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THE TEXTBOOK OF

Mathematics

For Class 11

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PREFACE

The Sindh Textbook Board, is assigned with preparation and publication of the textbooks to equip our new generation with knowledge, skills and ability to face the challenges of new millennium in the fields of Science, Technology and Humanities. The textbooks are also aimed at inculcating the ingredients of universal brotherhood and to reflect the valiant deeds of our forebears and portray the illuminating patterns of our rich cultural heritage and tradition.

The aims and objectives of the book are to provide the students with sound basis for studying mathematics at higher secondary stage in such a way that even an average student may understand and use it without any difficulty. The language is simple and easily understandable. It is designed to guide and subsequently provide a strong base to the readers in order to grasp the fundamental ideas and also master themselves with the elementary techniques to excel in further professional studies.

The Sindh Textbook Board has taken great pains and incurred expenditure in publishing this book inspite of its limitations. A textbook is indeed not the last word and there is always room for improvement. While the authors have tried their level best to make the most suitable presentation, both in terms of concept and treatment, there may still have some deficiencies and omissions. Learned teachers and worthy students are, therefore, requested to be kind enough to point out the shortcomings of the text or diagrams and to communicate their suggestions and objections for the improvement of the next edition of this book.

In the end, I am thankful to our learned authors, editors and specialist of Board for their relentless service rendered for the cause of education.

Chairman
Sindh Textbook Board

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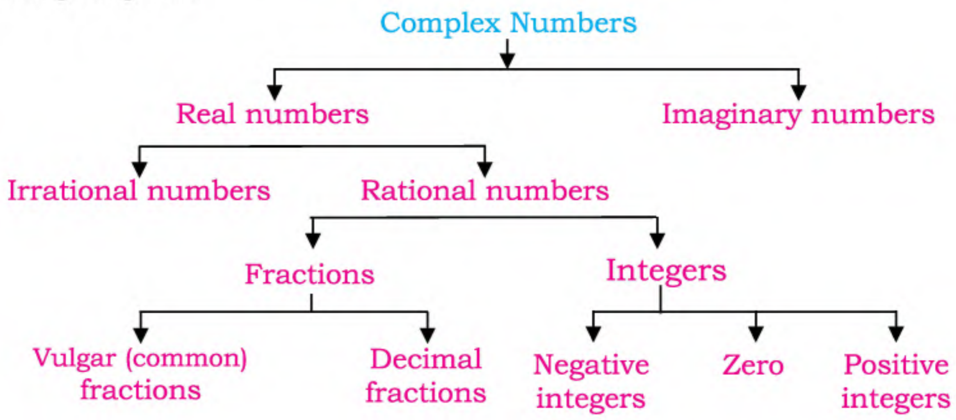
Complex Numbers

Unit 1

1.1 Complex Numbers and Geometrical Representation of Complex Number

We are already familiar with the system of real numbers. But the system of real numbers is not sufficient to solve all algebraic equations. Thus, real numbers provide inadequate solutions when we have to find the solution of the algebraic equations like $x^2 = -1, x^4 + 4 = 0$, etc., because no real number satisfies these equations. Similarly, there are so many other equations like $x^2 + x + 1 = 0$, $x^2 + 5x + 10 = 0$ which have no real roots. To overcome this inadequacy of real numbers, imaginary numbers were introduced. Later on, complex numbers were defined. The relationship of numbers is shown in the following diagram.



1.1.1 Recall complex number z represented by an expression of the form $z = a + ib$ or of the form (a, b) where a and b are real numbers and $i = \sqrt{-1}$

A complex number is the sum of a real number and an imaginary number. It is represented by an expression of the form $a + ib$ or (a, b), where a and b are real numbers, and 'i' is called imaginary unit and $i = \sqrt{-1}$. Complex number is usually denoted by z.

Note: The set of complex numbers is denoted by $\mathbb{C}$ i.e., $\mathbb{C} = \{(a,b) \mid a,b \in \mathbb{R}\}$.

1.1.2 Recognize a as real part of z and b as imaginary part of z

As we have mentioned (a,b) is a complex number. In this complex number a is called real part and b is called imaginary part. Real and imaginary parts of complex number z are denoted by $R_e(z)$ and $I_m(z)$ respectively.
For example, in the complex number $z = (3,2)$, 3 is real part and 2 is imaginary part.

1.1.3 Know the condition for equality of complex numbers

Two complex numbers are said to be equal if and only if they have same real parts and same imaginary parts.
i.e., Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal iff $a_1 = a_2$ and $b_1 = b_2$.

Example: Which of the following pairs of complex numbers are equal.

(i) $z_1 = (6-1) - (2\times 3)i$ and $z_2 = (7-2) + 6(\sqrt{-1})$

(ii) $z_1 = 2^3 - (2^3 - 1)i$ and $z_2 = (10-2) - (3^2 - 2)i$

Solution: (i) $z_1 = (6-1) - (2\times 3)i$ and $z_2 = (7-2) + 6(\sqrt{-1})$

or $z_1 = 5 - 6i$ and $z_2 = 5 + 6i$

Here $z_1 \neq z_2$ because imaginary parts are not equal.

(ii) $z_1 = 2^3 - (2^3 - 1)i$ and $z_2 = (10-2) - (3^2 - 2)i$

or $z_1 = 8 - (8-1)i = 8 - 7i$ and $z_2 = 8 - (9-2)i = 8 - 7i$

Here $z_1 = z_2$ because real and imaginary parts are equal.

1.1.4 Carryout basic operations on complex numbers

Basic operations on complex numbers are addition, subtraction, multiplication and division.

(i) Addition of Complex Numbers

Sum of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is obtained by adding their real and imaginary parts respectively.
i.e. $z_1 + z_2 = (a+c) + i(b+d) = (a+c,\ b+d)$
Hence $(a,b) + (c,d) = (a+c,\ b+d)$

Example 1.

Simplify: $(3 + 7i) + (6 + 9i)$

Solution:

$$(3+7i)+(6+9i) = 9+16i$$

Example 2.

Simplify: $(2,3) + (1,-6)$

Solution:

$$(2,3) + (1,-6) = (3,-3)$$

(ii) Subtraction of Complex Numbers

Difference of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is obtained by subtracting their real and imaginary parts respectively.

i.e., $z_1 - z_2 = (a+ib) - (c+id)$

$$= (a-c) + i(b-d) = (a-c, b-d)$$

Hence $(\boldsymbol{a}, \boldsymbol{b}) - (\boldsymbol{c}, \boldsymbol{d}) = (\boldsymbol{a}-\boldsymbol{c}, \boldsymbol{b}-\boldsymbol{d})$.

Example 1. Simplify: $(6+5i) - (4+3i)$

Solution: $(6+5i) - (4+3i)$

$$= 6 + 5i - 4 - 3i$$

$$= 2 + 2i$$

Example 2. Simplify: $(7,8) - (5,6)$

Solution: $(7,8) - (5,6)$

$$= (2,2)$$

(iii) Multiplication of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$z_1 z_2 = (a+ib)(c+id)$$

$$= ac + i^2 bd + iad + ibc$$

$$= (ac - bd) + i(ad + bc) = (ac - bd, ad + bc)$$

Hence $(a,b)(c,d) = (ac - bd, ad + bc)$

Example 1. Find $z_1\, z_2$, where $z_1 = (4,5)$ and $z_2 = (6,7)$

Solution:

$\because \quad z_1 = (4,5)$ and $z_2 = (6,7)$

$\therefore \quad z_1 z_2 = (4 \cdot 6 - 5 \cdot 7,\ 4 \cdot 7 + 5 \cdot 6)$

$$= (24 - 35, \quad 28 + 30)$$

$$= (-11, 58)$$

Example 2. Find $z_1\, z_2$, where $z_1 = 1 + 5i$ and $z_2 = 4 + 3i$

Solution:

$$z_1\, z_2 = (1+5i)(4+3i)$$

$$= 4 + 3i + 20i + 15i^2$$

$$= 4 + 23i - 15 \qquad (\because i^2 = -1)$$

$$= -11 + 23i$$

(iv) Division of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ where $z_2 \neq (0,0)$

Then, $$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$= \frac{a(c-id) + ib(c-id)}{(c+id)(c-id)}$$

$$= \frac{ac - iad + ibc - i^2bd}{c^2 - i^2d^2}$$

$$= \frac{ac - iad + ibc + bd}{c^2 + d^2} \qquad (\because\ i^2 = -1)$$

$$= \frac{ac + bd + i(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

Hence $(a, b) \div (c, d) = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$

Example 1. Find $\frac{z_1}{z_2}$ where $z_1 = 3 - 7i$ and $z_2 = 2 + 6i$

Solution:

Here, $$\frac{z_1}{z_2} = \frac{3-7i}{2+6i} = \frac{3-7i}{2+6i} \times \frac{2-6i}{2-6i}$$

$$= \frac{6 - 18i - 14i + 42i^2}{4 - 36i^2}$$

$$= \frac{(6 - 42) + i(-14 - 18)}{4 + 36} \qquad [\because\ i^2 = -1]$$

$$= \frac{-9}{10} - i\,\frac{4}{5}$$

Example 2. Find $\frac{z_1}{z_2}$ when $z_1 = (1, -3)$ and $z_2 = (2,5)$

Solution:

$$\frac{z_1}{z_2} = \frac{(1, -3)}{(2, 5)}$$

$$= \left(\frac{(1)(2) + (-3)(5)}{(2)^2 + (5)^2}, \frac{(-3)(2) - (1)(5)}{(2)^2 + (5)^2}\right)$$

$$= \left(\frac{-13}{29}, \frac{-11}{29}\right)$$

1.1.5 Define $\bar{z} = a - ib$ as the complex conjugate of $z = a + ib$

If $z = a + ib$ then the conjugate of z, denoted as $\bar{z}$, is defined by $\bar{z} = a - ib$.

Example:

(i) The conjugate of $z = 5 + 2i$ is $\bar{z} = 5 - 2i$

(ii) The conjugate of $z = (7, -9)$ is $\bar{z} = (7,9)$

(iii) The conjugate of $z = (3,0)$ is $\bar{z} = (3,0)$

1.1.6 Define $|z| = \sqrt{a^2 + b^2}$ as the absolute value or modulus of a complex number $z = a + ib$

The absolute value or modulus of a complex number $z = a + ib$ is denoted by mod (z) or $|z|$ or $|a + ib|$ and is defined as

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$

Example: Find modulus of the following complex numbers:

$$6 + 8i \text{ and } 5 - i\sqrt{7}$$

(i) $\because z = 6 + 8i$

$\therefore |z| = \sqrt{6^2 + 8^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$= 10$

(ii) $\because z = 5 - i\sqrt{7}$

$\therefore |z| = \sqrt{5^2 + \left(-\sqrt{7}\right)^2}$

$= \sqrt{25 + 7}$

$= \sqrt{32}$

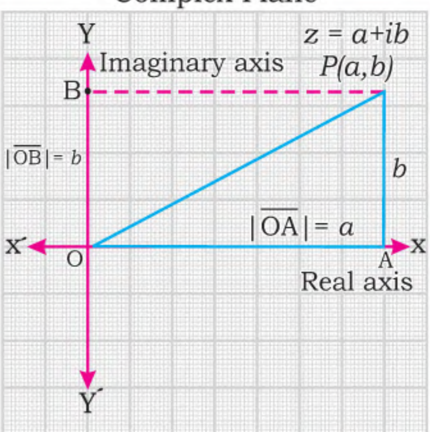
$= 4\sqrt{2}$

1.1.7 Geometrical representation of complex number z as a pair of real numbers (a, b).

The complex number $z = a + ib$ can be represented geometrically by the point whose cartesian coordinates are (a, b) in a plane where real part of z is taken along x-axis (real axis) of the plane and imaginary part of z is taken along y-axis (imaginary axis). This plane is called Argand diagram or complex plane.

In the figure 1.1, the complex number $z = a + ib$ is represented by the point $P(a, b)$ where $a = |\overline{OA}|$ and $b = |\overline{OB}|$.

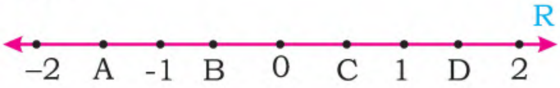
Complex Plane



(Fig. 1.1)

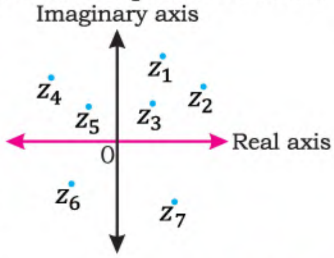
1.1.8 The order relation of complex numbers

Real numbers can be represented in either increasing or decreasing order. Numbers on the right side are greater than those on the left on a number line. Let A,B,C and D are the points representing real numbers on number line as shown in the figure (1.2a). They have increasing and decreasing order.



(Fig 1.2a)

On the other hand, number line cannot represent all complex numbers. The complex numbers $z_1, z_2, z_3, z_4, z_5, z_6, z_7$ etc are represented on the plane but cannot be written in increasing or decreasing order because all of them do not lie on the same line as shown in figure (1.2b).

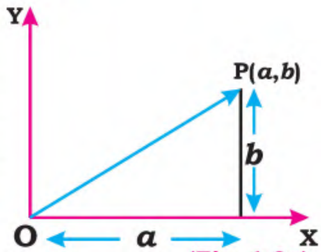


(Fig. 1.2b)

So, the complex numbers cannot be arranged in an order but moduli of complex numbers are real numbers and can be written as increasing and decreasing order on a number line. Hence there is no order relation for all complex number but their moduli follow order relation.

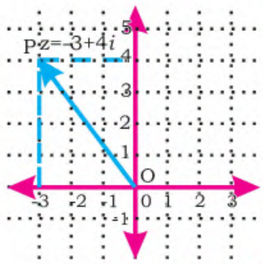
1.1.9 Vector representation of complex numbers

We represented a complex number $z = a + ib$ as the point $P(a, b)$ in the complex plane. The complex number (a, b) is interpreted as vector whose initial point is O and terminal point P as shown in Fig 1.3a. The length of the vector $\overrightarrow{OP}$ is the distance from the tail O of the vector to the tip P.



(Fig. 1.3a)

The vector representing the complex number $z = -3 + 4i$ is shown in Fig 1.3b.



(Fig. 1.3b)

Example: What are the lengths of the vectors representing the complex numbers

$z_1 = -3 + 4i$ and $z_2 = 2 - 7i$?

Solution:

Here $|Z_1| = \sqrt{9 + 16} = 5$

and $|Z_2| = \sqrt{4 + 49} = \sqrt{53} = 7.28$

The lengths of the vectors that represent z_1 and z_2 are 5 and 7.28 units respectively.

Exercise 1.1

1. Evaluate: (i) $i^{10} \cdot (-i^{12})$ (ii) $(-i)^{16} \cdot (-i)^6$ (iii) $(i)^{11} \cdot (-i)^{14}$

2. For what value of n, i^n is equal to 1, i, -1 or $-i$, where $1 \le n \le 4$

3. Simplify the following:

(i) $(-6, 3) \cdot (4, -2)$ (ii) $(8, -4) \div (-2, 2)$

(iii) $(2, 3) \div (4, 5)$ (iv) $(6 + 5i) - (4 + 3i)$

(v) $(5 - 6i) + (3 + 4i) - (5i - 7)$ (vi) $(4 + 5i)(6 + 7i)$

(vii) $(7 + 4i) \div (8 + 5i)$ (viii) $(5, -6) + (4, 8) - (3, -2)$

4. Simplify:

(i) $(2 - i)^4$ (ii) $(-1 - i\sqrt{3})^2$ (iii) $(-1 + i\sqrt{3})^2$ (iv) $(1 + i)^3$

(v) $\frac{\sqrt{2}+i}{\sqrt{2}-i}$ (vi) $\frac{1+i}{1-i} \cdot \frac{2-i}{1-i}$ (vii) $\frac{(2+i)^2}{3-4i}$ (viii) $\frac{1}{(2-i)^2}$

5. Show that $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$

6. Find the conjugate and absolute value of the following:

(i) $4 + 5i$ (ii) $-1 + 7i$ (iii) $\sqrt{3}i$

(iv) $\sqrt{7} - 3i$ (v) $-3 - 4i$ (vi) $(5 - 4i)^2$

(vii) $\frac{2}{3} - \sqrt{\frac{-9}{16}}$ (viii) $\frac{(1+i)(1+2i)}{3+i}$

7. Find real and imaginary parts of:

(i) $2i(3-5i)$ (ii) $\frac{\sqrt{5}+i}{\sqrt{5}-i}$

8. If $z = x + iy$ where $Re(z) = 0$ and $|z| = 2$ then find Z.

9. Solve the following complex equations:

(i) $(x, y)(2, 3) = (-4, 7)$ (ii) $(x + 3i) = 2yi$

10. Represent the following complex numbers on complex plane.

(i) $(2, -3)$ (ii) $(3, 4)$ (iii) $(-5, 7)$

(iv) $(-6, -2)$ (v) $(0, 6)$ (vi) $(-5, 0)$

11. Find the length of vector representing the complex numbers.

(i) $-5 + 2i$ (ii) $\frac{7}{3} + \frac{8}{3}i$ (iii) $\frac{1+i}{\sqrt{2}}$ (iv) $\frac{1+\sqrt{3}i}{2}$

1.2 Properties of Complex Numbers

1.2.1 Describe algebraic properties of complex numbers (e.g. commutative, associative and distributive) with respect to '+' and '×'

(i) Commutative property w.r.t. addition

Let z_1 and z_2 are two complex numbers then commutative property with respect to addition is defined as $z_1 + z_2 = z_2 + z_1$.

Example: Let $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$, then verify commutative property with respect to addition.

Verification: We have to verify that $z_1 + z_2 = z_2 + z_1$

L.H.S $= z_1 + z_2$

$= 2 + 3i + 4 + 5i$

$= 6 + 8i$

R.H.S $=\quad + z$

$= 4 + 5i + 2 + 3i$

$= 6 + 8i$

∵ L.H.S = R.H.S

∴ commutative property w.r.t addition is verified.

(ii) Commutative Property w.r.t. Multiplication

Let z_1 and z_2, are two complex numbers then commutative property under multiplication is defined as $z_1z_2 = z_2z_1$.

Example: Let $z_1 = 4 + 5i$ and $z_2 = 3 + 2i$ then verify commutative property with respect to multiplication.

Verification: We have to verify that $z_1z_2 = z_2z_1$

L.H.S = $z_1z_2 = (4, 5)(3, 2) = (4 \times 3 - 5 \times 2, 4 \times 2 + 5 \times 3) = (2, 23)$

R.H.S = $z_2 z_1 = (3,2)(4,5) = (3 \times 4 - 2 \times 5, 2 \times 4 + 3 \times 5) = (2,23)$
∵ L.H.S = R.H.S
∴ commutative property w.r.t. multiplication is verified.

(iii) Associative Property w.r.t. Addition

Let z_1, z_2 and z_3 are three complex numbers, then associative property under addition is defined as:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

Example: Let $z_1 = 4 + 5i$, $z_2 = 3 + 2i$ and $z_3 = 2 + 7i$ then verify associative property w.r.t addition.

Verification: We have to verify that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

L.H.S $= (z_1 + z_2) + z_3$
$= \{(4 + 5i) + (3 + 2i)\} + (2 + 7i)$
$= 7 + 7i + 2 + 7i$
$= 9 + 14i$

R.H.S $= z_1 + (z_2 + z_3)$
$= (4 + 5i) + \{(3 + 2i) + (2 + 7i)\}$
$= 4 + 5i + 5 + 9i$
$= 9 + 14i$

∵ L.H.S = R.H.S
∴ associative property of addition is verified.

(iv) Associative property w.r.t. Multiplication

Let z_1, z_2 and z_3 are three complex numbers, then associative property under multiplication is defined as:

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

Example: Let $z_1 = 2 + 3i$, $z_2 = 4 + 5i$ and $z_3 = 1 + i$ then verify associative property w.r.t multiplication.

Verification: We have to verify that $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

L.H.S $= (z_1 z_2) z_3$
$= \{(2 + 3i) \cdot (4 + 5i)\} \cdot (1 + i)$
$= (8 + 10i + 12i - 15)(1 + i)$
$= (-7 + 22i) \cdot (1 + i)$
$= -7 - 7i + 22i - 22$
$= -29 + 15i$

R.H.S $= z_1 (z_2 z_3)$
$= (2 + 3i) \cdot \{(4 + 5i) \cdot (1 + i)\}$
$= (2 + 3i)(4 + 4i + 5i - 5)$
$= (2 + 3i) \cdot (-1 + 9i)$
$= -2 + 18i - 3i - 27$

$= -29 + 15i$

$\because$ L.H.S = R.H.S

$\therefore$ associative property of multiplication is verified.

(v) Distributive Property of multiplication over addition

Let z_1, z_2 and z_3 are three complex numbers, then distributive property of multiplication over addition is defined as:

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

Example: Let, $z_1 = 2 + 3i$, $z_2 = 4 + 5i$ and $z_3 = 1 + i$ then verify distributive property of multiplication over addition.

Verification: We have to verify that $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

L.H.S $= z_1(z_2 + z_3)$

$= (2 + 3i)\{(4 + 5i) + (1 + i)\}$

$= (2 + 3i)(5 + 6i)$

$= 10 + 12i + 15i - 18$

$= -8 + 27i$

R.H.S $= z_1z_2 + z_1z_3$

$= \{(2 + 3i) \cdot (4 + 5i)\} + \{(2 + 3i) \cdot (1 + i)\}$

$= (8 + 10i + 12i - 15) + (2 + 2i + 3i - 3)$

$= (-7 + 22i) + (-1 + 5i)$

$= -8 + 27i$

$\because$ L.H.S = R.H.S

$\therefore$ distributive property of multiplication over addition is verified.

1.2.2 Know additive identity and multiplicative identity for the set of complex numbers

(i) Additive identity

Let $z = (a, b)$ be any complex number then there exists a complex number $(0, 0)$ such that,

$$z + (0, 0) = (a, b) + (0,0) = (a, b) = z$$

and $(0,0) + z = (0, 0) + (a, b) = (a, b) = z$

Thus, (0,0) is additive identity in set of complex numbers.

(ii) Multiplicative identity

Let $z = (a, b)$ be any complex number then there exists a complex number $(1, 0)$ such that,

$$z\,(1, 0) = (a, b)\,(1, 0) = (a, b) = z$$

and $(1, 0)\,z = (1, 0)(a, b) = (a, b) = z$

Thus, $(1, 0)$ is multiplicative identity in set of complex numbers.

1.2.3 Find additive inverse and multiplicative inverse of a Complex number z

(i) Additive Inverse of Complex Numbers

A complex number (c, d) is called the additive inverse of the complex number (a, b) if $(a, b) + (c, d) = (0, 0)$.

$\because \quad (a, b) + (c, d) = (0, 0)$

$\therefore \quad (a + c, b + d) = (0, 0)$

$\Rightarrow \quad a + c = 0 \quad \text{and} \quad b + d = 0$

$\Rightarrow \quad c = -a \quad \text{and} \quad d = -b$

Therefore, $\quad (c, d) = (-a, -b)$

So, the additive inverse of $a + ib$ is $-a - ib$

Note: (i) Additive inverse of (0,0) is (0,0).

(ii) Additive inverse of (a, b) is $(-a, -b)$.

Example:

(i) The additive inverse of the complex number $6 - 4i$ is $-6 + 4i$.

(ii) The additive inverse of the complex number $-14i$ is $14i$.

(ii) Multiplicative Inverse of a complex number:

The multiplicative inverse of a non-zero complex number $\boldsymbol{z}$, denoted as z^{-1} or $\frac{1}{z}$, is a complex number such that

$$zz^{-1} = z^{-1}z = (1, 0)$$

As the multiplicative inverse of a complex number z is $\frac{1}{z}$ where $z \neq (0, 0)$

So, $\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$ (By Rationalizing the denominator)

$$= \frac{a - ib}{(a + ib)(a - ib)}$$

$$= \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}$$

Hence the multiplicative inverse of a complex number (a, b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

Example 1.

Find the multiplicative inverse of a complex number $z = 5 + 3i$

Solution: $\because \; z = 5 + 3i$

$$\therefore z^{-1} = \left(\frac{5}{(5)^2 + (3)^2}, \frac{-3}{(5)^2 + (3)^2}\right) = \left(\frac{5}{25 + 9}, \frac{-3}{25 + 9}\right) = \left(\frac{5}{34}, \frac{-3}{34}\right)$$

Example 2. Find the multiplicative inverse of complex number z = 4

Solution:

$$\because \quad z = 4 \text{ or } (4,0)$$

$$\therefore \quad z^{-1} = \left(\frac{4}{(4)^2+(0)^2}, \frac{0}{(4)^2+(0)^2}\right) = \left(\frac{1}{4}, 0\right)$$

Example 3. Prove that $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is the multiplicative inverse of (a, b) where $(a,b) \neq (0,0)$.

Proof:

$$(a,b)\cdot\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \left(\frac{a^2}{a^2+b^2} - \frac{-b^2}{a^2+b^2}, \frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right) = \left(\frac{a^2+b^2}{a^2+b^2}, 0\right) = (1,0)$$

i.e., $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)\cdot(a,b) = (1,0)$, hence proved.

Note: The multiplicative inverse of (0,0) does not exist.

1.2.4 Demonstrate the following properties

(i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$,

(ii) $\bar{\bar{z}} = z, z\bar{z} = |z|^2, \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(iii) Triangle inequality of complex numbers

(iv) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

Property (i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

Proof: Let $z = a + ib$ then $|z| = |a+ib| = \sqrt{(a)^2+(b)^2} = \sqrt{a^2+b^2}$ ...(a)

and $|-z| = |-a-ib| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$ ... (b)

Aslo, $|\bar{z}| = |a-ib| = \sqrt{(a)^2+(-b)^2} = \sqrt{a^2+b^2}$ ... (c)

and $|-\bar{z}| = |-a+ib| = \sqrt{(-a)^2+(b)^2} = \sqrt{a^2+b^2}$ ... (d)

From the results (a), (b), (c) and (d), we get

$|z| = |-z| = |\bar{z}| = |-\bar{z}|$, hence proved.

Property (ii) (a) $\bar{\bar{z}} = z$, (b) $z\bar{z} = |z|^2$, (c) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

(a) $\bar{\bar{z}} = z$

Proof:

Let $z = a + ib$ then $\bar{z} = a - ib$

Now, $\bar{\bar{z}} = \overline{(\bar{z})} = \overline{(a-ib)} = a + ib = z$

Thus it is proved that $\bar{\bar{z}} = z$

(b) $z\bar{z} = |z|^2$

Proof: Let $z = a + ib$ then $\bar{z} = a - ib$

Now, $z\bar{z} = (a+ib)(a-ib) = a^2 - i^2b^2 = a^2 + b^2$ ...(i) and $|z| = |a+ib| = \sqrt{a^2+b^2}$

So, $|z|^2 = |a+ib|^2 = a^2+b^2$ ...(ii)

From (i) and (ii) we have,

$z\bar{z} = |z|^2$, hence proved. ...(b)

(c) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

Proof: Let $z_1 = a+ib$ and $z_2 = c+id$

L.H.S $= \overline{z_1+z_2} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)}$

$= (a+c) - i(b+d)$

Thus $\overline{z_1+z_2} = (a-ib)+(c-id)$ ...(i)

Now, R.H.S $= \overline{z_1} + \overline{z_2} = \overline{(a+ib)} + \overline{(c+id)}$

Thus $\overline{z_1} + \overline{z_2} = (a-ib)+(c-id)$ ...(ii)

From (i) and (ii), we have $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$, hence proved.

Property (iii)

Triangle Inequality of complex numbers:

If $z_1, z_2 \in \mathbb{C}$, then $|z_1| - |z_2| \leq |z_1+z_2| \leq |z_1| + |z_2|$, which is called Triangle inequality.

Geometrical Proof

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be represented by vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively as shown in (Fig 1.4)

Considering the figure, by completing the parallelogram $OP_1\,P_3\,P_2$.

In ΔOP_1P_3, $\overrightarrow{OP_3} = z_1 + z_2$

Since sum of the lengths of any two sides of a triangle is greater than the length of the third side.

$$|\overrightarrow{OP_1}| + |\overrightarrow{OP_2}| \geq |\overrightarrow{OP_3}| \qquad (\because\ \overrightarrow{P_1P_3} = \overrightarrow{OP_2})$$

So $|z_1| + |z_2| \geq |z_1+z_2|$

...(i)

Again, in ΔOP_1P_2

$$|\overrightarrow{OP_2}| + |\overrightarrow{P_2P_1}| \geq |\overrightarrow{OP_1}|$$

$\Rightarrow$ $|\overrightarrow{P_2P_1}| \geq |\overrightarrow{OP_1}| - |\overrightarrow{OP_2}|$

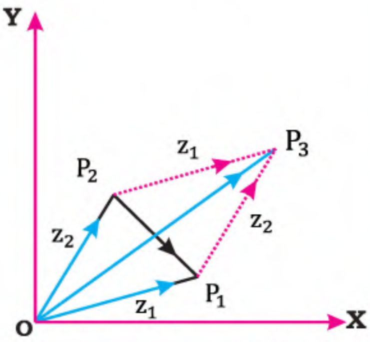
$\Rightarrow$ $|z_1 - z_2| \geq |z_1| - |z_2|$

$\Rightarrow$ $|z_1| - |z_2| \leq |z_1 - z_2|$

Replacing z_2 by $-z_2$ we get

$$|z_1| - |-z_2| \leq |z_1 - (-z_2)|$$

$\Rightarrow$ $|z_1| - |z_2| \leq |z_1 + z_2|$ ...(ii)



(Fig. 1.4)

Combining inequalities (i) and (ii), and rearranging

$|z_1| - |z_2| \leq |z_1+z_2| \leq |z_1| + |z_2|$, $\forall\, z_1, z_2 \in \mathbb{C}$. Hence proved.

Property (iv)

(a) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (b) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

Proof: (a) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

Let $z_1 = a + ib$ and $z_2 = c + id$

then $z_1 z_2 = (a + ib)(c + id)$

or $z_1 z_2 = (ac - bd) + (bc + ad)i$

L.H.S $= \overline{z_1 z_2} = \overline{(ac - bd) + (bc + ad)i} = (ac - bd) - (bc + ad)i$ ... (i)

R.H.S $= \overline{z_1}\, \overline{z_2} = (a - ib)(c - id) = (ac - bd) - (bc + ad)i$ ...(ii)

From the results (i) and (ii), we get $\overline{z_1 z_2} = \overline{z_1} . \overline{z_2}$, hence proved.

Proof: (b) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$ where, $z_1 = a + ib$ and $z_2 = c + id$.

Now, $\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \frac{(a + ib)(c - id)}{c^2 + d^2}$

Thus $\frac{z_1}{z_2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

L.H.S $= \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}} = \frac{(ac+bd)}{c^2+d^2} - i\frac{(bc-ad)}{c^2+d^2}$ ... (i)

R.H.S $= \frac{\overline{z_1}}{\overline{z_2}} = \frac{a-ib}{c-id} = \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$

$= \frac{(a - ib)(c + id)}{c^2+d^2} = \frac{ac + iad - ibc - i^2bd}{c^2+d^2}$

$= \frac{(ac + bd)}{c^2+d^2} + i\frac{(ad - bc)}{c^2+d^2} = \frac{(ac + bd)}{c^2+d^2} - \frac{i(bc - ad)}{c^2+d^2}$ ... (ii)

From (i) and (ii), we get $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$. Hence proved.

1.2.5 Find real and imaginary parts of the following type of complex numbers

Type I: $(x + iy)^n$,

Type II: $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2 + iy_2 \neq 0$ where, $n = \pm 1$ and ± 2

Type: (I) $(x + iy)^n$ where $n = \pm 1$ and ± 2

(a) Let, $n = 1$

then $(x + iy)^n = (x + iy)^1 = x + iy$

Here, x is real part and y is imaginary part.

(b) Let, $n = -1$

then $(x+iy)^n = (x+iy)^{-1} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2-i^2y^2} = \frac{x-iy}{x^2+y^2}$

Thus, $(x+iy)^{-1} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

Here, $\frac{x}{x^2+y^2}$ is real part and $-\frac{y}{x^2+y^2}$ is imaginary part

(c) Let, $n = 2$

then $(x+iy)^n = (x+iy)^2 = x^2 + 2xyi + i^2y^2 = (x^2-y^2) + 2xyi$

Here, real part is $x^2 - y^2$ and $2xy$ is imaginary part.

(d) Let, $n = -2$

$$(x+iy)^{-2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2+2xyi+i^2y^2}$$

$$= \frac{1}{(x^2-y^2)+2xyi} = \frac{\{(x^2-y^2)-2xyi\}}{\{(x^2-y^2)+2xyi\}\times\{(x^2-y^2)-2xyi\}}$$

$$= \frac{(x^2-y^2)-2xyi}{(x^2-y^2)^2+4x^2y^2i^2} = \frac{(x^2-y^2)-2xyi}{(x^2+y^2)^2}$$

Thus, $(x+iy)^{-2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2xyi}{(x^2+y^2)^2}$

Here, $\frac{x^2-y^2}{(x^2+y^2)^2}$ is real part and $-\frac{2xy}{(x^2+y^2)^2}$ is imaginary part.

