $f(c) = 0$, then c **is our approximate root.** This is shown in the Figure 12.2.

Example 2 Perform two iterations of the regula-falsi method to approximate the actual root r of the non-linear equation $f(x)=\sin x-e^{-x}$ (x is in radians) in the interval [0.5, 0.7].

> **Remember**
> To find approximate root of an equation $f(x)=0$ in the interval $[a, b]$. Proceed with the method only if $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs.

Solution Reset the given interval to obtain the initial interval $[a_0,b_0]=[0.5,0.7]$ and compute the function $f(x)=\sin x-e^{-x}$ values at $x=a_0=0.5$ and $x=b_o=0.7$ to obtain:

$$\left.\begin{aligned} f(a_o)&=f(0.5)=\sin 0.5-e^{-0.5}=-0.127<0\\ f(b_o)&=f(0.7)=\sin 0.7-e^{-0.7}=+0.148>0\end{aligned}\right\}\textit{opposite in signs}$$

The function values $f(a_o)$ and $f(b_o)$ are opposite in signs, so the actual root r of $f(x)$ lies in the interval [0.5,0.7]. Equation (i) is used to obtain

$$c_o=a_o-\frac{(a_o-b_o)f(a_o)}{f(a_o)-f(b_o)}=0.5-\frac{(0.5-0.7)(-0.127)}{-0.127-0.148}=0.592364$$

That provides the function value at $c_o=0.592364$: $f(c_o)=\sin(0.592364)-e^{-0.592364}=+0.081$

i. The function values at $x=a_0=0.5$, $x=b_0=0.7$, $c_0=0.592364$ are the following:

$$\left.\begin{aligned} f(a_o)&=\sin 0.5-e^{-0.5}=-0.127<0\\ f(b_o)&=\sin 0.7-e^{-0.7}=+0.148>0\\ f(c_o)&=\sin 0.592364-e^{-0.592364}=+0.081>0\end{aligned}\right\}$$

The function values $f(a_o)$ and $f(c_o)$ are opposite in signs, so the approximation to actual root r of $f(x)$ lies in the interval [0.5, 0.592364] and discard $b_0=0.7$. The initial interval is reset to obtain the first interval $[a_1,b_1]=[0.5,0.592364]$.

Equation (i) is used to obtain

$$c_1=a_1-\frac{(a_1-b_1)f(a_1)}{f(a_1)-f(b_1)}=0.5-\frac{(0.5-0.592364)(-0.127)}{-0.127+0.081}=0.755$$

that provides the function value at $c_1=0.755$: $f(0.755)=\sin(0.755)-\ e^{0.755}=0.215$

ii. The function values at $x=a_1=0.5$, $x=b_1=0.592364$, $c_1=0.755$ are the following:

$$\left.\begin{aligned} f(a_1)&=\sin 0.5-e^{-0.5}=-0.127<0\\ f(b_1)&=\sin 0.592364-e^{-0.592364}=+0.081>0\\ f(c_1)&=\sin 0.755-e^{-0.755}=+0.215>0\end{aligned}\right\}$$

The function values $f(a_1)$ and $f(c_1)$ are opposite in signs, so the approximation to actual root r of $f(x)$ lies in the interval [0.5, 0.755] and discard $b_1=0.592364$. The first interval is reset to obtain the second interval $[a_2, b_2]=[0.5, 0.755]$.

After second iteration of the regula-falsi, the point $c_1=0.755$ is declared as approximation to actual root $r\approx c_1$. The approximate value of a function $f(x)=sinx-e^{-x}$ at approximate root $r\approx c_1=0.755$ is $f(0.755)=0.215$.

(c) Newton-Raphson-method

Another numerical procedure is the Newton-Raphson method under the umbrella of **iterative methods**. The Newton-Raphson method is one-point **iterative method** which requires **one** previous approximation in contrast of bracketing methods which require **two** in computing the successive approximation.

The Newton-Raphson method uses the **slopes of the tangent lines to the graph of a function $f(x)$** to approximate roots of the equation $f(x)=0$.

If $f(x)$ and $f'(x)$ are continuous near actual roots, then this extra information regarding the nature of $f(x)$ can be used to develop a sequence of iterates $\{x_n\}$ that will converge faster to actual root than either the bisection and regula-falsi methods.

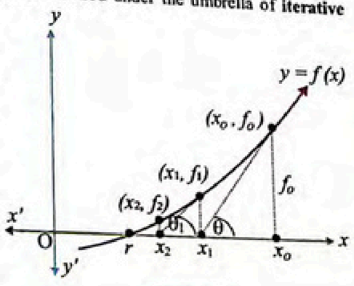


Figure 12.3

If r is any actual root of an equation of the form $f(x)=0$, and x_0 is an initial approximation to the actual r, then the tangent line on a curve $f(x)$ at a point (x_0, f_0) crosses the x-axis at a point $(x_1, 0)$. The point $(x_1, 0)$ is the point of intersection of the tangent line and the x-axis, and the x-coordinate x_1 of the point of intersection $(x_1, 0)$ is our first approximation to the actual root r of an equation $f(x)=0$. This is shown in the Figure 12.3.

The slope of the tangent line on a curve $y=f(x)$ at a point (x_0, f_0) is used to obtain the first approximation (iterate) x_1:

$$\left.\begin{aligned}\tan\theta&=\frac{f(x_o)}{x_o-x_1}\\ f'(x_o)&=\frac{f(x_o)}{x_o-x_1}\end{aligned}\right\}\Rightarrow x_o-x_1=\frac{f(x_o)}{f'(x_o)}\Rightarrow x_1=x_o-\frac{f(x_o)}{f'(x_o)} \qquad \text{(i)}$$

Similarly, the slope of the tangent line on a curve $y=f(x)$ at a point (x_1, f_1) is used to obtain the second iterate x_2:

$$\left.\begin{aligned}\tan\theta_1&=\frac{f(x_1)}{x_1-x_2}\\ f'(x_1)&=\frac{f(x_1)}{x_1-x_2}\end{aligned}\right\}\Rightarrow x_1-x_2=\frac{f(x_1)}{f'(x_1)}\Rightarrow x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

This procedure of slope finding method is continued till it reaches the $(i+1)$-th iterate:

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)},\ i=0, 1, 2, \ldots \qquad \text{(ii)}$$

which is called the **Newton-Raphson method.**

> **Remember**
> To find approximate root of an equation of the form $f(x)=0$ with initial iterate x_o, the iteration method
> $$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)},\ i=0,1,2,3,\ldots,\ i=1,2,3,\ldots$$
> develops a sequence of successive iterates $\{x_n\}$ that will converge faster to actual root r than either the bisection and regula-falsi methods.

Example 3 Use Newton-Raphson iterative method to approximate the actual root $r = 0.438447$ of the non-linear equation $f(x) = x^2 - 5x + 2$ with initial start $x_o = 0.4$ that must be accurate to six decimal places.

Solution The given non-linear equation $f(x) = x^2 - 5x + 2$ and its derivative $f'(x) = 2x + 5$ are used in the Newton-Raphson iterative method to obtain the successive iterates

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0,1,2$$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)} = \frac{x_o^2 - 2}{2x_o - 5} = \frac{(0.4)^2 - 2}{2(0.4) - 5} = 0.438095, \quad i = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{x_1^2 - 2}{2x_1 - 5} = \frac{(0.438095)^2 - 2}{2(0.438095) - 5} = 0.438447, \quad i = 1$$

that converge the actual root $r = 0.438447$.

The second iterate $x_2 = 0.438447$ agrees to six decimal accuracy of actual root $r = 0.438447$. We achieved in just two iterates of the Newton-Raphson method the six decimal accuracy.

12.1.4 MAPLE command "fsolve" to find numerical solution of an equation and demonstrate through examples

The use of MAPLE command "fsolve" to find the approximate solution of given function is illustrated in the following example.

Example 4 Use maple command "fsolve" to solve.

(a). Linear equation $x^2 - 5x + 6 = 0$ with initial start $x_0 = 1.8$.

(b). Nonlinear equation $x^2 - 5x + 6 = 0$ with initial start $x_0 = 0.5$.

Solution The command below will show you full detail of the approximate root of linear and non-linear equations on line by typing without initial start:

a.

```
> ?fsolve
> fsolve(x^2 - 5·x + 6)                      2.000000000, 3.000000000
```

The quadratic function $f(x)$ is also a second degree polynomial. The numerical solution through polynomial is:

```
> Polynomial := x^2 - 5·x + 6                Polynomial := x^2 - 5x + 6
> fsolve(Polynomial)                         2.000000000, 3.000000000
```

b.

```
> fsolve(sin(x) - exp(-x))                   0.5885327440
```

Context Menu:

```
> sin(x) - exp(-x)
> fsolve( sin(x)-exp(-x) )                   0.5885327440
```

This result is obtained through right-click on the last end of the expression by selecting "Solve < Numerically Solve" on the context menu.

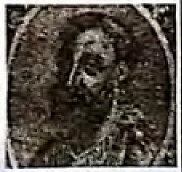
Exercise 12.1

1. Find an interval $a \le x \le b$ at which $f(a)$ and $f(b)$ have opposite signs for the following functions:
   a. $f(x) = e^x - 2 - x$
   b. $f(x) = \cos x + 1 - x$, x is in radians
   c. $f(x) = \ln(x) - 5 + x$
   d. $f(x) = x^2 - 10x + 23$
2. Compute four iterates of the bisection method for the following functions with indicated interval $[a_0, b_0]$:
   a. $f(x) = e^x - 2 - x$, $[1, 1.8]$
   b. $f(x) = \cos x + 1 - x$, $[0.8, 1.6]$ x is in radians
   c. $f(x) = \ln(x) - 5 + x$ $[3.2, 4]$
   d. $f(x) = x^2 - 10x + 23$ $[3.2, 4]$
3. Compute four iterates of the regula-falsi method for the following functions with indicated interval $[a_0, b_0]$:
   a. $f(x) = e^x - x$, $[-2.4, -1.6]$
   b. $f(x) = \cos x + 1 - x$, $[0.8, 1.6]$, x is in radians
   c. $f(x) = x^2 - 10x + 23$, $[-2.4, -1.6]$
4. What will happen if the bisection method is used with the function $f(x) = \frac{1}{(x-2)}$ for the following intervals:
   a. $[3, 7]$
   b. $[1, 7]$
5. Find iterate x_3 of Newton-Raphson iterative method for the following functions with initial start x_o:
   a. $f(x) = x^3 - 3$, $x_o = 1$
   b. $f(x) = \sin(x)$, $x_o = 1$
   c. $f(x) = x^3 + 2x - 1$, $x_o = 0$
   d. $f(x) = \sin x$, $x_0 = -2$
6. Use Newton-Raphson iterative method to approximate the actual r of the following non-linear equations with indicated interval:
   a. $f(x) = x^3 + 3x - 1 = 0$ on $(0,1)$
   b. $f(x) = x^3 + 2x^2 - x + 1 = 0$ on $(-3, -2)$
   c. $\sqrt[3]{x-3} = x + 1$ on $[-3, -2]$

   Continue the process until two consecutive iterates will agree to three decimal places.
7. Use MAPLE command 'fsolve' to solve $3x^2 + 4x - 3 = 0$ with initial start at $x_0 = 0.5$

History

Qusta Ibn Luqa was a Syrian mathematician, astronomer and philosopher. He contribute in many fields of science, medicine, astronomy. His translation on the difference between the spirit and the soul was one of the few works not attributed to Aristotle that was included in a list of books to be read or lectured on. He was the first person to write the double false position in 10^{th} century. He justified the technique by a formal, Euclidean-style geometric proof, within the tradition of Muslim mathematics. Double false position was known as Hisab-al-Khata'ayn. It was used for centuries to solve practical problems such as commercial and recreational problems.



Qusta Ibn Luqa (820-912)

12.2 Numerical Quadrature

Numerical integration is a primary tool used by engineers and scientists to obtain approximate solutions for definite integrals that cannot be solved analytically. For example, the integral

$$I = \int_0^1 e^{x^2} dx \qquad \text{(i)}$$

has no **actual** solution. This means that there is no any integral formula that could be used directly to obtain the actual solution. The only way is to find out the approximate solution that can be found by using some numerical procedures, such, as numerical integration.

