- The normal equation to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is: $\frac{y-y_1}{y_1/b^2}=\frac{x-x_1}{x_1/a^2}$

Hyperbola:

- a. The standard form of the equation of a hyperbola with center at the origin and the x–axis as the transverse axis is: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

  b. The standard form of the equation of a hyperbola with center at the origin and the y–axis as the transverse axis is: $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

- a. For horizontal standard form hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes are the lines: $y=\pm\frac{b}{a}x$

  b. For vertical standard form hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the asymptotes are the lines: $y=\pm\frac{b}{a}x$

- a. The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h+a,k)$ and $V_2(h-a,k)$, and foci at $F_1(h-c,k)$ and $F_2(h+c,k)$ is: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

  b. The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h,k+a)$ and $V_2(h,k-a)$, and foci at $F_1(h,k+c)$ and $F_2(h,k-c)$ is: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

- a. For translating the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, horizontally, the asymptotes are the lines:

  $(y-k)=\pm\frac{b}{a}(x-h)$

  b. For translating the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, vertically, the asymptotes are the lines:

  $(y-k)=\pm\frac{a}{b}(x-h)$

- A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.
- A function of the form $y=C/x$ is an inverse function called the rectangular hyperbola.
- The equation of the tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is: $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$
- The equation of any tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in the slope–form is:

  $y=mx\pm\sqrt{a^2m^2-b^2}$

- The line y=mx+c should touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ under condition: $c=\pm\sqrt{a^2m^2-b^2}$
- The normal equation to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is: $\frac{-(x-x_1)}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$

Unit 10 DIFFERENTIAL EQUATIONS

By the end of this unit, the students will be able to:

10.1 Introduction

i. Define ordinary differential equation (DE), order of a DE, degree of a DE, solution of a DE – general solution and particular solution.

10.2 Formation of differential equations

i. Demonstrate the concept of formation of a differential equation.

10.3 Solution of differential equation

i. Solve differential equations of first order and first degree of the form:
   - separable variables,
   - homogeneous equations,
   - equations reducible to homogeneous form.

ii. Solve real life problems related to differential equations.

10.4 Orthogonal Trajectories

i. Find orthogonal trajectories (rectangular coordinates) of the given family of curves.

ii. Use MAPLE graphic commands to view the graphs of given family of curves and its orthogonal trajectories.

Introduction

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are describe only by equations that relates rates at which quantities change.

Suppose the solution of problems concerning the motion of objects, the flow of charged particles, heat transport, etc often involves discussion of relations of the form

$$\frac{dx}{dt}=f(x,t) \quad \text{or} \quad \frac{dq}{dt}=g(q,t)$$

In the first equation, x might represent distance. For this case, $\frac{dx}{dt}$ is the rate of change of distance with respect to time t that is speed. In the second equation, q might be a charge and $\frac{dq}{dt}$ is the rate of flow of charge that is current. These are examples of different equations, so called because these are equations involving the derivatives of various quantities. Such equations arise out of situations in which change is occurring.

In engineering, differential equations are most commonly used to model dynamic systems. These are the systems which change with time. Examples include an electronic circuit with time-dependent currents and voltages, a chemical production line in which pressure, tank levels, flow rates, etc, vary with time.

There is a wide variety of differential equations which occur in engineering applications, and consequently there is a wide variety of solution techniques available.

10.1 Ordinary Differential Equations

A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).

*"If the unknown function depends on only one variable, then the derivative is an ordinary derivative, and the equation is then called an **ordinary differential equation.**"*

If the unknown function depends on more than one variable, then the derivative is partial derivative, and the equation is then called **partial differential equation.**

The following differential equations are the examples of ordinary differential equations with their corresponding unknown functions:

$$\frac{dy}{dx} = xy, \quad y(x) = ? \qquad \text{(i)}$$

$$\frac{dy}{dx} = x + y, \quad y(x) = ? \qquad \text{(ii)}$$

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \quad y(x) = ? \qquad \text{(iii)}$$

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 3, \quad y(x) = ? \qquad \text{(iv)}$$

$$\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3, \quad y(x) = ? \qquad \text{(v)}$$

(a) Order of a differential equation

The order of a differential equation is the order of the highest-order derivative occurring in the equation e.g.

i. $\frac{dy}{dx} = xy$ is first order differential equation.

ii. $\frac{d^2y}{dx^2} + (x^2 + 2x)y = 7$ is second order differential equation.