Example: Find the real and imaginary parts of the following complex numbers:

(i) $(2-5i)^2$ (ii) $(3-4i)^{-1}$

(i) Solution:

$$(2-5i)^2 = (2)^2 - 2(2)(5i) + (5i)^2 = -21 - 20i$$

Therefore, real part is -21 and imaginary part is -20.

(ii) Solution:

$$(3-4i)^{-1} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}i$$

we have real part $\frac{3}{25}$ and imaginary part $\frac{4}{25}$.

Note: We can solve the above example directly by using the derived formula as done below.

(i) real part $= x^2 - y^2 = (2)^2 - (-5)^2 = 4 - 25 = -21$

and imaginary part $= 2xy = 2(2)(-5) = -20$.

(ii) real part $= \frac{x}{x^2+y^2} = \frac{3}{(3)^2+(-4)^2} = \frac{3}{9+16} = \frac{3}{25}$

and imaginary part $= \frac{-y}{x^2+y^2} = \frac{-(-4)}{(3)^2+(-4)^2} = \frac{4}{9+16} = \frac{4}{25}$

Type: (II) $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2+iy_2 \neq 0$ where $n = \pm 1$ and ± 2

For $n = 1$

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^1$$

$$= \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2)^2-i^2y_2^2}$$

$$= \frac{x_1x_2-(x_1y_2-x_2y_1)i-(-1)y_1y_2}{x_2^2+y_2^2} = \frac{(x_1x_2+y_1y_2)-(x_1y_2-x_2y_1)i}{x_2^2+y_2^2}$$

Here, $\frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2}$ and $\frac{-(x_1y_2-x_2y_1)}{x_2^2+y_2^2}$ are the real and imaginary parts respectively.

For $n = -1$,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-1} = \frac{1}{\frac{x_1+iy_1}{x_2+iy_2}} = \frac{x_2+iy_2}{x_1+iy_1} = \frac{x_2+iy_2}{x_1+iy_1} \times \frac{x_1-iy_1}{x_1-iy_1}$$

$$= \frac{x_1x_2+(x_1y_2-x_2y_1)i-(-1)y_1y_2}{(x_1)^2-i^2y_1^2} = \frac{x_1x_2+y_1y_2}{x_1^2+y_1^2} + \frac{(x_1y_2-x_2y_1)i}{x_1^2+y_1^2}$$

Here, $\frac{x_1x_2+y_1y_2}{x_1^2+y_1^2}$ is real part and $\frac{(x_1y_2-x_2y_1)}{x_1^2+y_1^2}$ is imaginary part.

For $n = 2$

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^2 = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2} = \frac{(x_1)^2+2x_1y_1i+i^2y_1^2}{(x_2)^2+2x_2y_2i+i^2y_2^2} = \frac{x_1^2+2x_1y_1i+(-1)y_1^2}{(x_2)^2+2x_2y_2i+(-1)y_2^2}$$

$$= \frac{(x_1^2-y_1^2)+2x_1y_1i}{(x_2^2-y_2^2)+2x_2y_2i} = \frac{\{(x_1^2-y_1^2)+2x_1y_1i\}\{(x_2^2-y_2^2)-2x_2y_2i\}}{\{(x_2^2-y_2^2)+2x_2y_2i\}\{(x_2^2-y_2^2)-2x_2y_2i\}}$$

$$= \frac{\{(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1y_1x_2y_2\}+\{2x_1y_1(x_2^2-y_2^2)-2x_2y_2(x_1^2-y_1^2)\}i}{x_2^4+y_2^4-2x_2^2y_2^2+4x_2^2y_2^2}$$

Therefore, the real part is $\frac{(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1y_1x_2y_2}{(x_2^2+y_2^2)^2}$

and the imaginary part is $\frac{2x_1y_1(x_2^2-y_2^2)-2x_2y_2(x_1^2-y_1^2)}{(x_2^2+y_2^2)^2}$

For $n = -2$

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-2} = \frac{(x_1+iy_1)^{-2}}{(x_2+iy_2)^{-2}} = \frac{(x_2+iy_2)^2}{(x_1+iy_1)^2} = \frac{(x_2)^2+2x_2y_2i+i^2y_2^2}{(x_1)^2+2x_1y_1i+i^2y_1^2}$$

$$=\frac{(x_2)^2+2x_2y_2i+(-1)y_2^2}{(x_1)^2+2x_1y_1i+(-1)y_1^2}=\frac{\left(x_2^2-y_2^2\right)+2x_2y_2i}{\left(x_1^2-y_1^2\right)+2x_1y_1i}$$

$$=\frac{\left\{\left(x_2^2-y_2^2\right)+2x_2y_2i\right\}\left\{\left(x_1^2-y_1^2\right)-2x_1y_1i\right\}}{\left\{\left(x_1^2-y_1^2\right)+2x_1y_1i\right\}\left\{\left(x_1^2-y_1^2\right)-2x_1y_1i\right\}}$$

$$=\frac{\left\{\left(x_1^2-y_1^2\right)\left(x_2^2-y_2^2\right)+4x_1y_1x_2y_2\right\}+\left\{2x_2y_2\left(x_1^2-y_1^2\right)-2x_1y_1\left(x_2^2-y_2^2\right)\right\}i}{\left(x_1^4+y_1^4-2x_1^2y_1^2+4x_1^2y_1^2\right)}$$

Therefore, the real part is $\frac{\left(x_1^2-y_1^2\right)\left(x_2^2-y_2^2\right)+4x_1y_1x_2y_2}{\left(x_1^2+y_1^2\right)^2}$

and the imaginary part is $\frac{2x_2y_2\left(x_1^2-y_1^2\right)-2x_1y_1\left(x_2^2-y_2^2\right)}{\left(x_1^2+y_1^2\right)^2}$

Example: Find the real and imaginary parts of the following.

(i) $\left(\frac{2-i}{4-3i}\right)^2$ (ii) $\left(\frac{1-i}{2-i}\right)^{-2}$

(i) Solution: $\left(\frac{2-i}{4-3i}\right)^2$

$$=\frac{4-4i+i^2}{16-24i+9i^2}$$

$$=\frac{3-4i}{7-24i}$$

$$=\frac{3-4i}{7-24i}\times\frac{7+24i}{7+24i}$$

$$=\frac{21+72i-28i+96}{49+576}$$

$$=\frac{117+44i}{625}$$

$$=\frac{117}{625}+\frac{44}{625}i$$

So, real part $=\frac{117}{625}$ and imaginary part $=\frac{44}{625}$.

(ii) Solution: $\left(\frac{1-i}{2-i}\right)^{-2}$

$$=\left(\frac{2-i}{1-i}\right)^2$$

$$=\frac{4-4i+i^2}{1-2i+i^2}$$

$$=\frac{3-4i}{-2i}$$

$$= \frac{3-4i}{-2i} \times \frac{i}{i}$$
$$= \frac{4+3i}{2}$$
$$= 2 + \frac{3}{2}i$$

So, real part = 2 and imaginary part = $\frac{3}{2}$.

Note: We can also solve the above example by using the derived formulas.

Exercise 1.2

1. Let $z_1 = 3 - 4i,\quad z_2 = 4 + 5i$ and $z_3 = -5 + 6i$

Verify the following:

(i) Addition of complex numbers is commutative
(ii) Multiplication of complex numbers is commutative
(iii) Addition of complex numbers is associative
(iv) Multiplication of complex numbers is associative
(v) Multiplication of complex numbers is distributive over addition.

2. If z, z_1 and z_2 are complex numbers then prove that:

(a) $z + \bar{z}$ is real. (b) $z - \bar{z}$ is imaginary.
(c) $|z_1 \cdot z_2| = |z_1||z_2|$

3. Find the additive inverses of the following complex numbers:

(i) $3 + 5i$ (ii) $6 - 5i$ (iii) $(11, 0)$ (iv) $\left(\frac{2}{3}, 6\right)$
(v) $\left(\frac{8}{9}, \frac{-4}{5}\right)$ (vi) $\left(0, \frac{3}{8}\right)$

4. Find the multiplicative inverses of the following complex numbers.

(i) $3 + 5i$ (ii) $8i$ (iii) $(10, 4)$ (iv) $(12, -7)$
(v) $(-8, 0)$ (vi) $\left(0, \frac{-3}{4}\right)$

5. Find the additive and multiplicative inverses by definition of the following:

(i) $(2, 3)$ (ii) $(-4, 5)$

6. Verify the following properties with $z = 4 - 3i, z_1 = 3 - 2i$ and $z_2 = 2 + 3i$

(i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$ (ii) $\bar{\bar{z}} = z$ (iii) $z\,\bar{z} = |z|^2$
(iv) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$ (v) $\overline{z_1 z_2} = \bar{z_1} \cdot \bar{z_2}$ (vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$

7. If $z_1 = 3 - 2i$ and $z_2 = 2 - 3i$ then express the following in the form of $a + ib$.

(i) $\frac{z_1\overline{z_2}}{\overline{z_1}}$ (ii) $\frac{\overline{z_1}z_2}{\overline{z_2}}$ (iii) $\frac{\overline{z_1 z_2}}{z_1}$

Also verify Triangle inequality of complex numbers.

8. Find real and imaginary parts of each of the following by using any method:

(i) $(\sqrt{2}+i)^{-2}$ (ii) $(1-\sqrt{5}i)^{-1}$ (iii) $(\sqrt{3}-i)^{-2}$ (iv) $(2i-\sqrt{3})^{-1}$

(v) $\left(\frac{1}{4i-5}\right)^{-1}$ (vi) $\left(\frac{3+4i}{5i-4}\right)^{-2}$ (vii) $\left(\frac{3i-2}{2-3i}\right)^{-2}$

1.3 Solution of complex equations

In this section, we shall find the solution of equations with complex coefficients by using different methods.

1.3.1 Solve the simultaneous linear equations with complex coefficients. For example,

$$5z - (3+i)w = 7 - i$$
$$(2-i)z + 2iw = -1 + i$$

Example: Solve $5z - (3+i)w = 7 - i$

$(2-i)z + 2iw = -1 + i$

Where w and z are complex numbers.

Solution:

Here, Linear equations are

$$5z - (3+i)w = 7 - i \quad \ldots\text{(i)}$$
$$(2-i)z + 2iw = -1 + i \quad \ldots\text{(ii)}$$

In order to equate the coefficients of z, we multiply equation (i) by $(2-i)$ and equation (ii) by 5.

$$(2-i)5z - (2-i)(3+i)w = (2-i)(7-i) \quad \ldots\text{(iii)}$$
$$5(2-i)z + 5 \times 2iw = 5(-1+i) \quad \ldots\text{(iv)}$$

Subtracting equation (iv) from equation (iii), we get

$$-w(7+9i) = 18 - 14i$$

or $$w = \frac{18-14i}{-(7+9i)} = \frac{(-18+14i)\times(7-9i)}{(7+9i)(7-9i)} = \frac{-126+162i+98i+126}{130} = 2i$$

Now substituting $w = 2i$ in equation (i),

we get $5z - (3+i)2i = 7 - i$

$\Rightarrow \qquad 5z - 6i + 2 = 7 - i$

$\Rightarrow \qquad z = \frac{5+5i}{5}$

$\Rightarrow \qquad z = 1 + i$

Thus, $w = 2i$ and $z = 1 + i$.

1.3.2 Represent Polynomial P(z) as a product of linear factors

For example: (a) $z^2 + a^2 = (z + ia)(z - ia)$

(b) $z^3 - 3z^2 + z + 5 = (z + 1)(z - 2 - i)(z - 2 + i)$

(a) $z^2 + a^2 = z^2 - (-1)a^2 = z^2 - i^2a^2 = (z)^2 - (ia)^2$

$= (z - ia)(z + ia)$

Thus, factors of $z^2 + a^2$ are $(z - ia)$ and $(z + ia)$.

(b) Let $p(z) = z^3 - 3z^2 + z + 5$ is a polynomial.

The factors of 5 are ± 1 and ± 5.

Here, $\qquad p(z) = z^3 - 3z^2 + z + 5$

For $\quad z = 1, \quad p(1) = (1)^3 - 3(1)^2 + (1) + 5 \neq 0$

For $\quad z = -1, \quad p(-1) = (-1)^3 - 3(-1)^2 + (-1) + 5 = 0$

As, $\quad z = -1$, so, $(z + 1)$ is a factor of $p(z) = z^3 - 3z^2 + z + 5$.

By synthetic division

−1	1	−3	1	5
		−1	4	−5
	1	−4	5	0

we get $(z^3 - 3z^2 + z + 5) \div (z + 1) = (z^2 - 4z + 5)$

Hence, $\qquad (z^3 - 3z^2 + z + 5) = (z + 1)(z^2 - 4z + 5)$

Now we can easily find the factors of $z^2 - 4z + 5$ by using quadratic formula.

In order to find the factors of $z^2 - 4z + 5$, let $z^2 - 4z + 5 = 0$.

Here, $a = 1$, $b = -4$ and $c = 5$, by using quadratic formula

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

Thus, $z = 2 + i$ and $z = 2 - i$ or $z - 2 - i = 0$ and $z - 2 + i = 0$

Hence the three factors of the given polynomial are $(z+1), (z-2-i)$ and $(z-2+i)$

1.3.3 Solve quadratic equation of the form $pz^2 + qz + r = 0$ by completing the squares, where p, q and r are real numbers and z a complex number

For example: Solve $z^2 - 2z + 5 = 0$

$\Rightarrow (z - 1 - 2i)(z - 1 + 2i) = 0$

$\Rightarrow z = 1 + 2i, 1 - 2i$

We recall the method of completing the squares by solving the following standard form of quadratic equation.

The quadratic equation in standard form is:

$$pz^2 + qz + r = 0 \qquad \forall\, p, q, r \in \mathbb{R}$$

$$\Rightarrow \quad pz^2 + qz = -r \qquad \text{(Shifting constant on R.H.S)}$$

$$\Rightarrow \quad z^2 + \frac{q}{p}z = -\frac{r}{p} \qquad \text{(Dividing by the coefficient of } z^2\text{)}$$

$$\Rightarrow \quad (z)^2 + 2\left(\frac{q}{2p}\right)z + \left(\frac{q}{2p}\right)^2 = -\frac{r}{p} + \left(\frac{q}{2p}\right)^2 \qquad \left[\text{Adding } \left(\frac{q}{2p}\right)^2 \text{ to both sides}\right]$$

$$\Rightarrow \quad \left(z + \frac{q}{2p}\right)^2 = -\frac{r}{p} + \frac{q^2}{4p^2} = \frac{q^2 - 4pr}{4p^2}$$

$$z + \frac{q}{2p} = \pm\sqrt{\frac{q^2 - 4pr}{4p^2}} = \pm\frac{\sqrt{q^2 - 4pr}}{2p} \qquad \text{(Taking square root)}$$

$$\Rightarrow \quad z = \pm\frac{\sqrt{q^2 - 4pr}}{2p} - \frac{q}{2p} = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$$

Example: Solve $z^2 - 2z + 5 = 0$ by completing square method.

Solution: We have $z^2 - 2z + 5 = 0$

or $\quad z^2 - 2z = -5$

By adding 1 on both sides

we get $\quad z^2 - 2z + 1 = -5 + 1$

$\Rightarrow \quad (z - 1)^2 = -4$

By taking square root of both sides

$$z - 1 = \pm\sqrt{-4}$$

$$z - 1 = \pm 2i$$

$$z = 1 \pm 2i$$

Exercise 1.3

1. Solve the following simultaneous linear equations with complex coefficients for w and z.

(i) $3z + (2+i)w = 11 - i$
$(2-i)z - w = -1 + i$

(ii) $2z + (3+i)w = 9 - i$
$(3-i)z - iw = -1 + i$

2. Write the following polynomials as the product of linear factors:

(i) $z^2 + 81b^2$ (ii) $z^3 + 3z^2 + z - 5$ (iii) $4z^2 + 9b^2$

(iv) $z^3 + 3z^2 + 4z + 2$ (v) $z^3 - 7z^2 + 19z - 13$

(vi) $z^3 + 3z^2 + 19z + 17$ (vii) $z^3 - 3z^2 + 4z - 2$

3. Solve the following quadratic equations by completing the squares, where z is a complex number.

(i) $z^2 - 4z + 5 = 0$ (ii) $z^2 + 12z + 52 = 0$

(iii) $34z^2 - 6z = -1$ (iv) $z^2 - 6z + 34 = 0$

(v) $z^2 - 6z = -13$ (vi) $z^2 + 64 = 0$

Review Exercise 1

1. Select the correct option.

i. For any complex number z, $|z|$ is equal to

(a) $|\bar{z}|$ (b) $|-z|$ (c) $|-\bar{z}|$ (d) all of these

ii. If z_1 and z_2 are any two complex numbers, then

(a) $|z_1 + z_2| < |z_1| + |z_2|$ (b) $|z_1 + z_2| \le |z_1| + |z_2|$

(c) $|z_1 - z_2| < |z_1| - |z_2|$ (d) $|z_1 + z_2| \ge |z_1| + |z_2|$

iii. If $z = 3i - 4$, then $z + \bar{z}$ = --------- (a) 8 (b) $-3i$ (c) -8 (d) $3i$-8

iv. If $a > 0$ and $b < 0$, then (a) $ab > 0$ (b) $ab < 0$ (c) $ab = 0$ (d) all of these

v. If n is an even integer, then $(i)^n$ is equal to:

(a) i (b) $-i$ (c) 1 or -1 (d) i or $-i$

vi. If z is any real number x, then its conjugate is:

(a) x (b) $-x$ (c) xi (d) $-xi$

vii. $(-1)^{\frac{-21}{2}}$ is equal to: (a) i (b) $-i$ (c) 1 (d) -1

viii. If $z = (0,1)$ then z^2 is: (a) 1 (b) -1 (c) i (d) $-i$

ix. The product of two conjugate complex numbers is:

(a) A real number (b) An imaginary number

(c) Always zero (d) Not defined

x. Real part of $\frac{2+i}{i}$ is equal to: (a) 1 (b) 2 (c) -1 (d) $\frac{1}{2}$

xi. The plane on which complex numbers are shown is called:
(a) Coordinate plane (b) Complex plane
(c) Cartesian plane (d) Real plane

xii. The modulus of a complex number $z = x + iy$ is the distance of $P(x, y)$ from:
(a) x – axis (b) y – axis (c) Origin (d) (x, y)

xiii. If $z_1 = 2 + 3i$, $z_2 = 1 - i$ then $|z_1 z_2| =$ --------------
(a) $\sqrt{13}$ (b) $\sqrt{26}$ (c) $\sqrt{15}$ (d) 26

xiv. If $|x + 5i| = 3$, then x is equal to :(a) ± 4 (b)$\pm 4i$ (c)$\pm 22i$ (d)None of these

xv. Multiplicative inverse of (3,4) is:
(a) $\left(\frac{3}{25}, \frac{4}{25}\right)$ (b) $\left(\frac{3}{25}, \frac{-4}{25}\right)$ (c) $\left(\frac{-3}{25}, \frac{-4}{25}\right)$ (d) (1,0)

xvi. If $z = -3i + 4$, then $\bar{z} =$ --------------
(a) $-3i - 4$ (b)$-3i + 4$ (c) $3i + 4$ (d) $3i - 4$

xvii. Real and imaginary part of $i(3 - 2i)$ are respectively:
(a) $2 \& -3$ (b) 2 & 3 (c) $3 \& -2$ (d)$-3 \& -2$

xviii. If $z = a + ib$, then $|z| =$: --------------
(a) $\sqrt{a - b}$ (b) $\sqrt{a^2 - b^2}$ (c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a - b}$

xix. $\left(\frac{1+i}{1-i}\right)^{12} =$ --------------
(a) -1 (b) 1 (c) i (d) $-i$

xx. The value of $i^{-7}$is: --------------
(a) 1 (b) i (c) -1 (d) $-i$

2. Simplify:
(i) $(5 - 6i) + (5i) + (7 + 6i)$ (ii) $(4 - i) + (-9 + 6i)$
(iii) $(9 + 11i) - (3 + 5i)$ (iv) $(-2 - 15i) - (-12 + 13i)$
(v) $(-3 + 2i)(3 - 8i)$ (vi) $(3 - i)(4 + 3i)(5 - 2i)$
(vii) $\frac{4-i}{6-3i}$ (viii) $\frac{8+6i}{6-2i}$

3. **(a)** Find each of the following:
(i) $\bar{z}$ for $z = 3 - 15i$ (ii) $\overline{z_1 - z_2}$ for $z_1 = 5 + i$ and $z_2 = -8 + 3i$
(iii)$\overline{z_1 z_2}$ for $z_1 = -10 + 5i, z_2 = 5 - 10i$
(iv) $\frac{z}{\bar{z}}$ for $z = 15 - 3i$ (v) $\bar{z_1} - \bar{z_2}$ for $z_1 = 5 + 4i$ and $z_2 = -8 + 5i$

(b) Find absolute value of the following complex numbers:
(i) $-3 - 6i$ (ii) $-6 + 3i$ (iii) $\frac{6+3i}{10+2i}$ (iv) $\frac{1+i}{1-i}$

4. Find the values of x and y if:

(i) $5x + 3iy = -x + 2iy$ (ii) $x^2 - 7x + 9iy = iy^2 + 20i - 12$

5. Find real and imaginary parts of the following:

(i) $z = i - 42i - 3$ (ii) $\frac{3z+i}{z+4}$ for $z = 3 + 2i$

6. Find the additive and multiplicative inverse of each of the following:

(i) $3 - 7i$ (ii) $-2 + i$ (iii) $-2 - 5i$

(iv) $\frac{1}{2} + \frac{3}{2}i$ (v) $-4 + \sqrt{7}i$ (vi) $3 + 4i$

7. Solve the following equation:

$$z^2 - (2 - 3i)z - 5 + i = 0$$

8. Verify the following:

(i) $(2 + 3i)^2 = -5 + 12i$ (ii) $2(5 - 2i)^2 = 42 - 40i$

(iii) $z^3 + i = (z - i)(z^2 + iz - 1)$.

9. Solve the quadratic equations by completing the squares.

(i) $z^2 - 5z + 7 = 0$ (ii) $2z^2 - 6z + 9 = 0$ (iii) $4z^2 + 25 = 0$

(iv) $z^2 - 10z + 41 = 0$ (v) $4z^2 + 4z + 5 = 0$

Matrices and Determinants

Unit 2

2.1 Matrices

Matrix is a Latin word which means a place where something develops or originates.

J. J. Sylvester (1814-1897), was the first British mathematician, who formed squares containing rows and columns which he extracted from a rectangular arrangement of objects and called it a matrix (plural matrices).

Arthur Cayley (1821-1895) developed the theory of matrices and used it in algebra of matrices.

Matrices are used to solve the system of linear equations. These have wide applications in the fields of mathematics, statistics, engineering, physical and social sciences also in other various disciplines.

2.1.1 Recall the concept of

- **a matrix and its notation**
- **order of a matrix**
- **equality of two matrices**

(a) A matrix and its notation

A rectangular array of numbers, symbols, or expressions, arranged in rows and columns is called a matrix.

Matrices are generally denoted by capital letters of English alphabet. Small letters of English alphabet or numbers generally denote its elements or entries.

The following notations are used to enclose the elements of a matrix.

$$[\quad] \;,\; (\quad)$$

Following are the examples of matrices:

$$A = \begin{bmatrix} a \\ b \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 6 \\ 8 & 7 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} i & 2i \\ 3i & -4i \end{bmatrix} \text{ and } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The general form of a matrix 'A' with 'm' rows and 'n' columns is represented as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

(j^{th} column; i^{th} row)

It is noted that the element a_{ij} is lying on the intersection of the i^{th} row and j^{th} column of matrix A. It is referred to as the $(i, j)th$ element. Hence, the above given matrix A can be represented by $A = [a_{ij}]_{(m,n)}$

where $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$

2.1.1 (b) Order of a matrix

The order or dimension of a matrix having m rows and n columns is denoted by $m \times n$ (read as m by n).

A matrix of order $m \times n$ can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots \; \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}$$

Examples:

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2×2 matrix or matrix of order 2.

(ii) $\begin{bmatrix} i & 4i & 7i \\ 2i & 5i & 8i \\ 3i & 6i & 9i \end{bmatrix}$ is a 3×3 matrix or matrix of order 3.

(iii) $\begin{bmatrix} i \\ 2i \\ 3i \end{bmatrix}$ is a matrix of order 3×1.

(iv) $[1 \quad 2 \quad 3]$ is a matrix of order 1×3.

2.1.1 (c) Equality of two matrices

Two matrices A and B are said to be equal if they have the same order or dimension and the corresponding elements are equal.

For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2-1 & 1+1 \\ 4-1 & 3+1 \end{bmatrix}$ are equal matrices.

but, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ are not equal, because $a_{22} \neq c_{22}$ i.e. $4 \neq 5$

Similarly, $\begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 4 & -3 & 5 \end{bmatrix}$ because of different order.

2.1.2 Know row matrix, column matrix, square matrix, rectangular matrix, zero/null matrix, diagonal matrix, scalar matrix, identity matrix

(i) Row Matrix

A matrix having only one row is called a row matrix. i.e. matrix of order $\mathbf{1} \times \boldsymbol{n}$ is row matrix.

For example,

$A = \begin{bmatrix} a & b \end{bmatrix}$ is a row matrix of order 1×2;

$B = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}$ is a row matrix of order 1×3;

The matrix A has two columns and B has three columns but both have one row.

(ii) Column Matrix

A matrix having only one column is called a column matrix. i.e. matrix of order $m \times 1$ is a column matrix.

For example:

$A = \begin{bmatrix} a \\ b \end{bmatrix}$ is a column matrix of order 2×1;

$B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ is a column matrix of order 3×1;

The matrix A has two rows and B has three rows but both have one column.

(iii) Square Matrix

A matrix in which the number of rows and columns are equal is called a square matrix. i.e., matrix of order $m \times n$ is square matrix if $m = n$.

Example: $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ are square matrices.

The order of matrix B is 2×2 and the order of matrix C is 3×3.

(iv) Rectangular Matrix

A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix. i.e., matrix of order $m \times n$ is rectangular matrix if $m \neq n$.

Example: $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ -3 & 5 \\ 6 & 0 \end{bmatrix}$ are rectangular matrices.

(v) Zero/Null Matrix

A matrix in which every element is zero is called a zero or null matrix.

Symbolically, a null matrix of order $m \times n$ is denoted by $0_{m,n}$.

So, $0_{m,n} = [0]_{(m,n)}$ i.e., $a_{ij} = 0$

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix of order 2 and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null matrix of order 2×3.

(vi) Diagonal Matrix

A square matrix is said to be a diagonal matrix if all the non-diagonal elements of the matrix are zero and at least one diagonal element is non-zero, i.e., $\boldsymbol{a_{ij} = 0}$ where $\boldsymbol{i \neq j}$ and at least one $\boldsymbol{a_{ij} \neq 0}$ where $\boldsymbol{i = j}$.

Example: $\mathbf{D} = \begin{bmatrix} \boldsymbol{d_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{d_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{d_3} \end{bmatrix}$ is a diagonal matrix of order 3.

The entries d_1, d_2, d_3 are of principal or leading or main diagonal of the matrix D and these entries are called diagonal elements.

The matrix D can also be denoted as; $D = diag\,(d_1, d_2, d_3)$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are few examples of diagonal matrix of order 3.

(vii) Scalar Matrix

A diagonal matrix, in which all the diagonal elements are equal, is called a scalar matrix. i.e., $a_{ij} = 0$ where $i \neq j$, and $a_{ij} = k$ where $i = j$ and k is a scalar. For example, $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ is a scalar matrix of order 2.

and $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ is a scalar matrix of order 3.

(viii) Unit or Identity Matrix

A diagonal matrix in which each diagonal element is 1, is called a unit or identity matrix. The unit matrix of order $n \times n$ is denoted by I_n.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: Every scalar matrix is also a diagonal matrix.
Every identity matrix is also a scalar matrix.

2.1.3 Define upper and lower triangular matrix, transpose of a matrix, symmetric matrix and skew-symmetric matrix, Idempotent, Nilpotent, Involutory, Periodic, Hermitian matrix and Skew Hermitian matrix of order up to 4

(i) Upper Triangular Matrix

A square matrix, whose all elements below the main diagonal are zero, is called upper triangular matrix,

i.e., $a_{ij} = 0, \ \forall\, i > j$

For example, $\begin{bmatrix} 5 & 8 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

is an upper triangular matrix of order 3.

(ii) Lower Triangular Matrix

A square matrix, whose all elements above the main diagonal are zero, is called lower triangular matrix,

i.e., $a_{ij} = 0, \ \forall\, i < j.$

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

is a lower triangular matrix of order 3.

Note: If a matrix is upper triangular or lower triangular then it is said to be a triangular matrix.

(iii) Transpose of a Matrix

The matrix obtained from any given matrix A by interchanging its rows and columns is called transpose of A. It is denoted by $\mathbf{A}^t$; read as "**A** transpose".
i.e., $\mathbf{A}^t$ of order $\boldsymbol{n} \times \boldsymbol{m}$ is the transpose of matrix A of order $\boldsymbol{m} \times \boldsymbol{n}$.
Symbolically, if $\mathrm{A} = [a_{ij}]_{(m,n)}$ then $\mathrm{A}^t = [a_{ji}]_{(n,m)}$

For example, If $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \end{bmatrix}$, then $\mathrm{A}^t = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}$.

(iv) Symmetric Matrix and Skew-Symmetric Matrix

A square matrix **A** is called symmetric matrix if $\mathbf{A}^t = \mathbf{A}$.

For example, $\mathrm{A} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is a symmetric matrix because $\mathrm{A}^t = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \mathrm{A}$

$\mathrm{B} = \begin{bmatrix} a & d & c \\ d & b & f \\ c & f & e \end{bmatrix}$ is also a symmetric matrix because $\mathrm{B}^t = \begin{bmatrix} a & d & c \\ d & b & f \\ c & f & e \end{bmatrix} = \mathrm{B}$

A square matrix **A** is called skew-symmetric matrix if $A^t = -A$

For example, If $A = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

then $A^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$

$= -\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} = -A$

i.e., $A^t = -A$

So, A is skew-symmetric matrix.

Note: In skew-symmetric matrix, all the diagonal elements are always zero.

(v) Idempotent matrix

A square matrix **A** is called idempotent if $A^2 = A$.

Example: Show that the following matrix is an idempotent matrix.

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Since $A^2 = A$

Therefore, A is idempotent matrix.

Note: Matrix multiplication will be discussed in detail in section 2.2.1.

(vi) Nilpotent matrix

A nilpotent matrix is a square matrix A such that $A^p = 0$ for some positive integer p. The smallest p is called the index or degree of nilpotent matrix.

Example: Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} (1)(1)+(2)(1)+(3)(-1) & (1)(2)+(2)(2)+(3)(-2) & (1)(3)+(2)(3)+(3)(-3) \\ (1)(1)+(2)(1)+(3)(-1) & (1)(2)+(2)(2)+(3)(-2) & (1)(3)+(2)(3)+(3)(-3) \\ (-1)(1)+(-2)(1)+(-3)(-1) & (-1)(2)+(-2)(2)+(-3)(-2) & (-1)(3)+(-2)(3)+(-3)(-3) \end{bmatrix}$$

$$\Rightarrow \qquad A^2 = \begin{bmatrix} 1+2-3 & 2+4-6 & 3+6-9 \\ 1+2-3 & 2+4-6 & 3+6-9 \\ -1-2+3 & -2-4+6 & -3-6+9 \end{bmatrix}$$

$$\Rightarrow \qquad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

i.e., $A^2 = O$, so A is a nilpotent matrix of index 2.

(vii) Involutory matrix

A square matrix A is said to be involutory matrix if $A^2 = I$.

Example:

Let, $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24 & 40-40 & 0 \\ -15+15 & -24+25 & 0 \\ -5+6-1 & -8+10-2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

i.e., $A^2 = I$, so A is an involutory matrix.

(viii) Periodic matrix

A square matrix A is called a periodic matrix if $A^{k+1} = A$ for some positive integer $k \geq 1$ where k is called the period of A.

Example: Show that $\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is a periodic matrix of period 2.

Let $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+6-12 & -2-4-0 & -6-18+18 \\ -3-6+18 & 6+4+0 & 18+18-27 \\ 2-0-6 & -4+0+0 & -12+0+9 \end{bmatrix} = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}$$

again we multiply by A

So, $A^2 \times A = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

$$\text{or}\quad A^3 = \begin{bmatrix} -5+18-12 & 10-12-0 & 30-54+18 \\ 9-30+18 & -18+20+0 & -54+90-27 \\ -4+12-6 & 8-8-0 & 24-36+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = A$$

i.e., $A^3 = A$, so A is periodic matrix of period 2. Hence shown.

(ix) Hermitian Matrix and Skew Hermitian Matrix

A square matrix over $\mathbb{C}$ is called Hermitian matrix if $(\bar{A})^t = A$

Whereas a square matrix over $\mathbb{C}$ is called Skew Hermitian matrix if $(\bar{A})^t = -A$

Note: $(\bar{A})^t = \overline{A^t}$

Example 1. Show that matrix A is Hermitian matrix.

where $$A = \begin{bmatrix} 1 & i & -2i & 9i \\ -i & 5 & 5i & 2i \\ 2i & -5i & 8 & -i \\ -9i & -2i & i & 4 \end{bmatrix}$$

Solution:

Here $$A = \begin{bmatrix} 1 & i & -2i & 9i \\ -i & 5 & 5i & 2i \\ 2i & -5i & 8 & -i \\ -9i & -2i & i & 4 \end{bmatrix}$$

Now $$\bar{A} = \begin{bmatrix} 1 & -i & 2i & -9i \\ i & 5 & -5i & -2i \\ -2i & 5i & 8 & i \\ 9i & 2i & -i & 4 \end{bmatrix}$$

and $$(\bar{A})^t = \begin{bmatrix} 1 & i & -2i & 9i \\ -i & 5 & 5i & 2i \\ 2i & -5i & 8 & -i \\ -9i & -2i & i & 4 \end{bmatrix} = A$$

$\because$ $(\bar{A})^t = A$

$\therefore$ A is Hermitian matrix. Hence shown.