We now approach the subject of numerical integration. The goal is to approximate the definite integral of $f(x)$

$$I = \int_a^b f(x)dx \qquad \text{(ii)}$$

over the interval $[a, b]$ by evaluating $f(x)$ at a finite number of equally spaced grid points:

$$\left.\begin{matrix} x: & a = x_o & x_1 \ldots & x_{n-1} & x_n = b \\ f(x): & f_o & f_1 \ldots f_{n-1} & f_n & \end{matrix}\right\}$$

12.2.1 Numerical quadrature

"If a set of points in the interval $[a, b]$ is $a = x_o < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, then an expression of the form $Q[f(x)] = \sum_{j=0}^{n} w_j f(x_j) = w_o f(x_0) + w_1 f(x_1) + \ldots + w_n f(x_n)$

with the property $\int_a^b f(x)dx = Q[f(x)] + E[f(x)]$ (iii)

*is called a numerical integration or **quadrature formula**."*

The term $E[f(x)]$ is called the truncation error for integration. The values $\{x_j\}_{j=0}^{n}$ are called the **quadrature nodes** and $\{w_j\}_{j=0}^{n}$ are called the **weights**.

Depend on the given numerical procedure, the grid points $\{x_j\}$ are chosen in various ways. For trapezoidal rule and Simpson's rule, the grid points are chosen to be **equally spaced**. Before discussion of trapezoidal rule and Simpson's rule it must be familiar about approximation by rectangles and approximate area by rectangles.

Approximate by rectangles

If $f(x) = 0$ is a function over the interval $[a, b]$, then the definite integral (ii) represents the actual area under the graph of $f(x)$ on the interval $[a, b]$. This is shown in the Figure 12.4.

For approximate area, the function $f(x)$ must be known at equally spaced grid points in the interval $[a, b]$, each of width $\Delta x = (b-a)/n$:

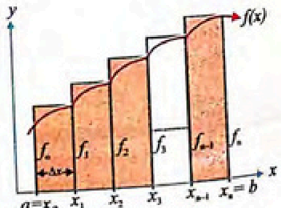


Figure 12.4

$$\left.\begin{matrix} x: a = x_o & x_1 & x_2 \ldots\ldots x_{n-1} & x_n = b \\ f(x): \quad f_o & f_1 & f_2 \ldots\ldots f_{n-1} & f_n \end{matrix}\right\}, \; x_k = x_o + k\Delta x, \; k = 0, 1, 2, \ldots, n$$

In light of above equally spaced grid points, the actual area (ii) under a curve $f(x)$ over the interval [a, b] is rearranged as under:

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx \qquad \text{(iv)}$$

Approximate Area by Rectangles

Consider the initial subinterval $[x_0, x_1]$. Let x_1 denote the right endpoint of the initial subinterval, and the base of the rectangle is of course the initial subinterval and its height is $f(x_0)$. The area of the rectangle in the initial subinterval is therefore $f(x_0)\Delta x$.

If $\int_{x_0}^{x_1} f(x)dx$ is the actual area in the initial subinterval, then the approximate area $f(x_0)\Delta x$ is of course the area of the rectangle lies in the initial subinterval $[x_0, x_1]$. Thus, the sum of the areas of all n rectangles is giving approximate area under the curve $f(x)$ to actual area represented by definite integral (iv).

$$I = \int_a^b f(x)dx \approx R_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \qquad \text{(v)}$$

This approximation improves as the number of rectangles increases, and we can estimate the integral to any desired degree of accuracy by taking n large enough. However, because fairly large values of n are usually required to achieve reasonable accuracy, approximation by rectangles is rarely used in practice.

i. **The Trapezoidal rule**

The accuracy of the approximation can be improved if trapezoids are used instead of rectangles. Figure 12.5 shows the area approximated by n trapezoids instead of n rectangles.

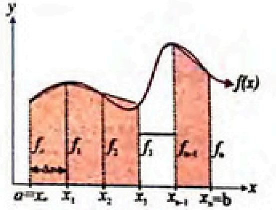


Figure 12.5

If $\int_{x_0}^{x_1} f(x)dx$ is the actual area in the initial subinterval.

then the approximate area $\left[\dfrac{f(x_0)+f(x_1)}{2}\right]\Delta x$ is of course the area of the trapezoid lies in the initial subinterval $[x_0, x_1]$. Thus, the sum of the areas of all n trapezoids is giving approximate area under the curve $f(x)$ to actual area represented by definite integral (iv):

$$\int_a^b f(x)dx \approx T_n$$

$$= \frac{1}{2}[f(x_o)+f(x_1)]\Delta x + \frac{1}{2}[f(x_1)+f(x_2)]\Delta x + \ldots\ldots + \frac{1}{2}[f(x_{n-1})+f(x_n)]\Delta x$$

$$= \frac{1}{2}[f_0 + 2f_1 + 2f_2 + \ldots\ldots + 2f_{n-1} + f_n]\Delta x \qquad \text{(vi)}$$

In general, If $f(x)$ is continuous on $[a, b]$, then the trapezoidal rule is

$$I = \int_a^b f(x) \approx T_n = \frac{\Delta x}{2}[f_0 + 2f_1 + 2f_2 + \ldots\ldots + 2f_{n-1} + f_n].$$

The n^{th} subinterval is $x_n = x_o + n\Delta x \Rightarrow b = a + n\Delta x$ that gives $\Delta x = \dfrac{(b-a)}{n}$. Moreover, the larger the value of n, the better the approximation.

Example 5 Approximate the definite integral $I=\int_{-1}^{2} x^2 dx$ for $n = 4$ subintervals and then compare your approximate answer with the actual value of the definite integral that must be accurate to 5 decimal places.

Solution The given interval is $[a, b] = [-1, 2]$ and the width for $n = 4$ subintervals is

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{4} = \frac{3}{4} = 0.75$$

The trapezoidal rule (vi) is used for $\Delta x = 0.75$ and $n = 4$ to obtain:

$$\int_{-1}^{2} x^2 dx \approx T_4 = \frac{1}{2}[f_0+2f_1+2f_2+2f_3+f_4](0.75) \qquad \text{(i)}$$

The function values f_0, f_1, f_2, f_3, f_4 at grid points x_0, x_1, x_2, x_3, x_4 are

$x_o = a = -1,$ $\qquad f_0 = f(-1) = (-1)^2 = 1$

$x_1 = a + 1\Delta x = -1 + \frac{3}{4} = -\frac{1}{4},$ $\qquad f_1 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16} 0.0625$

$x_2 = a + 2\Delta x = -1 + 2\frac{3}{4} = \frac{1}{2},$ $\qquad f_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$

$x_3 = a + 3\Delta x = -1 + 3\frac{3}{4} = \frac{5}{4},$ $\qquad f_3 = \left(\frac{5}{4}\right)^2 = \frac{25}{16} = 0 = 1.5625$

$x_4 = a + 4\Delta x = -1 + 4\frac{3}{4} = 2,$ $\qquad f_4 = (2)^2 = 4$

used in (i) to obtain: $\int_{-1}^{2} x^2 dx \approx T_4 = \frac{1}{2}[f_0+2f_1+2f_2+2f_3+f_4]\Delta x$

$$= \frac{1}{2}[1+2(0.0625)+2(0.25)+2(1.5625)+4](0.75) = 3.28125$$

The exact value of the definite integral is: $I = \int_{-1}^{2} x^2 dx = \left|\frac{x^3}{3}\right|_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = \frac{9}{3} = 3$

The trapezoidal approximation T_4 developed an error, which we denote by E_4:

$E_4 = Actual - Approximation = I - T_4 = 3 - 3.28125 = -0.28125$

The negative sign indicates that the trapezoidal formula overestimated the true value of the definite integral.

In terms of numerical quadrature notation, $Q[f(x)] = T_4 = 3.28125$ and $E_4[(x)] = -0.28125$

with nodes $x_o = -1,\ x_1 = \frac{-1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{5}{4},\ x_4 = 2.$

ii. **The Simpson's rule**

The accuracy of the approximation can be improved if instead of trapezoidal strips, the parabolic strips are used in three consecutive equally spaced grid points. The name given to the procedure in which the approximating strip has a parabolic arc is the **Simpson's rule.**

Mathematically, the parabolic arc is represented by a second degree polynomial $p(x) = Ax^2 + Bx + C.$

Simpson's rule approximates the actual area in an interval $[a, b]$ by parabolic arc if $f(x)$ is replaced by second degree polynomial $p(x)$

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \int_a^b (Ax^2+Bx+C)dx = \left(\frac{b-a}{6}\right)\left[p(a)+4p\left(\frac{a+b}{2}\right)+p(b)\right] \qquad \text{(i)}$$

that requires three consecutive grid points in an even interval $[a, b]$. The definite integral of the second degree polynomial in equation (i) is simplified by a rule called **prismoidal rule.** This rule is valid for a polynomial $p(x)$ of degree **less than or equal to 3.**

For Simpson's rule, the function $f(x)$ is known at equally even spaced grid points in the interval $[a, b]$:

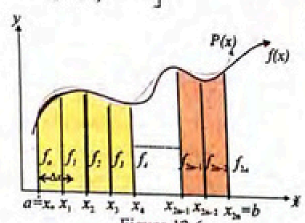


Figure 12.6

$$\left.\begin{array}{llllllll} x: a = x_o & x_1 & x_2 \ldots x_{2n-2} & x_{2n-1} & x_{2n} = b \\ f(x): \quad f_o & f_1 & f_2 \ldots f_{2n-2} & f_{2n-1} & f_{2n} \end{array}\right\}, \text{ with}$$

$$x_{2n} = x_o + 2n\Delta x,\ n = 0, 1, 2, \ldots$$

$$b = a + 2n\Delta x,\ x_{2n} = b,\ x_0 = a \ \Rightarrow \ \Delta x = \frac{b-a}{2n} \qquad \text{(ii)}$$

In light of even spaced grid points, the definite integral (iv) is rearranged as under:

$$I = \int_a^b f(x)dx = \int_{x_o}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx \qquad \text{(iii)}$$

If $\int_{x_0}^{x_2} f(x)dx$ is the actual area in the initial even subinterval $[x_0, x_2]$, then the parabolic arc in the subinterval $[x_0, x_2]$ represents the approximate area:

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx,\ a = x_0,\ b = x_2,$$

$$= \left(\frac{x_2 - x_0}{6}\right)\left[p(x_0) + 4p\left(\frac{x_0+x_2}{2}\right) + p(x_2)\right] \qquad \text{(iv)}$$

If the width of one subinterval is $\Delta x = \frac{(b-a)}{n}$, then the width of two consecutive subintervals is $x_2 - x_0 = 2\Delta x$. The subintervals are of equal width that gives $\frac{x_2 + x_0}{2} = x_1$. Using these in equation (iv) to obtain

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx = \frac{2\Delta x}{6}\left[p(x_0) + 4p\left(\frac{x_0+x_2}{2}\right) + p(x_2)\right] = \frac{\Delta x}{3}[p(x_0) + 4p(x_1) + p(x_2)] \qquad \text{(v)}$$

Since the polynomial passes through the three consecutive grid points on the curve, the best approximations would be the function itself: that is $p(x_0) = f(x_0)$, and $p(x_1) = f(x_1)$, and $p(x_2) = f(x_2)$. These are substituted in equation (v) to obtain:

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx = \frac{\Delta x}{3}[p(x_0) + 4p(x_1) + p(x_2)] = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + f(x_2)] \qquad \text{(vi)}$$

Thus, the sum S_{2n} of the areas of all n parabolic strips is giving the approximate area under the curve $f(x)$ to actual area represented by definite integral (ii):

$$S_{2n}=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)\right]+\frac{\Delta x}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$$

$$+\ldots+\frac{\Delta x}{3}\left[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})\right]$$

$$=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+\ldots+f_{2n-1})+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}\right] \qquad \text{(vii)}$$

In general: If $f(x)$ is continuous on $[a, b]$, then the Simpson's rule is

$$I=\int_a^b f(x)\approx S_{2n}=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+\ldots+f_{2n-1})+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}\right]$$

The even n^{th} subinterval is $x_{2n}=x_0+2n\Delta x \Rightarrow b=a+2n\Delta x$ that gives $\Delta x=\frac{(b-a)}{2n}$.

Moreover, the larger the value for n, the better the approximation.

Example 6 Approximate the definite integral $I=\int_{-1}^{2}x^2dx$ for $n=2$ subintervals and then compare your approximate answer with the actual value of the definite integral that must be accurate to 5 decimal places.