(b) Degree of a differential equation

The degree of a differential equation is the power of the highest-order derivative occurring in the equation.

i. $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 - 8\frac{dy}{dx} + 2y = 8$ is an equation having degree is 1.

Example 1 Determine the order and degree of the following ordinary differential equations:

(a). $\frac{dy}{dx} = \frac{x+y}{x-y}$ *(b).* $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 3$ *(c).* $\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3$

Solution Differential equation *(a)* is an ordinary differential equation of order 1 and degree 1, since the highest ordinary derivative is of order 1 and the exponent of the highest ordinary derivative is 1. Differential equation *(b)* is an ordinary differential equation of order 2 and degree 1, while Differential equation *(c)* is an ordinary differential equation of order 3 and degree 2.

(c) Solution of a differential equation

A solution of an equation in a single variable is a **number** which satisfies the equation. In similar fashion, solutions of the differential equations are **functions**, rather than numbers, which satisfy the differential equation. The variables which appear in equations are called **"unknowns."** Exactly, the only dependent variable in differential equations is referred to as **"unknown."**

For illustration, a solution of the differential equation $\frac{dy}{dx} = 1$ is an expression of the unknown dependent variable y in terms of the independent variable x.

"A solution of an ordinary differential is any function y = f(x) or f(x, y) which when substituted in the differential equation , reduces the differential equation to an identity; that is, it satisfies the equation."

Example 2 Show that $y = x + A$ is a solution of the first order differential equation $\frac{dy}{dx} = 1$.

Solution The given function $y = x + A$ and its derivative $\frac{dy}{dx} = 1$ is used in the differential equation $\frac{dy}{dx} = 1$ to obtain:

$$\frac{dy}{dx} = 1$$

$$1 = 1, \textit{ identity left side = right side}$$

This shows that $y = x + A$ is a solution of the ordinary differential equation $\frac{dy}{dx} = 1$.

(d) General and particular solution

*The solution of a differential equation when depends on a single arbitrary constant quantity, is called the **general solution** of the first order differential equation. If we give particular steps for value to a single arbitrary constant quantity, then the solution to obtain is called the **particular solution.***

Do You Know?

The particular solution is also known as specific solution or exact solution or actual solution.

Graphically,

- the general solution of a first order deferential equation represents a family of curves for any choice of arbitrary constant quantity.
- The particular solution of a first order differential equation is a particular curve chosen from a family of curves (general solution) for a particular value of a constant quantity.

Example 3 Graphically, show that $y = x + A$ is a general solution of the first order differential equation $\frac{dy}{dx} = 1$. Find a particular solution, when $x = 0$ and $y = 1$.

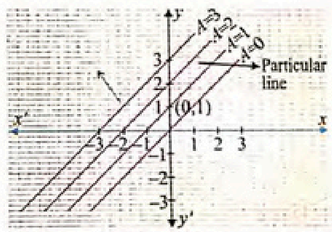


Figure 10.1

Solution The general solution $y = x + A$ of a first order differential equation $\frac{dy}{dx} = 1$, represents a family of parallel straight lines for different values of arbitrary constant quantities A = 0, 1, 2,...

The particular value for the particular line that passes through a point $P(0, 1)$ can be found from the general solution $y = x + A$ by putting $x = 0, y = 1$:

$$y = x + A \Rightarrow 1 = 0 + A \Rightarrow A = 1$$

Use this particular value of A = 1 in general solution $y = x + A$ to obtain a particular solution (line) $y = x + 1$.

Note

If we are to determine the solutions of a differential equation subject to conditions on the unknown function and its derivatives specified for one value of the independent variable, the conditions are then called **conditions** and the related differential equation is called an **initial value problem** "IVP".

Thus, the problem of example 3 is the initial value problem that leads the notation:

$$\frac{dy}{dx} = 1, \quad y(0) = 1 \qquad \text{IVP}$$

Example 4 Determine a particular solution for the first order differential equation $\frac{ds}{dt}=-32\,ft/sec$ that satisfies the initial condition $s=0$, when $t=0$.