Note: In Hermitian matrix elements of main diagonal are real numbers, and symmetric elements are conjugate to each other.

Example 2. Show that A is skew Hermitian matrix.

Where $$A = \begin{bmatrix} 0 & i & 2i & 9i \\ i & 0 & 5i & 2i \\ 2i & 5i & 0 & i \\ 9i & 2i & i & 0 \end{bmatrix}$$

Solution:

Here, $$A = \begin{bmatrix} 0 & i & 2i & 9i \\ i & 0 & 5i & 2i \\ 2i & 5i & 0 & i \\ 9i & 2i & i & 0 \end{bmatrix}$$

Now $$\bar{A} = \begin{bmatrix} 0 & -i & -2i & -9i \\ -i & 0 & -5i & -2i \\ -2i & -5i & 0 & -i \\ -9i & -2i & -i & 0 \end{bmatrix}$$

and $$(\bar{A})^t = \begin{bmatrix} 0 & -i & -2i & -9i \\ -i & 0 & -5i & -2i \\ -2i & -5i & 0 & -i \\ -9i & -2i & -i & 0 \end{bmatrix}$$

$\Rightarrow$ $$(\bar{A})^t = -\begin{bmatrix} 0 & i & 2i & 9i \\ i & 0 & 5i & 2i \\ 2i & 5i & 0 & i \\ 9i & 2i & i & 0 \end{bmatrix}$$

i.e., $(\bar{A})^t = -A$

Hence, A is skew Hermitian matrix.

Note: In Skew Hermitian matrix elements of main diagonal are always zero, and symmetric elements are conjugate to each other.

Exercise 2.1

1. Specify the type of each of the following matrices.

(i) $\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}$ (ii) $\begin{bmatrix} \sqrt{3} \\ \frac{4}{7} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{i}{2} & 0 & \frac{i}{3} \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

(vii) $\begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{7} \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & i & 2i \\ -i & 0 & -4i \\ -2i & 4i & 0 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & -i & 5i \\ i & 3 & 7i \\ -5i & -7i & 4 \end{bmatrix}$

2. A newspaper agent of a town records the number of papers sold on each day of one week as follows:

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Daily Dawn	80	90	100	95	85	75	70
Daily Jang	100	110	90	95	105	85	80

Write this information in a matrix form and write its order.

3. Find the values of the unknowns in each of the following.

(i) $\begin{bmatrix} a & -4i \\ 8i & 6i \end{bmatrix} = \begin{bmatrix} 7i & b \\ c & d \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ a & 9 & 0 \\ b & c & -1 \end{bmatrix} = \begin{bmatrix} d & e & g \\ -2 & f & 0 \\ -4 & 7 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y & 0 & z \\ 9 & 2x+y & 6 \\ a & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3x \\ 9 & 4 & 6 \\ \frac{y}{2} & 1 & 3 \end{bmatrix}$

4. Find the transpose of each of the following matrices.

(i) $\begin{bmatrix} -4 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 2i & 5i & -3i \\ 0 & -6i & 2i \end{bmatrix}$ (iii) $\begin{bmatrix} 8 \\ 3 \\ -4 \end{bmatrix}$

(iv) $\begin{bmatrix} -2i & i \\ 5i & 7i \\ 2i & -5i \end{bmatrix}$ (v) $\begin{bmatrix} -8 & 7 \\ 3 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} -7 & -10 & 8 \\ 4 & 5 & 9 \\ -1 & 2 & 3 \end{bmatrix}$

5. Write down in tabular form:

(i) $A = [a_{ij}]_{(2,3)}$ (ii) $X = [x_{ij}]_{(3,4)}$ (iii) $B = [b_{ik}]_{(4,4)}$

6. Which of the following are symmetric or skew-symmetric matrices.

(i) $\begin{bmatrix} 0 & -5 & -6 \\ 5 & 0 & 7 \\ 6 & -7 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 5 & 8 \\ 5 & -1 & 6 \\ 8 & 6 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & -7 \\ 5 & -7 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$

7. Find the index of the following nilpotent matrices.

(i) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

8. Find the period of the following periodic matrix.

$$\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

9. Which of the following is idempotent or involutory matrix.

(i) $\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

10. Which of the following is Hermitian or Skew Hermitian matrix or neither.

(i) $\begin{bmatrix} 3 & 1-2i & 4+7i \\ 1+2i & -4 & -2i \\ 4-7i & 2i & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -i & 3i & i \\ 3i & 7i & -5i \\ i & -5i & -i \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$

11. Find real numbers x, y, z such that matrix A is Hermitian matrix.

$$A = \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$$

2.2 Algebra of Matrices

2.2.1 Carryout scalar multiplication, addition/ subtraction of matrices, multiplication of matrices with real and complex entries (3 by 3)

(i) Scalar Multiplication of a Matrix

Let $A = [a_{ij}]$ is a matrix and k is a scalar then the scalar multiplication of the matrix A denoted by kA is defined as:

$$kA = k[a_{ij}] = [ka_{ij}]; \quad \forall_{i,j}$$

In other words, for a matrix A and a number k (also called a scalar), the matrix kA is obtained by multiplying each element of A by k.

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$.

(ii) Addition of Matrices

If A and B are two matrices of the same order (dimension) $m \times n$ then their sum $A + B$ is the matrix of the same order obtained by adding each element of A with the corresponding element of B.

Thus, if $A = [a_{ij}]_{(m,n)}$ and $B = [b_{ij}]_{(m,n)}$, then $A + B = [a_{ij} + b_{ij}]_{(m,n)}$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix}$. Find A + B.

Solution:

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+7 & 3+8 \\ 4+3 & 5+2 & 6+1 \\ 7+9 & 8+5 & 9+6 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 11 \\ 7 & 7 & 7 \\ 16 & 13 & 15 \end{bmatrix}$$

(iii) Subtraction of Matrices

If A and B are two matrices of the same order (dimension) $m \times n$ then their difference A – B is the matrix of the same order obtained by subtracting the elements of B from the corresponding elements of A.

Thus, if $A = [a_{ij}]_{(m,n)}$ and $B = [b_{ij}]_{(m,n)}$, then $A - B = [a_{ij} - b_{ij}]_{(m,n)}$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix}$; Find A – B

Solution:

$$A - B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1-4 & 2-7 & 3-8 \\ 4-3 & 5-2 & 6-1 \\ 7-9 & 8-5 & 9-6 \end{bmatrix} = \begin{bmatrix} -3 & -5 & -5 \\ 1 & 3 & 5 \\ -2 & 3 & 3 \end{bmatrix}$$

(iv) Multiplication of Matrices

Let $A = [a_{ij}]$ be a matrix of order $m \times p$ and $B = [b_{ij}]$ be a matrix of order $p \times n$. Then their product A·B or AB is the matrix $C = [c_{ij}]$ of the order $m \times n$ where $$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ip}b_{pj}$$

The following points may be followed in matrix multiplication.

(i) The product AB is defined only if the number of columns of matrix A is equal to the number of rows of matrix B.

(ii) The elements in the (i,j)th place of AB is the sum of the products of the corresponding elements of i^{th} row of A and j^{th} column of B.

Example 1.

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 2 \\ 8 & 6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 10 & 1 \\ 5 & 8 & 3 \\ 2 & 7 & 2 \end{bmatrix}$, compute AB and BA.

Solution:

$$AB = \begin{bmatrix} (2\times6)+(3\times5)+(1\times2) & (2\times10)+(3\times8)+(1\times7) & (2\times1)+(3\times3)+(1\times2) \\ (5\times6)+(7\times5)+(2\times2) & (5\times10)+(7\times8)+(2\times7) & (5\times1)+(7\times3)+(2\times2) \\ (8\times6)+(6\times5)+(4\times2) & (8\times10)+(6\times8)+(4\times7) & (8\times1)+(6\times3)+(4\times2) \end{bmatrix} = \begin{bmatrix} 29 & 51 & 13 \\ 69 & 120 & 30 \\ 86 & 156 & 34 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} (6\times2)+(10\times5)+(1\times8) & (6\times3)+(10\times7)+(1\times6) & (6\times1)+(10\times2)+(1\times4) \\ (5\times2)+(8\times5)+(3\times8) & (5\times3)+(8\times7)+(3\times6) & (5\times1)+(8\times2)+(3\times4) \\ (2\times2)+(7\times5)+(2\times8) & (2\times3)+(7\times7)+(2\times6) & (2\times1)+(7\times2)+(2\times4) \end{bmatrix} = \begin{bmatrix} 70 & 94 & 30 \\ 74 & 89 & 33 \\ 55 & 67 & 24 \end{bmatrix}$$

Example 2. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I_3 = O_3$.

Solution:

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_3$$

i.e., $A^2 - 4A - 5I_3 = O_3$. Hence, shown.

2.2.2 Show that commutative property:

(i) holds under addition i.e., $A + B = B + A$
(ii) does not hold under multiplication, in general

(i) Commutative property holds under addition i.e., $A + B = B + A$

If the matrices A and B are conformable for addition then commutative property under addition holds i.e.,

$$A + B = B + A$$

Example: If $A = \begin{bmatrix} i & 5i \\ 9i & -7i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3i \\ 8i & i \end{bmatrix}$, verify the commutative property under addition.

Solution:

$$A + B = \begin{bmatrix} i & 5i \\ 9i & -7i \end{bmatrix} + \begin{bmatrix} 0 & 3i \\ 8i & i \end{bmatrix} = \begin{bmatrix} i+0 & 5i+3i \\ 9i+8i & -7i+i \end{bmatrix} = \begin{bmatrix} i & 8i \\ 17i & -6i \end{bmatrix} \quad \ldots(i)$$

and $$B + A = \begin{bmatrix} 0 & 3i \\ 8i & i \end{bmatrix} + \begin{bmatrix} i & 5i \\ 9i & -7i \end{bmatrix} = \begin{bmatrix} 0+i & 3i+5i \\ 8i+9i & i-7i \end{bmatrix} = \begin{bmatrix} i & 8i \\ 17i & -6i \end{bmatrix} \quad \ldots(ii)$$

From (i) and (ii), we get $A + B = B + A$, Hence verified.

(ii) Commutative property does not hold under multiplication, in general

If the matrices A and B are conformable for multiplication then commutative property under multiplication does not hold in general

i.e., $AB \neq BA$

Example: If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, show that $AB \neq BA$

Solution:

$$AB = \begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}\begin{bmatrix}3 & 4 & 1\\1 & 2 & 1\\2 & 1 & 1\end{bmatrix} = \begin{bmatrix}3+4+14 & 4+8+7 & 1+4+7\\6+5+16 & 8+10+8 & 2+5+8\\9+6+18 & 12+12+9 & 3+6+9\end{bmatrix} = \begin{bmatrix}21 & 19 & 12\\27 & 26 & 15\\33 & 33 & 18\end{bmatrix} \ldots(i)$$

$$\text{Now } BA = \begin{bmatrix}3 & 4 & 1\\1 & 2 & 1\\2 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}$$

$$= \begin{bmatrix}3+8+3 & 12+20+6 & 21+32+9\\1+4+3 & 4+10+6 & 7+16+9\\2+2+3 & 8+5+6 & 14+8+9\end{bmatrix} = \begin{bmatrix}14 & 38 & 62\\8 & 20 & 32\\7 & 19 & 31\end{bmatrix} \quad \ldots(ii)$$

From (i) and (ii),

we get $AB \neq BA$. Hence shown.

2.2.3 Verify that $(AB)^t = B^tA^t$ (3 by 3)

If the matrices are conformable for multiplication then, we can verify: $(AB)^t = B^tA^t$

$$\text{Let} \quad A = \begin{bmatrix}2 & -3 & 4\\5 & 7 & 9\\-2 & 3 & -4\end{bmatrix} \text{ and } B = \begin{bmatrix}3 & 4 & 5\\-5 & -7 & -9\\2 & 3 & 6\end{bmatrix}$$

$$AB = \begin{bmatrix}2 & -3 & 4\\5 & 7 & 9\\-2 & 3 & -4\end{bmatrix}\begin{bmatrix}3 & 4 & 5\\-5 & -7 & -9\\2 & 3 & 6\end{bmatrix}$$

$$= \begin{bmatrix}6+15+8 & 8+21+12 & 10+27+24\\15-35+18 & 20-49+27 & 25-63+54\\-6-15-8 & -8-21-12 & -10-27-24\end{bmatrix} = \begin{bmatrix}29 & 41 & 61\\-2 & -2 & 16\\-29 & -41 & -61\end{bmatrix}$$

$$\text{Thus, } (AB)^t = \begin{bmatrix}29 & -2 & -29\\41 & -2 & -41\\61 & 16 & -61\end{bmatrix} \quad \ldots(i)$$

$$\text{Now,} \quad A^t = \begin{bmatrix}2 & 5 & -2\\-3 & 7 & 3\\4 & 9 & -4\end{bmatrix} \text{ and } B^t = \begin{bmatrix}3 & -5 & 2\\4 & -7 & 3\\5 & -9 & 6\end{bmatrix}$$

$$B^t.A^t = \begin{bmatrix}3 & -5 & 2\\4 & -7 & 3\\5 & -9 & 6\end{bmatrix}\begin{bmatrix}2 & 5 & -2\\-3 & 7 & 3\\4 & 9 & -4\end{bmatrix} = \begin{bmatrix}6+15+8 & 15-35+18 & -6-15-8\\8+21+12 & 20-49+27 & -8-21-12\\10+27+24 & 25-63+54 & -10-27-24\end{bmatrix}$$

$$= \begin{bmatrix}29 & -2 & -29\\41 & -2 & -41\\61 & 16 & -61\end{bmatrix} \quad \ldots(ii)$$

From (i) and (ii), we get $(AB)^t = B^t.A^t$. Hence verified.

Properties of matrix operations

(i) Properties of Matrix Addition

Following properties are satisfied by the matrices, A, B and C of the same order w.r.t matrix addition.

(i) A + B is also a matrix of the same order.

(ii) A + B = B + A

(iii) (A + B) + C = A + (B + C)

(iv) For any matrix A, there exist a matrix of the same order, that is null matrix 0, such that A + 0 = 0 + A = A

(v) For any matrix A, there exists a matrix B of the same order, such that A + B = B + A = 0

where O is the null matrix of same order. The matrix B is called the additive inverse of A and is denoted by – A.

(ii) Properties of Scalar Multiplication

Following properties are satisfied by the matrices A and B of the same order and two scalars with respect to scalar multiplication.

(i) k_1A is also a matrix of same order.

(ii) $(k_1k_2)A = k_1(k_2A)$

(iii) $(k_1 + k_2)A = k_1A + k_2A$ and $k_1(A + B) = k_1A + k_1B$

(iv) $1A = A$ and $-1A = -A$

(v) $oA = 0 = Ao$ and $k_10 = 0k_1 = 0$

(iii) Properties of Matrix Multiplication

If the matrices A, B and C are conformable for addition and multiplication, then

(i) $(AB)C = A(BC)$.

(ii) $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$.

(iii) $AI = IA = A$ where A and I are of the same order.

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

(v) Let A be a square matrix of order n, there exist a matrix B of the same order n, such that $AB = BA = I_n$, then B is called an inverse of A and is written as $B = A^{-1}$.

(iv) Properties of Transposed Matrices

If two matrices A and B are conformable for addition and multiplication, then

(i) $(A \pm B)^t = A^t \pm B^t$

(ii) $(kA)^t = kA^t$ where k is scalar

(iii) $(A^t)^t = A$

(iv) $(AB)^t = B^tA^t$

Exercise 2.2

1. If $A = \begin{bmatrix} 1+i & 2 & 3i \\ 4i & 5-i & 2+3i \\ 0 & 5 & 1-i \end{bmatrix}$ and $B = \begin{bmatrix} 2+i & 3-i & 4 \\ i & 0 & 5-i \\ 6+i & 2 & 2+3i \end{bmatrix}$ then find:

(i) $A+B$ (ii) $A-B$ (iii) $2A-B$ (iv) $3A+B+\text{I}$

where I is unit matrix of order 3.

2. Let $A = \begin{bmatrix} 3 & -4 & 1 \\ 4 & 5 & 7 \\ -2 & -3 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & -1 \\ 2 & -3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & 4 \\ 6 & -3 \\ -2 & 7 \end{bmatrix}$

wherever possible, compute the following:

(i) AC (ii) BC (iii) AB (iv) BA (v) A^2

(vi) CB (vii) $(AB)C$ (viii) C^tB^t (ix) C^tA^t (x) $(CA)\,B$

3. Let $\text{X} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, \text{Y} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & -1 \\ 3 & -5 & -1 \end{bmatrix}$ and $\text{Z} = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

then show that XY = XZ.

4. If $\text{A} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 4 & 3 \\ 0 & 8 & 5 \end{bmatrix}$ then find: $\text{A}^2 - 5\text{A} + 4\text{I}$.

5. Prove the identity: $\left\{\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

6. If $\text{A} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -2 & 5 \\ -1 & 0 & -7 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 9 & 4 & 6 \\ 7 & -8 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ then verify:

(i) commutative property under addition

(ii) $(\text{AB})^t = \text{B}^t\text{A}^t$

2.3 Determinants

2.3.1 Describe determinant of a square matrix, minor and cofactor of an element of a matrix

(i) Determinant of a square matrix

Determinant is the number which is associated with any square matrix. Determinant of a square matrix A is denoted by det(A), det A, or $|\text{A}|$.

For 2×2 matrix $\text{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the number $a_{11}a_{22} - a_{12}a_{21}$ is its determinant. Similarly,

for 3×3 matrix $\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ (Expansion by first row)}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Example: Let $A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$, Find its determinant.

Solution: $|A| = \begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 5 & 7 \\ 1 & 6 \end{vmatrix} - 9\begin{vmatrix} 3 & 7 \\ 8 & 6 \end{vmatrix} + 2\begin{vmatrix} 3 & 5 \\ 8 & 1 \end{vmatrix}$

$= 4\,(30 - 7) - 9(18 - 56) + 2(3 - 40) = 360.$

(ii) Minors and Cofactors of an element of a Matrix

(a) Minor

The minor of an element a_{ij} of a matrix A is the determinant of a square sub-matrix, obtained by deleting ith row and jth column. It is denoted by M_{ij}.

(b) Cofactor

The co-factor of an element a_{ij} of a matrix A, denoted as A_{ij}, is defined as $A_{ij} = (-1)^{i+j}M_{ij}$

Example 1. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

Find M_{12}, A_{12}, M_{31} and A_{31}.

Solution:

M_{12}= Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

A_{12} = Co-factor of $a_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{31} = Minor of $a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

A_{31} = Co-factor of $a_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$.

Example 2. Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & -4 \\ 0 & 2 & 3 \end{bmatrix}$. Compute M_{23} and A_{32}.

Solution:

$M_{23} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = (2) \times (1) - (4) \times (0) = 2.$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = (-1)\{(-4) + (4)\} = 0$

2.3.2 Evaluate determinant of square matrix using cofactors

Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & -4 \\ 0 & 2 & 3 \end{bmatrix}$, Using cofactors for each element of 1st row.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = [(3)(3) - (-4)(2)] = 17$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix} = -[(2)(3) - (-4)(0)] = -6$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = [(2)(2) - (3)(0)] = 4$$

A_{11}, A_{12}, and A_{13} are three cofactors.

Now, $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1(17) + (4)(-6) + (-2)(4) = -15$

Similarly, we can also calculate det A by using other rows,

$$\det A = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = -15 \text{ (By } R_2\text{)}$$

$$\det A = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = -15 \text{ (By } R_3\text{)}$$

we can also calculate det A by using columns.

Using cofactors for each element of 1st column

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = [(3)(3) - (-4)(2)] = 17$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} = -[(4)(3) - (-2)(2)] = -16$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & -2 \\ 3 & -4 \end{vmatrix} = [(4)(-4) - (-2)(3)] = -10$$

A_{11}, A_{21}, and A_{31} are three cofactors.

Now, $\det A = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = 1(17) + (2)(-16) + (0)(-10) = -15$

Similarly, we can also calculate det A by using other columns,

$$\det A = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = -15 \text{ (By } C_2\text{)}$$

$$\det A = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = -15 \text{ (By } C_3\text{)}$$

In general, det A can be calculated by using any row or column. The evaluation of a determinant with the help of cofactors is known as Laplacian expansion.

2.3.3 Define singular and non-singular matrices

(a) Singular Matrix:

A square matrix A is said to be a singular or non-invertible matrix if its determinant is zero.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$.

$\because |A| = 0$

$\therefore$ A is a singular matrix.

(b) Non-Singular Matrix:

A square matrix A is said to be a non-singular or invertible matrix if its determinant is not equal to zero.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$.

$\because |A| \neq 0$

$\therefore$ A is a non-singular matrix.

2.3.4 Describe the Adjoint of a square matrix and a diagonal matrix

(a) Adjoint of square matrix

The adjoint of a square matrix A is the transpose of the matrix formed by all the cofactors of the corresponding elements of A and is denoted by *adj* A,

Symbolically, $adj\ A = [A_{ij}]^t = [A_{ji}]$

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The matrix of the co-factors of the above matrix is

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \text{ and adj } A = [A_{ij}]^t = [A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Example: Find adjoint of matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 9 & 8 & 6 \end{bmatrix}$

Solution: We know that adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$, so we find all the cofactors.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 7 \\ 8 & 6 \end{vmatrix} = (-1)^{1+1}(-32) = -32$; $A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ 9 & 6 \end{vmatrix} = (-1)^{1+2}(-51) = 51$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 9 & 8 \end{vmatrix} = (-1)^{1+3}(-20) = -20$; $A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 5 \\ 8 & 6 \end{vmatrix} = (-1)^{2+1}(-22) = 22$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ 9 & 6 \end{vmatrix} = (-1)^{2+2}(-39) = -39$; $A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 9 & 8 \end{vmatrix} = (-1)^{2+3}(-19) = 19$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} = (-1)^{3+1}(1) = 1$; $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} = (-1)^{3+2}(-3) = 3$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (-1)^{3+3}(-2) = -2$

$$\text{Hence, adj } A = \begin{bmatrix} -32 & 51 & -20 \\ 22 & -39 & 19 \\ 1 & 3 & -2 \end{bmatrix}^t = \begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix}$$

(b) Adjoint of diagonal matrix

Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

The matrix of cofactors of matrix A is: $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$

where $A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & 0 \\ 0 & a_{33} \end{vmatrix} = a_{22}a_{33}$

$A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & 0 \\ 0 & a_{33} \end{vmatrix} = a_{11}a_{33}$

$A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}$

Hence, $\text{adj } A = \begin{bmatrix} a_{22}a_{33} & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}^t = \begin{bmatrix} a_{22}a_{33} & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$

Example: Find adjoint of matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The matrix of cofactors of matrix A is: $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$

where $A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 0 \\ 0 & 1 \end{vmatrix} = 5 \times 1 = 5$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \times 1 = 2$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 2 \times 5 = 10$

Hence, $\text{adj } A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}^t = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

2.3.5 Use adjoint method to calculate inverse of a square matrix and verify

The inverse of a square matrix A, denoted by A^{-1}, is another matrix such that the product of A and A^{-1} is the identity matrix.

i.e., $AA^{-1} = A^{-1}A = I$ (where I is identity matrix of the same order)

A^{-1} exists only if A is non singular matrix.

Note: (i) If $B = A^{-1}$, then $B^{-1} = A$

(ii) $(A^{-1})^{-1} = A$, i.e., inverse of the inverse of a matrix A is A itself.

Adjoint Method for computing A^{-1}

The inverse of a matrix A by adjoint method is defined as

$$A^{-1} = \frac{\text{adj } A}{|A|}; \text{ where } |A| \neq 0$$

Example 1. Find the inverse of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by adjoint method.

Solution: Here $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$

Thus, A is non-singular, so its inverse exists.

We know that for 2×2 matrix A, $\text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^t = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

So, $\text{adj } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

One can verify that $A\,A^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

Example 2. Find the inverse of $A = \begin{bmatrix} 9 & 8 & 6 \\ 2 & 4 & 7 \\ 1 & 3 & 5 \end{bmatrix}$ by adjoint method.

Solution: We know that, $A^{-1} = \frac{adj\ A}{|A|}$, where $|A| \neq 0$.

Here, $|A| = \begin{vmatrix} 9 & 8 & 6 \\ 2 & 4 & 7 \\ 1 & 3 & 5 \end{vmatrix} = 9(20 - 21) - 8(10 - 7) + 6(6 - 4) = -21 \neq 0.$

Cofactors of A:

$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 7 \\ 3 & 5 \end{vmatrix} = (-1)^{1+1}(-1) = -1;$ $A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ 1 & 5 \end{vmatrix} = (-1)^{1+2}(3) = -3$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = (-1)^{1+3}(2) = 2;$ $A_{21} = (-1)^{2+1}\begin{vmatrix} 8 & 6 \\ 3 & 5 \end{vmatrix} = (-1)^{2+1}(22) = -22$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 9 & 6 \\ 1 & 5 \end{vmatrix} = (-1)^{2+2}(39) = 39;$ $A_{23} = (-1)^{2+3}\begin{vmatrix} 9 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^{2+3}(19) = -19$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 8 & 6 \\ 4 & 7 \end{vmatrix} = (-1)^{3+1}(32) = 32;$ $A_{32} = (-1)^{3+2}\begin{vmatrix} 9 & 6 \\ 2 & 7 \end{vmatrix} = (-1)^{3+2}(51) = -51$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 9 & 8 \\ 2 & 4 \end{vmatrix} = (-1)^{3+3}(20) = 20$

So, $\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & -22 & 32 \\ -3 & 39 & -51 \\ 2 & -19 & 20 \end{bmatrix}$

Thus $A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-21}\begin{bmatrix} -1 & -22 & 32 \\ -3 & 39 & -51 \\ 2 & -19 & 20 \end{bmatrix}$

2.3.6 Verify the result $(AB)^{-1} = B^{-1} A^{-1}$

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, then show that $(AB)^{-1} = B^{-1} A^{-1}$.

Solution: L.H.S = $(AB)^{-1}$

First we find the product of A and B

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Now, $|AB| = \begin{vmatrix} 19 & 22 \\ 43 & 50 \end{vmatrix} = (19)(50) - (22)(43) = 4$

We know that $(AB)^{-1} = \frac{adj(AB)}{|AB|}$

So, $$(AB)^{-1} = \frac{1}{4}\begin{bmatrix} 50 & -22 \\ -43 & 19 \end{bmatrix} = \begin{bmatrix} \frac{50}{4} & \frac{-22}{4} \\ \frac{-43}{4} & \frac{19}{4} \end{bmatrix} = \begin{bmatrix} \frac{25}{2} & \frac{-11}{2} \\ \frac{-43}{4} & \frac{19}{4} \end{bmatrix}$$

R.H.S = $B^{-1} A^{-1}$

First we find A^{-1} and B^{-1}.

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3)$

$= 4 - 6 = -2 \neq 0$

$adj\ A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Now, $A^{-1} = \frac{adj\ A}{|A|} = \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

$|B| = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = (5)(8) - (6)(7)$

$= 40 - 42 = -2 \neq 0$

$adj\ B = \begin{bmatrix} 8 & -6 \\ -7 & 5 \end{bmatrix}$

Now, $B^{-1} = \frac{adj\ B}{|B|} = \frac{-1}{2}\begin{bmatrix} 8 & -6 \\ -7 & 5 \end{bmatrix}$

$= \begin{bmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{bmatrix}$

So, $$B^{-1} A^{-1} = \begin{bmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 8+\frac{9}{2} & -4-\frac{3}{2} \\ -7-\frac{15}{4} & \frac{7}{2}+\frac{5}{4} \end{bmatrix} = \begin{bmatrix} \frac{25}{2} & \frac{-11}{2} \\ \frac{-43}{4} & \frac{19}{4} \end{bmatrix}$$

∵ L.H.S = R.H.S

∴ $(AB)^{-1} = B^{-1} A^{-1}$.

Hence verified.

Exercise 2.3

1. Evaluate the following determinants:

(i) $\begin{vmatrix} 5 & 4 & 3 \\ 3 & -4 & 0 \\ 2 & 3 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} -2 & 4 & 3 \\ 5 & 4 & -2 \\ 2 & 7 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 2c & c & c \\ a & a & 2a \\ b & 2b & b \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & 0 & 1-i \\ 0 & 1 & i \\ 1+i & i & 1 \end{vmatrix}$ (v) $\begin{vmatrix} 7 & -2 & 1 \\ 2 & 2 & 4 \\ 4 & 3 & 7 \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}$

2. Identify the singular or non-singular matrix.

(i) $\begin{bmatrix} 4 & 0 & 1 \\ 7 & 5 & 5 \\ -12 & -6 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 2 & 0 \\ 3 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 13 & -5 & 4 \\ 8 & 1 & 3 \\ 7 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 6 & 3 \\ -2 & 1 & 0 \\ 6 & 4 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 20 & 10 & 30 \\ 2 & 1 & 3 \\ 0 & 20 & 1 \end{bmatrix}$

3. Find the value of x for which the following matrices are singular.

(i) $\begin{bmatrix} 10 & 5 \\ 6 & x \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 4 \\ x & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & 3 & 7 \\ 3 & -4 & 2 \\ 5 & x & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} x & -2 & 1 \\ 2 & -3 & 4 \\ x & -2 & -1 \end{bmatrix}$

4. Find the adjoint of the following matrices.

(i) $\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -0.3 & 0.5 \\ 1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 6 & 8 \\ 1 & 3 & 2 \\ 2 & 7 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 1-i \\ 1+i & i & 1 \\ 0 & 1 & -i \end{bmatrix}$ (v) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

5. If $A = \begin{bmatrix} 11 & 10 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ then verify:

(i) $(A^{-1})^{-1} = A$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$ (iii) $\text{adj}\,(AB) = (\text{adj}\,B)(\text{adj}\,A)$

6. Verify the following:

(i) $\left[\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}$

(ii) $\left[\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1}$

7. Use adjoint method to calculate the inverse of the following square matrices, if possible.

(i) $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 3 \\ 9 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 0 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ -5 & 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

2.4 Properties of Determinants

2.4.1 State and verify the properties of determinants

The properties given in this section are very useful in evaluating the determinants. The properties of determinants of order three are also valid for determinants of any order. All the properties which hold for rows are also valid for columns.

Property 1. The values of the determinants of any matrix A and its transpose are always same.

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$. Verify that $|A| = |A^t|$.

Solution: $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$. Expanding $|A|$ by R_1, we get

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = (2-6) - 0 + 3(6-4) = 2.$$

Now, $A^t = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ then $|A^t| = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 3 & 2 & 2 \end{vmatrix}$. Expanding $|A|$ by R_1, we get

$$|A^t| = 1\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = (2-6) - 2(0-9) + 4(0-3) = 2.$$

So, $|A| = |A^t|$. Hence verified.

Property 2. The interchange of any two rows of a matrix A changes the sign of its determinant without altering its numerical value.

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$. Verify that $|B| = -|A|$, where B is a matrix obtained by interchanging any two rows of A.

Solution: Interchanging any two rows, say second and third, we get:

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix},$$

Now $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2$ ...(i)

and $|B| = \begin{vmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = -2$ ...(ii)

From (i) and (ii), we get $|B| = -|A|$.

Hence verified.

Property 3. If two rows of a matrix are identical, then its determinant is zero.

Example 1. Show that $|A| = 0$, where $A = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 3 & 2 \\ 1 & 3 & 0 \end{bmatrix}$.

Solution: $|A| = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 3 & 2 \\ 1 & 3 & 0 \end{bmatrix}$.

Expanding $|A|$ by R_1, we get

$$|A| = 1\begin{vmatrix}3 & 2\\3 & 0\end{vmatrix} - 3\begin{vmatrix}4 & 2\\1 & 0\end{vmatrix} + 0\begin{vmatrix}4 & 3\\1 & 3\end{vmatrix}$$
$$= 1(0-6) - 3(0-2) + 0 = 0$$

So, $|A| = 0$. Hence shown.

Alternatively, $\because$ Two rows are identical

$\therefore |A| = 0.$

Example 2. Show that $|A| = 0$, where $A = \begin{bmatrix}1 & 3 & 1\\2 & 2 & 2\\4 & 0 & 4\end{bmatrix}$.

Solution: Here $|A| = \begin{bmatrix}1 & 3 & 1\\2 & 2 & 2\\4 & 0 & 4\end{bmatrix}$.

Expanding $|A|$ by R_1 , we get

$$|A| = 1\begin{vmatrix}2 & 2\\0 & 4\end{vmatrix} - 3\begin{vmatrix}2 & 2\\4 & 4\end{vmatrix} + 1\begin{vmatrix}2 & 2\\4 & 0\end{vmatrix}$$
$$= 1(8-0) - 3(8-8) + 1(0-8) = 0$$

So, $|A| = 0$

Alternatively, $\because$ Two columns are identical

$\therefore |A| = 0$

Property 4. If all the elements of a row of a square matrix are zero, then its determinant is zero.