Solution The given interval is $[a, b]=[-1, 2]$ and the width for $n=2$ subintervals is

$$\Delta x=\frac{b-a}{2n}=\frac{2-(-1)}{4}=\frac{3}{4}=0.75.$$

The Simpson's rule (vii) is used for $\Delta x=0.75$ and $n=2$ to obtain:

$$\int_{-1}^{2}x^2dx\approx S_4=\frac{0.75}{3}\left[f_0+4(f_1+f_3)+2(f_2)+f_4\right] \qquad \text{(i)}$$

The function values f_0, f_1, f_2, f_3, f_4 at grid points x_0, x_1, x_2, x_3, x_4 are

$x_0=a=-1,$ $\qquad f_0=f(-1)=(-1)^2=1$

$x_1=a+1\Delta x=-1+\frac{3}{4}=-\frac{1}{4},$ $\qquad f_1=\left(-\frac{1}{4}\right)^2=\frac{1}{16}0.0625$

$x_2=a+2\Delta x=-1+2\frac{3}{4}=\frac{1}{2},$ $\qquad f_2=\left(\frac{1}{2}\right)^2=\frac{1}{4}=0.25$

$x_3=a+3\Delta x=-1+3\frac{3}{4}=\frac{5}{4},$ $\qquad f_3=\left(\frac{5}{4}\right)^2=\frac{25}{16}=0=1.5625$

$x_4=a+4\Delta x=-1+4\frac{3}{4}=2,$ $\qquad f_4=(2)^2=4$

These function values are used in equation (i) to obtain:

$$\int_{-1}^{2}x^2dx\approx S_4=\frac{0.75}{4}\left[f_0+4(f_1+f_3)+2f_2+f_4\right]$$

$$=\frac{0.75}{3}\left[1+4(0.0625+1.5625)+2(0.25)+4\right]$$

$$=0.25(1+6.5+0.50+4)=3$$

The exact value of the integral is:

$$I=\int_{-1}^{2}x^2dx=\left|\frac{x^3}{3}\right|_{-1}^{2}=\frac{8}{3}+\frac{1}{3}=3$$

The error term is therefore:

$$E_2=I-S_4=3-3=0$$

This develops the idea that Simpson's rule is much more accurate than trapezoidal rule.

12.2.2 MAPLE commands "trapezoid" for trapezoidal rule and "Simpson" for Simpson's rule and demonstrate through examples

The use of MAPLE command "trapezoid" and "Simpson" is illustrated in the following example.

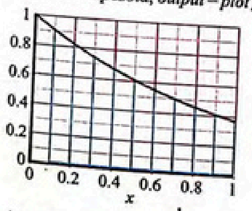
Example 7 Use of MAPLE command to find the approximate area $\int_0^1 e^{(-x)}dx$ in the interval [0, 1] by

(a). Trapezoidal rule. (b). Simpson rule

Solution

a. The command below will show you full detail about Trapezoidal rule on line by typing:

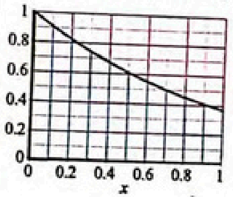
```
?trapezoid
with(Student[Calculus1]) :
ApproximateInt(exp(-x), x=0..1,
method=trapezoid, output=plot);
```



An approximation of $\int_0^1 f(x)\,dx$ using trapezoid rule, where $f(x)=e^{-x}$ and the partition is uniform. The approximate value of the integral is 0.6326472382. Number of subintervals used: 10.

b. The command below will show you full detail about Simpson rule on line by typing:

```
?simpson
with(Student[Calculus1]) :
ApproximateInt(exp(-x), x=0..1,
method=simpson, output=plot);
```



An approximation of $\int_0^1 f(x)\,dx$ using Simpson's rule, where $f(x)=e^{-x}$ and the partition is uniform. The approximate value of the integral is 0.6321205808. Number of subintervals used: 10.

Exercise 12.2

1. Use the trapezoidal rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your results with the exact value of the definite integral:

   a. $I=\int_1^3 x^2dx,\ n=4$ b. $I=\int_0^1\left(\frac{x^2}{2}+1\right)dx,\ n=4$

   c. $I=\int_1^3 \frac{dx}{x},\ n=6$ d. $I=\int_0^2\sqrt{1+x^2}\,dx,\ n=6$

2. Use Simpson's rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your results with the exact value of the definite integral:

   a. $I=\int_2^4 x^2dx,\ n=3$ b. $I=\int_2^3\left(\frac{x^2}{3}-1\right)dx,\ n=4$

   c. $I=\int_1^3 \frac{dx}{x},\ n=3$ d. $I=\int_0^1 e^{2x}dx,\ n=4$

3. A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0\le x\le 1$, which means that: $\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4},\ n=4$

   Approximate the definite integral on the left by trapezoidal rule that equals the right side when $\pi=3.1$.

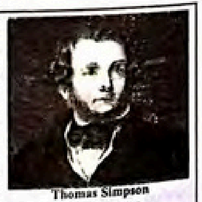
4. A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0\le x\le 1$, which means that $\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4},\ n=4$

   Approximate the definite integral on the left by Simpson's rule that equals the right side when $\pi=3.1$.

5. Use MAPLE commands to find the approximate area in the interval [0, 2] by

   a. Trapezoidal rule b. Simpson's rule

History



Thomas Simpson (1710-1761)

Thomas Simpson was a British mathematician and creator of Simpson's rule for approximate definite integrals. This rule was known and used earlier by Bonoventura Cavalieri in 1639 and later by James Gregory. The long popularity of Simpson's textbook invites this association with his name therefore many readers would have learnt it from them. A challenge proposed by 'Pierre de Fermat' to find a point 'D' such that the sum of the distances to three given points A, B and C is least. This challenge was popularized in Italy Sampson treats this challenge in this first part of Doctrine and application of Fluxions (1750) by the description of circular arc at which the edges of the triangle ABC subtend and angle of $\frac{\pi}{3}$. In second part of the book he extends this geometrical method. Sampson's several textbooks contains most of the optimization problems treated by simple techniques.

Review Exercise 12

1. **Choose the correct option.**

i. When $f(x)=0$, the roots of the equation $f(x)=6x^2+5x-6$ are:
(a). $\left\{\frac{2}{3},-\frac{2}{3}\right\}$ (b). $\left\{\frac{2}{3},-\frac{3}{2}\right\}$ (c). $\left\{\frac{3}{2},-\frac{2}{3}\right\}$ (d). $\left\{-\frac{3}{2},-\frac{2}{3}\right\}$

ii. When $f(x)=0$, then the value of x in the equation $f(x)=x^{-2}+6x-5$ is:
(a). $x\approx-3.7187\ldots$ (b). $x\approx0.37187\ldots$ (c). $x\approx-0.37187\ldots$ (d). $x=4.7847\ldots$

iii. The Newton Raphson method is also known as:
(a). tangent method (b). second method
(c). diameter method (d). chord method

iv. The iterative formula for newton Raphson method $x_3=$
(a). $x_1-\frac{f(x_2)}{f'(x_2)}$ (b). $x_2-\frac{f(x_3)}{f'(x_3)}$ (c). $x_2-\frac{f(x_2)}{f'(x_2)}$ (d). $x_0+\frac{f(x_2)}{f'(x_2)}$

v. If the function values $f(a)$ and $f(c)$ at $x=a$ and $x=c$ have the opposite signs, then the roots lies in the interval:
(a). $[a, b]$ (b). $[a, c]$ (c). (a, b) (d). (a, c)

vi. The Newton Raphson method fails at:
(a). stationary point (b). floating point
(c). continuous point (d). non-of these

vii. The positive roots for $3t-\cos t-1$ by using the Regula Falsi method and correct up to 3 decimal places are:
(a). 0.507 (b). 0.670 (c). 0.570 (d). 0.607

viii. The MAPLE command to solve equation $x^2-5x+3=0$ is:
(a). solve $(x^2-5x+3)=0$ (b). fsolve $(x^2-5x+3=0)$
(c). Simplon $(x^2-5x+3=0)$ (d). Psolve $(x^2-5x+3=0)$

ix. The roots of the equation $x-\sin x-\frac{1}{2}=0$ by using the bisection method between $1<x<2$:
(a). 1.88 (b). 1.48 (c). 1.49 (d). 1.897

x. The bisection method is also known as:
(a). Newton method (b). Binary Chopping method
(c). Quaternary method (d). none of these

Project

i. Create a quadratic function and find its roots by using Newton-Raphson method. Also plot its graph by using maple commands.

ii. Write a function and find its integral limit –1 to 2. Use trapezoidal rule for approximate value.

Summary

- If $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs, then the bisection method will be used to approximate the actual root r of the non-linear equation $f(x)=0$ in the interval $[a, b]$. In bisection method, the approximate root of r will always be the midpoint of the interval $[a, b]$.
- If $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs, then the regula-falsi method will be used to approximate the actual root r of the non-linear equation $f(x)=0$ in the interval $[a, b]$. In Regula-Falsi method, the approximate root of r will not be the midpoint of the interval $[a, b]$ that should be anywhere in the interval $[a, b]$.
- If $f(x)$ and its derivatives are continuous functions and x_0 is the initial iterate, then the Newton-Raphson method $x_{i+1}=x_i-\dfrac{f(x_i)}{f'(x_i)}$, $i=0,1,2,3,\ldots, i=1,2,3,\ldots$ produces a sequence of successive iterates $\{x_n\}$ that will converge faster to r than either the bisection and regula-falsi methods.
- If $f(x)$ is continuous on $[a, b]$, then the triangle rule
$$R_n=f(x_1)\Delta x+f(x_2)\Delta x+\ldots+f(x_n)\Delta x$$
is used to approximate the definite integral $\int_a^b f(x)dx$.
- If $f(x)$ continuous on $[a, b]$, then the trapezoidal rule is
$$T_n=\frac{\Delta x}{2}[f_0+2f_1+2f_2+\ldots\ldots+2f_{n-1}+f_n],\ \Delta x=\frac{(b-a)}{n}$$
is used to approximate the definite integral $\int_a^b f(x)dx$.
- If $f(x)$ is continuous on $[a, b]$, then the Simpson's rule
$$S_{2n}=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+\ldots+f_{2n-1})+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}\right]$$
with $\Delta x=\dfrac{(b-a)}{2n}$ is used to approximate the definite integrel $\int_a^b f(x)dx$.

History

Carl Runge was a German mathematician and physicist. He was co-developer of the runge – kutta method in the field of numerical analysis. He received his Ph.D degree in mathematics at Berlin in 1880. In 1886 he became the professor in Hanover, Germany. He also studied spectral lines of various elements and was very interested in the application of this work to astronomical spectrosphy. Runge reached to his retirement in 1923 but he continued to run his institute until his successor Gustave Herglotz, arrived in 1925.

Carl David Telme Runge (1710-1761)

Answers

Unit-2

Exercise 2.1

1. a. Domain: $(-\infty,\infty)$ or $-\infty<x<\infty$
Range: $(-\infty,5)$

b. Domain: $(-\infty,\infty)$ or $-\infty<x<\infty$
Range: $(-\infty,\infty)$ or $-\infty<y<\infty$

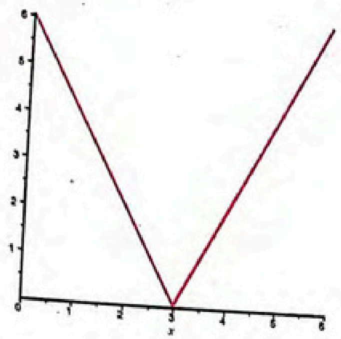
c. Domain: $(-\infty,-2)\cup(-2,\infty)$
Range: $(-\infty,0)\cup(0,\infty)$

d. Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$

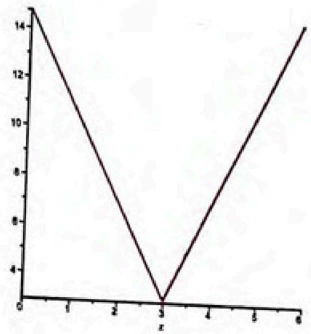
e. Domain: $(-\infty,\infty)$
Range: $[0,\infty)$

f. Domain: $(-\infty,\infty)$
Range: $[0,\infty)$

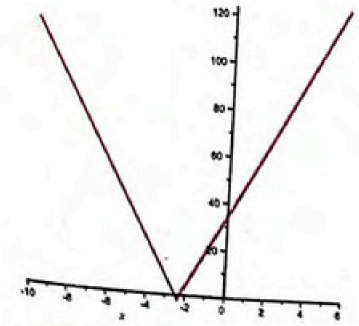
2. a.