Solution This information develops the initial value problem $\frac{ds}{dt}=-32,\ s(0)=0$ for which the solution is the unknown function $s(t)$ that can be found by integrating directly the first order differential equation with respect to t:

$$\frac{ds}{dt}=-32 \Rightarrow \int\frac{ds}{dt}=\int-32dt+c \Rightarrow s(t)=-32t+c$$

The general solution $s(t)=-32t+c$ at a point P(0, 0) is giving $c=0$. Use this $c=0$ in general solution to obtain the particular solution $s(t)=-32t$.

10.2 Formation of Differential Equation

In most of the physical situations, we can observe the process but can not worked out directly to the differential equation. As, a result, we have a general solution at our disposal before we know the equation of which it is the solution. Let's begin with the step for forming with differential equation.

i. Discover the differential equation that describes a specified physical situation.
ii. Find either exactly or approximately, the appropriate solution of that equation.
iii. Interpret the solution that is found.

Look at the following examples for the concept of formation of a differential equation.

Example 5 The rate at which the distance travels by Ali is 30 mph. Find the total distance travels by Ali at a time t hours.

Solution If S(t) is the unknown distance travel by you w.r.t. 't' number of hours, then, the rate at which the distance travels is the first derivative of S(t) with respect to t: $\frac{dS}{dt}=30$, the per hour speed

Integrating with respect to t to obtain S(t) $\int\frac{dS}{dt}=\int30dt+c \Rightarrow S(t)=30t+c$,

the distance travels by Ali with respect to 't' number of hours and the constant quantity c is the fixed distance in this situation.

Example 6 The rate at which the animal population is growing at a constant rate 4%. The habitat will support no more than 10,000 animals. There are 3000 animals present now. Find an equation that gives the animal population y w.r.t. x number of years.

Solution If $P(x)$ is the unknown animal population w.r.t. x number of years, then, the rate at which the animal population grows is the first derivative of $P(x)$ w.r.t. x:

$$\frac{dP}{dx}=k(N-P)$$
$$=0.04(10,000-P)$$
$$\Rightarrow \frac{dP}{(10,000-P)}=0.04dx$$

Here $k=0.04$ is the constant growth, $N=10,000$, is the total size of animal population in that habitat. Integrating with respect to x to obtain P(x),

$$\int\frac{dP}{(10,000-P)}=\int0.04dx+c$$

the total population and c, the fixed population that depends on P = 3000 when $x=0$. This problem is the IVP problem with the initial condition P(0) = 3000.

Exercise 10.1

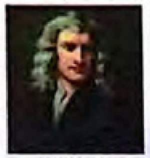
1. Find the order and degree of each the following ordinary differential equations:
   a. $\frac{dy}{dx}=x^2+y$ b. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+11y=3x$
   c. $\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)^3-y=0$ d. $y\frac{d^2y}{dx^2}+5x\frac{dy}{dx}=4$
2. In each case, show that the indicated function is a solution of the differential equation:
   a. $y=e^x+e^{2x},\ \frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ b. $y=x-x\ln x,\ x\frac{dy}{dx}+x-y=0$
   c. $y=(x+c)e^{-x},\ \frac{dy}{dx}+y=e^{-x}$ d. $y=e^x+e^{-x},\ \frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$
3. In each case, use the initial condition and the general solution of the differential equation to determine a particular solution:
   a. $xy=c,\ y(2)=1$ b. $y=x-x\ln x+c,\ y(1)=2$
   c. $\sin(xy)+y=c,\ y\left(\frac{\pi}{4}\right)=1$ d. $\frac{y^2}{x}=\frac{x^2}{2}+c,\ y(1)=1$
4. Solve the following initial value problems:
   a. $\frac{dy}{dx}=\cos x,\ y(0)=1$ b. $\frac{dy}{dx}=x^2,\ y(0)=1$
   c. $\frac{dy}{dx}=\frac{1}{x^2},\ y(2)=0.$ d. $\frac{dy}{dx}=2xy^2,\ y(3)=-1$
5. Suppose a student carrying Corona Virus returns to an isolated college campus of 1000 students. If it is considered that the rate at which the virus spreads is proportional not only to the number 'x' of infected students but also to the number of students not infected. Find the number of infected students after 6 days. If it is further observed after 4 days = 50.