Example: Show that $|A| = 0$, where $A = \begin{bmatrix}2 & -3 & 5\\0 & 0 & 0\\9 & -4 & 2\end{bmatrix}$.

$$|A| = 2\begin{vmatrix}0 & 0\\-4 & 2\end{vmatrix} + 3\begin{vmatrix}0 & 0\\9 & 2\end{vmatrix} + 5\begin{vmatrix}0 & 0\\9 & -4\end{vmatrix}$$
$$= 2\,(0) + 3(0) + 5(0) = 0$$

Alternatively, $\because$ each element of R_2 is zero.

$\therefore |A| = 0.$

Property 5. If every element in a row of matrix A is multiplied by the same number k, then $|A|$ gets multiplied by k.

Example: Show that $5\begin{vmatrix}1 & 2 & 3\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix} = \begin{vmatrix}5 & 10 & 15\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix}$.

Solution: R.H.S $= \begin{vmatrix}5 & 10 & 15\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix}$

$$= 5\begin{vmatrix}3 & 7\\0 & 1\end{vmatrix} - 10\begin{vmatrix}2 & 7\\4 & 1\end{vmatrix} + 15\begin{vmatrix}2 & 3\\4 & 0\end{vmatrix} = 5\left(1\begin{vmatrix}3 & 7\\0 & 1\end{vmatrix} - 2\begin{vmatrix}2 & 7\\4 & 1\end{vmatrix} + 3\begin{vmatrix}2 & 3\\4 & 0\end{vmatrix}\right)$$

$$= 5\begin{vmatrix}1 & 2 & 3\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix} = \text{L.H.S}$$

$\because$ L.H.S = R.H.S

$\therefore \quad 5\begin{vmatrix}1 & 2 & 3\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix} = \begin{vmatrix}5 & 10 & 15\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix}$. Hence shown.

Property 6. If every element a row of a matrix A be expressed as the sum of two terms then |A| can be expressed as the sum of determinants of two matrices differing in the elements of that row but with remaining rows as the same as those of |A|.

Example: If

$$A = \begin{bmatrix}16 & 3 & 0\\20 & 5 & 1\\17 & 7 & 2\end{bmatrix}, B = \begin{bmatrix}16 & 3 & 0\\16 & 5 & 1\\16 & 7 & 2\end{bmatrix}, C = \begin{bmatrix}0 & 3 & 0\\4 & 5 & 1\\1 & 7 & 2\end{bmatrix},$$

then show that $\quad |A| = |B| + |C|$

Solution: L.H.S

$$|A| = \begin{vmatrix}16 & 3 & 0\\20 & 5 & 1\\17 & 7 & 2\end{vmatrix} = \begin{vmatrix}16+0 & 3 & 0\\16+4 & 5 & 1\\16+1 & 7 & 2\end{vmatrix} = \begin{vmatrix}16 & 3 & 0\\16 & 5 & 1\\16 & 7 & 2\end{vmatrix} + \begin{vmatrix}0 & 3 & 0\\4 & 5 & 1\\1 & 7 & 2\end{vmatrix} = |B| + |C| = \text{R.H.S}$$

Property 7. If the elements of one row of a matrix A are k times the corresponding elements of its another row, then $|A| = 0$.

Let $\quad A = \begin{bmatrix}2 & 5 & -3\\k(2) & k(5) & k(-3)\\7 & -2 & 11\end{bmatrix}$ where $R_2 = kR_1$

Then $\quad |A| = \begin{vmatrix}2 & 5 & -3\\k(2) & k(5) & k(-3)\\7 & -2 & 11\end{vmatrix} = k\begin{vmatrix}2 & 5 & -3\\2 & 5 & -3\\7 & -2 & 11\end{vmatrix}$ by property 5

$= k(0)$ by property 3 i.e., $R_1 = R_2$

$= 0$

Corollary:

If $\quad A = \begin{bmatrix}k_1a_{21} + k_2a_{31} & k_1a_{22} + k_2a_{32} & k_1a_{23} + k_2a_{33}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$, then $|A| = 0$.

Hint: $|A| = |B| + |C|$,

where $B = \begin{bmatrix}k_1a_{21} & k_1a_{22} & k_1a_{23}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$ and $C = \begin{bmatrix}k_2a_{31} & k_2a_{32} & k_2a_{33}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$

Property 8. If to each element of a row of a matrix A is added, a constant multiple of the corresponding element of another row, then the value of |A| is unaltered.

Let $\quad A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$

and $\quad B = \begin{bmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

where (R_1 of B) = (R_1 of A) + k(R_2 of A).

Then $|B| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{21} & ka_{22} & ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, By using property 6

$= |A| + k\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, By property 5

$= |A| + k(0) = |A|$ By using the property 3 ($R_1 = R_2$)

Similarity, it can be shown that

$$\begin{vmatrix} a_{11}+k_1a_{21}+k_2a_{31} & a_{12}+k_1a_{22}+k_2a_{32} & a_{13}+k_1a_{23}+k_2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$$

2.4.2 Evaluate the determinant without expansion (i.e., using properties of determinants)

Example 1. Without expanding, show that $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$ vanishes.

Solution: Let $\Delta = \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$.

Adding C_2 to C_3, we get:

$\Delta = \begin{vmatrix} 1 & x & x+y+z \\ 1 & y & x+y+z \\ 1 & z & x+y+z \end{vmatrix}$;

$\Delta = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$ [Taking $(x+y+z)$ common from C_3],

$= (x+y+z) \times 0 = 0$; [By using property 3]

Example 2. Without expanding, show that $|A| = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$.

Solution:

$|A| = \begin{vmatrix} 1 & \omega & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1+\omega+\omega^2 \\ \omega^2 & 1 & 1+\omega+\omega^2 \end{vmatrix}$; [By adding C_1 and C_2 to C_3]

$$= \begin{vmatrix} 1 & \omega & 0 \\ \omega & \omega^2 & 0 \\ \omega^2 & 1 & 0 \end{vmatrix}; \qquad [\because 1 + \omega + \omega^2 = 0]$$

$= 0$ [By using property 4]

Example 3. Without expanding, show that $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ vanishes.

Solution: Let $\Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$\Delta = \Delta_1 - \Delta_2$ (say) ... (i) [By using property 6]

Now $\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$; [Multiplying R_1, R_2, R_3, by a, b, c respectively]

$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$; [Taking *abc* common from C_3]

$= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$ [By using property 2]

$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \Delta_1$ [By using property 2]

From (i), $\Delta = \Delta_1 - \Delta_2 = \Delta_1 - \Delta_1 = 0$ Hence shown. $[\because \Delta_2 = \Delta_1]$

Exercise 2.4

1. Let $A = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 4 & 6 \\ 1 & -5 & 1 \end{bmatrix}$, verify that $|A| = |A^t|$

2. Without expanding, prove, each of the following:

(i) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

(iii) $\begin{vmatrix} k & b & a & c+d \\ k & b & c & a+d \\ k & d & c & b+a \\ k & a & d & b+c \end{vmatrix} = 0$

3. Without expanding determinants, prove that

(i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x+1 & x+3 & x+5 \\ x+4 & x+6 & x+8 \\ x+7 & x+9 & x+11 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} x+1 & x+3 & x+5 \\ x+4 & x+6 & x+8 \\ x+7 & x+9 & x+11 \end{vmatrix} = 0$

(iv) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(v) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$

(vi) $\begin{vmatrix} \alpha & \beta\gamma & \alpha\beta\gamma \\ \beta & \gamma\alpha & \alpha\beta\gamma \\ \gamma & \alpha\beta & \alpha\beta\gamma \end{vmatrix} = \begin{vmatrix} \alpha & \alpha^2 & \alpha^3 \\ \beta & \beta^2 & \beta^3 \\ \gamma & \gamma^2 & \gamma^3 \end{vmatrix}$

4. Without expanding the determinants, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

5. If a, b, c are different and $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$,

then show that $1 + abc = 0$.

2.5 Row and Column Operations

Row and column operations are very useful in many applications in matrix theory, specially solving the homogenous and non-homogenous systems of linear equations.

2.5.1 Describe the elementary row and column operations on matrices

(a) Row operations on matrices:

If A is $m \times n$ matrix, then $m \times n$ matrix B obtained from A by performing elementary row operations on A is called row equivalent to A. Symbolically, we write B~A and read as "B is row equivalent to A."

Similarly, we can define column equivalent matrices, that is replacing the word "row" by "column" in the above definition. We also write B~A to denote B is column equivalent to A.

There are three elementary row operations:

(i) Interchange of any two rows. This is usually denoted by R_{ij} which means interchanging of R_i with R_j.

For example, $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix}$ by R_{12}

(ii) Multiplication of a row by a non-zero scalar. This is usually denoted by kR_i which means R_i multiplied by k.

For example, $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 8 & -6 \\ 0 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ by $2R_1$

(iii) Addition of any multiple of one row to another row of the matrix. This is usually denoted by $R_i + kR_j$ which means kR_j is added to R_i.

For example, $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 4 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ by $R_2 + 2R_1$

(b) Column operations on matrices

Three elementary column operations with notation are given as below:

(i) Interchanging any two columns, i.e., C_{ij} .

(ii) Multiplication of a column by any non-zero scalar k i.e., kC_i.

(iii) Addition of any multiple of one column to another column i.e., $C_i + kC_j$, where C_i , C_j are any two columns and k is any non-zero scalar.

2.5.2 Define echelon and reduced echelon form of a matrix

(a) Echelon form of a matrix

A matrix A of order $m \times n$ is called (row) echelon form, if it has the following structure.

(i) The first non-zero entry in any row is 1 that is leading entry.

(ii) All entries below the leading entry must be zeros.

(iii) Every non-zero row in a matrix precedes every zero row, if there is any

For example, $\begin{bmatrix} 0 & 1 & -3 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 5 & -1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form.

But $\begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -5 \\ 0 & 0 & -7 \\ 0 & 0 & 4 \end{bmatrix}$ are not in echelon form.

(b) Reduced Echelon form of a matrix

A matrix A of order $m \times n$ is called reduced echelon form if it is in echelon form, additionally all the elements of column which contain leading entry 1 are zero except that leading entry.

For example, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are in reduced echelon form.

But $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not in reduced echelon form.

2.5.3 Reduce a matrix to its echelon and reduced echelon form

Method of reducing a matrix in echelon form is explained with the help of the following example.

Example: Reduce the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -1 & -2 & -3 \\ 3 & -1 & 3 & 2 \end{bmatrix}$ to (row) echelon form.

Solution:

$$A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -1 & -2 & -3 \\ 3 & -1 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 2 & 3 & -1 & 4 \\ 3 & -1 & 3 & 2 \end{bmatrix} \quad \text{by } R_{12}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 5 & 3 & 10 \\ 0 & 2 & 9 & 11 \end{bmatrix} \quad \text{by } R_2 + (-2)R_1 \text{ and } R_3 + (-3)R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 2 & 9 & 11 \end{bmatrix} \quad \text{by } \frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 0 & \frac{39}{5} & 7 \end{bmatrix} \quad \text{by } R_3 + (-2)R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & \frac{-3}{5} & 2 \\ 0 & 0 & 1 & \frac{35}{39} \end{bmatrix} \quad \text{by } \frac{5}{39}R_3$$

It is an echelon form of the given matrix.

(ii) Reduced Echelon form of a matrix

Example: Find the reduced echelon form of the matrix $A = \begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}, \quad R_{13}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 6 & 3 & -4 \end{bmatrix}, \quad R_2 + 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix}, \quad R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -\frac{26}{9} \\ 0 & -9 & 26 \end{bmatrix}, \quad \frac{1}{9}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{26}{9} \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} R_1 - 2R_2 \\ R_3 + 9R_2 \end{matrix}$$

It is the reduced echelon form of A.

2.5.4 Recognize the rank of a matrix

The number of non-zero rows in echelon form/reduced echelon form of a matrix is called rank of that matrix.

For example, $\begin{bmatrix} 1 & 1 & -2 & 6i \\ 0 & 0 & 1 & 3i \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has rank 2, because there are 2 non-zero rows in echelon form.

Whereas, $\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has rank 3, because there are 3 non-zero rows in reduced echelon form.

Note: 1. If $|A| \neq 0$, then rank (A) = order of the matrix A.
2. Rank (A) ≥ 0.
3. Rank of a non-zero row or column matrix is 1.

2.5.5 Use row operations to find the inverse and the rank of a matrix

Let A be a non-singular matrix. If the application of elementary row operations in succession reduce A to I then same sequence of operations reduces I to A^{-1}. i.e. $[A:I] \sim [I:A^{-1}]$

Example 1. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{bmatrix}$ by using row operations.

Solution: $|A| = \begin{vmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{vmatrix}$

$= 2(-8-4) - 5(-6-2) - 1(6-4) = -24 + 40 - 2$

$= 40 - 26 = 14$

As $|A| \neq 0$, so A is non-singular and its inverse exists.

Appending I_3 on the right of the matrix A, we have $\begin{bmatrix} 2 & 5 & -1 & : & 1 & 0 & 0 \\ 3 & 4 & 2 & : & 0 & 1 & 0 \\ 1 & 2 & -2 & : & 0 & 0 & 1 \end{bmatrix}$

$$\overset{\qquad A \qquad\qquad\qquad I_3}{\sim \begin{bmatrix} 1 & 2 & -2 & : & 0 & 0 & 1 \\ 3 & 4 & 2 & : & 0 & 1 & 0 \\ 2 & 5 & -1 & : & 1 & 0 & 0 \end{bmatrix}} \text{ by } R_{13}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & : & 0 & 0 & 1 \\ 0 & -2 & 8 & : & 0 & 1 & -3 \\ 0 & 1 & 3 & : & 1 & 0 & -2 \end{bmatrix} \text{ by } \begin{matrix} R_2 + (-3)R_1 \\ R_3 + (-2)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & : & 0 & 0 & 1 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & : & 1 & 0 & -2 \end{bmatrix} \text{ by } \left(-\frac{1}{2}\right)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & 0 & 1 & -2 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 7 & : & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \text{ by } \begin{matrix} R_3 + (-1)R_2 \\ R_1 + (-2)R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & 0 & 1 & -2 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & : & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \text{ by } \frac{1}{7}R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & -\frac{6}{7} & \frac{4}{7} & 1 \\ 0 & 1 & 0 & : & \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ 0 & 0 & 1 & : & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \text{ by } \begin{matrix} R_1+(-6)R_3 \\ R_2+4R_3 \end{matrix}$$

Thus, $$A^{-1}=\begin{bmatrix} -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix}.$$

Example 2. Find the rank of $A=\begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$ by reducing it to echelon form.

Solution: $$A=\begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$$

$$\sim\begin{bmatrix} -1 & -5 & -3 \\ -3 & 5 & 6 \\ 5 & 9 & 3 \end{bmatrix} \text{ by } R_{13}$$

$$\sim\begin{bmatrix} 1 & 5 & 3 \\ -3 & 5 & 6 \\ 5 & 9 & 3 \end{bmatrix} \text{ by } (-1)\,R_1$$

$$\sim\begin{bmatrix} 1 & 5 & 3 \\ 0 & 20 & 15 \\ 5 & 9 & 3 \end{bmatrix} \text{ by } R_2+3R_1$$

$$\sim\begin{bmatrix} 1 & 5 & 3 \\ 0 & 20 & 15 \\ 0 & -16 & -12 \end{bmatrix} \text{ by } R_3-5R_1$$

$$\sim\begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & -16 & -12 \end{bmatrix} \text{ by } \frac{1}{20}\,R_2$$

$$\sim\begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_3+16R_2$$

This is an echelon form of the matrix A and number of its non-zero rows is 2. Hence, the rank of the matrix A is 2.

Example 3. Find the rank of the matrix $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix}, \text{ by } R_2 + (-2)R_1, \text{ by } R_3 + (-3)R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 4 & 6 & -2 \end{bmatrix}, \text{ by } \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 + (-4)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_1 + R_2$$

This is in reduced echelon form and the number of non-zero rows are 2, so the rank of the given matrix A is 2.

Exercise 2.5

1. Reduce the following matrices into echelon form using elementary row operations.

(i) $\begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -4 & 0 & 9 \\ 2 & 4 & -1 & 0 \\ 10 & 0 & -2 & -4 \end{bmatrix}$

2. Reduce the following matrices into reduced echelon forms using elementary row operations.

(i) $\begin{bmatrix} 3 & 5 & 4 \\ 4 & 1 & 5 \\ 7 & 6 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -2 \\ 2 & -4 & 6 \\ 2 & 3 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 1 & -3 \\ -1 & -1 & 1 & -3 \end{bmatrix}$

3. Find the rank of the following matrices using elementary row operations.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -7 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

4. Find the inverse of the following matrices using elementary row operations.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

2.6 Solving System of Linear Equations

A system of linear equations is a collection of two or more linear equations for solving same set of variables.

For example,

$$a_1x + b_1y = k_1 \qquad \text{and} \qquad a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y = k_2 \qquad\qquad\quad a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

are the systems of linear equations in two and three variables respectively. An ordered triple (t_1, t_2, t_3) is called a solution of given system of three linear equations and three variables if all equations are satisfied by these values, the set of all solutions of linear system is called the solution set.

2.6.1 Distinguish between homogeneous and non-homogeneous systems of linear equations in 2 and 3 unknowns

Consider a system of three linear equations in three variables:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \qquad \dots (1)$$

The system (1) can be written as

$$AX = B, \qquad \dots (2)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The matrix A is called the matrix of the coefficients of the system of equations, X is the column matrix of unknowns and B is the column matrix of constants.

In the above system (2), if $B = 0$, then the system is called homogeneous, otherwise non-homogeneous.

For example, the system of equations:

$$-3x_1 + 2x_2 = 0,$$
$$7x_1 - 5x_2 = 13$$

can be written as $AX = B$

where $A = \begin{bmatrix} -3 & 2 \\ 7 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \neq 0$ (i.e $B = 0$)

Thus, the system is a non-homogeneous system of two linear equations in two unknowns.

Similarly, the system of equations:

$$x_1 + 7x_2 - 3x_3 = 0,$$
$$11x_1 - 5x_2 + 2x_3 = 0,$$
$$-x_1 + 2x_2 + 3x_3 = 0 \quad \text{can be written as } AX = B,$$

where $A = \begin{bmatrix} 1 & 7 & -3 \\ 11 & -5 & 2 \\ -1 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$ (i.e $B = O$)

Hence, it is a homogeneous system of three linear equations with three unknowns.

2.6.2 Solve a system of three homogeneous linear equations in three unknowns

A system of homogeneous linear equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

is always satisfied by $x = 0, y = 0$ and $z = 0$. The solution (0,0,0) of the above system is called trivial solution or zero solution. Any other non-zero solution of the above system of equations is called a non-trivial solution.

We usually convert the matrix of the coefficients to echelon form by using elementary row operations to get simplified form of the system.
Finally, with help of free variable(s), we get non-zero solutions, if possible.

Note: If AX = O is homogenous system of linear equations with "n" unknowns, then:
(i) it has only trivial solution if rank of A = n or $|A| \neq 0$
(ii) it has trivial as well as infinitely many non-trivial solutions iff rank (A) $< n$ or $|A| = 0$

Example 1. Solve the following system of homogeneous linear equations:

$$x + y + z = 0$$
$$4x + 5y + 2z = 0$$
$$2x + 3y = 0$$

for non-trivial solutions if possible.

Solution: We change the above system of equations to the matrix form $AX = O$

i.e $$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$ by $\begin{matrix} R_2 - 4R_1 \\ R_3 - 2R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ by $R_3 - R_2$

Thus, we have $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ... (A)

Here, Rank=2, n=3. Since Rank $(A) = 2 < 3$ so system has infinitely many solutions.

From system (A)

We get $x + y + z = 0$ ...(i)

$y - 2z = 0$ ...(ii)

From equation (ii), we get: $y = 2z$

Put $y = 2z$ in equation (i), we get: $x + 2z + z = 0$

or $x + 3z = 0$ or $x = -3z$

Here, we have only two equations with three variables, so we take one variable as a free-variable. The value of free-variable will be assumed as a non-zero real number.

Therefore we consider $z = k,\ k \neq 0$ as free variable.

By using above equations we get: $y = 2k$ and $x = -3k$

Now the solutions are $x = -3k, y = 2k$ and $z = k$, $k \in \mathbb{R} - \{0\}$.

By putting different values of k, we will get the different solutions.

If $k = 1$, then $x = -3,\ y = 2$ and $z = 1$.

If $k = 2$, then $x = -6,\ y = 4$ and $z = 2$, and so on.

So non trivial solutions are (-3,2,1), (-6,4,2) at $k = 1$ and $k = 2$ respectively.

2.6.3 Define a consistent and inconsistent system of linear equations and demonstrate through examples

Consistent and Inconsistent Systems

1. Consistent System:

A system of equations is said to be consistent if it has one or more solutions, for example

$\left.\begin{matrix} x + 2y = 4 \\ 3x + 2y = 2 \end{matrix}\right\}$ is consistent because it has a unique solution $\left(-1, \frac{5}{2}\right)$

and $\left.\begin{matrix} x + 2y = 4 \\ 3x + 6y = 12 \end{matrix}\right\}$ is also consistent because it has infinite solutions $(0,2), (-2,3), (-4,4), \ldots$

2. Inconsistent System:

If a system of equations has no solution, it is said to be inconsistent, for example

$$\left.\begin{array}{r} x + 2y = 4 \\ 3x + 6y = 5 \end{array}\right\} \text{ is inconsistent because it has no solution.}$$

Demonstration for consistency of a system of linear equations

Augmented Matrix:

Augmented matrix of the system of equations $AX = B$ is obtained by adding constant matrix as the last column of the coefficient matrix and it is denoted by A_B. Consider a system of a non-homogeneous linear equations in three variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix form, the above system of equations can be written as $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Augmented matrix is obtained by adding the constant terms as the last column of the coefficient matrix.

We have $$A_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Consistency Criteria

i. If rank of A = rank of $A_B = n$ then system has unique solutions

ii. If rank of A = rank of $A_B < n$ then system has infinite solutions

where n is the number of unknowns of the system of equations.

Inconsistency Criterion

If rank of $A \neq$ rank of A_B then the system has no solution.

Example: Check the following system of linear equations to be consistent or inconsistent.

$$x - y + 2z = 5,$$
$$3x + y + z = 8,$$
$$2x - 2y + 3z = 7$$

Solution: We write the above system of equations in matrix form:

$$AX = B$$

i.e., $$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

Now we reduce the augmented matrix to echelon form to check consistency.

$$\text{Augmented Matrix} = A_B = \begin{bmatrix} 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 2 & -2 & 3 & 7 \end{bmatrix} \text{by } R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 0 & -1 & -3 \end{bmatrix} \text{by } R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{by } (-1)R_3$$

Here, we can see rank of A = rank of $A_B = 3$ =number of unknowns; so, the system has unique solutions. Hence the system is consistent.

2.6.4 Solve a system of 3 by 3 non-homogeneous linear equations using:

(i) matrix inversion method,

(ii) Cramer's rule

(iii) Gauss elimination method (echelon form)

(iv) Gauss-Jordan method (reduced echelon form)

(i) Matrix Inversion Method

We can solve the following system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

by using matrix inversion method which has the following steps:

- Write the system of linear equations in the matrix form AX = B.
- Find A^{-1} if exists.
- Find X by using $X = A^{-1}B$.

Note: The "Matrix Inversion Method" works when the given system is consistent and also has unique solution.

Example: Use matrix inversion method to solve the following system of linear equations, if possible.

$$3x_1 + 2x_2 - x_3 = 4$$
$$2x_1 - x_2 + 2x_3 = 10$$
$$x_1 - 3x_2 - 4x_3 = 5$$

Solution: We write the system of linear equations to the matrix form AX = B,

where, $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 10 \\ 5 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{vmatrix} = 3(4+6) - 2(-8-2) - 1(-6+1) = 55$$

For $adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{31} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$

We find cofactors of A:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = 10,$$
$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} = 10,$$
$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -5,$$
$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ -3 & -4 \end{vmatrix} = 11,$$
$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = -11,$$
$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = 11,$$
$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3,$$
$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} = -8,$$
$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7.$$

Now, $$adj\ A = \begin{bmatrix} 10 & 10 & -5 \\ 11 & -11 & 11 \\ 3 & -8 & -7 \end{bmatrix}^t = \begin{bmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{bmatrix}$$

Thus, $$A^{-1} = \frac{1}{|A|}\ adj\ A = \frac{1}{55}\begin{bmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{bmatrix}$$

Now, $$X = A^{-1}B = A^{-1}\begin{bmatrix} 4 \\ 10 \\ 5 \end{bmatrix}$$

i.e $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{55}\begin{bmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 4 \\ 10 \\ 5 \end{bmatrix} = \frac{1}{55}\begin{bmatrix} 40+110+15 \\ 40-110-40 \\ -20+110-35 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $x_1 = 3, x_2 = -2$ and $x_3 = 1$ is the required solution.

(ii) Cramer's rule

Consider the system of linear equations,

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \quad \ldots \text{(i)}$$

We write the above system of linear equations in matrix form as

$$AX = B \quad \ldots \text{(ii)}$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

To solve the system of linear equations by Cramer's rule, following steps are used.

i. Calculate $|A|$, if $|A| \neq 0$ then go to step (ii) otherwise the method fails.

ii. Calculate $|A_1|$, $|A_2|$ and $|A_3|$ where,

$$A_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \ A_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

iii. Now find x, y and z by using $x = \frac{|A_1|}{|A|}, y = \frac{|A_2|}{|A|}$ and $z = \frac{|A_3|}{|A|}$

Note: This method works only when the given system has non-singular coefficient matrix .

Example: Use Cramer's rule to solve the following system of linear equations.

$$x + 3y + 2z = 19 \quad \ldots \text{(i)}$$
$$2x + y + z = 13 \quad \ldots \text{(ii)}$$
$$4x + 2y + 3z = 31 \quad \ldots \text{(iii)}$$

Solution:

Write the system of linear equations to the matrix form AX = B

i.e., $$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 13 \\ 31 \end{bmatrix}$$

Now, we find the determinants of A, A_1, A_2 and A_3.

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 1 - 6 + 0 = -5 \neq 0.$$

$$|A_1| = \begin{vmatrix} 19 & 3 & 2 \\ 13 & 1 & 1 \\ 31 & 2 & 3 \end{vmatrix} = 19\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 13 & 1 \\ 31 & 3 \end{vmatrix} + 2\begin{vmatrix} 13 & 1 \\ 31 & 2 \end{vmatrix} = 19 - 24 - 10 = -15$$

$$|A_2| = \begin{vmatrix} 1 & 19 & 2 \\ 2 & 13 & 1 \\ 4 & 31 & 3 \end{vmatrix} = 1\begin{vmatrix} 13 & 1 \\ 31 & 3 \end{vmatrix} - 19\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 13 \\ 4 & 31 \end{vmatrix} = 8 - 38 + 20 = -10$$

$$|A_3| = \begin{vmatrix} 1 & 3 & 19 \\ 2 & 1 & 13 \\ 4 & 2 & 31 \end{vmatrix} = 1\begin{vmatrix} 1 & 13 \\ 2 & 31 \end{vmatrix} - 3\begin{vmatrix} 2 & 13 \\ 4 & 31 \end{vmatrix} + 19\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 5 - 30 + 0 = -25.$$

By Cramer's rule $x = \frac{|A_1|}{|A|} = \frac{-15}{-5} = 3$;$y = \frac{|A_2|}{|A|} = \frac{-10}{-5} = 2$ and $z = \frac{|A_3|}{|A|} = \frac{-25}{-5} = 5.$

Thus, $x = 3, y = 2$ and $z = 5$ is the required solution.

(iii) Gauss elimination method (echelon form)

Gauss elimination method is an algorithm for solving system of linear equations. It is usually understood as a sequence of operations performed on the corresponding matrix of coefficients. The method is named after Carl Friedrich Gauss (1777-1855).

This method can be used to solve the non-homogeneous system of linear equations.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Following are the steps of Gauss elimination method:

- Write the system of linear equations to the matrix form AX = B.
- Form the augmented matrix by including the constant elements as an extra column in the coefficient matrix.
- Convert augmented matrix into echelon form by using elementary row operations.
- Find X by detaching the last column back to its original position i.e., on the right-hand side of the equivalent matrix equation to AX = B.

Example: Use Gauss elimination method to solve the following system of non-homogeneous linear equations:

$$x + 5y + 2z = 9$$
$$x + y + 7z = 6$$
$$-3y + 4z = -2$$

Solution:

We write the system of linear equations in matrix form AX = B.

where, $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ -2 \end{bmatrix}$

Forming the augmented matrix by including the constant elements as an extra column in the coefficient matrix.

$$A_B = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix}$$

Converting augmented matrix into echelon form by using elementary row operations.

$$A_B \sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{3}{4} \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } \left(-\frac{1}{4}\right) R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \text{ by } R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } 4R_3$$

From the above matrix, we get equivalent matrix equation

$$x + 5y + 2z = 9 \quad \ldots \text{(i)}$$

$$y - \frac{5}{4}z = \frac{3}{4} \quad \ldots \text{(ii)}$$

$$z = 1 \quad \ldots \text{(iii)}$$

From (iii), we get $z = 1$. Putting $z = 1$ in (ii) we get $y = 2$ then from (i), we get:

$$x + 5(2) + 2 = 9 \implies x + 12 = 9 \implies x = -3$$

Thus, $x = -3$, $y = 2$ and $z = 1$ is the required solution.

(iv) Gauss - Jordan Method (reduced echelon form)

Gauss-Jordan method is the modified form of Gauss elimination method in which the augmented matrix is converted into the reduced echelon form.

Example: Use Gauss-Jordan method to solve the system of linear equations:

$$x + 5y + 2z = 9$$
$$x + y + 7z = 6$$
$$-3y + 4z = -2$$

Solution:

Changing the system of linear equations in the form AX = B.

where, $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ -2 \end{bmatrix}$.

Here

$$\text{Augmented matrix} = A_B = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -1 & 1 & -1 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_1 - 5R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_1 - 7R_3$$

Find X by detaching the last column back to its original position from the above matrix.

We get $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$x + 0y + 0z = -3$ ... (i)

or $0x + y + 0z = 2$ ... (ii)

$0x + 0y + z = 1$ ... (iii)

From (i), (ii) and (iii) equations, we directly get $x = -3, y = 2$ and $z = 1$ as the required solution.