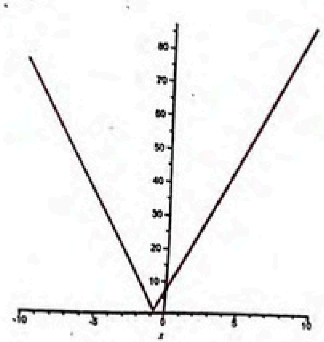
b.



c.



d.



3. a. $f[g(x)]=4x^2+1, g[f(x)]=2x^2+2$ b. $f[g(x)]=\sin(1-x^2), g[f(x)]=1-\sin^2 x=\cos^2 x$

c. $f[g(x)]=x, g[f(x)]=x$ d. $f[g(x)]=Sin(2x+3), g[f(x)]=2Sinx+3$

4. a. $h^{-1}(x)=f^{-1}(g(x))=x-1$ b. $h^{-1}(x)=f^{-1}(g(x)=\frac{x-7}{4}$

$k^{-1}(x)=g^{-1}f(x)=x-1$ $k^{-1}(x)=g^{-1}(f(x))=\frac{x-14}{4}$

c. $h^{-1}(x)=f^{-1}[g(x)]=x+3$ d. $h^{-1}(x)=f^{-1}[g(x)]=x$ self-inverse

$k^{-1}(x)=g^{-1}[f(x)]=\frac{2x-3}{2}$ $k^{-1}(x)=g^{-1}[f(x)]=x$ self-inverse

Exercise 2.2

1. a. Trigonometric b. Algebraic c. Inverse trigonometric

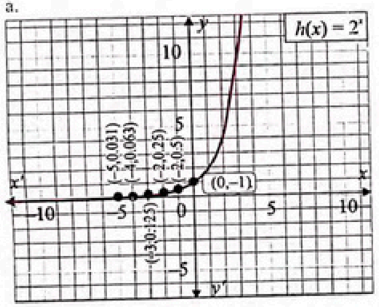
d. Logarithmic e. Implicit f. Hyperbolic g. Explicit

2. a. 1 in gram 100 b. 0.7089 in gram 70.9g

c. 0.5025 in gram 50.3g d. 0.11648 in gram 11.6g

c. The amount of radium in the box decreases with the passage of time.

3. a.



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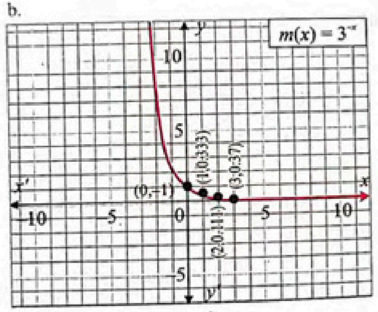
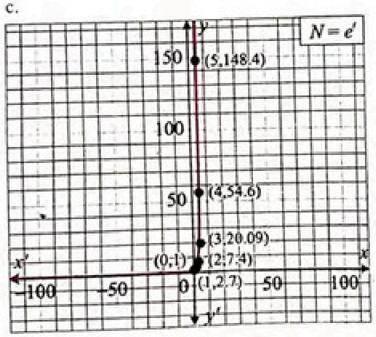


c.



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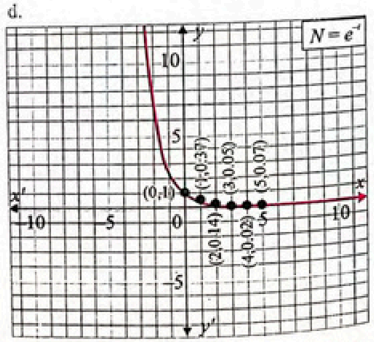
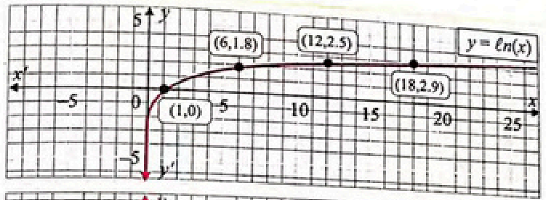


4. a.



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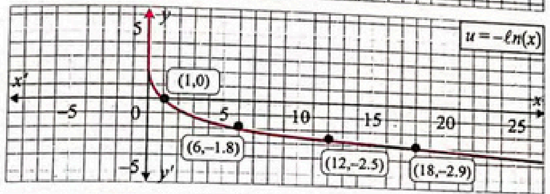

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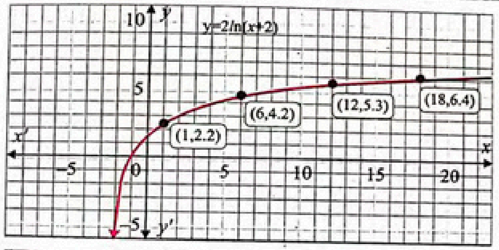

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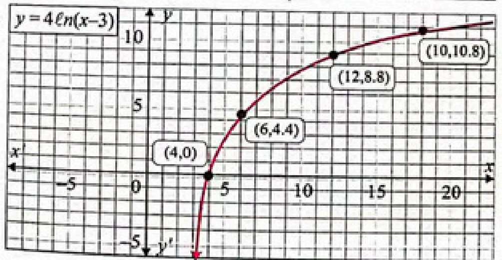
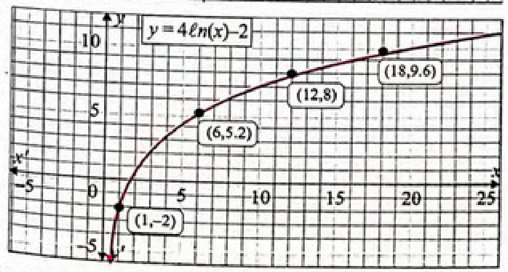
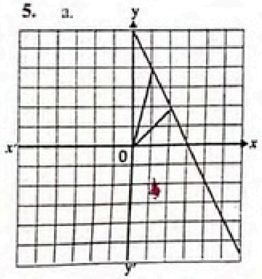


e.



5. a.



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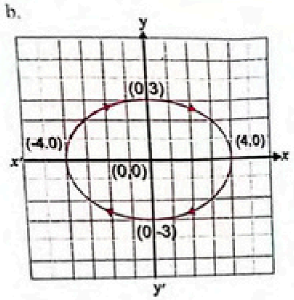
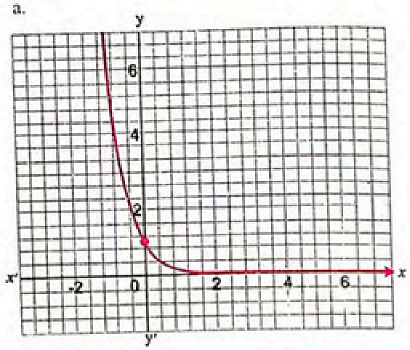
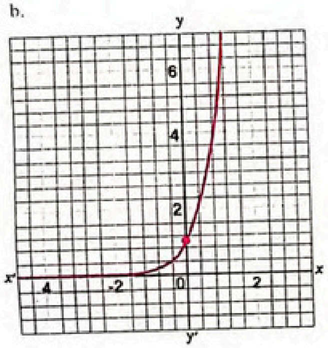


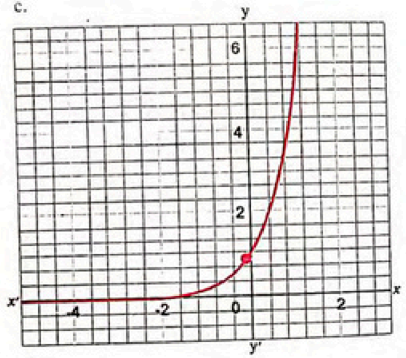
6. a.



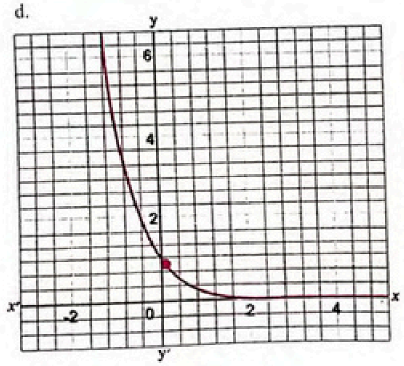
b.



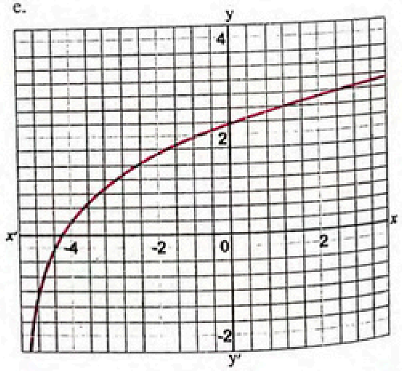
c.



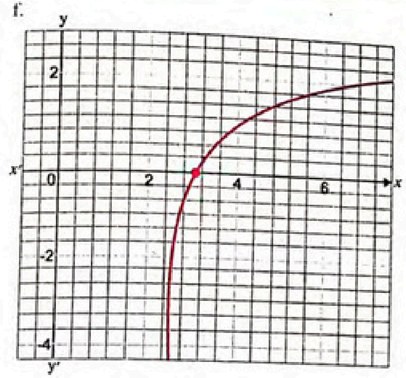
d.



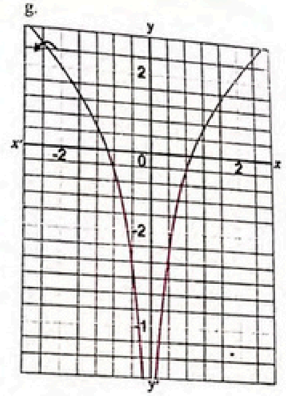
e.



f.

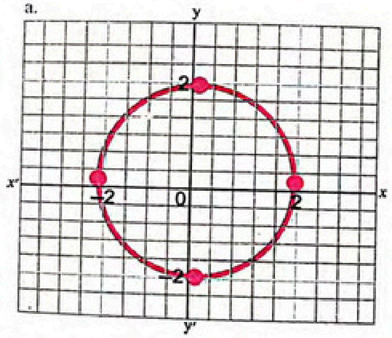


g.

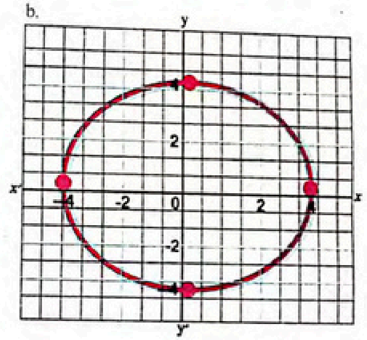


7.

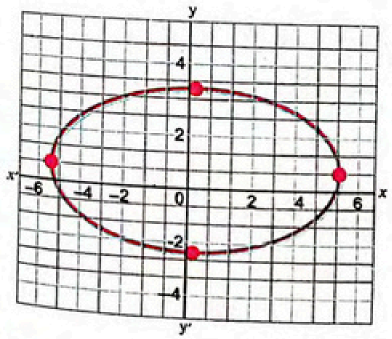
a.



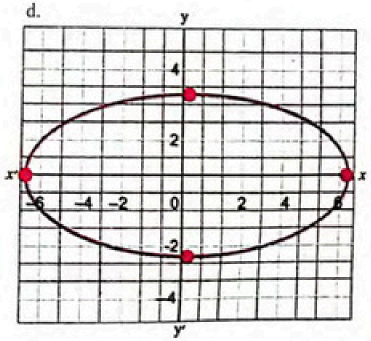
b.



c.



d.