History

Gottfried Wilhelm Leibniz (1646-1716)

Leibniz was the German Mathematician and philosopher. He introduced and published the concept of differential equation in (1684). In most of the documents indicated that he knew how to solve the differential equations in 1666.

Sir Isaac Newton was an English Mathematician, Physicist and Astronomer. He never published his "Method of fluxions" but it is claimed that he discovered it in 1665 to 1667. Which is known as exact by the modern classification.



Isaac Newton (1643-1727)

10.3 Solution of differential equations

If the solution of a first order differential equation is not possible by direct integration, then, the integral process (in case of difficulties) for obtaining the solution of a differential equation indicates the actual concept of a differential equation.

10.3.1 Solution of first order and first degree differential equations

We examine techniques for solving first order differential equations. For this unit, the recommended techniques for solving the differential equations are the separation of variables, reducible to separation form, homogeneous and equations reducible to homogeneous form.

i. **Separable variables**

If the solution of a differential equation is not possible by direct integration, then the integral technique called **separation of variables** will be used for solving the differential equation. Separation of variables is a technique commonly used to solve first order differential equations. It is so called because we try to rearrange the equation to be solved in such a way that all terms involving the dependent variable (y say) appear on one side of the equation, and all terms involving the independent variable (x, say) appear on the other side. It is not possible to rearrange all first order differential equations in this way so this technique is not always appropriate. Further, it is not always possible to perform the integration even if the variables are separable.

In general, a differential equation of the form $\dfrac{dy}{dx}=\dfrac{f(x)}{g(y)},\ g(y)\neq 0$ (i)

that by shifting x on one side and y on the other side $g(y)dy=f(x)dx$, SDE (ii)

is giving a **separable differential equation**. The solution to separable differential equation (ii) can be found by integrating left hand side w.r.t. y and right hand side w.r.t. x.

Example 7 Find the general solution of the linear differential equation $\dfrac{dy}{dx}=y$.

Solution The solution of the given differential equation is not possible by direct integration. The separable form of the given first order differential equation is obtained by shifting y on the left and x on the right: $\dfrac{1}{y}dy=dx$

Integrating both sides

$$\int\frac{1}{y}dy=\int dx \Rightarrow \ln y=x+c$$

$$\Rightarrow y=e^{x+c}=e^{x}e^{c}=c_1e^{x},\quad c_1=e^{c}$$

is giving the general solution of the first order differential equation. This general solution represents a family of exponential functions as shown in the Figure 10.2.

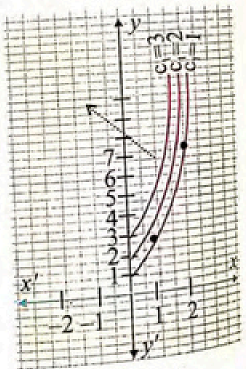


Figure 10.2

Remember

If the solution of the differential equation is not possible by separable form, then the given differential equation can be reduced in separable form by substitution. This substitution changes the dependent variable form y to a new variable, say, u and keeps x as the independent variable.

Example 8 Find the general solution of the non-linear differential equation $\dfrac{dy}{dx}=(x+y)^2$.

Solution The given non-linear differential equation is not separable differential equation, but can be reduced into separable form by substitution: $x+y=u(x)$

that on differentiation w.r.t. x is giving:

$$\frac{d}{dx}(x+y)=\frac{du}{dx}\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-1$$

Use $x+y=u$ and $\dfrac{dy}{dx}=\left(\dfrac{du}{dx}\right)-1$ in the given differential equation to obtain separable differential equation in variable u and its derivative $\dfrac{du}{dx}$:

$$\frac{du}{dx}-1=u^2\Rightarrow\frac{du}{dx}=1+u^2\Rightarrow\frac{du}{1+u^2}=dx \quad \text{(i)}$$

Integrating equation (i) to obtain the general solution of ordinary differential equation (i).

$$\int\frac{du}{1+u^2}=\int dx\Rightarrow\tan^{-1}u=x+c\Rightarrow u=\tan(x+c)$$

that by back substitution of $u=x+y$ is giving

$$x+y=\tan(x+c)\Rightarrow y=-x+\tan(x+c)$$

the general solution of the given ordinary differential equation that depends on a single arbitrary constant c.

ii. **Homogeneous equations**

The homogeneous differential equations are related to homogeneous functions.