Exercise 2.6

1. Solve the following homogeneous system of linear equations for non-trivial solutions, if possible.

(i) $x + 2y - 2z = 0$, $2x + y + 5z = 0$, $5x + 4y + 8z = 0$

(ii) $x + 4y + 2z = 0$, $2x + y - 3z = 0$, $3x + 2y - 4z = 0$

2. Determine the consistency of non-homogeneous system of linear equations.

(i) $x - 2y - 2z = -1$, $2x + 3y + z = 1$, $5x - 4y - 3z = 1$

(ii) $x + 2y + z = 2$, $2x + y + 2z = -1$, $2x + 3y - z = 9$

3. Solve the non-homogeneous system of linear equations using matrix inversion method.

(i) $x + 2y + z = 8$, $2x - y + z = 3$, $x + y - z = 0$

(ii) $2x - y + 2z = 4$, $x + 10y - 3z = 10$, $-x + y + z = -6$

4. Solve the non-homogeneous system of linear equations using Gauss elimination method.

(i) $-x + y + z = 0$, $x + 2y = 5$, $-3x + 2y - z = -2$

(ii) $x + 2y + z = 8$, $2x - y + z = 3$, $x + y - z = 0$

5. Solve the non-homogeneous system of linear equations using Gauss-Jordan- method.

(i) $x - y + 4z = 4$, $2x + 2y - z = 2$, $3x - 2y + 3z = -3$

(ii) $2x + 2y - z = 4$, $x - 2y + z = 2$, $x + y = 0$

6. Solve the non-homogeneous system of linear equations using Cramer's Rule.

(i) $x - 2y + z = 2$, $2x + 2y - z = 4$, $x + y + 0.z = 0$

(ii) $x - 2y + 0.z = -4$, $3x + y + 0.z = -5$, $2x + 0.y + z = -1$

Review Exercise 2

1. Select correct option.

i. If a matrix **A** has m row and n column, then order of **A** is:

(a) $m \times n$ (b) $n \times m$ (c) mn (d) m^n

ii. Any matrix of order $m \times 1$ is called:

(a) Row matrix (b) Column matrix
(c) Square matrix (d) Zero matrix

iii. For the square matrix $A = [a_{ij}]$. If all $a_{ij} = 0, i \neq j$ and all $a_{ij} = k$ (non zero) for $i = j$, then A is called:
(a) Rectangular matrix (b) Scalar matrix
(c) Identity matrix (d) Null matrix

iv. The matrix [7] is:
(a) Square matrix (b) Row matrix
(c) Column matrix (d) all of these

v. $(k\,ABC)^t =$
(a) $kA^tB^tC^t$ (b) $kC^tB^tA^t$ (c) $k(BA)^t$ (d) $k^t(AB)$

vi. $\begin{vmatrix} 1 & 0 \\ 5 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$:______
(a) 4 (b) 8 (c) −2 (d) 10

vii. If AB = BA , then which one is true, where A and B are square matrices:
(a) A and B are multiplicative inverses of each other
(b) One of A or B is null matrix.
(c) One of A or B is identity matrix. (d) all of these

viii. If $A = [-7]$, then $|A|$ is equal to:
(a) 7 (b) −7 (c) 0 (d) Not possible

ix. Let $A = [a_{ij}]$ be a square matrix. Then cofactor of a_{ij} is equal to:
(a) M_{ij} (b) $(-1)^{i+j}M_{ij}$ (c) $(-1)^{ij}M_{ij}$ (d) $(-1)^{i+j}a_{ij}$

x. For any triangular matrix A, |A| is equal to:
(a) Product of leading diagonal elements
(b) Sum of leading diagonal elements
(c) Sum of square of diagonal elements
(d) All of these

xi. A square matrix $A = [a_{ij}]$ for which all $a_{ij} = 0,\ i < j,$ then A is called:
(a) Upper triangular (b) Lower triangular
(b) Symmetric (d) Hermitian

xii. A triangular matrix is always a:
(a) Diagonal matrix (b) Scalar matrix
(c) Square matrix (d) all of these

xiii. A square matrix A is skew symmetric if:
(a) $A^t = A$ (b) $A^t = -A$ (c) $(\bar{A})^t = A$ (d) None

xiv. A square matrix A is Hermitian matrix if:
(a) $A^t = A$ (b) $A^t = -A$ (c) $(\bar{A})^t = A$ (d) $(\bar{A})^t = -A$

xv. Each diagonal element of main diagonal of a skew Hermitian matrix must be:
(a) 1 (b) 0 (c) Any non-zero number (d) Any complex number

xvi. If $\begin{vmatrix} a & b \\ 0 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & -9 \end{vmatrix}$ then

(a) $a = -3$ (b) $a = b$ (c) $a = \frac{1}{3}$ (d) $a = \frac{-1}{3}$

xvii. The number of non-zero rows in echelon form of a matrix is called:

(a) Order of a matrix (b) Rank of a matrix

(c) Leading Column (d) Leading row

xviii. If A is any square matrix and $A = -A^t$ then A is a:

(a) Symmetric matrix (b) Skew symmetric matrix

(c) Hermitian matrix (d) Skew Hermitian matrix

xix. If A is idempotent matrix then:

(a) $A^2 = I$ (b) $A^2 = 0$

(c) $A^2 = A$ (d) $A^2 = A^t$

xx. The cofactor A_{22} of $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 5 \\ 0 & 1 & -1 \end{bmatrix}$ is:

(a) 0 (b) −1 (c) 1 (d) 2

2. Find the values of x and y if: $\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 8 & 14 \end{bmatrix}$.

3. Calculate AC, BC and (A + B)C. Also verify that (A + B)C = AC + BC for

$A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $A^2 - 5A + 7I = 0$.

5. Find the degree or index of the nilpotent matrix $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$

6. Show that matrix $\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is periodic matrix of period 2.

7. Which of the following is idempotent or involutory matrix

(i) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

8. Which of the following matrices are Hermitian or Skew Hermitian

(i) $\begin{bmatrix} 1 & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 1-i & 7 \\ 1+i & 6 & -i \\ 7 & i & 5 \end{bmatrix}$

9. Let A be a square matrix. Show that

(a) $A + (\bar{A})^t$ is Hermitian,

(b) $A - (\bar{A})^t$ is skew Hermitian

10. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

11. Show that $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

12. Solve the following system of linear equations by Cramer's rule, Gauss elimination, Gauss-Jordan and matrix inversion methods.

$2x + 4y - z = 0;$ $x - 2y - 2z = 2$ and $-5x - 8y + 3z = -2.$

Vectors

Unit 3

3.1 Vectors in Plane

There are many quantities in daily life which need direction for their specification. Let us take an example, in the army, when they are launching missiles, they first need the direction and distance as to know their target and the impact, it is going to cause. This kind of quantity which needs magnitude and direction, is called vector. Furthermore, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors.

3.1.1 Define a scalar and a vector

The quantity that is completely specified only by its magnitude with an appropriate unit, is called scalar quantity or scalar, for example mass, volume, area, distance, energy etc.

The quantity that has direction as well as magnitude with appropriate unit, is called vector quantity, or vector for example weight, force, displacement, velocity etc.

3.1.2 Give geometrical representation of a vector

A vector in a plane is represented by a directed line segment. In Fig 3.1, a vector AB is shown which is denoted as $\overrightarrow{AB}$ or $\vec{u}$. The endpoints of the segment are called the initial point and the terminal point of the vector.

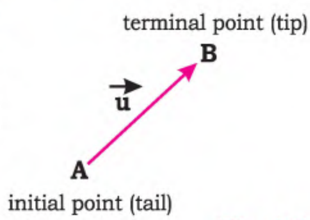


Fig. 3.1

The length of the segment represents the magnitude whereas the arrow indicates the direction of the vector.

3.1.3 Give the following fundamental definitions using geometrical representation:

i. magnitude of a vector,
ii. equal vectors
iii. negative of a vector,
iv. unit vector,
v. zero/null vector,
vi. position vector,
vii. parallel vectors,
viii. addition and subtraction of vectors,
ix. triangle, parallelogram and polygon laws of addition,
x. scalar multiplication

i. Magnitude of a vector

The length of a vector is called its magnitude. The magnitude of vector $\vec{v}$ is denoted by $|\vec{v}|$ or v as shown in Fig. 3.2.

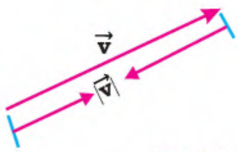


Fig 3.2

ii. Equal vectors

Two vectors $\vec{u}$ and $\vec{v}$ are said to be equal if they have the same magnitude and direction (Fig. 3.3). Symbolically, it is written as $\vec{u} = \vec{v}$.

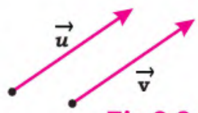


Fig 3.3

iii. Negative of a vector

The negative of a vector $\vec{v}$ is denoted by $-\vec{v}$. It has the same length as $\vec{v}$ but opposite in direction as shown in Fig. 3.4.

Fig 3.4

iv. Unit vector

A vector whose magnitude is 1 is called a unit vector. The unit vector of $\vec{v}$ is denoted by $\hat{v}$ as shown in Fig. 3.5.

Note: "$\hat{v}$" is pronounced as "v cap".

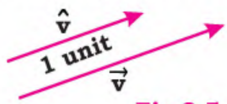


Fig 3.5

v. Zero or Null vector

A vector whose initial point and terminal point are same is called zero or null vector, it is denoted by $\vec{0}$.

vi. Position vector

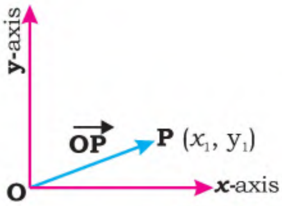


Fig 3.6

A vector whose initial point is origin and terminal point is P, is called position vector of point P and it is denoted as $\overrightarrow{OP}$ as shown in Fig. 3.6. Its magnitude represents the distance of the point P from the origin.

vii. Parallel vectors

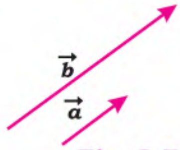


Fig. 3.7

Two vectors are said to be parallel if they have the same direction. In figure 3.7, $\vec{a}$ and $\vec{b}$ are parallel vectors, we denote as $\vec{a} \parallel \vec{b}$.

viii. Addition and subtraction of vectors

Addition of vectors:

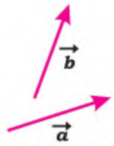


Fig. 3.8(a)

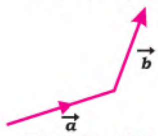


Fig. 3.8(b)

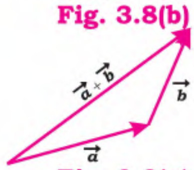


Fig. 3.8(c)

Consider two vectors $\vec{a}$ and $\vec{b}$ as shown in Fig. 3.8 (a). In order to add these two vectors, we first place $\vec{b}$ in such a way that its tail coincides with tip or head of $\vec{a}$ keeping its length and direction same, as shown in Fig. 3.8 (b).

Now, the sum or resultant of $\vec{a}$ and $\vec{b}$, denoted as $\vec{a}+\vec{b}$, is the vector whose initial point is the tail of $\vec{a}$ and terminal point is the head of $\vec{b}$ as shown in Fig. 3.8(c).

The above method of addition of vectors is called Head and Tail rule.

Subtraction of vectors:

Consider two vectors $\vec{a}$ and $\vec{b}$ as shown in Fig. 3.9 (a). The difference $\vec{a}-\vec{b}$ of these two vectors $\vec{a}$ and $\vec{b}$ is defined as the sum of the vectors $\vec{a}$ and $(-\vec{b})$, i.e., $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

Thus, in order to obtain the difference $\vec{a}-\vec{b}$, we just add $\vec{a}$ with the negative vector of $\vec{b}$ by Head and Tail rule as shown in Fig. 3.9(b).

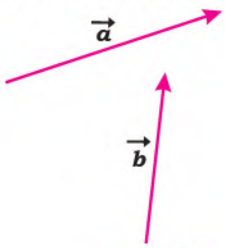


Fig. 3.9a

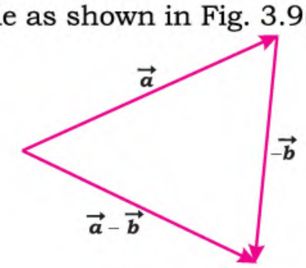


Fig. 3.9b

ix. Triangle, Parallelogram and Polygon Laws of Addition

Triangle law of addition:

Triangle law of vector addition states that when two vectors $\vec{a}$ and $\vec{b}$ are represented by two sides of a triangle in same order, then the third side $\vec{c}$ of the triangle represents the resultant vector of the vectors $\vec{a}$ and $\vec{b}$ taken in the opposite order, as shown in (Fig. 3.10).

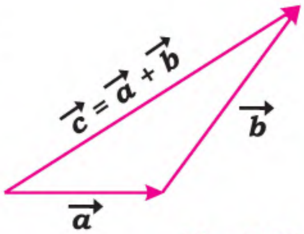


Fig. 3.10

$$\vec{c} = \vec{a} + \vec{b}$$

Parallelogram law of addition:

Parallelogram law of vector addition states that when two vectors $\vec{a}$ and $\vec{b}$ of same initial point represented by two adjacent sides of a parallelogram then the resultant of these vectors is $\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$ represented by the diagonal of the parallelogram starting from the same initial point of the vectors $\vec{a}$ and $\vec{b}$, as shown in (Fig. 3.11).

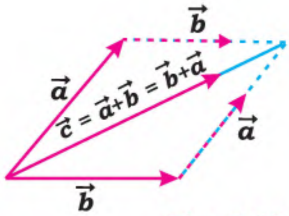


Fig. 3.11

Polygon law of addition:

Polygon law of vector addition states that if a number of vectors are represented in magnitude and direction by the consecutive sides of a polygon taken in the same order, then their resultant is represented by the closing side of the polygon taken in the opposite order. In Fig. 3.12, the sides of given polygon are represented by the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_{n-1}$, the closing side is represented by vector $\vec{a}_n$ which is the resultant.

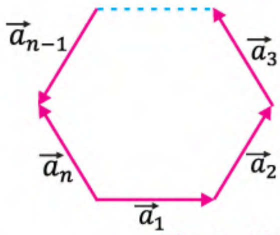


Fig. 3.12

i.e., $\vec{a}_n = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \cdots + \vec{a}_{n-1}$.

Example: Prove pentagon law of vector addition.

Proof: **Pentagon law of vector addition:**

Pentagon law of vector addition states that, if four vectors are represented by four consecutive sides of a pentagon in same order then the last side is the resultant in opposite order.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the four vectors represented by the sides $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DE}$ respectively of pentagon ABCDE as shown in Fig 3.13. By pentagon law of vectors, we have to prove that $\overrightarrow{AE} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$

In Δ ABC, by triangle law of addition

$\overrightarrow{AC} = \vec{a} + \vec{b}$ ... (i)

In Δ ACD, by triangle law of addition

$\overrightarrow{AD} = \overrightarrow{AC} + \vec{c}$

i.e., $\overrightarrow{AD} = \vec{a} + \vec{b} + \vec{c}$ ... (ii) (Using equation (i))

In Δ ADE, by triangle law of addition

$\overrightarrow{AE} = \overrightarrow{AD} + \vec{d}$

i.e., $\overrightarrow{AE} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$ (Using equation (ii)).

Hence proved.

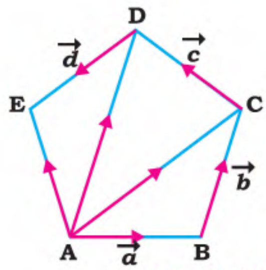


Fig. 3.13

x. Scalar Multiplication of vector

When we multiply a vector $\vec{v}$ by a scalar k, the result is a vector $k\vec{v}$. The scalar multiplication by a positive number other than 1 changes the magnitude of the vector but not its direction. The scalar multiplication by negative number other than -1 changes its magnitude and reverses its direction as shown in the Fig. 3.14, whereas multiplication by -1 only reverses the direction.

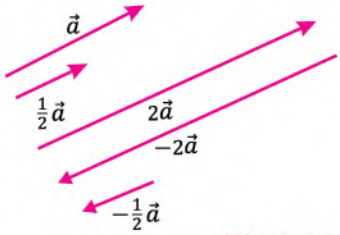


Fig 3.14

3.1.4 Represent a vector in a Cartesian plane by defining fundamental unit vectors *i* and *j*.

The algebra of vectors is based on representing each vector in terms of components parallel to the Cartesian coordinate axes and writing each component as an appropriate multiple of a unit vector along either of axes.

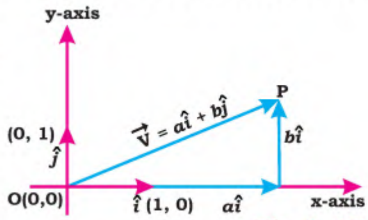


Fig 3.15

The unit vector along x-axis is the vector $\hat{\imath}$ determined by the directed line segment from (0,0) to (1,0). The basic or unit vector along y-axis is the vector $\hat{\jmath}$ determined by the directed line segment from (0,0) to (0,1).

Figure 3.15 shows a vector $\vec{v} = \overrightarrow{OP}$ resolved into its $\hat{\imath}$ and $\hat{\jmath}$ components as the sum: $\vec{v} = a\hat{\imath} + b\hat{\jmath}$

Here $a\hat{\imath}$, where a is a scalar, represents a vector of length $|a|$, parallel to the x-axis, pointing to the right if, $a > 0$ and to the left if, $a < 0$. Similarly, $b\hat{\jmath}$ is a vector of length $|b|$, parallel to the y-axis, pointing up if, $b > 0$ and down wards if, $b < 0$.

The numbers 'a' and 'b' are the scalar components of $\vec{v}$ in the directions of $\hat{\imath}$ and $\hat{\jmath}$ respectively.

The vector $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ can be written as $\begin{pmatrix} a \\ b \end{pmatrix}$ or $[a, b]$ or (a, b).

3.1.5 Recognize all above definitions using analytical representation

i. Magnitude of a vector

If a vector $\vec{v}$ is given in terms of components as $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ then its magnitude is obtained by $|\vec{v}| = \sqrt{a^2 + b^2}$.

Example 1. If $\vec{v} = 7\hat{\imath} - \hat{\jmath}$, find $|\vec{v}|$.

Solution: Here $a = 7$ and $b = -1$

Now, $|\vec{v}| = \sqrt{(7)^2 + (-1)^2} = \sqrt{50}$ units

Let P, Q are the points in the xy-plane with Cartesian coordinates (x_1, y_1) and (x_2, y_2) respectively, then the vector from P to Q written as $\overrightarrow{PQ}$, is:

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$$

By using distance formula between two points.

Magnitude of $\overrightarrow{PQ}$ is: $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 2. If P(7,0) and Q(13,8) are two points in Cartesian plane then, find magnitude of vector $\overrightarrow{PQ}$

Solution: Here $(x_1, y_1) = (7, 0)$ and $(x_2, y_2) = (13, 8)$

Magnitude of vector $\overrightarrow{PQ} = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(13 - 7)^2 + (8 - 0)^2}$

$= \sqrt{(6)^2 + (8)^2} = 10$ units

ii. Equal vectors

We know that two vectors $\vec{p}$ and $\vec{q}$ are said to be equal if they have same magnitude and the same direction.

In terms of components if, $\vec{p} = a\hat{\imath} + b\hat{\jmath}$ and $\vec{q} = a'\hat{\imath} + b'\hat{\jmath}$

then $\vec{p} = \vec{q}$ if and only if $a = a'$ and $b = b'$

Example: If $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\overrightarrow{CD} = \frac{1}{2}\begin{pmatrix} 12 \\ 6 \end{pmatrix}$ then show that $\overrightarrow{AB} = \overrightarrow{CD}$

Solution:

Here , $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 6\hat{\imath} + 3\hat{\jmath}$ and $\overrightarrow{CD} = \frac{1}{2}\begin{pmatrix} 12 \\ 6 \end{pmatrix} = 6\hat{\imath} + 3\hat{\jmath}$.

$\because$ coefficient of $\hat{\imath}$ and $\hat{\jmath}$ are equal.

$\therefore$ $\overrightarrow{AB} = \overrightarrow{CD}$

iii. Negative of a vector

Negative of vector $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ is $-\vec{a} = -a_1\hat{\imath} - a_2\hat{\jmath}$

Let, $\vec{a} = 6\hat{\imath} + \hat{\jmath}$ then $-\vec{a} = -(6\hat{\imath} + \hat{\jmath}) = -6\hat{\imath} - \hat{\jmath}$

iv. Unit Vector

We have already studied that a vector having magnitude 1 is called unit vector.

Let $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ then, $|\vec{v}| = \sqrt{a^2 + b^2}$

The unit vector of $\vec{v}$ is denoted by $\hat{v}$ and is obtained by

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{a\hat{\imath} + b\hat{\jmath}}{\sqrt{a^2 + b^2}}$$

Example: If $\vec{v} = \hat{\imath} - 7\hat{\jmath}$ then find $\hat{v}$.

Solution:

Here $a = 1, \quad b = -7$ then, $|\vec{v}| = \sqrt{(1)^2 + (-7)^2} = 5\sqrt{2}$ units

Now $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{\imath} - 7\hat{\jmath}}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}\hat{\imath} - \frac{7}{5\sqrt{2}}\hat{\jmath}$

v. Null vector or Zero vector

A vector whose magnitude is zero is called Zero vector or Null vector. It is denoted by $\vec{0}$.

Zero vector in plane, in terms of component, is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

vi. Position vector

The vector from origin $\mathbf{O}(0,0)$ to a point $P(x, y)$ is vector $\overrightarrow{OP}$ and is called position vector of the point P.

The position vector of $P(x, y)$ is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = x\hat{\imath} + y\hat{\jmath}$

Example: Write down the position vector of P(3,4) and Q(5,2).

Solution: The position vector of P(3,4) is $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\hat{\imath} + 4\hat{\jmath}$.

and the position vector of Q(5,2) is $\overrightarrow{OQ} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 5\hat{\imath} + 2\hat{\jmath}$.

vii. Parallel vectors

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if there exists a non-zero real number k such that $\vec{b} = k\vec{a}$.

Note: (i) $\vec{b}$ has a magnitude $|k|$ times the magnitude of $\vec{a}$.
(ii) if $k < 0$ then $\vec{a}$ is antiparallel to $\vec{b}$.

Example: If $\vec{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 10 \\ -15 \end{pmatrix}$ are three vectors then show that, $\vec{a}$ is parallel to $\vec{b}$. Also show that $\vec{c}$ is antiparallel to $\vec{a}$.

Solution:

Here, $\vec{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$;

and $\vec{b} = \begin{pmatrix} -6 \\ 9 \end{pmatrix} = 3\begin{pmatrix} -2 \\ 3 \end{pmatrix} = 3\vec{a}$

$\because \quad \vec{b} = 3\vec{a}$

$\therefore$ $\vec{b}$ is parallel to $\vec{a}$

and the magnitude of $\vec{b}$ is 3 times the magnitude of $\vec{a}$.

Also $\quad \vec{c} = \begin{pmatrix} 10 \\ -15 \end{pmatrix} = -5\begin{pmatrix} -2 \\ 3 \end{pmatrix} = -5\vec{a}$

$\because \quad \vec{c} = -5\vec{a}$

$\therefore \quad \vec{c}$ is antiparallel to $\vec{a}$.

viii. Addition and subtraction of vectors

Vector Addition:

If $\quad \vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath}$

then, $\vec{a} + \vec{b} = (a_1 + b_1)\hat{\imath} + (a_2 + b_2)\,\hat{\jmath}$

For example, if $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{b} = \begin{pmatrix} 8 \\ -5 \end{pmatrix} = 8\hat{\imath} - 5\hat{\jmath}$

then $\quad \vec{a} + \vec{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$

or $\quad \vec{a} + \vec{b} = 3\hat{\imath} + 4\hat{\jmath} + 8\hat{\imath} - 5\hat{\jmath} = 11\hat{\imath} - \hat{\jmath}$

Vector Subtraction:

If $\vec{a} = a_1 i + a_2 j$ and $\vec{b} = b_1 i + b_2 j$

then $\quad \vec{a} - \vec{b} = (a_1 - b_1)\hat{\imath} + (a_2 - b_2)\,\hat{\jmath}$

For example, if $\vec{a} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} = 7\hat{\imath} - 4\hat{\jmath}$ and $\vec{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 5\hat{\imath} + 3\hat{\jmath}$

then, $\quad \vec{a} - \vec{b} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$

or $\quad \vec{a} - \vec{b} = 7\hat{\imath} - 4\hat{\jmath} - 5\hat{\imath} - 3\hat{\jmath} = 2\hat{\imath} - 7\hat{\jmath}$

ix. Scalar Multiplication of vector

If k is a scalar and $\vec{v} = a\hat{\imath} + b\hat{\jmath}$ is a vector, then

$$k\vec{v} = k\,(a\hat{\imath} + b\hat{\jmath}) = (ka)\hat{\imath} + (kb)\hat{\jmath}$$

The length of $k\vec{v}$ is $|k|$ times the length of $\vec{v}$:

i.e., $\quad |k\vec{v}| = |(ka)\hat{\imath} + (kb)\hat{\jmath}|$

$$= \sqrt{(ka)^2 + (kb)^2} = \sqrt{k^2(a^2 + b^2)} = |k||\vec{v}|.$$

Example: If $\quad k = -2$ and $\vec{v} = -3\hat{\imath} + 4\hat{\jmath}$, Find $|k\vec{v}|$.

Solution:

Here $\vec{v} = -3\hat{\imath} + 4\hat{\jmath}$, so $|\vec{v}| = |-3\hat{\imath} + 4\hat{\jmath}| = \sqrt{(-3)^2 + (4)^2} = 5$

Now, $\quad |k\vec{v}| = |-2\vec{v}| = |-2||\vec{v}|$

$= 2 \times 5$

$= 10$

3.1.6 Find a unit vector in the direction of another given vector

The unit vector in the direction of a given vector $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$ or $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$.

Example: Find unit vector in the direction of $\vec{v} = 3\hat{\imath} - 4\hat{\jmath}$ and verify.

Solution:

Here, $|\vec{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = 5$

Thus, unit vector in the direction of $\vec{v}$ is: $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{3\hat{\imath}-4\hat{\jmath}}{5} = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$

Verification:

$$|\hat{v}| = \left|\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}\right| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$$

Hence verified.

3.1.7 Find the position vector of a point which divides the line segment joining two points in a given ratio

Vector in terms of position vectors

Let X and Y be the end points of a vector XY. Now taking position vectors of points X and Y, we have

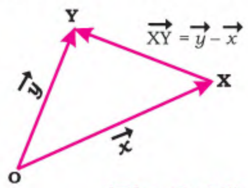
$\overrightarrow{OX} = \vec{x}$ and $\overrightarrow{OY} = \vec{y}$ as shown in the Fig. 3.16.

Now applying triangle law of vector addition, we get

$$\overrightarrow{OX} + \overrightarrow{XY} = \overrightarrow{OY}$$

$$\Rightarrow \quad \overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$$

$$\text{or} \quad \overrightarrow{XY} = \vec{y} - \vec{x}$$



(Fig. 3.16)

Thus, if $\vec{x}$ and $\vec{y}$ are position vectors of points X and Y then $\overrightarrow{XY} = \vec{y} - \vec{x}$.

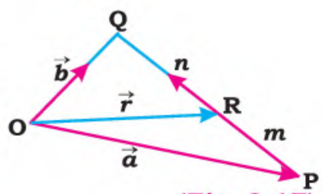
Position vector of point of division of line segment

Consider two points P and Q having position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ with respect to origin O. The line segment connecting P and Q is divided by a point R lying on $\overrightarrow{PQ}$. The point R can divide the line segment PQ in two ways, viz. internally and externally as shown in Fig. 3.17 and Fig. 3.18.

(i) Line segment PQ is divided by R internally:

Consider the point R which divides the line segment PQ internally in the ratio $m:n$, given that m and n are positive scalar quantities (Fig. 3.17)

So, $\frac{|\overrightarrow{PR}|}{|\overrightarrow{RQ}|} = \frac{m}{n}$



(Fig. 3.17)

$\Rightarrow \quad m|\overrightarrow{RQ}| = n|\overrightarrow{PR}|$

$\Rightarrow \quad m\overrightarrow{RQ} = n\overrightarrow{PR} \quad \ldots(i)$

Let $\vec{a}$, $\vec{b}$ and $\vec{r}$ be position vectors of points P, Q and R respectively.

i.e., $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$ and $\overrightarrow{OR} = \vec{r}$

Using equation (i)

We have $\quad m(\vec{b} - \vec{r}) = n(\vec{r} - \vec{a})$

$$\Rightarrow \quad m\vec{b} - m\vec{r} = n\vec{r} - n\vec{a}$$

$$\Rightarrow \quad \vec{r}(m + n) = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad \vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

Hence, the position vector of point R dividing $\overline{PQ}$ internally in the ratio $m{:}n$ is given by:

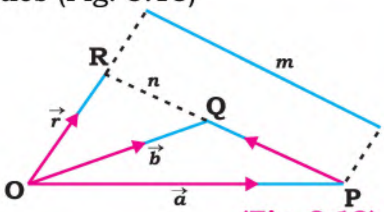
$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

If R is the midpoint, then R divides the line segment PQ in the ratio 1:1, i.e., $m{:}n = 1{:}1$. Then position vector of point R will be as under

$$\vec{r} = \frac{\vec{b} + \vec{a}}{2}$$

(ii) Line segment PQ which is divided by R externally:

Consider the point R which divides the line segment PQ externally in the ratio $m{:}n$, given that m and n are positive scalar quantities (Fig. 3.18)



(Fig. 3.18)

So, $\quad \dfrac{|\overrightarrow{PR}|}{|\overrightarrow{QR}|} = \dfrac{m}{n}$

$\Rightarrow \quad n|\overrightarrow{PR}| = m|\overrightarrow{QR}|$

$\Rightarrow \quad n\overrightarrow{PR} = m\overrightarrow{QR}$

$\quad n\overrightarrow{PR} = -m\overrightarrow{RQ}$

or $\quad m\overrightarrow{RQ} = -n\overrightarrow{PR} \quad \ldots(ii)$

Let $\vec{a}$, $\vec{b}$ and $\vec{r}$ be position vectors of points P, Q and R respectively.

i.e., $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$ and $\overrightarrow{OR} = \vec{r}$

Using equation (ii) $\quad m(\vec{b} - \vec{r}) = -n(\vec{r} - \vec{a})$

$$\Rightarrow \quad m\vec{b} - m\vec{r} = -n\vec{r} + n\vec{a}$$

$$\Rightarrow \quad \vec{r}(m - n) = m\vec{b} - n\vec{a}$$

$$\Rightarrow \quad \vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

Hence, the position vector of point R dividing $\overline{PQ}$ externally in the ratio $m{:}n$ is given by:

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

Example: Find the position vector of the point dividing the join of two points with position vectors $3\hat{\imath} + 4\hat{\jmath}$ and $8\hat{\imath} - 5\hat{\jmath}$ in the ratio of 1:3.

Solution:

Let, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$; $\vec{b} = 8\hat{\imath} - 5\hat{\jmath}$ and $m : n = 1:3$

Let $\vec{r}$ be the required position vector

So, $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$
$$= \frac{1(8\hat{\imath} - 5\hat{\jmath}) + 3(3\hat{\imath} + 4\hat{\jmath})}{1+3}$$
$$= \frac{8\hat{\imath} - 5\hat{\jmath} + 9\hat{\imath} + 12\hat{\jmath}}{4}$$
$$\vec{r} = \frac{17}{4}\hat{\imath} + \frac{7}{4}\hat{\jmath}$$

3.1.8 Use vectors to prove simple theorems of descriptive geometry

Theorem 1:

If the position vectors of points X and Y are $\vec{x}$ and $\vec{y}$ respectively and M be the midpoint of $\overrightarrow{XY}$ then position vector of point M is $\frac{\vec{x}+\vec{y}}{2}$

Proof:

Since $\vec{x}$ and $\vec{y}$ are the position vectors of X and Y respectively.

Therefore $\overrightarrow{OX} = \vec{x}$ and $\overrightarrow{OY} = \vec{y}$.

Let $\vec{m}$ be the position vector of midpoint M as shown in the Fig. 3.19.

So, $\vec{m} = \overrightarrow{OM}$

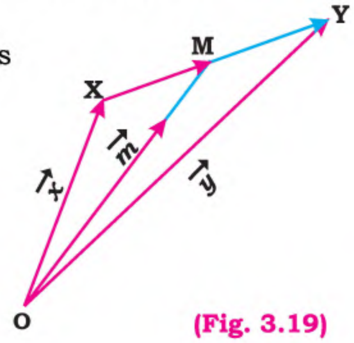
Now applying triangle law of vector addition, we get

$$\overrightarrow{OM} = \overrightarrow{OX} + \overrightarrow{XM} \quad \ldots(i)$$

Also, $$\overrightarrow{OM} = \overrightarrow{OY} - \overrightarrow{MY} \quad \ldots(ii)$$

By adding eq. (i) and (ii)

$$2\overrightarrow{OM} = \overrightarrow{OX} + \overrightarrow{XM} + \overrightarrow{OY} - \overrightarrow{MY}$$
$$2\vec{m} = \vec{x} + \overrightarrow{XM} + \vec{y} + \overrightarrow{YM}$$



(Fig. 3.19)

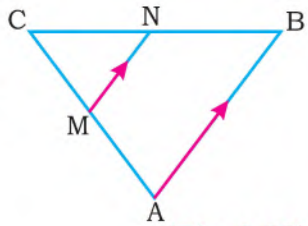
Since M is the mid-point of the line segment XY therefore, $\overrightarrow{XM}$ and $\overrightarrow{YM}$ are equal in magnitude but opposite in direction and cancel each other, so we get

$$2\vec{m} = \vec{x} + \vec{y}$$

i.e., $\vec{m} = \frac{\vec{x}+\vec{y}}{2}$, Hence proved.

Theorem 2:

The line segment joining the midpoints of two sides of a triangle is parallel and half of the length of the third side.



(Fig. 3.20)

Proof: Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of vertices A, B and C of Δ ABC respectively as shown in Fig 3.20.

By using Theorem 1, the position vector of midpoint M of $\overline{AC}$ is $\frac{\vec{a}+\vec{c}}{2}$.

Similarly, the position vector of mid-point N of $\overline{BC}$ is $\frac{\vec{b}+\vec{c}}{2}$.

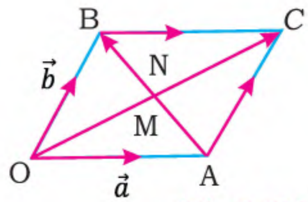
Now $\overrightarrow{MN} = \frac{\vec{b}+\vec{c}}{2} - \frac{\vec{a}+\vec{c}}{2} = \frac{1}{2}(\vec{b} - \vec{a}) = \frac{1}{2}\overrightarrow{AB}$.

So, $\overrightarrow{MN} || \overrightarrow{AB}$ and $\overrightarrow{MN}$ is half of length of $\overrightarrow{AB}$.

Hence proved.

Theorem 3:

The diagonals of a parallelogram bisect each other



(Fig 3.21)

Proof:

Consider a parallelogram OACB as shown in Fig 3.21, where $\vec{a}, \vec{b}$ be the position vectors of the vertices A and B respectively.

then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

In ΔOAC, by triangle law of addition of vectors

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b} \qquad [\because \overrightarrow{AC} = \overrightarrow{OB} = \vec{b}]$$

The midpoint M of the diagonal $\overrightarrow{OC}$ has the position vector

$$\vec{m} = \frac{\overrightarrow{OC}}{2} = \frac{\vec{a}+\vec{b}}{2} \qquad \ldots(i)$$

Using Theorem 1, the midpoint N of the diagonal $\overrightarrow{AB}$ has the position vector $\frac{\vec{a}+\vec{b}}{2}$.

$$\text{i.e.,} \qquad \vec{n} = \frac{\vec{a}+\vec{b}}{2} \qquad \ldots(ii)$$

From (i) and (ii) we have $\vec{m} = \vec{n}$

Hence the midpoints of both the diagonal are same

Thus, the diagonals of the parallelogram bisects each other. Hence proved.