Exercise 2.3

1. a. $-\frac{1}{4}$ b. $\frac{9}{4}$ c. $\frac{1}{2}$ d. -1 e. 1 f. 1 g. 0 h. 0
2. a. does not exist b. $-\frac{1}{9}$ c. $\frac{1}{2\sqrt{5}}$
3. a. 5 b. -7 c. 0
4. a. 5900 b. 5514.29 c. 5200
6. a. $\frac{1}{2\sqrt{x}}$ b. $\frac{n}{m}a^{n-m}$ c. $\frac{p^2}{q^2}$ d. does not exist c. e^{-1} *or* $\frac{1}{e}$ f. $e^{1/2}$ g. e^{-1}

Exercise 2.4

1. a. *Continuous on the whole set* R *of real numbers*
   b. *Continuous at every real value of* x, *except* $x=5$
   c. *Continuous at every real value of* x , *except the values* $x=3$ *and* $x=-2$
2. $f(x)$ *is continuous at* $x=0$
3. a. $g(x)$*is continuous on the open interval* $(-1, 2)$
   b. $g(x)$*is continuous from the right at* $x=-1$
   c. $g(x)$*is continuous from the left at* $x=2$
   d. $g(x)$*is continuous on the closed interval*$[-1, 2]$
4. a. $f(x)$*is continuous on the open interval*$(0, 3)$
   b. $f(x)$*is continuous from the right at* $x=0$
   c. $f(x)$*is continuous from the left at* $x=3$
   d. $f(x)$*is continuous on the closed interval*$[0, 3]$
5. a. $f(x)$*is discontinuous at* $x=1$ b. $f(x)$*is discontinuous at* $x=0$
6. a. The graph is as under: b. $\lim_{s \to 10,000} E(s)=$ *\$1,000 E(10,000)= \$1,000*
   c. $\lim_{s \to 20,000} E(s)$ does not exist; E(20,000)=\$2,000. d. Yes at s=10,000, no at s=20,000
7. true

Review Exercise 2

1. (i). a (ii). d (iii). a (iv). d (v). c (vi). b (vii). b (viii). c (ix). a (x). c (xi). d (xii). c (xiii). a (xiv). a (xv). c

ANSWERS

Unit-3

Exercise 3.1

1. a. 5 b. $\frac{1}{3}$ c. 21 d. 0.25
2. a. 2 b. $\Delta x=0.1$ c. 4.1π d. $\frac{1}{7}$
3. a. $\Delta t=0.1$; 14.4 units/second b. $\Delta t=0.01$; 15.84 units/second
4. $\Delta t=2$; 25; the average rate of inflation is Rs.25 per year.
5. $\Delta t=30$; 1170; the profit is increasing at the rate of Rs.1170 per acre, when the number of acres changes from 20 to 50.
6. a. 3 b. $\frac{5}{2\sqrt{5x+6}}$ c. $2x$ d. $-2x$ e. $32x-7$ f. $\frac{-7}{x^2}$
7. a. 3 b. $y=3x-19$ c. slope = 5
8. a. 1, $y=x+9$ b. 1, $y=x-16$ c. -6, $y=-6x-10$ d. 7, $y=7x-6$

Exercise 3.2

1. a. $\frac{d}{dx}[f(x)+g(x)]=12x+5$, $\frac{d}{dx}[f(x)-g(x)]=-12x+1$
   b. $\frac{d}{dx}[f(x)+g(x)]=\frac{104}{3}x+\frac{1}{2}$, $\frac{d}{dx}[f(x)-g(x)]=\frac{100}{3}x-\frac{1}{2}$
   c. $\frac{d}{dx}[f(x)+g(x)]=12x^2-8x$, $\frac{d}{dx}[f(x)-g(x)]=-6x^2+8x$
   d. $\frac{d}{dx}[f(x)+g(x)]=12x^2+\frac{6x}{5}-7$, $\frac{d}{dx}[f(x)-g(x)]=12x^2-\frac{6}{5}x-3$
2. a. $\frac{dy}{dx}=9x^2+2x-6=12x^2+\frac{6x}{5}-7$ b. $\frac{dy}{dx}=42x^5-112x^3+6x^2-8$
   c. $\frac{dy}{dx}=3\sqrt{x}-\frac{3}{2\sqrt{x}}-2$ d. $\frac{dy}{dx}=\frac{15}{\sqrt{x}}-12$
3. a. $\frac{dy}{dx}=-7(x-4)^{-2}$ b. $\frac{dy}{dx}=\frac{4x^4-48x^3-2x+6}{(4x^3+1)^2}$
   c. $\frac{dy}{dx}=\frac{5x-6}{2x^{\frac{3}{2}}}$ d. $f'(p)=\frac{24p^2+32p+29}{(3p+2)^2}$
4. a. $y=3x-7$ b. $3x-4y+1=0$ c. $x+25y-7=0$ d. $y=-2x+9$
5. a. $y=\frac{Px}{x-p}$ b. $\frac{dy}{dx}=\frac{-P^2}{(x-p)^2}$

Exercise 3.3

1. a. $20(x^3-4x+2)^4(3x^2-4)$ b. $\frac{11}{5}(4x-x^3)^{10}(4-3x^2)$
   c. $-2t(1-3t^2)^{-\frac{2}{3}}$ d. $\frac{-21}{(3t+1)^8}$
2. a. $f'(x)=(2x-5)^2(40x-67)$ b. $f'(x)=\frac{(x^2-2x-8)}{(x-1)^2}$
   c. $f'(x)=\frac{-12(2x-5)^3}{(x-4)^5}$ d. $f'(x)=\frac{4x^2+11}{(2x^2+11)^{\frac{1}{2}}}$
3. a. $y'=\frac{3}{2}t+2$ b. $y'=\frac{12}{a}t^2$ c. $y'=\frac{b(t^2-1)}{2at}$ d. $y'=\frac{[2t-t^4]}{[1-2t^3]}$
4. 4222.8 dollars/hour
5. a. $-\frac{x}{y}$ b. $-\frac{3y^2+4xy}{2x^2+6xy}$ c. $-\frac{(x+y)^2}{(x+y)^2+1}$ d. $-\frac{y^2}{x^2}$
6. a. $-\frac{2}{3}x(x^2+1)^{-\frac{4}{3}}$ b. $\frac{x^2+4x}{(x+2)^2}$ c. $\frac{1}{(5-x)^2}$ d. $\frac{-3}{(x-1)^2}$
7. a. $\frac{du}{dv}=-a^2v/b^2u$ b. $\frac{dv}{du}=-b^2u/a^2v$
8. $\text{slope}=9/7$
9. 25 miles per hour.

Exercise 3.4

1. a. $2Cosx(2x)$ b. $-3\operatorname{cosec}^2(3x)$ c. $3\left[\sec^2(3x)-\sin(3x)\right]$
   d. $-2\cot(x).\operatorname{cosec}^2(x)$ e. $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$ f. $3\sin^2(x).\cos(x)$
2. a. $-3x^2\operatorname{cosec}^2(3x)+2x\cot(3x)$ b. $2(\sin 2x+\cot 3x)(2\cos 2x-3\operatorname{cosec}^2 3x)$
   c. $-8\operatorname{cosec}2x.\cot 2x$ d. $4(x+3)\sec^2(x+3)^2$
   e. $\frac{\sqrt{x}\cos\sqrt{x}\sec^2 x+\tan x\sin\sqrt{x}}{2\sqrt{\tan x}\cos^2\sqrt{x}}$ f. $\frac{2\sec^2 2x+3\cot 3x(1+\tan 2x)}{\operatorname{cosec}3x}$
3. a. $\frac{dy}{dx}=\frac{-1}{\sqrt{a^2-x^2}}$ b. $\frac{dy}{dx}=\frac{p}{p^2+x^2}$ c. $\frac{dy}{dx}=\frac{a}{x^2+a^2}$
   d. $\frac{dy}{dx}=\frac{-1}{1+x^2}$ e. $\frac{dy}{dt}=\frac{-1}{(t+3)\sqrt{t^2+6t+8}}$ f. $\frac{dy}{dx}=\frac{-1}{x(x^2+1)}+\frac{1}{x^2}\tan^{-1}\left(\frac{x+1}{x-1}\right)$
4. a. $P'(t)=\frac{5\pi}{26}\sin\left(\frac{\pi}{26}t\right)$ b. $P'(8)=50$ dollars/ week, $P'(26)=0$ *dollars / week*, $P'(50)=-14$ *dollars / week*

5. a. $V'(t)=\frac{7\pi}{40}\sin\left(\frac{\pi t}{2}\right)$
   b. $V'(3)=-0.5498\approx-0.55$ cubic units/ sec, $V'(3)=0$ *cubic units / sec* $V'(3)=0.5498\approx 0.55$ *cubic units / sec*

Exercise 3.5

1. a. $2e^{2x}$ b. e^{3x} c. $\frac{5}{4}x.e^{x^2+1}$ d. $2^x.\ln(2)$ e. $256.4^t.\ln 4$
   f. $\frac{1}{(x+1)}$ g. $\frac{2}{x\ln a}$ h. $\frac{3}{x+2}$
2. a. $-20x^4 11^{-4x^5+3}.\ln(11)$ b. $\frac{1}{2\sqrt{x}}e^{\sqrt{x}-5}$ c. $(5x^4-x^3)e^{\frac{1}{x}}$
   d. $\frac{2(e^{2x}+2)}{(e^{-2x}+1)^2}$ e. $\frac{me^{mx}+me^{-mx}}{e^{mx}-e^{-mx}}$ f. $\frac{ae^{ax}+ae^{-ax}}{e^{ax}+e^{-ax}}-\frac{(e^{ax}-e^{-ax})(ae^{ax}-ae^{-ax})}{(e^{ax}+e^{-ax})^2}$
3. a. $\frac{x^2}{2}(1+3\ln x)$ b. $\frac{x(1+4\ln x)}{2\sqrt{\ln x}}$ c. $\frac{2x}{1-x^4}$ d. $\frac{-1}{\sqrt{x^2+1}}$
   e. $-2e^{-2x}[\sin(2x)+\cos(2x)]$ f. $x(x.\cos x+2)e^{\sin x}$
4. a. $2.\cosh(2x)$ b. $3.\sinh(3x)$ c. $\frac{-\sin x}{\sqrt{1+\cos^2 x}}$
   d. $\frac{2}{4-x^2}$ e. $-2\operatorname{cosec}(h2x)$ f. $\cosh^{-1}(x)$
5. 2.27 mm of mercury/yr; 0.81 mm of mercury/yr; 0.41 mm of mercury/yr.
6. $A'(t)\approx 10,000(\ln 2)2^{2t}$; $A'(1)=27,726$ bacteria/hr
   $A'(5)=7,097,827$ bacteria/hr

Review Exercise 3

1. (i) b (ii) d (iii) b (iv) c (v) b (vi) d (vii) a (viii) c (ix) c (x) d

Unit-4

Exercise 4.1

1. a. $f''(x)=18x$ b. $f'''(x)=-\frac{6}{x^4}$ c. $s''(t)=\frac{-25}{4}(5t+7)^{\frac{-3}{2}}$ d. $\frac{4}{(x-1)^3}$
2. a. $y''=-\frac{b^4}{a^2y^3}$ b. $y''=-\frac{r^2}{y^3}$ c. $y''=0$ d. $\frac{(e^y-e^x)(e^{x+y}-1)}{(e^y+1)^3}$
3. a. $y''=\frac{9}{32t}$ b. $y''=\frac{4}{3a^2}$ c. $y''=\frac{-b}{a^2\sin^3 2t}$
   d. $\frac{d^2y}{dx^2}=\frac{(1+t^3)[6a(3a-6at^3)(1-7t^3+t^6)+54a^2t^3(2-t^3)^2]}{(3a-6at^3)^3}$

4. a. $24(3x^2+4)^3+864(x^3+4x-5)(3x^2+4)\cdot x+1296(x^3+4x-5)^2\cdot x^2+288(x^3+4x-5)^2\cdot(3x^2+4)$

b. $16(1+\tan^2 x)\cdot\tan x+24\tan^3 x(1+\tan^2 x)$ c. $\frac{729}{2}e^{9x}\cdot\ln|e|^3$ d. $24x\cdot\ln(x^2)+52x$

5. a. $\sqrt{\sec(2x)}\cdot\tan(2x)^3+7\sqrt{\sec(2x)}\tan(2x)(2+2\tan(2x))^2$

b. $-\cos(\sin(x))\cos(x)^3+3\sin(\sin(x))\cos(x)\sin(x)-\cos(\sin(x))\cos(x)$

6. a. $1663200\times 3^7(3x+2)^4$ b. $-2^{11}(181440)(2x-4)^{-10}$

c. $(3)^6 4\cdot\cos(3x+8+3\pi)$ d. $(5)^{12}\cdot 7\,e^{5x+4}$

Exercise 4.2

1. a. $y=\frac{(x+5)}{4}$ b. $y=-2x$ c. $y=\frac{x}{e}$ d. $y=\frac{1}{2}$

2. a. $y=\frac{-1}{2e}x+\left(\frac{1+2e^2}{2e}\right)$ b. $y=\frac{-1}{12}x+1$ c. $-x+\frac{\pi}{2}$ d. $y=-x+1$