"A function $f(x, y)$ is homogeneous function of degree n in variables x and y if and only if for all values of the variables x, and y and for every positive value of λ, the identity is true."

$$f(\lambda x,\lambda y)=\lambda^n f(x,y),\quad n=1,2,3,\ldots \quad \text{(i)}$$

For illustration, the function $f(x,y)=x^2+y^2$ is homogeneous function of degree 2, since the identity (i) is true:

$$f(\lambda x,\lambda y)=(\lambda x)^2+(\lambda y)^2=\lambda^2\left(x^2+y^2\right)=\lambda^2 f(x,y) \qquad x=\lambda x,\ y=\lambda y$$

The identity (i) is not true for a function $f(x,y)=x^2+y^2+1$, since the function is not homogeneous.

"The differential equation $\dfrac{dy}{dx}=\dfrac{f(x,y)}{g(x,y)}$ (ii) is called a homogenous differential equation, if it defines a homogenous function of degree zero."

The homogeneous differential equation (ii) can be reduced to separable form by introducing a new variable:

$$u(x)=\frac{y}{x}\ \text{or}\ y=ux \quad\text{and}\quad \frac{dy}{dx}=\frac{d}{dx}(ux)=u+x\frac{du}{dx} \quad \text{(iii)}$$

The substitution of (iii) in equation (ii) automatically converts the homogeneous differential equation in separable differential equation.

Example 9 Find the general solution of the homogeneous differential equation: $\frac{dy}{dx}=\frac{y-x}{y+x}$

Solution The given differential equation defines a homogeneous function of degree zero, when the function on the right of the given differential equation defines a homogeneous function of degree zero:

$$\frac{dy}{dx}=\frac{y-x}{y+x}=\frac{\lambda y-\lambda x}{\lambda y+\lambda x}$$

$$=\frac{\lambda(y-x)}{\lambda(y+x)}=\frac{\lambda}{\lambda}\left[\frac{y-x}{y+x}\right]=\lambda^{1-1}\left[\frac{y-x}{y+x}\right]=\lambda^{0}\left[\frac{y-x}{y+x}\right]=\left[\frac{y-x}{y+x}\right],\ HDE$$

The given homogeneous differential equation is used for the assumptions

$y=ux,\ \frac{dy}{dx}=u+x\frac{du}{dx}$ to obtain a separable differential equation of the form:

$$u+x\frac{du}{dx}=\frac{ux-x}{ux+x}=\frac{u-1}{u+1}$$

$$\Rightarrow\quad x\frac{du}{dx}=\frac{u-1}{u+1}-u=\frac{u-1-u^2-u}{u+1}=\frac{-(u^2+1)}{u+1}\qquad \text{(i)}$$

$$\Rightarrow\quad -\frac{(u+1)}{u^2+1}du=\frac{dx}{x},\ \text{SDE}\qquad \text{(ii)}$$

Integrating *SDE* (ii) to obtain the general solution of the *SDE* (ii):

$$-\int\frac{(u+1)du}{u^2+1}=\int\frac{dx}{x}\ \Rightarrow\ -\int\frac{2(u+1)du}{2(u^2+1)}=\int\frac{dx}{x},\ \textit{Multiply and divide out by } 2$$

$$\Rightarrow\ -\frac{1}{2}\int\frac{2udu}{u^2+1}-\int\frac{du}{u^2+1}=\ln x+c,\ \Rightarrow\ -\frac{1}{2}\ln\left(u^2+1\right)-\tan^{-1}u=\ln x+\ln c$$

$$\Rightarrow\ -\ln\sqrt{u^2+1}-\tan^{-1}u=\ln cx$$

$$\Rightarrow\ -\tan^{-1}u=\ln\sqrt{u^2+1}+\ln cx=\ln cx\sqrt{u^2+1}$$

$$\tan^{-1}u=-\ln cx\sqrt{u^2+1}\qquad \text{(iii)}$$

The back substitution $u=\frac{y}{x}$ is used in equation (iii) to obtain the general solution of the given homogeneous differential equation:

$$\tan^{-1}\frac{y}{x}=-\ln cx\sqrt{\frac{y^2}{x^2}+1}$$

$$\tan^{-1}\frac{y}{x}=-\ln c\sqrt{y^2+x^2}\Rightarrow\frac{y}{x}=\tan\left[-\ln c\sqrt{y^2+x^2}\right]$$

iii. **Equations reducible to homogenous form**

A given differential equation of the form $\frac{dy}{dx}=\frac{ax+by+c}{a_1x+b_1y+c_1}$, where $\frac{a}{a_1}\neq\frac{b}{b_1}$ can be reduced to the homogeneous form by taking new variable x and y such that $x=X+h$ and $y=Y+k$, where h and k are constants to be choosen as to make the given equation homogeneous.