Exercise 3.1

1. Classify the following quantities as scalars or vectors:

(i)	Force	(ii)	Speed	(iii)	Velocity
(iv)	Area	(v)	Acceleration	(vi)	Work
(vii)	Volume	(viii)	Weight	(ix)	Mass

2. Write the vector $\overrightarrow{PQ}$ in the form $x\hat{\imath} + y\hat{\jmath}$

(i) $P\,(4,6),\ Q(-4,5)$ (ii) $P\left(-\frac{5}{2}, 9\right), Q(10,8)$

(iii) $P(-3,6),\ Q\,(5,9)$ (iv) $P(-6,2), Q\,(0,-5)$

3. Find the following vectors if $\vec{p} = -5\hat{\imath} + 4\hat{\jmath}$, $\vec{q} = -3\hat{\imath} + 5\hat{\jmath}$ and $\vec{r} = -6\hat{\imath} + 7\hat{\jmath}$.

(i) $2\vec{p} + 3\vec{q}$ (ii) $3\vec{p} + \frac{1}{2}\vec{q} + \vec{r}$

(iii) $3\vec{p} + 5\vec{q} - 4\vec{r}$ (iv) $\frac{1}{2}\vec{p} - 2\vec{r} + 4\vec{q}$

4. $P(-2,5), Q\,(8,0), R\left(\frac{7}{2}, 6\right)$ and $S\,(-10,7)$ are the given points. Find the sum of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$.

5. Find unit vector in the direction of each of the following vector:

(i) $\vec{p} = -3\hat{\imath} + 8\hat{\jmath}$ (ii) $\vec{q} = 5\hat{\imath} - 9\hat{\jmath}$

(iii) $\vec{r} = 10\hat{\imath} - 12\hat{\jmath}$ (iv) $\vec{s} = 7\hat{\imath} + 5\hat{\jmath}$

6. Find the vectors in the direction of the following vectors with given magnitudes:

(i) $\vec{a} = 2\hat{\imath} - 3\hat{\jmath}$; magnitude 6 units

(ii) $\vec{b} = 10\hat{\imath} - 9\hat{\jmath}$; magnitude 7 units

(iii) $\vec{c} = 5\hat{\imath} + 6\hat{\jmath}$; magnitude 11 units

7. Find the values of unknowns if:

(i) $4\hat{\imath} + 5\hat{\jmath} = p\hat{\imath} + 5\hat{\jmath}$ (ii) $(5,7) = (5, -q)$

(iii) $\frac{7}{2}\hat{\imath} - \frac{3}{5}\hat{\jmath} = p\hat{\imath} - \frac{3}{\sqrt{25}}\hat{\jmath}$ (iv) $\begin{pmatrix} \frac{5}{2} \\ -7 \end{pmatrix} = \begin{pmatrix} p \\ -7 \end{pmatrix}$

8. Find the position vector of point dividing the join of A and B in the given ratio, where position vectors of A and B are given:

(i) $4\hat{\imath} - 5\hat{\jmath}$ and $2\hat{\imath} + 7\hat{\jmath}$ where ratio is $3:5$

(ii) $3\hat{\imath} + 5\hat{\jmath}$ and $11\hat{\imath} + 8\hat{\jmath}$ where ratio is $3:2$

(iii) $2\hat{\imath} - 3\hat{\jmath}$ and $2\hat{\imath} + 6\hat{\jmath}$ where ratio is $4:3$

9. Prove Hexagon law of vector addition.

10. If $A(3,-5), B(6,0)$ and $C(2,4)$ be vertices of a parallelogram, then using vectors find the coordinates of point D if:

(a) ABCD is the parallelogram (b) ADBC is the parallelogram

(c) ABDC is the parallelogram

11. Given that A is the point (1,3). $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $\binom{4}{1}$ and $\binom{2}{3}$ respectively. Find the coordinates of vertices B, C and D of the parallelogram ABCD.

12. Using vectors, prove that:

(i) If one pair of opposite sides of a quadrilateral are congruent and parallel then it is a parallelogram.

(ii) The line segments joining the mid points of the opposite sides of a quadrilateral bisect each other.

(iii) Medians of a triangle are concurrent.

(iv) The midpoint of the hypotenuse of a right-angled triangle is equidistant from its vertices.

13. Given that the points (1,1), (5,4), (8,9) and (0,3) represent position vectors of A,B,C and D respectively. Show that ABCD is a trapezium.

14. A, B, C are the points $\vec{a}, \vec{b}$ and $2\vec{a} - \vec{b}$ respectively. D divides $\overrightarrow{AC}$ in 2:3 and E divides $\overrightarrow{BD}$ in 4:1. Find the position vector of E.

15. In a parallelogram ABCD, X is the mid-point of $\overrightarrow{AB}$ and Y divides $\overrightarrow{BC}$ in 1:2. Show that if Z divides $\overrightarrow{DX}$ in 6:1 then it also divides $\overrightarrow{AY}$ in 3:4.

3.2 Vectors in Space

In the previous section we studied vectors in plane, now we study vectors in three-dimensional space.

3.2.1 Recognize rectangular coordinate system in space

In three-dimensional space, the rectangular coordinate system or Cartesian coordinate system is based on three mutually perpendicular lines which are called coordinate axes, viz. the x – axis, y – axis and z – axis as shown in the Fig 3.22 (a).

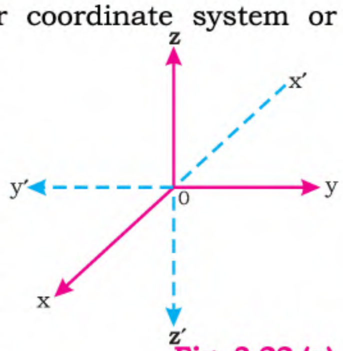


Fig. 3.22 (a)

These axes determine three mutually perpendicular planes which are called coordinate planes (yz –plane, zx –plane and xy –plane). The point of intersection of coordinate axes is called origin.

The origin along with the three axes, is said to form a Cartesian coordinate system for 3-space. It is of two types: left-handed and right-handed system.

A right-handed system has the property that when the fingers of the right-hand are curled from positive x-axis toward the positive y-axis, the thumb points roughly in the direction of the positive z-axis. The system which is not right-handed is called left-handed (Fig 3.22 (b). We shall follow the right-handed system.

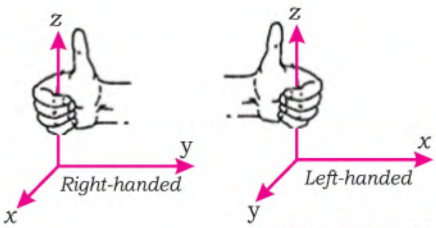


Fig. 3.22 (b)

To locate a point P in space, we draw perpendicular $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$ from P on yz-, zx- and xy -planes respectively. The lengths of these perpendiculars are taken as x, y and z. So coordinates of P are (x, y, z) as shown in Fig. 3.22(c).

It should be noted that

(i) for a point on x-axis, both $y = 0$ and $z = 0$

(ii) for a point on y -axis, both $z = 0$ and $x = 0$

(iii) for a point on z-axis, both $x = 0$ and $y = 0$

(iv) for a point on xy -plane, $z = 0$

(v) for a point on yz -plane, $x = 0$

(vi) for a point on zx -plane, $y = 0$

Furthermore, the coordinate planes divide the space into eight octants.

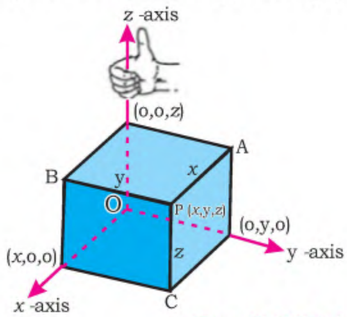


Fig. 3.22 (c)

3.2.2 Define unit vectors *i*, *j* and *k*

The vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit vectors in the direction of positive x, y, and z -axis, respectively. In terms of coordinates, we can write them as $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, and $\hat{k} = (0,0,1)$.

3.2.3 Recognize components of a vector

In three-dimensional space, vector $\vec{A}$ has vector components, $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ along x, y and z-axis respectively. Vector $\vec{A}$ is equal to the sum of its three component Fig. 3.23.

i.e., $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$

$\Rightarrow \quad \vec{A} = \vec{A}_x\hat{\imath} + \vec{A}_y\hat{\jmath} + \vec{A}_z\hat{k}$

Where, $\vec{A}_x = \vec{A}_x\hat{\imath}$, $\vec{A}_y = \vec{A}_y\hat{\jmath}$ and $\vec{A}_z = \vec{A}_z\hat{k}$.

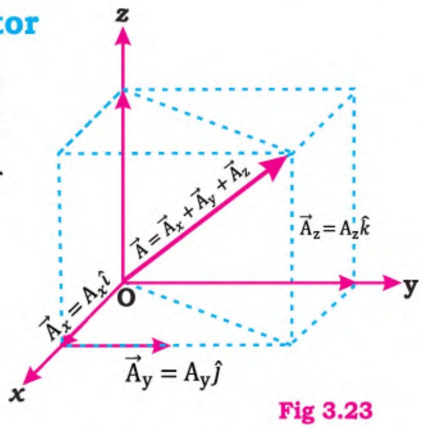


Fig 3.23

3.2.4 Give analytic representation of a vector

Let, $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be a vector in 3- space.

Here, a_1, a_2 and a_3 are real numbers. The notation (a_1, a_2, a_3) or $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ or $[a_1, a_2, a_3]$ is called the analytic representation of the vector $\vec{a}$ and the numbers a_1, a_2 and a_3 are called the *x*,y and *z* components of the vector in space respectively.

Example: In the given triangle of vertices P, Q and R with its coordinates $P(1, -1, 0)$, $Q(2,1, -1)$ and $R(-1, 1, 2)$ respectively. Find the components of the vectors $\overrightarrow{PQ}, \overrightarrow{QP}, \overrightarrow{PR}, \overrightarrow{RP}, \overrightarrow{QR}, \overrightarrow{RQ}$

Solution: Here $\overrightarrow{OP} = (1, -1, 0)$, $\overrightarrow{OQ} = (2, 1, -1)$ and $\overrightarrow{OR} = (-1, 1, 2)$ are the position vectors of points P, Q and R respectively.

Now,

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

i.e., $\overrightarrow{PQ} = (2 - 1,1 + 1, -1 - 0) = (1, 2, -1)$

$\Rightarrow$ $\overrightarrow{QP} = (-1, -2, 1)$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

i.e., $\overrightarrow{PR} = (-1 - 1,1 + 1,2 - 0) = (-2, 2, 2)$

$\Rightarrow$ $\overrightarrow{RP} = (2, -2, -2)$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

i.e., $\overrightarrow{QR} = (-1 - 2,1 - 1,2 + 1) = (-3,0, 3)$

$\Rightarrow$ $\overrightarrow{RQ} = (3,0, -3)$

3.2.5 Find magnitude of a vector

We have already studied that the length of a vector is its magnitude.

If $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$ is a vector in space, then its magnitude $|\vec{V}|$ is obtained by $|\vec{V}| = \sqrt{a^2 + b^2 + c^2}$.

Example 1. Find the magnitude of a vector $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

Solution: $\because$ $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

$\therefore$ $|\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$ units.

Example 2. Find the values of t if the vectors $3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$ and $5\hat{\imath} + 2\hat{\jmath} - t\hat{k}$ have same magnitude.

Solution: Let $\vec{a} = 3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$ and $\vec{b} = 5\hat{\imath} + 2\hat{\jmath} - t\hat{k}$

As both vectors have same magnitude i.e., $|\vec{a}| = |\vec{b}|$

$$\Rightarrow \quad \sqrt{(3)^2 + (5)^2 + (-7)^2} = \sqrt{(5)^2 + (2)^2 + (-t)^2}$$

$$\Rightarrow \quad \sqrt{9 + 25 + 49} = \sqrt{25 + 4 + t^2}$$

$$t = \pm 3\sqrt{6}$$

Example 3. Find the vector of magnitude 6 in the direction of vector $5\hat{i} + 2\hat{j} - 7\hat{k}$.

Solution: Here, $\vec{a} = 5\hat{i} + 2\hat{j} - 7\hat{k}$

So, $6\hat{a}$ is the required vector.

Now,
$$6\hat{a} = 6\frac{\vec{a}}{|\vec{a}|}$$
$$= 6\frac{(5\hat{i} + 2\hat{j} - 7\hat{k})}{\sqrt{78}}$$
$$= \frac{30\hat{i}}{\sqrt{78}} + \frac{12\hat{j}}{\sqrt{78}} - \frac{42\hat{k}}{\sqrt{78}}$$

3.2.6 Repeat all fundamental definitions for vectors in space which, in the plane, have already been discussed

(i) Equal vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal vectors if they have same corresponding components.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a} = \vec{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

(ii) Negative of a vector

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then $-\vec{a} = -a_1\hat{i} - a_2\hat{j} - a_3\hat{k}$ is negative of the given vector $\vec{a}$.

(iii) Unit vector

We have already studied that any vector whose length or magnitude is 1 is a unit vector. The vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are basic unit vectors in the direction of x, y and z -axis respectively.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a vector in space then its unit vector in the direction of $\vec{a}$ is $\hat{a}$ and defined as; $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{a_1\hat{i}+a_2\hat{j}+a_3\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(iv) Zero or Null vector

We know that a vector with magnitude zero, is called zero vector. In space $\begin{pmatrix}0\\0\\0\end{pmatrix}$ is zero vector or null vector.

(v) Position vector

Position vector is the vector which represents the position of a point in space with respect to the origin O.

It also represents the distance and direction of the point from the origin. If $P(x, y, z)$ is the point then its position vector is represented by

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

as shown in Fig. 3.24

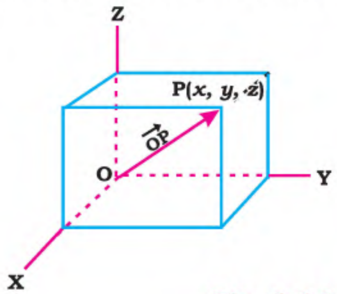


Fig. 3.24

Example:

The position vector of point P is $\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and the position vector of point Q is $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$, then find vector $\overrightarrow{PQ}$

Solution:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3-4 \\ -2-2 \\ 7+1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}$$

Therefore, $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}$

(vi) Parallel vectors

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are parallel if, and only if, they are scalar multiples of one another:

i.e., if $\vec{a} = k\vec{b}$ or $\vec{b} = h\vec{a}$ where k and h are non-zero real numbers.

In other words, if two vectors are parallel, then the ratios of each of their corresponding components are same.

i.e., $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

(vii) Addition and Subtraction of vectors

Addition of vectors

Two vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are added by adding their corresponding components.

i.e., $$\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

Subtraction of vectors

Difference of two vectors $\vec{a}$ and $\bar{b}$ is obtained by subtracting their corresponding components.

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are two vectors then the difference $\vec{a} - \vec{b}$ is obtained as;

$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

Example: If $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ then find

(i) $\vec{a} + \vec{b}$ (ii) $\vec{a} - \vec{b}$

Solution:

(i) $\vec{a} + \vec{b}$; $\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ 4 \end{pmatrix}$

(ii) $\vec{a} - \vec{b}$; $\vec{a} - \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

(viii) Scalar Multiplication

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ be a vector and $k \in \mathbb{R}$ then; $k\vec{a} = k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$

Example: If $\vec{a} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$ then find: (i) $-\vec{a}$ (ii) $3\vec{a}$

Solution:

(i) $-\vec{a} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}$

(ii) $3\vec{a} = 3\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -15 \\ 12 \end{pmatrix}$

Exercise 3.2

1. Let $\vec{u} = \hat{i} + 2\hat{j} + \hat{k}, \vec{v} = 3\hat{i} - 5\hat{j}$ and $\vec{w} = -8\hat{i} + 7\hat{j} - 2\hat{k}$. Find:

(i) $3\vec{u} + 2\vec{v}$ (ii) $\vec{v} - 2\vec{u}$ (iii) $\vec{u} - 3\vec{v} + 2\vec{w}$

(iv) $3\vec{u} + \vec{v} - \vec{w}$ (v) $-2\vec{u} + \frac{1}{2}\vec{v} - 3\vec{w}$

2. If $\vec{a} = -5\hat{i} + 3\hat{j} - 4\hat{k}$ then find:

(i) $-2\vec{a}$ (ii) $3|\vec{a}|$ (iii) $\frac{4}{5}\vec{a}$ (iv) $-\frac{1}{2}\vec{a}$

3. Let $\vec{A} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}, \vec{C} = 3\hat{i} - \hat{j} + 5\hat{k}$. Find

(i) $\vec{A} - 2\vec{B}$ (ii) $3\vec{B} + 2\vec{C}$ (iii) $3\vec{A} - (2\vec{B} + \vec{C})$

4. Let $\vec{u} = \hat{i} - 3\hat{j} + 2\hat{k}, \vec{v} = \hat{i} + \hat{j}$ and $\vec{w} = 2\hat{i} + 2\hat{j} - 4\hat{k}$. Find

(i) $|\vec{u} + \vec{v} - \vec{w}|$ (ii) $|\vec{u}| + |\vec{v}|$ (iii) $\left|\frac{1}{|\vec{w}|}\vec{w}\right|$

5. **(a)** The position vector of point P is $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ and the position vector of point Q is $\begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}$. Find $4\overrightarrow{PQ}$ and $|5\overrightarrow{PQ}|$.

(b) If $\vec{a} = 3\hat{i} - 5\hat{j} + 2\hat{k}, \quad \vec{b} = -\hat{i} - 2\hat{j} + 3\hat{k}$ then find

(i) $\hat{a}$ (ii) $\hat{b}$ (iii) $|\vec{a} + \vec{b}|$ (iv) $|\vec{a} - \vec{b}|$

6. Find vector A_1A_2 when:

(i) $A_1(0,0,0), \ A_2(-2,5,1)$ (ii) $A_1(2,1,-3), \ A_2(7,1,-3)$

(iii) $A_1(5,-2,1), \ A_2(2,4,2)$

7. (i) Find the initial point of the vector $\vec{a} = (-2,1,2)$ if the terminal point \`is $(4,0,-1)$.

(ii) Find the terminal point of the vector $\overrightarrow{A_1A_2} = (1,3,-3)$ if the initial point is $(-2,1,4)$.

8. The initial point of a vector $\vec{a}$ of magnitude 5 is $(1,-\sqrt{3},-5)$. Find k if the terminal point is:

(a) $(3,\sqrt{3},k)$ (b) $(2,-3\sqrt{3},-10k)$ (c) $(-2,k,-3)$

9. Find the coordinates of A, if $\overrightarrow{OA}$ is of length 6 units in the direction of $\overrightarrow{OB}$, where B is the point $(2,-1,4)$.

10. P,Q,R,S are the points with position vectors given by $(1,1,-1)$, $(1,-1,2), (0,1,1)$ and $(2,1,0)$ respectively.

(i) Find $|\overrightarrow{PQ}|$ and $|\overrightarrow{QS}|$.

(ii) Find the position vector of the point which:

(a) divides $\overline{QR}$ internally in the ratio 3: 2

(b) divides $\overline{PR}$ externally in the ratio 3: 2

(iii) If X and Y are the midpoints of $\overline{PR}$ and $\overline{RS}$. Show that $\overrightarrow{XY} = \frac{1}{2}\overrightarrow{PS}$.

11. Find the vector $\overrightarrow{OA}$, where:

(i) $|\overrightarrow{OA}| = 6$ and $\overrightarrow{OA}$ is in the direction of the vector $(2, -3, 6)$.

(ii) $|\overrightarrow{OA}| = 2$ and $\overrightarrow{OA}$ is in the opposite direction of the vector $(8, 1, -4)$.

12. **(a)** Find the magnitude of a vector $\vec{a} = -2\hat{i} + 3\hat{j} - 4\hat{k}$.

(b) Find the value of t if the vectors $-3\hat{i} + 5\hat{j} - 6\hat{k}$ and $2\hat{i} - 3\hat{j} + t\hat{k}$ have the same magnitude.

(c) Find the vector whose magnitude is 5 times the length of $\vec{a}$ and is in the opposite direction of $\vec{a} = 4\hat{i} - 5\hat{j} + 6\hat{k}$.

(d) Express $\vec{a} = -2\hat{i} + 4\hat{j} - 6\hat{k}$ as a product of its magnitude and direction.

13. Find the length of the median through point O of the triangle OCD, if C and D are $(2, 7, -1)$ and $(4, 1, 2)$ respectively.

14. If S and T are the mid points of $\overrightarrow{PR}$ and $\overrightarrow{QR}$ show that $\overrightarrow{ST} = \frac{1}{2}\overrightarrow{PQ}$.

3.3 Properties of Vector Addition

3.3.1 State and prove

i. commutative law for vector addition

ii. associative law for vector addition

(i) Commutative law for vector addition

This law states that the sum of two vectors remains same irrespective of their order.

Let $\vec{A}$ and $\vec{B}$ are two vectors then by commutative law for vector addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Consider two vectors $\vec{A}$ and $\vec{B}$, which represent two adjacent sides of a parallelogram OPQR as shown in Fig 3.25.

In ΔOPQ, by triangle law of vector addition

$$\vec{R} = \vec{B} + \vec{A} \quad \ldots(i)$$

In ΔOQR, by triangle law of vector addition

$$\vec{R} = \vec{A} + \vec{B} \qquad \ldots(ii)$$

From (i) and (ii), we get $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

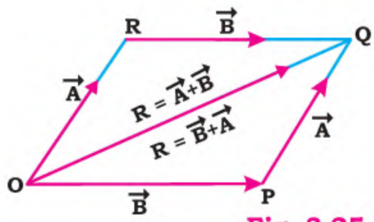


Fig. 3.25

(ii) Associative law for vector addition

This law states that the sum of three vectors remains same irrespective of their order or grouping in which they are arranged.

Let $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors then by the associative law of vector addition $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

Consider three vectors $\vec{A}, \vec{B}$ $\vec{C}$ and their resultant vector $\vec{R}$ represented by line segments $\overline{OP}$, $\overline{PQ}$, $\overline{QR}$ and $\overline{OR}$ respectively as shown in Fig. 3.26.

We apply triangle law of vector addition to obtain $(\vec{A}+\vec{B})$ and $(\vec{B}+\vec{C})$.

Now in ΔOPR; $\quad \vec{OR} = \vec{OP} + \vec{PR}$

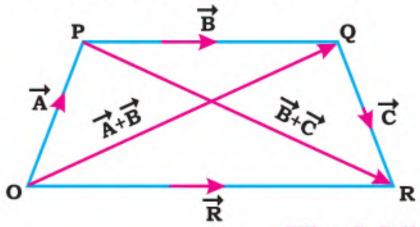
i.e., $\quad \vec{R} = \vec{A} + (\vec{B} + \vec{C}) \qquad \ldots(i)$

And in ΔOQR; $\quad \vec{OR} = \vec{OQ} + \vec{QR}$

i.e., $\quad \vec{R} = (\vec{A} + \vec{B}) + \vec{C} \qquad \ldots(ii)$

From (i) and (ii)

We get $\quad \vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$



(Fig. 3.26)

Example: $\vec{A} = 5\hat{i} + \hat{j} - \hat{k}$, $\vec{B} = \hat{i} + 5\hat{j} - 7\hat{k}$ and $\vec{C} = \hat{i} + \hat{j} - \hat{k}$ are three vectors then verify associative law of vector addition.

Verification:

Associative law of vector addition is:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

$$\begin{aligned} \text{L.H.S} = \vec{A} + (\vec{B} + \vec{C}) &= (5\hat{i} + \hat{j} - \hat{k}) + [(\hat{i} + 5\hat{j} - 7\hat{k}) + (\hat{i} + \hat{j} - \hat{k})] \\ &= (5\hat{i} + \hat{j} - \hat{k}) + (2\hat{i} + 6\hat{j} - 8\hat{k}) \\ &= (5 + 2)\hat{i} + (1 + 6)\hat{j} + (-1 - 8)\hat{k} \\ &= 7\hat{i} + 7\hat{j} - 9\hat{k} \end{aligned}$$

$$\begin{aligned} \text{R.H.S} = (\vec{A} + \vec{B}) + \vec{C} &= [(5\hat{i} + \hat{j} - \hat{k}) + (\hat{i} + 5\hat{j} - 7\hat{k})] + (\hat{i} + \hat{j} - \hat{k}) \\ &= (6\hat{i} + 6\hat{j} - 8\hat{k}) + (\hat{i} + \hat{j} - \hat{k}) \\ &= (6 + 1)\hat{i} + (6 + 1)\hat{j} + (-8 - 1)\hat{k} \\ &= 7\hat{i} + 7\hat{j} - 9\hat{k} \end{aligned}$$

$\because$ L.H.S = R.H.S

$\therefore$ $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

Hence verified.

3.3.2 Prove that

(i) $\vec{0}$ is the identity for vector addition

(ii) $-\vec{A}$ is the inverse for $\vec{A}$.

(i) $\vec{0}$ is the Identity for vector addition

Let V is the set of all vectors in space, containing vector $\vec{0}$ which satisfies the property: $\vec{0}+\vec{v}=\vec{v}+\vec{0}=\vec{v}$, for any vector $\vec{v}\in V$.

Therefore, $\vec{0}$ is the identity for vector addition.

Proof: Let $\vec{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

then there exist a vector $\vec{0}=0\hat{i}+0\hat{j}+0\hat{k}$

such that $\vec{v}+\vec{0}=(v_1\hat{i}+v_2\hat{j}+v_3\hat{k})+(0\hat{i}+0\hat{j}+0\hat{k})=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

So, $\vec{v}+\vec{0}=\vec{v}$

and similarly we can prove that $\vec{0}+\vec{v}=\vec{v}$

Hence, $\vec{0}$ is the additive identity or identity for vector addition.

(ii) $-\vec{A}$ is the additive inverse for vector $\vec{A}$.

Let $\vec{A}$ be a vector, the additive inverse $-\vec{A}$ which is also called negative vector of $\vec{A}$, is a vector when added to $\vec{A}$ gives the additive identity

i.e $\vec{A}+(-\vec{A})=\vec{0}$

Proof: Let $\vec{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ is a vector in space.

then $-\vec{A}=-a_1\hat{i}-a_2\hat{j}-a_3\hat{k}$

Now, $\vec{A}+(-\vec{A})=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}+(-a_1\hat{i}-a_2\hat{j}-a_3\hat{k})$

$=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}-a_1\hat{i}-a_2\hat{j}-a_3\hat{k} \quad =\vec{0}$

Thus, $-\vec{A}$ is the additive inverse of $\vec{A}$.

Example: If $\vec{A}=4\hat{i}+\hat{j}-\hat{k}$ and $\vec{B}=-4\hat{i}-\hat{j}+\hat{k}$ then show that $\vec{A}$ and $\vec{B}$ are additive inverses of each other.

Solution:

Here, $\vec{A}+\vec{B}=(4\hat{i}+\hat{j}-\hat{k})+(-4\hat{i}-\hat{j}+\hat{k})=(4-4)\hat{i}+(1-1)\hat{j}+(-1+1)\hat{k}=\vec{0}$

$\because \quad \vec{A}+\vec{B}=\vec{0}$

$\therefore$ $\vec{A}$ and $\vec{B}$ are additive inverses of each other. Hence shown.

3.4 Properties of Scalar Multiplication of vectors

3.4.1 State and verify

(i) commutative law for scalar multiplication: $m(\vec{a}) = (\vec{a})m$

(ii) associative law for scalar multiplication: $m\left(n(\vec{a})\right) = (mn)(\vec{a})$

(iii) distributive laws for scalar multiplication:

$m\left(\vec{a}+\vec{b}\right) = m\vec{a}+m\vec{b}$ and $(m+n)\vec{a} = m\vec{a}+n\vec{a}$

(i) Commutative law for scalar multiplication

Let $\vec{a}$ be a vector and m is any scalar such that $m \in \mathbb{R}$

Then $$m(\vec{a}) = (\vec{a})m$$

is called commutative law for scalar multiplication.

Proof:

Let $\vec{a} = [x,y,z]$ be a vector

then $m(\vec{a}) = m[x,y,z] = [mx,my,mz]$

and $= [xm,ym,zm]$

$= [x,y,z]m = (\vec{a})m$

Thus, $m(\vec{a}) = (\vec{a})m$. Hence Proved.

(ii) Associative law for scalar multiplication

Let $\vec{a}$ be a vector and m,n are scalars such that $m,n \in \mathbb{R}$

Then $$m(n(\vec{a})) = (mn)\vec{a}$$

is called associative law for scalar multiplication.

Proof:

Let $\vec{a} = [x,y,z]$ be a vector

then $n(\vec{a}) = n[x,y,z] = [nx,ny,nz]$

Now, $m\left(n(\vec{a})\right) = m[nx,ny,nz]$

$= [(mn)x,(mn)y,(mn)z]$

$= (mn)[x,y,z]$

$= (mn)\vec{a}$

Thus, $m\left(n(\vec{a})\right) = (mn)\vec{a}$ Hence proved.

(iii) Distributive laws for scalar multiplication

Let $\vec{a}$ and $\vec{b}$ are two vectors in space and $m,n \in \mathbb{R}$ are scalars then;

(i) $m\left(\vec{a}+\vec{b}\right) = m\vec{a}+m\vec{b}$ (ii) $(m+n)\vec{a} = m\vec{a}+n\vec{a}$

are called distributive laws for scalar multiplication.

(i) $m\left(\vec{a}+\vec{b}\right) = m\vec{a}+m\vec{b}$

Proof: Let $\vec{a} = [x,y,z]$ and $\vec{b} = [u,v,w]$ are two vectors in space and $m \in \mathbb{R}$ is scalar.

Then, $m(\vec{a}+\vec{b}) = m\{[x,y,z]+[u,v,w]\} = m[x+u, y+v, z+w]$
$= [mx+mu, my+mv, mz+mw]$
$= [mx, my, mz] + [mu, mv, mw]$
$= m[x,y,z] + m[u,v,w] = m\vec{a} + m\vec{b}$

Thus, $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$, Hence proved.

(ii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

Proof:

Let, $\vec{a} = [x,y,z]$ is a vector in space and $m, n \in \mathbb{R}$ are scalars,

Then, $(m+n)\vec{a} = (m+n)[x,y,z] = [(m+n)x, (m+n)y, (m+n)z]$
$= [(mx+nx), (my+ny), (mz+nz)]$
$= [mx, my, mz] + [nx, ny, nz]$
$= m[x,y,z] + n[x,y,z]$
$= m\vec{a} + n\vec{a}$

Thus, $(m+n)\vec{a} = m\vec{a} + n\vec{a}$, Hence proved .

Exercise 3.3

1. If $\vec{a} = -5\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 5\hat{k}$, then:
(i) verify: $\vec{0} + \vec{a} = \vec{a} + \vec{0} = \vec{a}$
(ii) find additive inverses of $\vec{a}$, $\vec{b}$ and $2\vec{a} + 3\vec{b}$
(iii) verify commutative law for vector addition

2. Let $\vec{u} = 3\hat{i} + \hat{j} + 5\hat{k}$, $\vec{v} = \hat{i} - 2\hat{j}$ and $\vec{w} = -\hat{i} + 3\hat{j} - \hat{k}$.
Verify associative law of vector addition.

3. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ is a vector and $m = 5$ is a scalar then verify commutative law of scalar multiplication.

4. Let $\vec{a} = 5\hat{i} + 2\hat{j} - 3\hat{k}$ is a vector and $m = 2$ and $n = 5$ then verify associative law of scalar multiplication.

5. Let $\vec{a} = 3\hat{i} + \hat{j} + 5\hat{k}$ and $\vec{b} = -5\hat{i} + 3\hat{j} - 4\hat{k}$ are two vectors and $m = 3$ and $n = 7$ are scalars, prove that;
(i) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$ (ii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

3.5 Dot or Scalar Product

3.5.1 Define dot or scalar product of two vectors and give its geometrical interpretation

Let $\vec{a}$ and $\vec{b}$ are two vectors with magnitudes $|\vec{a}| = a$ and $|\vec{b}| = b$ respectively then their dot or scalar product is denoted by $\vec{a} \cdot \vec{b}$ and is defined as: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

or $= a\,b\cos\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$ in the counter clockwise direction with $0 \le \theta \le \pi$

When two non-zero vectors $\vec{a}$ and $\vec{b}$ are placed so that their initial points coincide, they form an angle θ of measure $0 \le \theta \le \pi$ in the counter clockwise direction, whereas $\vec{b}\cos\theta$ is the component of vector $\vec{b}$ along vector $\vec{a}$ and its length $b\cos\theta$ is called projection of $\vec{b}$ onto $\vec{a}$ as shown in Fig.3.27.