3. a. $y=\frac{2}{3}x+\frac{13}{3}$ b. $y=(1+\pi)x+\pi$ c. $x=1$ d. $x=1$

5. a. $1-x+x^2-x^3+\dots\dots$ b. $x^2-\frac{1}{3}x^4+\frac{2}{45}x^6-\dots\dots$

c. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\dots$ d. $-4x-8x^2-\frac{64}{3}x^3-\dots\dots$

6. b. 2.7183 7. a. $\frac{\pi}{4}$ b. $\tan^{-1}\sqrt[3]{16}$ 8. $x=\frac{4+\sqrt{91}}{15}$ *and* $\frac{4-\sqrt{91}}{15}$

Exercise 4.3

1. a. i. critical point $(0,-2)$ ii. Increasing on $(-\infty,-2)\cup(0,+\infty)$; decreasing on $(-2,0)$

iii. $(0,1)$ is relative minimum; $(-2,5)$ is relative maximum

b. i. critical point $\left(\frac{5}{3},-25\right)$ ii. increasing on $(-\infty,-25)$ $\left(\frac{5}{3},\infty\right)$ decreasing on $\left(-25,\frac{5}{3}\right)$

iii. relative minimum $(-25,0)$ relative maximum $\left(\frac{5}{3},-9481\right)$

2. a. $x=-3,4$ b. $x=-3,5$ c. $x=0,1$ d. $x=0,1$

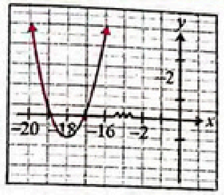
3. a. Relative minimum at $x=1$ b. relative minimum at $x=1$

c. neither maximum nor minimum d. Relative minimum at $x=4$

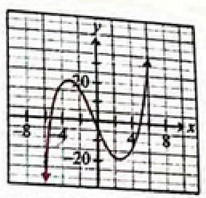
4. a: Critical point: $(-18,-1)$ is relative minimum; increasing on $(-18,+\infty)$; decreasing on $(-\infty,-18)$; concave up on $(-\infty,+\infty)$.

b. Critical points: $(-3,20)$ is relative maximum; $(3,-16)$ is relative minimum; increasing on $(-\infty,-3)\cup(3,+\infty)$; decreasing on $(-3,3)$; inflection point: $(0,2)$; concave up on $(0,+\infty)$; concave down on $(-\infty,0)$.

(a)



(b)



5. a. Relative maximum of 1 at $x=0$, relative minimum of -3 at $x=2$

b. Relative maximum of 2 at $x=-3$, relative minimum of -2 at $x=-1$

6. a. Relative maximum at $x=2$, relative minimum at $x=4$, neither at $x=1,-5$

b. Relative maximum at $x=-3$; relative minimum at $x=\frac{1}{2}$; neither at $x=1$

7. a. The expenditure on advertising that leads to maximum profit is $x=6$.

b. The maximum profit at $x=6$ is 512 hundred dollars.

8. The number of hundred thousands of tires is $x=1{,}000{,}000$ tires; the number of hundred thousands of tires x that developed the maximum profit is $P(10)=\$700=\$700{,}000$.

9. a. The drug concentration is increasing in the interval $(0,3)$ and is decreasing in the interval $(3,\infty)$.

b. The maximum drug concentration time is $x=3$.

c. The maximum drug concentration at a time $x=3$ is $K(3)=0.22\%=0.0022$.

d. The concentration is maximum at $x=3$ as $k(x)$ changes from increasing function to decreasing function. Maximum concentration $k(x)=0.22\%$

10. Hint: $C(x)=[G(x)][32]2.25]=\frac{1}{48}\left(\frac{300}{x}+2x\right)(32)(2.25)=1.5\left(\frac{300}{x}+2x\right)$

$C'(x)=1.5\left(\frac{-300}{x^2}+2\right)=0\Rightarrow x=\pm\sqrt{150}$

Minimum cost at $x=\sqrt{150}=12.2$ is $C(12.2)=\$73.48$.

Review Exercise 4

1.

(i) b	(ii) c	(iii) b	(iv) a	(v) a
(vi) b	(vii) c	(viii) d	(ix) c	(x) b

Unit-5

Exercise 5.1

1. a. $t\neq 0,\ t>0$ b. $t\neq 2,\ t\geq 0.$

c. $t\neq(2n-1)\frac{\pi}{2},\ n=0,\pm1,\pm2,\dots$ d. $t\neq n\pi,\ n=0,\pm1,\pm2,\dots$

2. a. $(7t-3)\hat{i}-10\hat{j}+\left(\frac{2t^2-3}{t}\right)\hat{k}$ b. $(2t+4)\hat{i}-15\hat{j}+(3t^2+\frac{4}{t})\hat{k}$

c. $(1-t)\sin t$ d. $-t^2e^t\hat{i}+t^2\sin t\,\hat{j}+\left(2te^t+5\sin t\right)\hat{k}$

3. a. $3\hat{i}+e^2\hat{j}$ b. $3\hat{i}-\frac{1}{3}\hat{j}+2\hat{k}$ c. $\hat{i}+e\hat{k}$ d. $\hat{i}+\ln(4)\hat{j}$

4. a. All values of t. b. All values of t except $t=0$ c. All values of t except $t\neq 0, t\neq -1$
   d. All values of t $\in$ positive real number R^+

Exercise 5.2

1. a. $F'(t)=\hat{i}+2t\hat{j}+(1+3t^2)\hat{k}$ b. $F'(s)=(1+4s)\hat{i}+(2s-1)\hat{j}+2s\hat{k}$
   c. $-\sin\theta\hat{i}+\cos\theta\hat{j}-3\sin\theta\hat{k}$

2. a. $2\hat{i}+18t\hat{j}-16\hat{k}$ b. $2\hat{i}-2\hat{j}+36s^2\hat{k}$ c. $-\frac{1}{x^2}\hat{i}-2\hat{k}$
   d. $2[\cos^2\theta-\sin^2\theta]\hat{i}+2[\cos^2\theta-\sin^2\theta]\hat{j}$

3. a. $-2x-9x^2$ b. $\frac{4x}{\sqrt{4x^2+1}}$

4. a. $v(t)=\hat{i}+2t\hat{j}+2\hat{k}$; $v(1)=\hat{i}+2\hat{j}+2\hat{k}$, $|v(1)|=3$
   in the direction of $\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$, $\vec{a}(t)=2\hat{j}$; $\vec{a}(1)=2\hat{j}$,

   b. $v(t)=-\sin t\hat{i}+\cos t\hat{j}+3\hat{k}$, $v\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\hat{i}+\frac{\sqrt{2}}{2}\hat{j}+3\hat{k}$
   $\left|v\left(\frac{\pi}{4}\right)\right|=\sqrt{10}$ *in the direction of* $-\frac{1}{2\sqrt{5}}\hat{i}+\frac{1}{2\sqrt{5}}\hat{j}+\frac{3}{\sqrt{10}}\hat{k}$
   $\vec{a}(t)=-\cos t\hat{i}-\sin t\hat{j}$, $\vec{a}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\hat{i}-\frac{\sqrt{2}}{2}\hat{j}$

   c. $v(t)=e^t\hat{i}-e^{-t}\hat{j}+2e^{2t}\hat{k}$, $v(\ln 2)=2\hat{i}-\frac{1}{2}\hat{j}+8\hat{k}$
   $|v(\ln 2)|=\frac{\sqrt{273}}{2}$ *in the direction of* $\frac{4}{\sqrt{273}}\hat{i}-\frac{1}{\sqrt{273}}\hat{j}+\frac{16}{\sqrt{273}}\hat{k}$
   $\vec{a}(t)=e^t\hat{i}+e^{-t}\hat{j}+4e^{2t}\hat{k}$; $\vec{a}(\ln 2)=2\hat{i}+\frac{1}{2}\hat{j}+16\hat{k}$

Review Exercise 5

1. (i) a (ii) b (iii) c (iv) d (v) a
   (vi) c (vii) a (viii) a (ix) d (x) b

Unit-6

Exercise 6.1

1. a. $\frac{x^4}{4}+C$ b. $(a+15)x$ *or* $ax+15x$ c. $\frac{1}{5}\ln|x|+c$
   d. x^3+2x^2-5x+c e. $\frac{t^4}{4}+\frac{3}{5}t^5+C$ f. $-\frac{1}{2}x^{-2}+C$

   g. $\frac{e^{5x}}{5}+C$ h. $5\tan^{-1}x$ i. $\frac{2}{9}t^{\frac{9}{2}}$

2. a. $\frac{1}{27}(3x+4)^9+C$ b. $\frac{(x^3-4)^2}{2}+C$ c. $\frac{(x^3+7x)^9}{9}+C$
   d. $-\frac{1}{3}(x^3-7x)^{-3}+C$ e. $2\left(\sqrt{3t^2-7}\right)$ f. $2\left(r^{\frac{1}{3}}+4\right)^{\frac{3}{2}}$
   g. $\frac{2}{3}x^{\frac{3}{2}}+\frac{6}{5}(x)^{\frac{5}{2}}+C$ h. $\frac{-1}{2\left(x^2+2x+2\right)}+C$

3. a. $e^{6t}+C$ b. $\frac{1}{10}e^{(5x^2+1)}$ c. $\frac{1}{3}e^{(x^3-6x+4)}$ d. $\frac{8^{(7-3x^2)}}{\ln 8}+C$

4. $y=x^4+2x^3-3$ 5. $y=\frac{\left(2x^2-1\right)^3}{12}-406.42$

6. a. $-e^{-0.01t}$ b. 0.095 7. a. $w=0.0005h^3$ b. 171.5 ponds 8. 19,400

Exercise 6.2

1. a. $\frac{\sin^5 x}{5}+C$ b. $\frac{3}{5}(\sin x)^{\frac{5}{3}}+C$ c. $-\frac{(\ln(\cos x))^2}{2}+C$
   d. $-\cos e^x+C$ e. $2\ln\left|\sin\sqrt{x}\right|+C$ f. $-\ln|\sin x+\cos x|+C$

2. a. $\frac{1}{4}tan^{-1}\left(\frac{x}{4}\right)+C$ b. $-\tan^{-1}(\cos x)+C$ c. $\frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right)+C$
   d. $\ln\left|x^2+4x+5\right|+\tan^{-1}(x+2)+C$ e. $\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)-(5-(x+1)^2)^{\frac{1}{2}}+C$

3. a. $x^2\frac{e^{2x}}{2}-x\frac{e^{2x}}{2}+\frac{e^{2x}}{8}+C$ b. $-x\sin x+\cos x+C$ c. $-\frac{1}{5}e^{2x}\cos x+\frac{2}{5}e^{2x}\sin x+C$
   d. $\frac{1}{2}e^x\sin x+\frac{1}{2}e^x\cos x+C$ e. $x\sin^{-1}x+\sqrt{1-x^2}+C$ f. $-\frac{1}{3}\cos(e^{3t})+C$

Exercise 6.3

1. a. $\frac{1}{3}\ln\left|\frac{x-3}{x}\right|+C$ b. $3x-\ln x+C$ c. $3\ln x+\ln(x+1)+\frac{1}{x}+\frac{2}{x+1}+C$
   d. $\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)\right)+C$
   e. $\left[\frac{x^2}{2}+x+2\ln(x-1)-2\ln(x)\right]$ f. $\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$
   g. $\ln\left|\frac{(2x+1)^{\frac{5}{6}}}{(x-1)^{\frac{4}{3}}}\right|+C$ h. $x+\ln\left|\frac{(x-5)^3}{(x+3)}\right|+C$

i. $\frac{1}{2}\tan^{-1}x+\ln\left[\frac{(x^2+1)^{\frac{1}{4}}}{(x+1)^{\frac{1}{2}}}\right]+C$ j. $\frac{3}{2}\tan^{-1}x+\frac{1}{4}\sin(2\tan^{-1}x)+C$

2. $1.89ml$ fourth hour 3. 45000 4. 109.86 feet

Exercise 6.4

1. a. 17.5 b. 34816 c. -2 d. 14 e. 15625 f. 0 g. $-2+\sqrt{2}$ h. 1
2. a. 4.357 b. 0.458
3. a. $\frac{23}{3}$ b. $\frac{29}{6}$ c. 8 d. $\frac{68}{6}$
4. a. $A=4.01$ square units b. $A=2.953$ square units c. $A=3.83$ square units d. $\frac{8}{3}$ square units
5. a. L_1 day $=414.44$ barrels b. L_2 day $=191.6$ barrels
   c. As is apparent from the amount of oil leaked on 1st and 2nd day, the number of barrel of oil leakage per day decreases with the passage of time.
6. a. $\frac{x^3}{3}+\frac{x^2}{2}-x+C$ b. $-\frac{e^{2x}\cos(x)}{5}-\frac{2e^{2x}\sin(x)}{5}+C$