By using the above substitutions, we get $dx=dX$ and $dy=dY$ implies that $\frac{dy}{dx}=\frac{dY}{dX}$

Therefore, the given equation becomes, $\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$

$$=\frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)}$$

Now, by choosing h and k such that $ah+bk+c=0$ and $a_1h+b_1k+c_1=0$

So, the differential equation becomes.

$$\frac{dY}{dX}=\frac{aX+bY}{a_1X+b_1Y},\ \text{which is a homogeneous equation.}$$

Example 10 Find the general solution of the differential equation $x\frac{dy}{dx}=x+y$.

Solution The given differential equation is not in the standard form of homogeneous differential equation, but it can be reduced in the standard form of homogenous differential equation by the following procedure:

Divide out by x to obtain the standard form of homogeneous differential equation:

$$\frac{dy}{dx}=\frac{x+y}{x},\ \text{HDE}\qquad \text{(i)}$$

Homogeneous differential equation (i) is used for the assumptions

$y=ux,\ \frac{dy}{dx}=u+x\frac{du}{dx}$ to obtain a separable differential equation of the form:

$$u+x\frac{du}{dx}=\frac{x+xu}{x}=1+u$$

$$x\frac{du}{dx}=1\Rightarrow du=\frac{dx}{x},\ \text{SDE}\qquad \text{(ii)}$$

Integrating the SDE (ii) to obtain the general solution of the SDE (ii):

$$\int du=\int\frac{dx}{x}\ \Rightarrow u=\ln x+\ln c=\ln cx$$

that by back substitution $u=\frac{y}{x}$ is giving $\frac{y}{x}=\ln cx\ \Rightarrow y=x\ln cx$

the general solution of the given homogeneous differential equation that depends on a single arbitrary constant c.

> **Remember**
> If the differential equation is not homogeneous differential equation, then it might be a nonhomogeneous differential equation.

10.3.2 Solve real life problems related to differential equation

Example 11 A certain bacteria grows at a rate that is proportional to the number present at a particular time. If the number of bacterial at a time $t=0$ is N_o and at time $t=1$ hour, the number of bacteria is $\frac{5N_o}{2}$. Determine the time necessary for the number of bacteria to be quadruple.

Solution If $N(t)$ is the unknown number of bacteria w.r.t time t hours, then, the rate at which bacterial grows, is represented by:

$$\frac{dN}{dt}\alpha N\Rightarrow\frac{dN}{dt}=kN$$

Reduce the differential equation to separable form $\frac{dN}{N}=k\,dt\qquad$ (i)

that on integration is giving the general solution of (i):

$$\int\frac{dN}{N}=\int k\ dt$$

$$\ln N=kt+c\qquad \text{(ii)}$$

The initial condition $N(0)=N_o$ is used in equation (ii) to obtain c:

$$\ln N_o = 0 + c \Rightarrow c = \ln N_o \qquad \text{(iii)}$$

Use c in equation (ii) to obtain a particular solution:

$$\ln N = kt + \ln N_o \Rightarrow \ln\frac{N}{N_o} = kt \Rightarrow \frac{N}{N_o} = e^{kt} \Rightarrow N = N_o e^{kt} \qquad \text{(iv)}$$

The condition $N(1)=\frac{5N_o}{2}$ is used in equation (iv) to obtain the value of k:

$$\frac{5}{2}N_o = N_o e^k \Rightarrow e^k = \frac{5}{2} \Rightarrow k = \ln\left(\frac{5}{2}\right) = 0.9163 \qquad \text{(v)}$$

Use the value of k in equation (iv) to obtain a particular solution (specific number of bacteria):

$$N = N_o e^{0.9163t} \qquad \text{(vi)}$$

The condition $N = 4N_o$ (when the bacterial have quadrupled) is used (vi) to obtain the time

$$4N_o = N_o e^{0.9163t}$$

$$4 = e^{0.9163t} \Rightarrow 0.9163t = \ln 4 \Rightarrow t = \frac{\ln(4)}{0.9163} = 1.51\ hr$$

at which the bacteria is four times of the original number of bacterial.