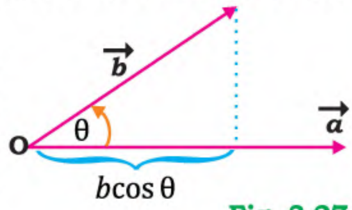


Fig. 3.27

Hence geometrically, the dot or scalar product of vector $\vec{a}$ and $\vec{b}$ is the product of magnitude of $\vec{a}$ with projection of $\vec{b}$ onto $\vec{a}$.

i.e., $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

3.5.2 Prove that:

(i) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(ii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Proof (i) : $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Since $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along coordinate axes, so they are mutually perpendicular vectors as shown in Fig. 3.28. By definition of dot product

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

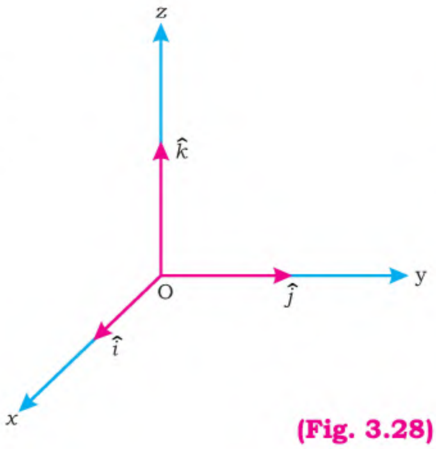
If $\vec{a} = \vec{b} = \hat{i}$, then $|\hat{i}| = 1$ and $\theta = 0°$

So, $\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0° = 1$ $[\because \cos 0° = 1]$

i.e., $\hat{i}\cdot\hat{i} = 1$

Similarly, we can prove that $\hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Hence $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$



(Fig. 3.28)

Proof (ii): $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

If $\bar{a} = \hat{i}$ and $\bar{b} = \hat{j}$, then $\theta = 90°$ and $|\hat{i}| = |\hat{j}| = 1$

So, $\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90° = 0$ $[\because \cos 90° = 0]$

i.e., $\hat{i}\cdot\hat{j} = 0$

Similarly, we can prove that $\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Hence $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

3.5.3 Express dot product in terms of components

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be two non-zero vectors. The dot product in term of component is defined as $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Proof:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Taking scalar product, we have

$$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k}) + a_2b_1(\hat{j}\cdot\hat{i}) + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) + a_3b_3(\hat{k}\cdot\hat{k})$$

Since $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Therefore, $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Hence, the dot product of two vectors is the sum of the product of their corresponding components.

Example: If $\vec{a} = \hat{i} - 4\hat{j} - 7\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$ then find $\vec{a}\cdot\vec{b}$

Solution:

Here, $\vec{a} = \hat{i} - 4\hat{j} - 7\hat{k}$ and $\bar{b} = 3\hat{i} - \hat{j} + \hat{k}$

then $\vec{a}\cdot\vec{b} = (1)(3) + (-4)(-1) + (-7)(1)$

$= 3 + 4 - 7 = 0$

So, $\vec{a}\cdot\vec{b} = 0$

3.5.4 Find the condition for orthogonality of two vectors

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$ as shown in Fig. 3.29.

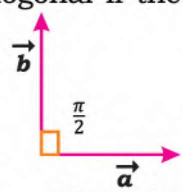


Fig. 3.29

For such vectors, we have $\vec{a}\cdot\vec{b} = 0$ because $\cos\frac{\pi}{2} = 0$

Conversely, if $\vec{a}$ and $\vec{b}$ are non-zero orthogonal vectors with $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1}0 = \frac{\pi}{2}$.

Hence the condition of orthogonality of two vectors is $\vec{a}\cdot\vec{b} = 0$.

Example 1. If $\vec{a} = 3\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$, then show that they are orthogonal.

Solution: Here, $\vec{a} = 3\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$,

Now, $\vec{a}\cdot\vec{b} = (3\hat{i} + \hat{j} + \hat{k})\cdot(\hat{i} - \hat{j} - 2\hat{k},) = (3)(1) + (1)(-1) + (1)(-2) = 0$

So, the vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Example 2. If $\vec{a} = 5\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$, find $\vec{a} \cdot \vec{b}$.
Are these vectors perpendicular to each other?

Solution:

Here, $\vec{a} = 5\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$

Now, $\vec{a} \cdot \vec{b} = (5\hat{i} + 3\hat{j} - 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 7\hat{k}) = (5)(3) + (3)(-2) + (-2)(7)$

$= 15 - 6 - 14 = -5 \neq 0$

$\because \vec{a} \cdot \vec{b} \neq 0$ the, vectors are not perpendicular.

3.5.5 Prove the commutative and distributive laws for dot product

(i) Commutative law for dot product

If $\vec{a}$ and $\vec{b}$ are two vectors, then commutative law for dot product is:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors,

then $\vec{a} \cdot \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= a_1b_1(\hat{i} \cdot \hat{i}) + a_2b_2(\hat{j} \cdot \hat{j}) + a_3b_3(\hat{k} \cdot \hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

similarly, $\vec{b} \cdot \vec{a} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \cdot (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$= b_1\, a_1(\hat{i} \cdot \hat{i}) + b_2a_2(\hat{j} \cdot \hat{j}) + b_3a_3(\hat{k} \cdot \hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

Thus, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Hence dot product obeys the commutative law.

(ii) Distributive law for dot product

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors, then distribution law for dot product is:

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three vectors,

then $\vec{a} \cdot (\vec{b} + \vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + c_1\hat{i} + b_2\hat{j} + c_2\hat{j} + b_3\hat{k} + c_3\hat{k})$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}\}$

$= a_1(b_1 + c_1)(\hat{i} \cdot \hat{i}) + a_2(b_2 + c_2)(\hat{j} \cdot \hat{j}) + a_3(b_3 + c_3)(\hat{k} \cdot \hat{k})$

$= a_1b_1 + a_1\, c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Thus, $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

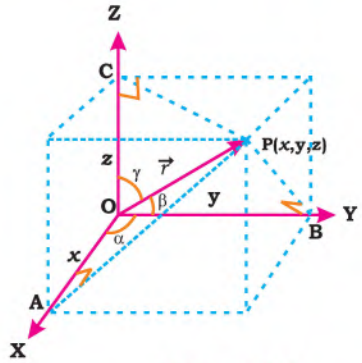
Hence dot product obeys the distributive law.

3.5.6 Explain direction cosines and direction ratios of a vector

Consider a vector $\overrightarrow{OP}$ passing through the origin O makes angles α, β and γ with the positive x, y and z -axis respectively as shown in Fig. 3.30. These angles are referred as the direction angles of $\overrightarrow{OP}$ and the cosines of these angles give us the direction cosines. These direction cosines are usually represented by l, m and n.

i.e., $\cos\alpha = \frac{x}{r} = l$, $\cos\beta = \frac{y}{r} = m$ and $\cos\gamma = \frac{z}{r} = n$,

where $|\overrightarrow{OP}| = r$



(Fig. 3.30)

These are called direction cosines.

Now, $x = r\cos\alpha$, $y = r\cos\beta$ and $z = r\cos\gamma$

These are called the direction ratios of vector $\overrightarrow{OP}$ or $\vec{r}$

where $r = |\overrightarrow{OP}| = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$

3.5.7 Prove that the sum of the squares of direction cosines is unity

Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then the square of direction cosine of $\vec{r} = 1$.

i.e., $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$

Proof: $\because$ $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$,

$\therefore$ $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$ ...(i)

We know that $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$

By squaring and adding above equations, we have

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$$

Hence, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$

Example 1. Find the direction cosines of;

(i) $\vec{r} = \hat{i}+3\hat{j}-2\hat{k}$ (ii) $\overrightarrow{AB}$; A(1,3,2) and B(5,−7,0)

Solution:

(i) $\vec{r} = \hat{i}+3\hat{j}-2\hat{k}$

$\because \quad \vec{r} = \hat{i} + 3\hat{j} - 2\hat{k}$

$\therefore \quad |\vec{r}| = r = \sqrt{(1)^2 + (3)^2 + (-2)^2} = \sqrt{14}$

Now, $\cos\alpha = \dfrac{x}{r} = \dfrac{1}{\sqrt{14}}$;

$$\cos\beta = \frac{y}{r} = \frac{3}{\sqrt{14}};$$

$$\cos\gamma = \frac{z}{r} = \frac{-2}{\sqrt{14}}$$

(ii) $\overrightarrow{AB}$; A(1,3,2) and B(5,−7,0)

Here, $\overrightarrow{AB} = (5-1)\hat{i} + (-7-3)\hat{j} + (0-2)\hat{k} = 4\hat{i} - 10\hat{j} - 2\hat{k}$

$$|\overrightarrow{AB}| = \sqrt{(4)^2 + (-10)^2 + (-2)^2} = \sqrt{120} = 2\sqrt{30}$$

$$\cos\alpha = \frac{x}{|\overrightarrow{AB}|} = \frac{4}{2\sqrt{30}} = \frac{2}{\sqrt{30}}$$

$$\cos\beta = \frac{y}{|\overrightarrow{AB}|} = \frac{-10}{2\sqrt{30}} = \frac{-5}{\sqrt{30}}$$

and $\cos\gamma = \dfrac{z}{|\overrightarrow{AB}|} = \dfrac{-2}{2\sqrt{30}} = \dfrac{-1}{\sqrt{30}}$

Example 2. If measures of two direction angles of a vector are $\alpha = 30°$ and $\beta = 60°$. Find measure of third direction angle.

Solution:

Let $\alpha = 30°$, $\beta = 60°$, $\gamma = ?$

We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\cos^2 30° + \cos^2 60° + \cos^2\gamma = 1$$

$$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (\cos\gamma)^2 = 1$$

$$(\cos\gamma)^2 = 1 - \frac{3}{4} - \frac{1}{4} = 0$$

$$\cos\gamma = 0$$

$$\gamma = \cos^{-1}(0) = 90° \text{ or } 270°$$

3.5.8 Use dot product to find the angle between two vectors

Let $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ are two vectors

By the definition of scalar product $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \qquad \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{(a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k})\cdot(b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k})}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

$$\theta = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Note:

(i) If $\theta = 0$ or π, then vectors $\vec{a}$ and $\vec{b}$ are collinear.

(ii) If $\theta = 90°$ or $\frac{\pi}{2}$, then vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Example 1. Find angle 'θ' between the vectors $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

Solution:

As, $\quad \vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k} \quad$ and $\quad \vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

So, $\quad \vec{a}\cdot\vec{b} = (3\hat{\imath} - 2\hat{\jmath} + \hat{k})\cdot(\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) = (3)(1) + (-2)(-2) + (1)(3) = 10$

Also, $\quad |\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$

So, $\quad \theta = \cos^{-1}\left(\frac{10}{\sqrt{14}\cdot\sqrt{14}}\right) = \cos^{-1}\left(\frac{10}{14}\right) = 44.4°$

Example 2. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $\vec{a}\cdot\vec{b} = 1$.

Solution:

$\because \quad \vec{a}\cdot\vec{b} = 1, \quad |\vec{a}| = 1 \quad$ and $\quad |\vec{b}| = 2$

$\therefore \quad \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60°$ or $\frac{\pi}{3}$

3.5.9 Find the projection of a vector along another vector

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are two vectors, then the projection of a vector $\vec{a}$ along a vector $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Consider two vectors $\vec{a}$ and $\vec{b}$, the angle between them is α. Draw a perpendicular $\overline{AB}$ on vector $\vec{b}$ from $\vec{a}$. The projection of $\vec{a}$ along $\vec{b}$ is

$|\overline{OB}| = |\vec{a}| \cos\alpha$ as shown in Fig 3.31.

In ΔOBA, we have

$$\frac{|\overline{OB}|}{|\overline{OA}|} = \cos\alpha$$

$$\Rightarrow \quad |\overrightarrow{OB}| = |\overrightarrow{OA}| \cos\alpha = |\vec{a}| \cos\alpha = \frac{|\vec{a}||\vec{b}|\cos\alpha}{|\vec{b}|}$$

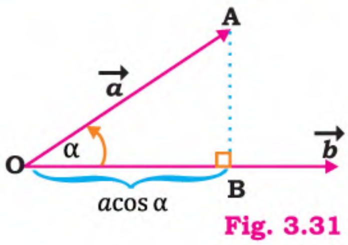


Fig. 3.31

By the definition of scalar product, we have

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha$$

So, $\quad |\overrightarrow{OB}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

This shows that the projection of $\vec{a}$ along $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Similarly, in ΔOCB (Fig 3.32), we have

The projection of vector $\vec{b}$ along $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

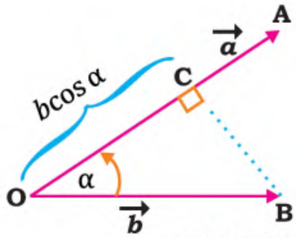


Fig.3.32

Example: Find the projection of $\hat{i} - 2\hat{j} + 3\hat{k}$ along $3\hat{i} - \hat{j} - 5\hat{k}$

Solution:

Let $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\bar{b} = 3\hat{i} - \hat{j} - 5\hat{k}$ then the projection of $\vec{a}$ along $\vec{b}$ is:

$$\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{(\hat{i} - 2\hat{j} + 3\hat{k})\cdot(3\hat{i} - \hat{j} - 5\hat{k})}{\sqrt{(3)^2 + (-1)^2 + (-5)^2}} = \frac{(1)(3) + (-2)(-1) + (3)(-5)}{\sqrt{9 + 1 + 25}} = \frac{-10}{\sqrt{35}}$$

3.5.10 Find the work done by a constant force in moving an object along a given vector

If a constant force $\vec{F}$ acts on an object for some time and it covers displacement $\vec{d}$, then by the definition, the work done is

$$W = \vec{F}\cdot\vec{d} = Fd\cos\theta\ ,$$

where θ is the angle between the force $\vec{F}$ and displacement $\vec{d}$.

Example: A particle moving in space undergoes a displacement $\vec{d} = 2\hat{i} + 2\hat{j} + 2\hat{k}$ as a constant force $\vec{F} = 7\hat{i} + 9\hat{j} - 11\hat{k}$ acts on the particle. Calculate;

(i) magnitude of force and displacement.

(ii) work done by the force.

Solution: Here $\quad \vec{F} = 7\hat{i} + 9\hat{j} - 11\hat{k}$ and $\vec{d} = 2\hat{i} + 2\hat{j} + 2\hat{k}$

(i) Magnitude of Force and displacement:

$$|\vec{F}| = \sqrt{(7)^2 + (9)^2 + (-11)^2} = \sqrt{251}$$

$$|\vec{d}| = \sqrt{(2)^2 + (2)^2 + (2)^2} = 2\sqrt{3}$$

(ii) Work done by the force:

$$W = \vec{F} \cdot \vec{d} = (7\hat{i} + 9\hat{j} - 11\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 2\hat{k}) = 14 + 18 - 22$$

$$W = 10 \text{ Joules}$$

3.5.11 Solve daily life problems based on work done

Example 1. A father pulls a child in a sleigh with a force of 150 N at an angle of 30 degrees with the ground. How much work is done over 2 km walk.

Solution: Work done = (Force).(Displacement)

$$= |\vec{F}|\,|\vec{d}| \cos\theta = (150)(2000) \cos 30^\circ$$

$$= 259807.6 \text{ Nm (Joules)}$$

Example 2. A box is dragged along the floor by the rope which makes an angle 60° with the horizontal. The force applied by the rope is 100 N and the work done by the force is 500 Joules. Find how much distance, the box is dragged.

Solution:

Here force and displacement are at an angle of 60°.

Now, Work done = Dot product of Force and Displacement

$$= |\vec{F}||\vec{d}| \cos\theta$$

$$500 = (100)(d) \cos 60^\circ$$

$$d = \frac{5}{0.5} = 10 \text{ meters.}$$

Exercise 3.4

1. If $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} - \hat{k}$ then find:

(i) $\vec{a} \cdot \vec{b}$ (ii) $\vec{a} \cdot (\vec{b} + \vec{c})$ (iii) $(2\vec{a} + 3\vec{b}) \cdot (\vec{a} - 2\vec{b})$

2. If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that:

(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (ii) Vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

3. Check whether the following vectors are orthogonal or not;

(i) $\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$, $\vec{b} = 5\hat{i} + 8\hat{j} + 7\hat{k}$

(ii) $\vec{a} = \hat{i} + 3\hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} + 3\hat{k}$

4. Verify the distributive laws for dot product for the following vectors.

(i) $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{c} = 5\hat{i} - \hat{j} + 2\hat{k}$

(ii) $\vec{a} = 3\hat{i} - 5\hat{j} + 7\hat{k}$, $\vec{b} = -4\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 6\hat{i} + \hat{j} - 2\hat{k}$

5. Find the angle between the given vectors;

(i) $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

(ii) $3\hat{i}+5\hat{j}-3\hat{k}$ and $\hat{i}-4\hat{j}+\hat{k}$

6. Find the direction cosines and direction angles for the following vectors.

(i) $2\hat{i}+\hat{j}-3\hat{k}$ (ii) $2\hat{i}-4\hat{j}+5\hat{k}$

(iii) $-3\hat{i}+4\hat{j}+5\hat{k}$

7. If α,β,γ are the direction angles of a vector then show that;
$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

8. If measures of two of the direction angles of a vector are 45° and 60°. Find measure of third angle.

9. Find the work done by the force $\vec{F}=7\hat{i}+9\hat{j}-11\hat{k}$ in moving an object along a straight line from (4,2,7) to (6,4,9).

10. Calculate the projection of $\bar{a}$ along $\bar{b}$ and projection of $\bar{b}$ along $\bar{a}$ when;

(i) $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-5\hat{k}$

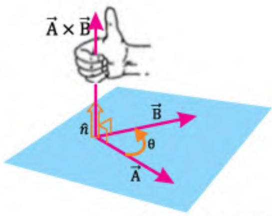
(ii) $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$

11. Calculate the work done by a Force $\vec{F}=3\hat{i}+4\hat{j}+5\hat{k}$ in displacing a body from position B to position A along a straight path. The position vectors of A and B are respectively given as $\vec{r_A}=2i+5j-2k$ and $\vec{r_B}=7i+3j-5k$.

3.6 Cross or Vector Product

3.6.1 Define cross or vector product of two vectors and give its geometrical interpretation

Let $\vec{A}$ and $\vec{B}$ are two non-zero vectors in space. Vectors $\vec{A}$ and $\vec{B}$ are not parallel as such these vectors determine a plane and $\hat{n}$ is a unit vector perpendicular to the plane as per the right-hand rule. The angle θ between $\vec{A}$ and $\vec{B}$ is taken positive as it is measured in anticlockwise direction as shown in Fig.3.33. So, the vector or cross product is defined as



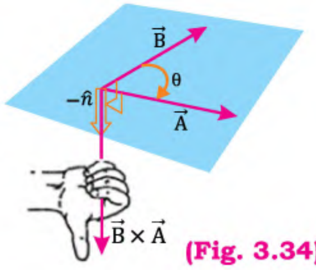
(Fig. 3.33)

$$\vec{A}\times\vec{B}=|\vec{A}||\vec{B}|\sin\theta\,\hat{n}\,,\quad (0\le\theta\le\pi)$$

Now, if we reverse the order of the vectors, it reverses the direction of the product. The angle is now from vector $\vec{B}$ to $\vec{A}$ and the unit normal vector is $-\hat{n}$ whereas $\vec{B}\times\vec{A}$ will be negative of $\vec{A}\times\vec{B}$ as shown in Fig. 3.34. Therefore the vector product $\vec{B}\times\vec{A}$ is:

$$\vec{B}\times\vec{A} = -(\vec{A}\times\vec{B}) \qquad \ldots(i)$$

Geometrically, $\vec{A}\times\vec{B}$ is a vector whose length is numerically equal to the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$. We will prove it in section 3.6.4.



(Fig. 3.34)

3.6.2 Prove that:

i. $\hat{\imath}\times\hat{\imath}=\hat{\jmath}\times\hat{\jmath}=\hat{k}\times\hat{k}=\vec{0}$,

ii. $\hat{\imath}\times\hat{\jmath}=-\hat{\jmath}\times\hat{\imath}=\hat{k}$,

iii. $\hat{\jmath}\times\hat{k}=-\hat{k}\times\hat{\jmath}=\hat{\imath}$,

iv. $\hat{k}\times\hat{\imath}=-\hat{\imath}\times\hat{k}=\hat{\jmath}$.

When we apply the definition of cross or vector product of two vectors to calculate the pair wise cross products of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, we find

i. $\hat{\imath}\times\hat{\imath}=\hat{\jmath}\times\hat{\jmath}=\hat{k}\times\hat{k}=\vec{0}$

ii. $\hat{\imath}\times\hat{\jmath}=-\hat{\jmath}\times\hat{\imath}=\hat{k}$

iii. $\hat{\jmath}\times\hat{k}=-\hat{k}\times\hat{\jmath}=\hat{\imath}$

iv. $\hat{k}\times\hat{\imath}=-\hat{\imath}\times\hat{k}=\hat{\jmath}$

Proof:

(i) $\hat{\imath}\times\hat{\imath}=\hat{\jmath}\times\hat{\jmath}=\hat{k}\times\hat{k}=\vec{0}$

As the angle between two same vectors is always zero i.e., $\theta = 0°$, so their cross product is calculated as;

$\hat{\imath}\times\hat{\imath} = |\hat{\imath}||\hat{\imath}|\sin 0°\,\hat{n}$

$\hat{\imath}\times\hat{\imath} = \vec{0}$ $\qquad$ ($\because |\hat{\imath}| = 1$ and $\sin 0° = 0$)

Similarly, $\hat{\jmath}\times\hat{\jmath}=\hat{k}\times\hat{k}=\vec{0}$

(ii) $\hat{\imath}\times\hat{\jmath}=-\hat{\jmath}\times\hat{\imath}=\hat{k}$

Since $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are unit vectors along co-ordinate axes, so these are mutually orthogonal vectors.

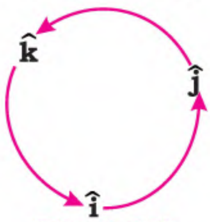
By definition of cross product of two vectors;

$\hat{\imath}\times\hat{\jmath} = |\hat{\imath}||\hat{\jmath}|\sin 90°\,\hat{n}$

$= \hat{n}$ $\qquad$ [$\because |\hat{\imath}| = |\hat{\jmath}| = 1$ and $\sin 90° = 1$]

$= \hat{k}$ $\qquad$ ($\because \hat{n} = \hat{k}$ in this case)

Also, $\hat{\imath}\times\hat{\jmath}=-\hat{\jmath}\times\hat{\imath}=\hat{k}$ $\qquad$ (Using (i) of section 3.6.1)



(Fig. 3.35)

Similarly,

(iii) $\hat{\jmath}\times\hat{k}=-\hat{k}\times\hat{\jmath}=\hat{\imath}$

(iv) $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

Fig. 3.35 helps to remember these results. In anticlockwise direction

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{k} \times \hat{i} = \hat{j}$$

Whereas, in clockwise direction

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{i} \times \hat{k} = -\hat{j} \text{ and } \hat{k} \times \hat{j} = -\hat{i}.$$

3.6.3 Express cross product in terms of components

If $\qquad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then

$$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) + a_2b_3(\hat{j} \times \hat{k}) + a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j})$$
$$= a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i}$$
$$= a_2b_3\hat{i} - a_3b_2\hat{i} - a_1b_3\hat{j} + a_3b_1\hat{j} + a_1b_2\hat{k} - a_2b_1\hat{k}$$
$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

This is the cross product of two vectors in terms of components and it can be written in determinant form as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: If $\vec{a} = -2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$ then find $\bar{a} \times \bar{b}$.

Solution: Here, $\quad \vec{a} = -2\hat{i} + 5\hat{j} - 3\hat{k}$, $\quad$ and $\quad \vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$

We know that;

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 5 & -3 \\ 3 & 2 & 1 \end{vmatrix}$$
$$= \hat{i}(5 + 6) - \hat{j}(-2 + 9) + \hat{k}(-4 - 15)$$
$$= 11\hat{i} - 7\hat{j} - 19\hat{k}$$

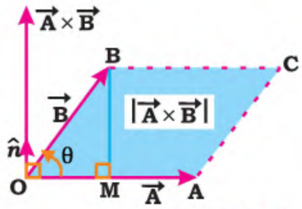
3.6.4 Prove that the magnitude of $\vec{A} \times \vec{B}$ represents the area of a parallelogram with adjacent sides $\vec{A}$ and $\vec{B}$

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector in the direction given by the right-hand rule, however, its magnitude is equal to the area of the parallelogram that is determined by $\vec{A}$ and $\vec{B}$ as shown in Fig 3.36.

i.e., $|\vec{A} \times \vec{B}|$ = Area of the parallelogram determined by $\vec{A}$ and $\vec{B}$.

Proof:

Let $\vec{A}$ and $\vec{B}$ are two non-zero vectors in space as the adjacent sides of the parallelogram OACB and θ is the angle between them measured from $\vec{A}$ to $\vec{B}$ as shown in Fig 3.36. The cross product of the vectors is defined by the formula



(Fig. 3.36)

$$\vec{A} \times \vec{B} = (|\vec{A}||\vec{B}|\sin\theta)\hat{n}$$

where, $\hat{n}$ is the unit vector in the direction of $\vec{A} \times \vec{B}$.

Magnitude of vector $\vec{A}$ is the base of the parallelogram. Draw a perpendicular from head of vector $\vec{B}$ to the base of $\vec{A}$. Now, $|\overline{BM}| = |\vec{B}|\sin\theta$, is the height of the parallelogram.

From elementary geometry, we have

Area of the parallelogram = base x height

$$= |\vec{A}||\vec{B}|\sin\theta$$

Hence, $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta$ = Area of the parallelogram

Area of Triangle

If $\vec{A}$ and $\vec{B}$ are two vectors as sides of triangle, also as adjacent sides of parallelogram and $|\vec{A}|, |\vec{B}|$ represent the length of adjacent sides of a parallelogram.

From elementary geometry

$$\text{Area of triangle} = \frac{1}{2}\text{ (area of parallelogram)}$$

$$= \frac{1}{2}|\vec{A} \times \vec{B}|$$

Example 1. Find the area of parallelogram with two adjacent sides represented by vectors.

$\hat{i} - \hat{j} + 3\hat{k}$ and $2\hat{i} - 5\hat{j} + 2\hat{k}$

Solution: Let $\vec{A} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - 5\hat{j} + 2\hat{k}$

Here, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -5 & 2 \end{vmatrix} = \hat{i}(-2+15) - \hat{j}(2-6) + \hat{k}(-5+2) = 13\hat{i} + 4\hat{j} - 3\hat{k}$

Now $|\vec{A} \times \vec{B}| = \sqrt{(13)^2 + (4)^2 + (-3)^2} = \sqrt{194} = 13.9$ square units.

Hence, area of the parallelogram is 13.9 square units.

Example 2. Find the area of triangle with vertices A(3,5,7), B(3,7,9) and C(5,3,2).

Solution: Here, $\overrightarrow{AB} = (3-3)\hat{i} + (7-5)\hat{j} + (9-7)\hat{k} = 2\hat{j} + 2\hat{k}$

$$\overrightarrow{AC} = (5-3)\hat{i} + (3-5)\hat{j} + (2-7)\hat{k} = 2\hat{i} - 2\hat{j} - 5\hat{k}$$

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 2 \\ 2 & -2 & -5 \end{vmatrix} = \hat{i}(-10+4) - \hat{j}(0-4) + \hat{k}(0-4) = -6\hat{i} + 4\hat{j} - 4\hat{k}$

So, $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-6)^2 + (4)^2 + (-4)^2} = \sqrt{68} = 2\sqrt{17}$

Area of triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{2\sqrt{17}}{2} = \sqrt{17}$ square units.

3.6.5 Find the condition for parallelism of two non-zero vectors

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors which are parallel vectors then the angle θ between them is zero or π.

We know that the cross product $\vec{a} \times \vec{b} = |\bar{a}||\bar{b}| \sin \theta° \hat{n}$, which is zero at $\theta = 0$ or π.

i.e., Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

This is the condition of parallelism of two vectors $\vec{a}$ and $\vec{b}$.

Example: Let $\vec{A} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{B} = 9\hat{i} - 3\hat{j} + 6\hat{k}$ are two vectors. Show that they are parallel.

Solution:

As non-zero vectors $\vec{A}$ and $\vec{B}$ are parallel if and only if $\vec{A} \times \vec{B} = \vec{0}$

Now, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 9 & -3 & 6 \end{vmatrix} = \hat{i}(-6+6) - \hat{j}(18-18) + \hat{k}(-9+9) = \vec{0}$

Hence, $\vec{A}$ and $\vec{B}$ are parallel.

3.6.6 Prove that $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

Let $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then, from the definition of cross product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

By using the property of determinant

$$\vec{A} \times \vec{B} = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}; \; R_{23}$$

Thus, $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

Example: If $\vec{A} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{B} = 3\hat{i} + 5\hat{j} - \hat{k}$ then verify that

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$$

Solution: We have $\vec{A} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{B} = 3\hat{i} + 5\hat{j} - \hat{k}$

Now, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 3 & 5 & -1 \end{vmatrix} = \hat{i}(1-5) - \hat{j}(-3-3) + \hat{k}(15+3) = -4\hat{i} + 6\hat{j} + 18\hat{k}$

and $\vec{B} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & -1 \\ 3 & -1 & 1 \end{vmatrix} = \hat{i}(5-1) - \hat{j}(3+3) + \hat{k}(-3-15)$

$$= -\left(-4\hat{i} + 6\hat{j} + 18\hat{k}\right) = -(\vec{A} \times \vec{B})$$

Thus, $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$ Hence verified.

3.6.7 Prove the distributive laws for cross product

Let $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors, then distributive law is defined as:

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Proof:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

then, $\vec{b} + \vec{c} = \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right)$

$$= (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

and $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ using property of determinant}$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Hence, the distributive property holds for the cross product.

Example: If $\vec{a} = \hat{i} - 2\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 2\hat{j}$

then verify that; $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Verification:

Here $\vec{a} = \hat{i} - 2\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 2\hat{j}$

$$\vec{b} + \vec{c} = \left(2\hat{i} + \hat{j} - 3\hat{k}\right) + (3\hat{i} - 2\hat{j}) = 5\hat{i} - \hat{j} - 3\hat{k}$$

and $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 5 & -1 & -3 \end{vmatrix} = \hat{i}(0-2) - \hat{j}(-3+10) + \hat{k}(-1-0)$

$$= -2\hat{i} - 7\hat{j} - \hat{k}$$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}(0+2) - \hat{j}(-3+4) + \hat{k}(1-0)$

$$= 2\hat{i} - \hat{j} + \hat{k}$$

and $\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{vmatrix} = \hat{i}(0-4) - \hat{j}(0+6) + \hat{k}(-2-0)$

$= -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\bar{a} \times \bar{b} + \bar{a} \times \bar{c} = 2\hat{i} - \hat{j} + \hat{k} - 4\hat{i} - 6\hat{j} - 2\hat{k} = -2\hat{i} - 7\hat{j} - \hat{k}$

Thus, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

3.6.8 Use cross product to find the angle between two vectors

Let $\vec{a}$ and $\vec{b}$ are two non-zero vectors then by definition of cross product;

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta^\circ\, \hat{n}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta^\circ$$

We can find the angle θ from: $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right)$$

Example: If $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + 7\hat{j} - 7\hat{k}$ then find angle between $\vec{a}$ and $\vec{b}$.

Solution:

Here, $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + 7\hat{j} - 7\hat{k}$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 1 & 7 & -7 \end{vmatrix} = \hat{i}(7-0) - \hat{j}(-21-0) + \hat{k}(21+1) = 7\hat{i} + 21\hat{j} + 22\hat{k}$

So, $|\vec{a} \times \vec{b}| = \sqrt{(7)^2 + (21)^2 + (22)^2} = = \sqrt{49 + 441 + 484} = \sqrt{974}$

$|\vec{a}| = \sqrt{(3)^2 + (-1)^2 + (0)^2} = \sqrt{10}$

and $|\vec{b}| = \sqrt{(1)^2 + (7)^2 + (-7)^2} = \sqrt{99}$

Now, $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{\sqrt{974}}{\sqrt{10}\sqrt{99}} = 0.99$

$\theta = \sin^{-1}(0.99) = 82^\circ$ approximately.

3.6.9 Find the vector moment of a given force about a given point

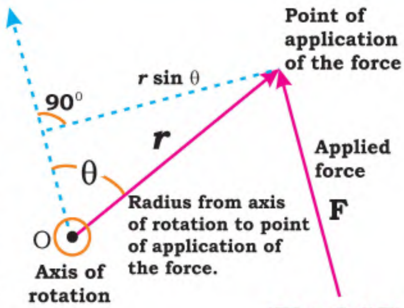
The moment M_o or torque of a force $\vec{F}$ about a point O or axis of rotation, is defined as the vector product of $\vec{r}$ and $\vec{F}$.

where, $\vec{r}$ is the radius vector or position vector, from point O of the axis of rotation to the point of application of the force $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ as shown in Fig. 3.37.

Mathematically, it can be written as

$$M_o = \vec{r} \times \vec{F}$$

Here, $|\vec{r}|\sin\theta$ is the perpendicular distance from the axis of rotation to the line of action of $\vec{F}$ called moment arm of the force, whereas θ is the angle between Axis of rotation and radius vector. The vector M_o is called vector moment of the given force $\vec{F}$ about point O.



(Fig. 3.37)

3.6.10 Application in daily life based on Cross or Vector Product

Example 1.

A force of 22N is applied to the end of 0.15 meter wrench at an angle of 75 degrees with the axis of rotation. Calculate the magnitude of the moment $\vec{M}_o$ produced by applied force.

Solution:

$$\vec{M}_o = \bar{r} \times \vec{F}$$

$$|\vec{M}_o| = |\bar{r}|\,|\vec{F}|\,Sin\,\theta$$

$$= (0.15)(22)Sin\,75° = 3.18Nm$$

Example 2. Find the moment about a point O $(1,-1,0)$ of the force $\vec{F} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ applied at $A(-3,1,2)$.