Review Exercise 6

1. i. (b) ii. (c) iii. (c) iv. (c) v. (a) vi. (b) vii. (b) viii. (c) ix. (a) x. (d)

Unit-7

Exercise 7.1

2. a. $(-1, 6)$ b. $(0, 0)$ c. $\left(\frac{1}{20},\frac{1}{28}\right)$
3. a. $\left(\frac{11}{6},\frac{17}{6}\right)$ b. $(-24,-2)$ c. $\left(-\frac{32}{9},\frac{41}{9}\right)$
4. a. 3:4 b. 14:11
5. a. $\left(\frac{7}{3},\frac{7}{3}\right)$ b. $\left(\frac{10}{3},\frac{10}{3}\right)$ c. $\left(1,\frac{4}{3}\right)$

Exercise 7.2

1. a. $x-y+5=0$ b. $3x+y-6=0$ c. $3x+4y+12=0$
2. a. $\frac{1}{4}$ b. $-\frac{4}{7}$ c. 2 d. $-\frac{1}{6}$
3. a. x–intercept, $a=-3$ and y–intercept, $b=6$
   b. x–intercept, $a=3$ and y–intercept, $b=9$
   c. x–intercept, $a=-2$ and y–intercept, $b=2$

4. a. perpendicular b. neither perpendicular nor parallel c. parallel d. parallel
5. a. $3x-y=0$ b. $5x-y-5=0$ c. $x-y=0$
6. a. $4x-y-2=0$ b. $x+2y+5=0$ c. $3x+y+4=0$ d. $5x-y-43=0$
7. a. $2x-y+2=0$ b. $8x-y+4=0$ c. $x-2y-8=0$
8. a. $y=\frac{7}{10}x+\frac{13}{7}$ b. $\frac{x-\frac{13}{7}}{\cos 35^0}=\frac{y+\frac{13}{10}}{\sin 35^0}$ c. *Normal form* $\theta=45^{\circ}$, $P=1$, $x\cos\theta+y\sin\theta=1$

Exercise 7.3

1. a. Below the line, on the opposite sides of the line
   b. Above the line, on the same side of the line
2. a. on the opposite sides b. on the same sides
3. a. $d=6$ b. $d=\frac{6}{\sqrt{13}}$ c. $d=1$
4. a. $45°$ b. $\tan^{-1}\left(\frac{1}{7}\right), 180-\tan^{-1}\left(\frac{1}{7}\right)$
5. a. $90°$ b. $\tan^{-1}\left(\frac{-7}{6}\right)$
6. a. $45°$ b. $\tan^{-1}(13)$
7. a. $\tan^{-1}\left(\frac{3}{19}\right)$ b. $18°-\tan^{-1}(9)$
8. a. $119x+102y-125=0$ b. $23x+23y-11=0$ c. $3x+4y-5a=0$ d. $x-3y=0$

Exercise 7.4

1. a. $(1, 2)$ b. $(-16, 12)$
2. a. concurrent; $(3, 1)$ b. concurrent; $(-1,2)$ c. not concurrent
4. a. 6 b. 15 c. $\frac{5}{2}$
5. a. 0 b. 0 c. yes
6. a. $3x-5y=0$ and $x+y=0$ b. $x-y=0$ and $4x-5y=0$
8. a. $2x^2+7xy+3y^2=0$ b. $x^2-2\tan\theta xy-y^2=0$ c. $ay^2-2hxy+bx^2=0$

Review Exercise 7

1. i. (b) ii. (c) iii. (a) iv. (a) v. (b)
   vi. (a) vii. (d) viii. (a) ix. (b) x. (d)

Unit-8

Exercise 8.1

1. a. $x^2+y^2=16$ b. $x^2+y^2-6x-4y+12=0$ c. $x^2+y^2+8x+6y+9=0$
   d. $x^2+y^2+2ax+2by-2ab=0$

ANSWERS

2. a. $x^2+y^2=25$ b. $x^2+y^2=36$ c. $x^2+y^2-12x+12y=0$
d. $x^2+y^2+18x+12y+117=317$ e. $(x+5)^2+(y-4)^2=16$
f. $(x-5)^2+(y-3)^2=25$

3. a. $center=c(4,3), r=4$ b. $center=c\left(-2,\frac{3}{2}\right), r=2\sqrt{2}$ c. $center=c(-2,3), r=0$
d. $center=c\left(\frac{1}{2},4\right), r=\sqrt{\frac{-7}{4}}$

4. a. $x^2+y^2-2x-21=0$ b. $x^2+y^2+4x+6y-72=0$ c. $x^2+y^2-22x-4y+25=0$

5. a. $x^2+y^2-2x+2y-48=0$ b. $x^2+y^2-6x-8y+15=0$

6. a. $5x^2+5y^2+12x-15y=0$ b. $x^2+y^2=3x+y=0$

7. a. $16x^2+16y^2+128x-56y+49=0$ b. $x^2+y^2-8x-4=0$
c. $x^2+y^2+6x-10y+9=0$

8. a. $x^2+y^2=25$ b. $x^2+y^2=20$

Exercise 8.2

1. a. equation of tangent : $x+2y=5$, equation of normal : $2x-y=0$
b. equation of tangent: $3x+2y+13=0$ equation of normal $2x+3y=0$
c. equation of tangent: $x+y-5=0$ equation of normal: $y-x=-3$
equation of tangent: $18x+18\sqrt{3}y=13$, equation of normal: $18\sqrt{3}x+18y=0$,

2. a. equation of tangent : $x+y-2\sqrt{2}=0$, equation of normal : $x-y=0$
b. equation of tangent : $\sqrt{3}x+y-2=0$, equation of normal : $x-\sqrt{3}y=0$

3. a. $n=\pm a\sqrt{m^2+l^2}$ b. $n=\pm 3\sqrt{2}$ c. $n=\pm 18\sqrt{2}$

4. a. $c=\pm a\sqrt{1+m^2}$ b. $c=\pm 3\sqrt{2}$ c. $c=\pm 9a\sqrt{2}$

5. a. $g^2m^2+f^2\ell^2-2\ell ng-2gf\ell m-n^2+2fnm-m^2c-c^2=0$

6. a. $n=7,-43$ b. $n=4.52,-13.52$ c. Does not touch the circle $C=\frac{-152\pm\sqrt{14721}}{32}$

10. a. $y=\frac{-1}{2}x+\sqrt{5}$ b. $3x+4y=25$

12. a. $y=x\pm 2$ b. $y=\frac{x}{\sqrt{3}}\pm\frac{2}{3}$ c. $y=\sqrt{3}x\pm\frac{\sqrt{13}}{2}$

Review Exercise 8

1. i. (d) ii. (d) iii. (c) iv. (a) v. (a) vi. (d) vii. (b) viii. (d) ix. (c) x. (b)

Unit - 9

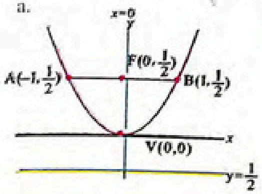
Exercise 9.1

1. a. Vertex (0,0), focus $\left(0,\frac{1}{2}\right)$, ends of latus rectum $A\left(-1,\frac{1}{2}\right), B\left(1,\frac{1}{2}\right)$, axis of symmetry; $x=0$ (y-axis)

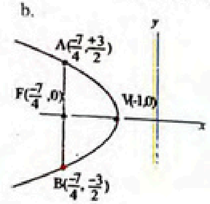
b. Vertex (−1,0), focus $\left(\frac{-7}{4},0\right)$, ends of latus rectum $A\left(\frac{-7}{4},\frac{+3}{2}\right), B\left(\frac{-7}{4},\frac{-3}{2}\right)$, axis of symmetry; $y=0$

c. Vertex (0,3), focus $\left(\frac{1}{4},3\right)$, ends of latus rectum $A\left(\frac{1}{4},\frac{5}{2}\right), B\left(\frac{1}{4},\frac{7}{2}\right)$, axis of symmetry; $y=3$

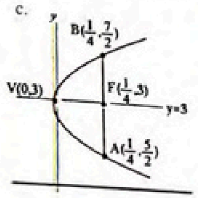
a.



b.



c.



2. a. $(x-2)^2=8(y+1)$ b. $y^2=24x$

3. a. $x^2=12y$ b. $y^2=16x$ c. $y^2=12x$ d. $x^2=-48y$
e. $(x-5)^2=\frac{64}{3}(y-1)$ f. $(y+7)^2=64(x-3)$

4. $3x+y+5=0$

5. $(y+2)^2=6\left(x-\frac{5}{2}\right)$

6. a. $P_1(1,3)$ and $P_2(4,6)$
b. $x=1\pm\sqrt{3}i$ which is imaginary so the line does not intersect the parabola.

7. a. $c=\frac{9}{4}$ b. $c=-\frac{1}{6}$

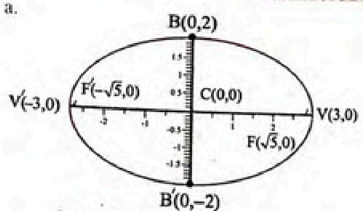
8. a. equation of tangent is $x-y+3=0$, equation of normal is $x+y-9=0$
b. equation of tangent is $4x+3y-1=0$, equation of normal is $8x+9y-1=0$

9. a. $4x+4y+1=0$ b. $y=\sqrt{3}x+\frac{1}{4\sqrt{3}}$

10. $x^2=-12(y+3)$

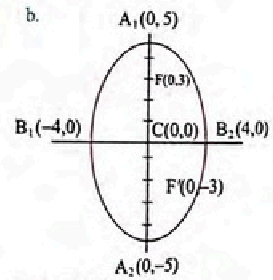
Exercise 9.2

1. a.



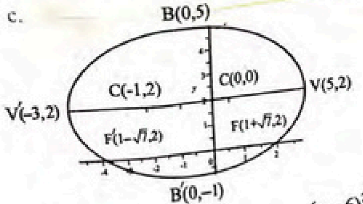
Centre C(0,0)
Foci $F(\pm\sqrt{5},0)$
End points of major axis $A(\pm 3,0)$
End Points of minor axis $B(0,\pm 2)$

b.



Centre C(0,0)
Foci $F(0,\pm 3)$
End points of major axis $A(0,\pm 5)$
End Points of minor axis $B(\pm 4,0)$

c.



Centre: The center is $C(h, k) = C(-1, 2)$

Foci: $F_1\left(-1+\sqrt{7},-1,2\right)$, $F_2\left(-1-\sqrt{7},-1,2\right)$

End points of major axis $A(5,2) A'(-3,2)$

End Points of minor axis $B(1,5) B'(1,-1)$

2. a. $\frac{x^2}{36}+\frac{y^2}{9}=1$ b. $\frac{(x-6)^2}{49}+\frac{(y+2)^2}{16}=1$ c. $\frac{(x-2)^2}{9}+\frac{y^2}{25}=1$

3. a. $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{1}=1$ b. $\frac{(x-8)^2}{16}+\frac{(y-2)^2}{4}=1$ c. $\frac{\left(x-\frac{23}{16}\right)^2}{\left(\frac{73}{16}\right)^2}+\frac{(y-3)^2}{(3)^5}=1$

d. $\frac{x^2}{4}+\frac{(y-5)^2}{9}=1$

4. a. $e=\frac{3}{5}$ b. $\frac{x^2}{25}+\frac{y^2}{16}=1$ c. $e=\frac{\sqrt{3}}{2}$

5. a. $c=\pm\sqrt{5}$ b. $c=\pm 4$ $c=\pm 6$

6. a. equation of tangent is $\frac{x}{4}+\frac{2y}{9}=1$ equation of normal $8x-9y+10=0$

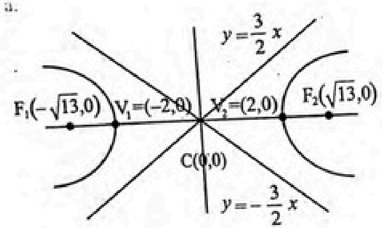
b. equation of tangent is $6x+21y-14=0$ equation of normal $7x-2y-9=0$

c. equation of tangent is $x+4y-4=0$ equation of normal $4x-y-3=0$

7. a. $8x-9y+30=0$ $6x+21y-144=0$

Exercise 9.3

1. a.