10.4 Orthogonal Trajectories

Our experience with first order differential equations has taught us that such equations often have general solutions containing a single arbitrary constant. Each such solution defines a corresponding set of integral curves. A nonempty set of plane curves defined by a differential equation involving just one parameter (single arbitrary constant) is commonly called a one-parameter family of curves. Of special importance in certain applications are those one-parameter families of curves which are orthogonal trajectories of one another.

"The curves of a family $F(x,y,c_1)$ are said to be orthogonal trajectories of curves of a family $G(x,y,c_2)$, if and only if each curve of either family is intersected by at least one curve of the other family and at every point of intersection of a curve of F with a curve of G, the two curves are perpendicular."

10.4.1 Orthogonal trajectories of the given family of curves

The two families of curves $F(x,y,c_1)$ and $G(x,y,c_2)$ are perpendicular at a point of intersection, if and only if their tangents are perpendicular at the point of intersection. If their tangent lines, say, L_1 and L_2, are perpendicular, then the product of their slopes equals -1:

$$m_1 m_2 = -1,\ m_1 \text{ and } m_2 \text{ are the slopes of the two tangent lines } L_1 \text{ and } L_2$$

$$m_1 = -\frac{1}{m_2} \Rightarrow \left(\frac{dy}{dx}\right)_G = -\frac{1}{\left(\frac{dy}{dx}\right)_F} \qquad \text{(i)}$$

This is called the **differential equation of orthogonal trajectories**. If one family of curves F is given, then the other family of curves G can be found by solving the differential equation of orthogonal trajectories (i).

Example 12 Determine the orthogonal trajectories of the family of curves (circles) $x^2+y^2=c$.

Solution To determine the orthogonal trajectories of the circles, we need to determine the slope (derivative) of the family of circles

$x^2+y^2=c$ with respect to x $\quad 2x + 2y\frac{dy}{dx} = 0$

$$\left(\frac{dy}{dx}\right)_F = -\frac{x}{y} \qquad \text{(ii)}$$

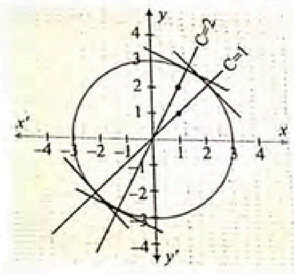


Figure 10.3

The differential equation of the orthogonal trajectories (i) with the slope of the given orthogonal trajectories (ii) is used to obtain the other family of curves G of orthogonal trajectories:

$$\left(\frac{dy}{dx}\right)_G = \frac{-1}{\left(\frac{dy}{dx}\right)_F}$$

$$\frac{dy}{dx} = \frac{-1}{-\frac{x}{y}} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}, SDE \Rightarrow \int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\ln y = \ln x + \ln C \Rightarrow y = Cx \qquad \text{(iii)}$$

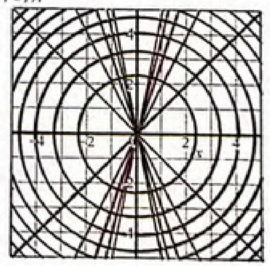
Thus, the family of curves G represents a family of homogeneous straight lines that pass through the origin. This is the result, we would expect, since the radii of a circle are the homogeneous lines $y = Cx$, C is any real number) perpendicular to the lines tangent to a circle.

10.4.2 MAPLE graphic commands to view the graphs of given family of curves and its orthogonal trajectories

The general solution $y(x) = sqrt\left[-x^2+c_1\right]$ of the above problem in example 12 is the first family of curves. This can also be written as $x^2+y^2=c$. The orthogonal trajectories of a first family of curves is the second family of curves represented by $y = Cx$.