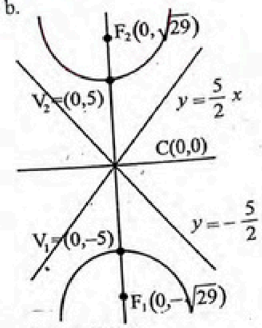
Centre C(0,0)

Vertices $V_1(-2,0)$, $V_2(2,0)$

Foci $F_1\left(-\sqrt{13},0\right)$, $F_2\left(\sqrt{13},0\right)$

Equation of asymptotes $y=\pm\frac{3}{2}x$

b.



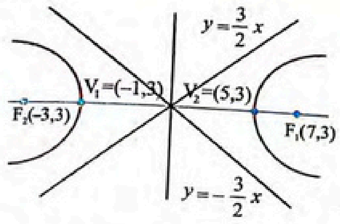
Centre C(0,0)

Vertices $V_1(0, 5)$, $V_2(0, -5)$

Foci $F_1\left(0,\sqrt{29}\right)$, $F_2\left(0,-\sqrt{29}\right)$

Equation of asymptotes $y=\pm\frac{5}{2}x$

c.



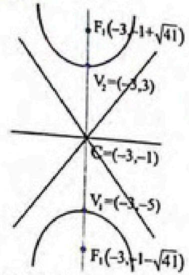
Centre C(2,3)

Vertices $V_1(-1, 3)$, $V_2(5,3)$

Foci $F_1(-3, 3)$, $F_2(7,3)$

Equation of asymptotes $y-3=\pm\frac{4}{3}(x-2)$

d.



Centre C(−3,−1)

Vertices $V_1(-3,-5)$, $V_2(-3,3)$

Foci $F_1\left(-3,-1-\sqrt{41}\right)$, $F_2\left(-3,-1+\sqrt{41}\right)$

Equation of asymptotes $y+1=\pm\frac{4}{5}(x+3)$

2. a. $\frac{x^2}{16}-\frac{y^2}{4}=1$ b. $\frac{y^2}{25}-\frac{x^2}{16}=1$ c. $\frac{(x-4)^2}{9}-\frac{4(y-2)^2}{9}=1$ d. $\frac{(y+2)^2}{9}-\frac{(x+5)^2}{16}=1$

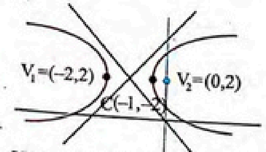
3. a. $\frac{y^2}{4}-\frac{x^2}{5}=1$ b. $\frac{x^2}{25}-\frac{y^2}{24}=1$ c. $9x^2-y^2=9$

d. $16x^2-9y^2=144$ e. $\frac{(x-1)^2}{4}-\frac{2}{9}(y+1)^2=1$

4. a. $\frac{x^2}{16}-\frac{y^2}{9}=1$ b. $\frac{y^2}{25}-\frac{x^2}{144}=1$

5. a. $\frac{(x+1)^2}{9}-\frac{(x+2)^2}{3}=1$ b. $\frac{(y-1)^2}{81}-\frac{(x+1)^2}{16}=1$

6. a.

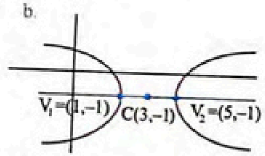


Vertices $V_1(-2,27)$, $V_2(0,2)$

Foci $F_1\left(-\sqrt{2}-1,2\right)$, $F_2\left(\sqrt{2}-1,2\right)$

The equation of asymptotes $y-2=\pm 1(x+1)$

b.



Vertices $V_1(1,-1)$, $V_2(5,-1)$

Foci $F_1\left(-\sqrt{2}+3,-1\right)$, $F_2\left(\sqrt{2}+3,-1\right)$

The equation of asymptotes $y+1=\pm(x-3)$

7. a. $p_1(4,1)$, $p_2(-1,-4)$ b. $p_1(0,1)$, $p_2(-\frac{8}{3},\frac{5}{3})$

8. a. $c=\pm 2\sqrt{3}$ b. $c=\pm\sqrt{7}$

9. a. Equation of tangent is $\sqrt{13}x+2y+4=0$. Equation of normal is $4x-2\sqrt{13}y+13\sqrt{13}=0$

b. Equation of tangent is $y=\frac{3x}{5}+\frac{9}{5}$. Equation of normal is $y=\frac{5x}{3}+\frac{125}{9}$

Exercise 9.4

1. a. $2X - Y = 0$ b. $X^2 + Y^2 = 4$ c. $X^2 + 2Y^2 = 16$ d. $X^2 + Y - 3XY + 1 = 0$
2. a. $2XY + a^2 = 0$ b. $X^2 = -4pY$ c. $3X^2 - Y^2 - 1 = 0$ d. $XY + 6X + 4Y - 1 = 0$
3. a. $3X^2 + 2Y^2 = 5$ b. $-X^2 + 4Y^2 = 1$ c. $6X^2 - 19Y^2 = 25$
4. a. $\theta = 36.87^\circ$ b. $\theta = -30^\circ$

Unit -10

Exercise 10.1

1. a. order = degree = 1, b. order = 2, degree = 1, c. order = 3, degree = 1, d. order = 2, degree = 1,
2. a. y is a solution, since the substitution of y and its derivative y' reduce the differential equation into an identity. b. y is a solution. c. y is a solution d. y is not a solution, since the substitution of y and its derivative y' does not reduce the differential equation into an identity.
3. a. $xy = 2$ b. $y = x - x\ln x + 1$ c. $\sin(xy) = 0.7071$ d. $\frac{y^2}{x} = \frac{x^2}{2} + \frac{1}{2}$
4. a. $y = \sin x + 1$ b. $f = \frac{1}{3}x^3 + 1$ c. $y = -\frac{1}{x} + \frac{1}{2}$ d. $y = \frac{1}{8 - x^2}$
5. 276 students

Exercise 10.2

1. a. $y = \pm\sin(x + c)$ b. $y = \frac{-1}{e^{-x} + c}$ c. $y = \sin(\sin^{-1} x + c)$ d. $y = \sin^{-1}\left[\frac{\sin 2x}{4} - \frac{x}{2} + c\right]$
2. a. $y = \tan(x + c) - x$ b. $y = \sin^{-1}(e^x c_1) - x$ c. $y = \frac{-x}{\ln x + c}$ d. $y = x\left(\frac{3x^2}{2} + c_2\right)^{\frac{1}{3}}$
3. a. $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln(x) + c$ b. $y = \frac{x}{\ell n(x) + c}$ c. $y = \pm x\sqrt{-1 + cx}$ d. $-\frac{x^2}{2y^2} - \frac{x}{y} + \ln(y) = c$
4. a. $y = \frac{1}{4}x^2 - 1$ b. $\frac{x^2 \cdot \ln(x)}{1 + \ln(x)}$
5. $y = \frac{x - 13}{x - 1}$ 6. $\left(\frac{2y^2}{x^2} + 1\right)^{\frac{1}{4}} = 9^{\frac{1}{4}} x$ 7. $\ln x + e^{\frac{-y}{x}} = 1$

8. $\frac{dx}{dt} = \frac{x}{2} - 3t$, $x(t) = 6t + 12 + ce^{\frac{t}{2}}$, $x(t) = 6t + 12 - 28e^{\left(\frac{t}{2} - 1\right)}$
9. General consumption of oil is $x(t) = 30e^{0.04t} + c$. At time $t = 0$ (1990), the oil consumption was $x = 0$, that gives $c = -30$. The actual oil consumption at time t is $x(t) = 30e^{0.04t} - 30$.
10. 110.825 items
11. a. $x^2 + 3y^2 = k$ b. $y^2 - x^2 = k$ c. $\frac{y^2}{2} = -x + \ell nk(x + 1)$ d. $yx = k$

Review Exercise 10

1. i. a ii. b iii. c iv. d v. b vi. b vii. a

Unit-11

Exercise 11.1

1. a. 0 b. 0 c. 0 d. $2t^3$ e. $t^4 + t^5$ f. $t - t^2$
2. a. 0 b. $1 - e^2$ c. $e^{-2} + 1$ d. $2x^3e^{2x} + 3x^2e^{2x} + 2x$ e. $e^2 + 2y$ f. $4z(z^2 - 2)$
3. a. $f_x = 2x\cos x^2 \cos y$, $f_y = -\sin x^2 \sin y$ b. $f_x = \frac{3x}{\sqrt{3x^2 + y^4}}$, $f_y = \frac{2y^3}{\sqrt{3x^2 + y^4}}$ c. $f_x = y^3 \tan^{-1} y$, $f_y = xy^2\left(\frac{y}{1 + y^2} + 3\tan^{-1} y\right)$ d. $f_x = 3x^2 + 2xy + y^2$, $f_y = x^2 + 2xy + 3y^2$ e. $f_x = \frac{y}{\sqrt{1 - x^2y^2}}$, $f_y = \frac{x}{\sqrt{1 - x^2y^2}}$ f. $f_x = x(x + 2)e^{x+y}\cos y =$, $f_y = x^2e^{x+y}(\cos y - \sin y)$
4. $f_x = 0.7x^{-0.3}y^{0.3}$, $f_y = 0.3x^{0.7}y^{-0.7}$
5. $f_x = 0.4yx^{-0.6}y^{0.6}$, $f_y = 0.6x^{0.4}y^{-0.4}$
6. $f_x = 2xy + y^2$, $f_y = x^2 + 2xy$

Exercise 11.2

1. a. Not homogeneous b. Homogeneous c. Homogeneous, degree=3 d. Homogeneous

Review Exercise 11

1. i. c ii. b iii. b iv. c v. c vi. c vii. c viii. c ix. a

Unit -12

Exercise 12.1

1. a. Hence the function takes values with opposite sign in the interval [1,2].
   b. Hence the function takes values with opposite sign in the interval [1,2].
   c. Hence the function takes values with opposite sign in the interval [1,4].
   d. Hence the function takes values with opposite sign in the interval [3,4].

2. a. $c_0 = 1.4, c_1 = 1.2, c_2 = 1.1, c_3 = 1.15$ b. $c_0 = 1.2, c_1 = 1.4, c_2 =$ [illegible]$, c_3 = 1.25$
   c. $c_0 = 3.6, c_1 = 3.8, c_2 = 3.7, c_3 = 3.65$ d. $c_0 = 3.6, c_1 = 3.4, c_2 = 3.5, c_3 = 3.55$

3. a. $c_o = -1.8300782,\ c_1 = -1.8409252,\ c_2 = -1.8413854,\ c_3 = -1.8414048$
   b. $c_o = 1.27,\ c_1 = 1.283,\ c_2 = 1.283412,\ c_3 = 1.283402$
   c. As both $f(a_0)$ and $f(b_0)$ are positive, there is no root in the given interval.

4. a. The bisection method is not valid, since the functions have not opposite signs in the interval [3,7].
   b. The bisection method is valid, since the functions have opposite signs in the interval [1,7].

5. a. $x_3 = 1.443$ b. $x_3 = 0$ c. $x_3 = 0.453$ d. $x_3 = -1.89706$

6. a. For 5 decimal accuracy, the forth iteration $x_4 = 0.324$
   b. For 5 decimal accuracy, the third iteration $x_3 = -2.5468$
   c. For 5 decimal accuracy, the third iteration is $x_3 = -2.79553$

Exercise 12.2

1. a. $T_4 = 8.75$ b. $T_4 = 1.17$ c. $T_6 = 1.11$ d. $T_6 = 2.92116$(approx)
2. a. $S_6 = 18.70$ b. $S_8 = 1.11$ c. $S_6 = 1.10$ d. $S_8 = 3.19$

Review Exercise 12

1. i. b ii. c iii. a iv. c v. b vi. a
   vii. d viii. b ix. c x. b

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Enjoys a rich and diversified experience of 36 years of curriculum development in the specialized areas of Pure Mathematics, Applied Mathematics, Statistics, Quantitative Research and Operational Research. Preparing and delivering lectures at undergraduate, graduate and post-graduate level in Engineering, Computer Science, Business Administration and Information Technology, supervising research at undergraduate, graduate, and post graduate level. He is author of many books being taught at graduate and undergraduate level. He remained associated with the University of Engineering & Technology Peshawar since 1976.

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