This equation can be viewed through command on line by typing MAPLE commands as,

```
> ?conturplot
> with(plots):
> F := contourplot(x^2 + y^2, x=-5..5, y=-5..5)
                                   F := PLOT(...)        (1)
> G := plot({seq(c·x, c=-5..5)}, x=-5..5, y=-5..5) :
> display({F, G});
```



Exercise 10.2

1. Find general solution of the following differential equations:

   a. $\left(\frac{dy}{dx}\right)^2 = 1 - y^2$ b. $e^x \frac{dy}{dx} + y^2 = 0$

   c. $\sqrt{1-x^2}\,dy = \sqrt{1-y^2}\,dx$ d. $\operatorname{cosec}^2 x\,dy + \sec y\,dx = 0$

2. Reduce the following differential equations in separable form and then solve:

   a. $y' = (y+x)^2$ b. $y' = \tan(x+y) - 1$

   c. $\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ d. $\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}$

3. Solve the following homogeneous differential equations:

   a. $\frac{dy}{dx} = \frac{x+y}{x-y}$ b. $\frac{dy}{dx} = \frac{xy-y^2}{x^2}$

   c. $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$ d. $\frac{dy}{dx} = \frac{xy+y^2}{x^2+xy+y^2}$

4. Reduce the following differential equations in the standard form of homogeneous form and then solve:

   a. $x\frac{dy}{dx} = y + \sqrt{x^2+y^2},\ y(4) = 3$ b. $(x^4+y^4)dx = 2x^3y\,dy,\ y(1) = 0$

5. The slope of a family of curves at a point $P(x, y)$ is $\frac{y-1}{1-x}$. Determine the equation of the curve that passes through the point P(4, –3).

6. Find the solution curve of the differential equation $xyy' = 3y^2 + x^2$ which passes through the point P(–1, 2).

7. Find the real portion from the solution curves of the differential equation $xe^{\frac{y}{x}}dx + ydx = xdy$ which passes through the point P(1, 0).

8. A particle moves along the *x-axis* so that its velocity at any point is equal to half its abscissa minus three times the time. At a time $t = 2, x = -4$, determine the motion of a particle along the *x-axis*.

9. The rate of consumption of oil (billions of barrels) is given by $\frac{dx}{dt} = 1.2e^{0.04t}$, Where t=0 correspond to 1990. Find the total amount of oil used from 1990 to year 1995. At this rate, how much oil will be used in (t = 8) years?

10. The rate of infection of a disease (in people per month) is given by: $\frac{dI}{dt} = \frac{100t}{t^2+1}$

    Where t is the time in months since the disease broke out. Find the total number of infected people over the first four months of the disease.

10. Determine the equations of the orthogonal trajectories of the following families of curves.

    a. $y = cx^3$ b. $xy = c$ c. $y = cxe^x$ d. $y^2 = x^2 + C$

Review Exercise 10

1. Choose the correct option.

i. The order of equation $3\frac{dy}{dx} + 2y = 5$ is:

   (a). 1   (b). 2   (c). 3   (d). 4

ii. The degree of the equation $\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0$ is:

   (a). 1   (b). 2   (c). 3   (d). 4

iii The solution that depends on an arbitrary constant quantity is called:

   (a). exact solution   (b). particular

   (c). general solution   (d). none of these

iv. The solution of $\frac{dy}{dx} = \sin(x)$ at $y(0) = 1$ is:

   (a). $y = \sin(x) + 2$ (b). $y = -\sin(x) + 2$

   (c). $y = \cos(x) + 2$ (d). $y = -\cos(x) + 2$

v. The solution of $y\frac{dy}{dx} = \cos(x)$ at $y(0) = -1$ is:

   (a). $y = -\sqrt{2\sin(x)+1}$ (b). $y = \sqrt{2\sin(x)+1}$

   (c). $y = -\sqrt{2\cos(x)+1}$ (d). $y = \sqrt{2\cos x+1}$

vi. The differential equation for the orthogonal trajectory of the family of curve $x^2 + y^2 = c$ is:

   (a). $\frac{dy}{dx} = cx$ (b). $y = cx$ (c). $\frac{dy}{dx} = ct$ (d). $cy = \frac{dy}{d\theta}$

vii. The Orthogonal trajectories for the family of $y^2 = 4ax$:

   (a). $y^2 = k - 2x^2$ (b). $x^2 = k - 2y^2$ (c). $x^2 = k + 2y^2$ (d). $y^2 = 2x^2 - k$