Solution:

Here $\vec{r}$ is the vector $\overrightarrow{OA}$, joining points O and A, point O is on the axis of rotation and A is the point of application of force. So moment $\vec{M}_o$ about O is given by

$$\vec{M}_o = \bar{r} \times \vec{F}$$

Here, $\vec{F} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$

and, $\vec{r} = \overline{OA} = (-3-1)\hat{\imath} + (1+1)\hat{\jmath} + (2-0)\hat{k} = -4\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

So, $\vec{M}_o = \bar{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & 2 & 2 \\ 1 & -1 & 3 \end{vmatrix} = \hat{\imath}(6+2) - \hat{\jmath}(-12-2) + \hat{k}(4-2)$

Thus, $\vec{M}_o = 8\hat{\imath} + 14\hat{\jmath} + 2\hat{k}$

Exercise 3.5

1. Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ if:
   (i) $\vec{a} = 4\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$
   (ii) $\vec{a} = -2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$
2. Verify that $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ for $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.
3. Find a unit vector which is orthogonal to both the vectors.
   $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$
4. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where; $\vec{a} = 3\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 5\hat{k}$.
5. Find the area of the parallelogram whose adjacent sides are represented by
   $\vec{a} = 5\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{k}$
6. Find the area of triangle with vertices $A(1,1,2)$, $B(2,3,5)$ and $C(1,5,5)$.
7. If $\vec{a} = 2\hat{i} + \hat{j} + 7\hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = -\hat{i} + 5\hat{j} + \hat{k}$ then verify that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
8. Using cross product, find the angle between $\hat{i} - 7\hat{j} + 7\hat{k}$ and $3\hat{i} - 2\hat{j} + 2\hat{k}$.
9. Find the moment about a point $A(1,3,5)$ of the force $\vec{F} = \hat{i} + 2\hat{j} - \hat{k}$ applied at $B(3,-2,5)$.
10. If you apply 40N force to the end of 0.3 meter wrench at an angle of 110 degrees. Calculate the magnitude of the moment $\vec{M}_o$ produced by applied force.

3.7 Scalar Triple Product

3.7.1 Define scalar triple product of vectors

The scalar triple product of three vectors is defined as a dot product of a vector with cross product of other two vectors. The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $\vec{a} \cdot (\vec{b} \times \vec{c})$. It can be written as $[\vec{a} \quad \vec{b} \quad \vec{c}]$ or $[\vec{a}\,,\ \vec{b},\ \vec{c}]$.

3.7.2 Express scalar triple product of vectors in terms of components (determinant form)

We know that if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three vectors then their scalar triple product is; $\vec{a} \cdot (\vec{b} \times \vec{c})$

By definition of cross product.

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)$$

and

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot [\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)]$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

This is scalar triple product of vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in components form which is written in determinant form as:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

Example: Find the scalar triple product of the vectors $3\hat{i} - 2\hat{j}$; $7\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} - \hat{j} + 5\hat{k}$.

Solution: Let $\vec{a} = 3\hat{i} - 2\hat{j}$, $\vec{b} = 7\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 5\hat{k}$ then;

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 0 \\ 7 & 1 & -3 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= 3(5 - 3) + 2(35 + 9) + 0$$

$$= 6 + 88 = 94$$

3.7.3 Prove that: (i) $\hat{i} \cdot \hat{j} \times \hat{k} = \hat{j} \cdot \hat{k} \times \hat{i} = \hat{k} \cdot \hat{i} \times \hat{j} = 1$ and (ii) $\hat{i} \cdot \hat{k} \times \hat{j} = \hat{j} \cdot \hat{i} \times \hat{k} = \hat{k} \cdot \hat{j} \times \hat{i} = -1$

Proof: (i) $\hat{i} \cdot \hat{j} \times \hat{k} = \hat{j} \cdot \hat{k} \times \hat{i} = \hat{k} \cdot \hat{i} \times \hat{j} = 1$

Here, $\hat{i} \cdot \hat{j} \times \hat{k} = \hat{i} \cdot \hat{i} = 1$ $[\because \hat{j} \times \hat{k} = \hat{i}$ and $\hat{i} \cdot \hat{i} = 1]$

Similarly, $\hat{j} \cdot \hat{k} \times \hat{i} = \hat{j} \cdot \hat{j} = 1$ $[\because \hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \cdot \hat{j} = 1]$

and $\hat{k} \cdot \hat{i} \times \hat{j} = \hat{k} \cdot \hat{k} = 1$ $[\because \hat{i} \times \hat{j} = \hat{k}$ and $\hat{k} \cdot \hat{k} = 1]$

so, $\hat{i} \cdot \hat{j} \times \hat{k} = \hat{j} \cdot \hat{k} \times \hat{i} = \hat{k} \cdot \hat{i} \times \hat{j} = 1$

or $[\hat{i}, \hat{j}, \hat{k}] = [\hat{j}, \hat{k}, \hat{i}] = [\hat{k}, \hat{i}, \hat{j}] = 1$

Proof: (ii) $\hat{i} \cdot \hat{k} \times \hat{j} = \hat{j} \cdot \hat{i} \times \hat{k} = \hat{k} \cdot \hat{j} \times \hat{i} = -1$

Here, $\hat{i} \cdot \hat{k} \times \hat{j} = \hat{i} \cdot (-\hat{i}) = -1$ $[\because \hat{k} \times \hat{j} = -\hat{i}]$

$\hat{j} \cdot \hat{i} \times \hat{k} = \hat{j} \cdot (-\hat{j}) = -1$ $[\because \hat{i} \times \hat{k} = -\hat{j}]$

$\hat{k} \cdot \hat{j} \times \hat{i} = \hat{k} \cdot (-\hat{k}) = -1$ $[\because \hat{j} \times \hat{i} = -\hat{k}]$

So, $\hat{i} \cdot \hat{k} \times \hat{j} = \hat{j} \cdot \hat{i} \times \hat{k} = \hat{k} \cdot \hat{j} \times \hat{i} = -1$

i.e., $[\hat{i}, \hat{k}, \hat{j}] = [\hat{j}, \hat{i}, \hat{k}] = [\hat{k}, \hat{j}, \hat{i}] = -1$

3.7.4 Prove that dot and cross are inter-changeable in scalar triple product

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be the three vectors then by definition of scalar triple product.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \text{[interchanging } R_1 \text{ and } R_2]$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \qquad \text{[interchanging } R_2 \text{ and } R_3]$$

$$= \vec{b}\cdot(\vec{c}\times\vec{a})$$

And $$\vec{b}\cdot(\vec{c}\times\vec{a}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \qquad \text{[interchanging } R_1 \text{ and } R_2]$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{[interchanging } R_2 \text{ and } R_2]$$

$$= \vec{c}\cdot(\vec{a}\times\vec{b})$$

Hence, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ or $[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b}]$

This shows that operation of dot and cross are interchangeable in scalar triple product.

3.7.5 Find the volume of :

- **a parallelepiped,**
- **a tetrahedron determined by three given vectors**

(i) Parallelepiped

Consider the parallelepiped as shown in Fig. 3.38, with three vectors $\vec{a},\vec{b}$ and $\vec{c}$ as its co-terminal edges. The volume of the parallelepiped is the area of the base times the height. From the geometric definition of the cross product, we know that the magnitude, $|\vec{b}\times\vec{c}|$, is the area of the parallelogram base, and that the direction of the vector $\vec{b}\times\vec{c}$ is perpendicular to the base.

The height h of the parallelepiped is the component of vector $\vec{a}$,

i.e., height $= h = |\vec{a}| \cos\theta$ in the direction normal to the base, i.e., in the direction of vector $\vec{b} \times \vec{c}$.,

where θ is the angle between $\vec{a}$ and $\vec{b} \times \vec{c}$.

The volume V of the parallelepiped is:

V = (base area of parallelepiped) .(height of parallelepiped)

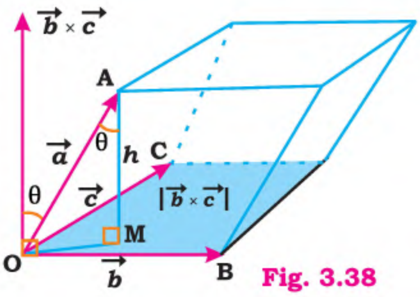


Fig. 3.38

$$= |\vec{b} \times \vec{c}|\,|\vec{a}| \cos\theta$$
$$= (\vec{b} \times \vec{c}) \cdot \vec{a}$$
$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$

So, we have volume of parallelepiped $= \vec{a} \cdot (\vec{b} \times \vec{c})$

Thus, $V = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example: Calculate the volume of the parallelepiped determined by the three vectors $-2\hat{i} - \hat{j} + 3\hat{k}$, $\hat{i} + 7\hat{j} - 2\hat{k}$ and $-\hat{j} + \hat{k}$

Solution: Let $\vec{a} = -2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} + 7\hat{j} - 2\hat{k}$ and $\vec{c} = -\hat{j} + \hat{k}$

We know that

Volume of parallelepiped $= \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} -2 & -1 & 3 \\ 1 & 7 & -2 \\ 0 & -1 & 1 \end{vmatrix} \text{ (by using components of vectors)}$$
$$= -2(7-2) + 1(1-0) + 3(-1-0)$$
$$= -10 + 1 - 3 = -12$$

$\because$ volume is always positive,

$$\therefore \text{ Required volume} = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |-12|$$
$$= 12 \text{ cubic units}$$

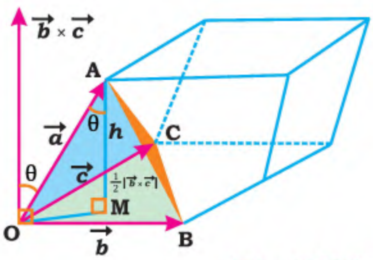
(ii) Tetrahedron

Consider a tetrahedron having vertices O, A, B and C such that

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ with respect to origin as shown in Fig. 3.39.

Area of base ΔOBC of tetrahedron $= \frac{1}{2}|\vec{b} \times \vec{c}|$

Let $\hat{n}$ be the unit vector perpendicular to



(Fig. 3.39)

the plane of ΔOBC.

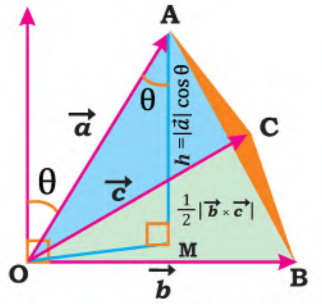
Let $\overline{AM}$ be perpendicular to $\overline{OM}$, is the height of the tetrahedron and $\angle OAM = \theta$ $|\vec{a}| \cos\theta$

So, h=$|\vec{a}| \cos\theta$, from Fig. 3.40

From elementary geometry,

Volume of tetrahedron

$$= \frac{1}{3}(\text{Base Area}) \times (\text{height of tetrahedron})$$



(Fig. 3.40)

Hence,

$$\text{Volume of tetrahedron} = \frac{1}{3}\left(\frac{1}{2}|\vec{b} \times \vec{c}|\right)(|\vec{a}| \cos\theta)$$

$$= \frac{1}{6}(\vec{b} \times \vec{c}) \cdot \vec{a} = \frac{1}{6}\vec{a} \cdot (\vec{b} \times \vec{c})$$

Hence, volume of tetrahedron $= \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$.

Example 1. Compute the volume of tetrahedron with $\vec{a} = (7,-1,5)$, $\vec{b} = (-3,1,0)$ and $\vec{c} = (3,-1,2)$ are its coterminal edges.

Solution: We know that

$$\text{Volume of tetrahedron} = \frac{1}{6}[\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$= \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} 7 & -1 & 5 \\ -3 & 1 & 0 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \frac{1}{6}\{7(2-0) + 1(-6-0) + 5(3-3)\}$$

$$= \frac{1}{6}(14 - 6 + 0)$$

$$= \frac{1}{6} \times 8 = 1.33 \text{ cubic units}$$

Example 2. Find the volume of tetrahedron whose vertices are $A(2,1,7), B(5,-1,3), C(4,3,5)$ and $D(0,2,3)$

Solution:

Here,

$$\overrightarrow{AB} = (5-2, -1-1, 3-7) = (3,-2,-4)$$
$$\overrightarrow{AC} = (4-2, 3-1, 5-7) = (2,2,-2)$$
$$\overrightarrow{AD} = (0-2, 2-1, 3-7) = (-2,1,-4)$$

we have, volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \ldots(i)$$

$$= \frac{1}{6}\begin{vmatrix} 3 & -2 & -4 \\ 2 & 2 & -2 \\ -2 & 1 & -4 \end{vmatrix}$$

$$= \frac{1}{6}[3(-8+2)+2(-8-4)-4(2+4)]$$

$$= \frac{1}{6}[-18-24-24] = \frac{1}{6}|-66|$$

So, volume of tetrahedron $= 11$ cubic units

3.7.6 Define Coplanar vectors and find the condition for coplanarity of three vectors

Coplanar Vectors

Vectors lying on the same plane are called coplanar vectors.

Condition for co-planarity of three vectors

If $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors, then $\vec{a} \times \vec{b}$, being perpendicular to the plane containing vectors $\vec{a}$ and $\vec{b}$ is also perpendicular to $\vec{c}$ as shown in Fig. 3.41. Since dot product of two perpendicular vectors is zero,

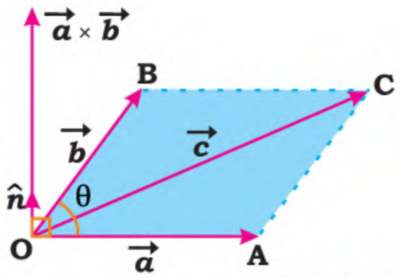


Fig. 3.41

Therefore, $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$, i.e., $[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$

Hence, three vectors are coplanar if their scalar triple product is zero. This is called the condition for co-planarity of three vectors.

Example 1. Show that $\vec{a} = \hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 6\hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} - 11\hat{k}$ are coplanar.

Solution:

Vectors $\vec{a}, \vec{b}, \vec{c}$ will be coplanar if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Now, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 3 & 5 \\ -2 & 4 & -6 \\ -3 & 1 & -11 \end{vmatrix}$

$$= 1(-44+6) - 3(22-18) + 5(-2+12)$$

$$= -38 - 12 + 50 = 0$$

Hence, vectors are coplanar.

Example 2. For what value of λ, the vectors $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{j}+4\hat{k}$ and $3\hat{i}+\hat{j}+\lambda\hat{k}$ are coplanar.

Solution:

Here, vectors are $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{j}+4\hat{k}$ and $3\hat{i}+\hat{j}+\lambda\hat{k}$

If the given vectors are coplanar, then $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \quad 1(-3\lambda-4)-2(2\lambda-12)+3(2+9)=0$ [Expanding by R_1]

$\Rightarrow -3\lambda-4-4\lambda+24+33=0$

$\Rightarrow \quad \lambda = \frac{53}{7}$

Exercise 3.6

1. Find the scalar triple product of vectors $\hat{i}+2\hat{j}+3\hat{k}, -\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.
2. Compute $[\hat{i}+\hat{j},\ \hat{i},\ \hat{i}-\hat{j}+\hat{k}]$.
3. Find the volume of parallelepiped whose edges are represented by the vectors:
   (i) $3\hat{i}+2\hat{j}-\hat{k}$; $\hat{i}-2\hat{j}+\hat{k}$; $\hat{i}+2\hat{j}-4\hat{k}$
   (ii) $\hat{i}-2\hat{j}+3\hat{k}$; $2\hat{i}-\hat{j}-\hat{k}$; $\hat{i}+2\hat{j}-4\hat{k}$
4. Find the volume of the tetrahedron whose vertices are; $A(2,1,8), B(3,2,9), C(2,1,4)$ and $D(3,3,10)$
5. Show that the vectors $4\hat{i}-\hat{j}+\hat{k}, 3\hat{i}-2\hat{j}-\hat{k}$ and $\hat{i}+\hat{j}+2\hat{k}$ are coplanar.
6. Find λ if the vectors $\hat{i}+\hat{j}+2\hat{k}, \lambda\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}-2\hat{j}-\hat{k}$ are coplanar.
7. Show that $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}-\hat{j}+3\hat{k}$ are not coplanar vectors.

Review Exercise 3

1. **Select true option.**

i. The unit vector in the direction of $\hat{r}=2\hat{i}+\hat{j}-\hat{k}$ is:

(a) $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}-\hat{k})$ (b) $\sqrt{6}(2\hat{i}+\hat{j}-\hat{k})$

(c) $6(2\hat{i}+\hat{j}-\hat{k})$ (d) $\frac{1}{6}(2\hat{i}+\hat{j}-\hat{k})$

ii. The magnitude of $\vec{a}\times\vec{b}$ represents the ________ of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$.

(a) opposite sides (b) diagonal (c) area (d) volume

iii. The volume of tetrahedron determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

(a) $[\vec{a}, \vec{b}, \vec{c}]$ (b) $\frac{1}{3}[\vec{a}, \vec{b}, \vec{c}]$ (c) $\frac{1}{6}[\vec{a}, \vec{b}, \vec{c}]$ (d) None

iv. $\vec{a} \cdot (\vec{b} \times \vec{c}) =$ ____________.

(a) $\vec{a}.\vec{b}.\vec{c}$ (b) $\vec{a} \times \vec{b} \times \vec{c}$ (c) $\vec{b} \cdot (\vec{a} \times \vec{c})$ (d) $(\vec{a} \times \vec{b}).\vec{c}$

v. If $\vec{a}$ and $\vec{b}$ are parallel then $\vec{a} \cdot \vec{b} =$ ____________

(a) 1 (b) -1 (c) 0 (d) ab

vi. If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] =$ ____________

(a) 1 (b) 0 (c) $\vec{c}$ (d) $\vec{a}$

vii. Direction cosines of the vector $\hat{i} + \hat{j} - \hat{k}$ are:

(a) $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (b) $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

(c) $1, 1, -1$ (d) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

viii. $\vec{a} \times \vec{b}$ is ______________ to plane of $\vec{a}$ and $\vec{b}$:

(a) Parallel (b) Perpendicular (c) Opposite (d) None of these

ix. If $\vec{a} = \overrightarrow{P_1P_2}$, where $P_1(0,0,1)$ and $P_2(-3,1,2)$, then $|\vec{a}| =$

(a) $\sqrt{12}$ (b) $\sqrt{10}$ (c) $\sqrt{13}$ (d) $\sqrt{11}$

x. If $\vec{p} = 4\hat{i} + 6\hat{k}$ and $\vec{q} = 6\hat{i} - 4\hat{j}$ then $|\vec{p} - \vec{q}|$ is

(a) $\sqrt{8}$ (b) $2\sqrt{14}$ (c) $2\sqrt{26}$ (d) $2\sqrt{3}$

xi. For non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b}$ is a unit vector and $|\vec{a}| = |\vec{b}| = \sqrt{2}$, then angle θ between vectors $\vec{a}$ and $\vec{b}$ is:

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{2}$

xii. Magnitude of a vector $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ is:

(a) 13 (b) $\sqrt{12}$ (c) $\sqrt{14}$ (d) $\sqrt{11}$

xiii. The position vector of the point (1, 0, 2) is:

(a) $\hat{i} + \hat{j} + 2\hat{k}$ (b) $\hat{i} + 2\hat{j}$ (c) $2\hat{i} + 3\hat{k}$ (d) $\hat{i} + 2\hat{k}$

xiv. If O be the origin and $\overrightarrow{OP} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\overrightarrow{OQ} = 5\hat{i} + 4\hat{j} - 3\hat{k}$, then $\overrightarrow{PQ}$ is equal to:

(a) $7\hat{i} + 7\hat{j} - 7\hat{k}$ (b) $-3\hat{i} + \hat{j} - \hat{k}$ (c) $3\hat{i} + \hat{j} + \hat{k}$ (d) $-7\hat{i} - 7\hat{j} + 7\hat{k}$

xv. $\hat{k} \times \hat{j} =$ ___________.

(a) $\hat{i}$ (b) $\hat{k}$ (c) $\hat{j}$ (d) $-\hat{i}$

xvi. If $|\vec{a} \times \vec{b}| = |\vec{a}.\vec{b}|$ then the angle between $\vec{a}$ and $\vec{b}$:

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

xvii. The distance of the point $(-3, 4, 5)$ from the origin is:

(a) 50 (b) $5\sqrt{2}$ (c) 6 (d) None of these

xviii. The vector having, initial and terminal points as (2, 5, 0) and $(-3, 7, 4)$ respectively is:

(a) $-\hat{i} + 12\hat{j} + 4\hat{k}$ (b) $-5\hat{i} + 2\hat{j} - 4\hat{k}$ (c) $-5\hat{i} + 2\hat{j} + 4\hat{k}$ (d) $-\hat{i} + \hat{j} + \hat{k}$

xix. If $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 0$, then the value of $|\vec{a} \times \vec{b}|$ is:

(a) 10 (b) 16 (c) 14 (d) 20

xx. The number of vectors of unit length perpendicular to the plane of the vectors

$\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ is:

(a) One (b) Two (c) Three (d) Infinite

2. Write $\overrightarrow{PQ}$ in the form $x\hat{i} + y\hat{j}$

(i) P (4,6), Q(−4,5) (ii) $P\left(-\frac{5}{2}, 9\right), Q(10, 8)$

3. Find the vector PQ joining the points P(3,4,2) and Q(−2,3,4) and also find direction cosines of $\overrightarrow{PQ}$.

4. Show that P(2,3,6), Q(3,7,4) and R(4,11,−2) are collinear.

5. Show that the vectors $\frac{1}{7}\ \hat{i} + \hat{j} + \hat{k}$, $2\hat{i} - 3\hat{j} + 3\hat{k}$ and $4\hat{i} + \hat{j} - 5\hat{k}$ are mutually perpendicular.

6. Find the area of parallelogram whose adjacent sides are $3\hat{i} - 5\hat{j} + 6\hat{k}$ and $\hat{i} + 3\hat{j} - 4\hat{k}$. Also find unit vector parallel to its diagonal through common initial point.

7. Prove that $[\vec{p} + \vec{q}, \vec{q} + \vec{r}, \vec{r} + \vec{p}] = 2\,[\vec{p}\ \vec{q}\ \vec{r}]$.

8. For the vectors $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 8\hat{j} + \hat{k}$, verify the distributive property of cross product.

Sequences and Series

Unit 4

4.1 Sequence

An ordered set of numbers, formed according to a definite rule, is called a sequence (or progression) and the individual members are called terms (or elements) of the sequence.

4.1.1 Define a sequence (progression) and its terms

We may also define a sequence of numbers as follows:

A function $a: \mathbb{N} \rightarrow \mathbb{R}$ or $\mathbb{C}$ is called a sequence, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are the set of natural, real and complex numbers respectively.

For any $n \in \mathbb{N}, a(n) \in \mathbb{R}$ $(or\ \mathbb{C})$ and is called the nth term or general term of the sequence.

We usually write $a(n)$ as a_n and the sequence as $\{a_n\}$ where $n \in \mathbb{N}$. A common representation of a finite sequence, one that has finite number of terms n is

$$a_1, a_2, a_3, \ldots, a_n$$

If the sequence has an unlimited number of terms, then it is called an infinite sequence, we may write

$$a_1, a_2, a_3, \ldots$$

or $\quad \{a_k\}$, where $k = 1,2,3, \ldots$

The nth term of a sequence is also represented by T_n (or t_n). The terms $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ (or t_1, t_2, t_3) will denote first, second and third terms of the sequence respectively. In T_n (or t_n), n indicates the position or rank of the term in the sequence.

Some examples of sequences along with their general terms are:

(i) $1, 2, 3, 4, \ldots;$ where $a_n = n$

(ii) $5, 9, 13 \ldots;$ where $a_n = 4n + 1$

(iii) $-1, 1, -1, 1, -1, \ldots;$ where $a_n = (-1)^n$

(iv) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots;$ where $a_n = \frac{n}{n+1}$

4.1.2 Know that a sequence can be constructed from a formula or an inductive definition

If we are able to find a pattern from the given initial terms of a sequence, then we can deduce a rule or formula for the terms of the sequence as explained in the following example.

Example: Find n^{th} term of the sequence:

$$6, 11, 16, 21, \ldots$$

Solution: Here, $a_1 = 6 = 5 \times 1 + 1$

$a_2 = 11 = 5 \times 2 + 1$

$a_3 = 16 = 5 \times 3 + 1$

$a_4 = 21 = 5 \times 4 + 1$

From the above pattern, we deduce that $a_n = 5n + 1 \quad (\forall n \in \mathbb{N})$

We can find any term of the sequence by giving corresponding value to n in the nth or general term a_n of a sequence. In this way, we can construct a sequence from a formula or an inductive definition as explained in the following examples.

Example 1. Find the sequence if $a_n = 2n - 1$

Solution: $a_n = 2n - 1$

For n =1, $a_1 = 2(1) - 1 = 1$

For n =2, $a_2 = 2(2) - 1 = 3$

For n =3, $a_3 = 2(3) - 1 = 5$

For n =4, $a_4 = 2(4) - 1 = 7$

Thus, the required sequence is 1, 3, 5, 7, ...

Example 2. Find the sequence if $a_n - a_{n-1} = n + 1$ and $a_4 = 14$

Solution: Putting $n = 2, 3, 4, 5$ in $a_n - a_{n-1} = n + 1$, we have,

$a_2 - a_1 = 3$ ...(i)

$a_3 - a_2 = 4$ ...(ii)

$a_4 - a_3 = 5$ ...(iii)

$a_5 - a_4 = 6$ ...(iv)

From (iv), $a_5 = a_4 + 6$

$= 14 + 6 = 20 \quad (\because a_4 = 14)$

From (iii), $a_3 = a_4 - 5$

$= 14 - 5 = 9 \quad (\because a_4 = 14)$

From (ii), $a_2 = a_3 - 4$

$= 9 - 4 = 5 \quad (\because a_3 = 9)$

and from (i), $a_1 = a_2 - 3$

$= 5 - 3 = 2 \quad (\because a_2 = 5)$

Thus, the sequence is $2, 5, 9, 14, 20, \ldots$

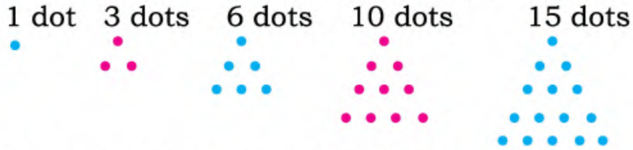
4.1.3 Recognize triangle, factorial and Pascal sequences as fractional form

(a) Triangular Sequence

Consider the following sequence;

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots$$

which simply represents the number of dots in each triangular pattern as shown in Fig. 4.1, so this is called triangular sequence:

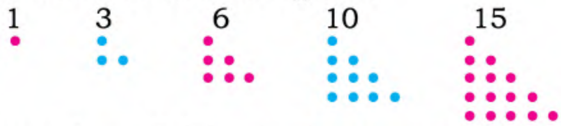


(Fig. 4.1)

By adding a row of dots in preceding pattern and counting all the dots, we can find the next term of the sequence. The first pattern has just one dot. The second pattern has another row with 2 extra dots, making $1 + 2 = 3$

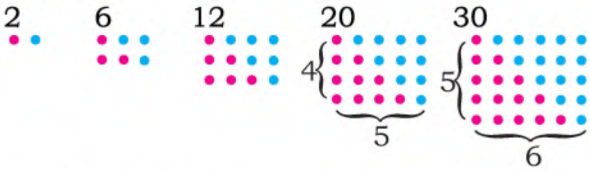
The third pattern has another row with 3 extra dots, making $1 + 2 + 3 = 6$. The fourth has $1 + 2 + 3 + 4 = 10$ etc.

We can make a "Rule" to calculate any term or triangular number. First, rearrange the dots as shown is Fig 4.2:



(Fig. 4.2)

Then double the number of dots, and arrange them into a rectangular pattern as shown in Fig 4.3



(Fig. 4.3)

Now number of dots in each rectangular pattern is:

$$\text{Number of dots} = n(n + 1)$$

Where n shows the position of term of the sequence.

But remember we doubled the number of dots, so number of dots in each term of triangular sequence is: $\frac{n(n+1)}{2}$.

Hence, $a_n = \frac{n(n+1)}{2}$ is the general term of the triangular sequence.

Example: Find the number of dots in triangular sequence for $n = 6$, 10, 13 and 16.

Solution: Using formula: $t_n = \frac{n(n+1)}{2}$

(i) When n = 6, then $t_6 = \frac{6(6+1)}{2} = 21$

(ii) When n = 10, then $t_{10} = \frac{10(10+1)}{2} = 55$

(iii) When n = 13, then $t_{13} = \frac{13(13+1)}{2} = 91$

(iv) When n = 16, then $t_{16} = \frac{16(16+1)}{2} = 136$

(b) Factorial Sequence

In mathematics, the factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For example, $3! = 3 \times 2 \times 1 = 6$ and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

We define $0! = 1$, a sequence involving factorial is called factorial sequence, for example, a sequence with $a_n = \frac{1}{n!}$

i.e., $\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots$

Example: Find the first four terms of the sequence with $a_n = \frac{2^n}{n!}$

Solution: Here, $a_n = \frac{2^n}{n!}$, so first four terms of the sequence are a_1, a_2, a_3 and a_4

Now,

$$a_1 = \frac{2^1}{1!} = 2$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2 \cdot 1} = 2$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{3 \cdot 2 \cdot 1} = \frac{8}{6} = \frac{4}{3}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{16}{24} = \frac{2}{3}$$

(c) Pascal Sequence

Pascal's Triangle

Pascal's triangle is a triangular arrangement of numbers which represent the coefficients of the expansion of power of binomial like $(a+b)^n$ as shown in Fig 4.4.

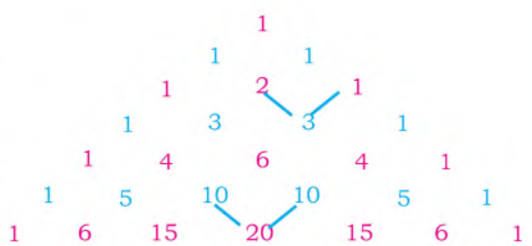


Fig. 4.4

Pascal Sequence

The number '1' is at the tip of Pascal's Triangle, which makes up the zeroth row. The first row (1 and 1) contains two 1's, both formed by adding the two numbers above them to the left and the right, in this case 1 and 0 (all numbers outside the triangle are 0's). By the same process we get the 2nd row (1, 2, 1): $0+1=1$; $1+1=2$; $1+0=1$. And the third row (1, 3, 3, 1): $0+1=1$; $1+2=3$; $2+1=3$; $1+0=1$. In this way, the rows of the triangle go on infinitely. And every row is called Pascal sequence. For finding terms of Pascal sequence, we use the formula: $\frac{n!}{r!(n-r)!}$; $r \leq n$

Where n denotes the row of Pascal triangle and r denotes its define column.

Example: Find the Pascal sequence when $n = 4$.

Solution: By using formula of Pascal sequence, when $n = 4$ and $r = 0,1,2,3,4$

$$r = 0;\quad \frac{4!}{0!(4-0)!} = \frac{4!}{1!(4-0)!} = \frac{4!}{4!} = 1 \qquad [\because\ 0! = 1]$$

$$r = 1;\,;\quad \frac{4!}{1!(4-1)!} = \frac{4!}{1!(3!)} = \frac{4\cdot 3!}{3!} = 4 \qquad [\because\ 1! = 1]$$

$$r = 2;\quad \frac{4!}{2!(4-2)!} = \frac{4!}{2(2!)} = \frac{4\cdot 3\cdot 2!}{2\cdot 2!} = 6$$

$$r = 3;\quad \frac{4!}{3!(4-3)!} = \frac{4\cdot 3!}{3!(1!)} = \frac{4}{1} = 4$$

$$r = 4;\quad \frac{4!}{4!(4-4)!} = \frac{4!}{4!(0!)} = 1$$

Hence the required Pascal sequence for $n = 4$ is (1, 4, 6,4,1).

Exercise 4.1

1. Find the nth term (rule of formation) of each of the following sequences:

(i) $2,4,6,\ldots$ (ii) $1^2, 2^2, 3^2, \ldots$ (iii) $3,9,27,\ldots$

(iv) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (v) $1, \frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \ldots$ (vi) $1,8,27,64,\ldots$

2. Find the first five terms of sequences with the given general terms.

(i) $\frac{2n}{3}(n+1)$ (ii) $(-1)^{n+1}\cdot 3^{n-1}$ (iii) $\frac{1}{4}n^2(n+1)^2$

(iv) $\frac{n}{3n+1}$ (v) $\frac{1}{6}n(n+1)(n+2)$ (vi) $\frac{1}{6}n(n+1)(2n+7)$

(vii) $a_{n+1} = 6 + a_n$, $a_1 = 2$, (viii) $a_n = \frac{n}{2}a_{n+1}$, $a_1 = 1$

3. Find the sequence by using

$T_{n+1} = (n+1)T_n$, where $T_1 = 2$,

4. Find the values of n^{th} term of triangular sequence when n = 7, 9, 12 and 16.

5. Find the first five terms of the sequence with general term: $a_n = \frac{(n+1)!}{2!}$.

6. Find the Pascal sequence when $n = 5, 6, 7$ and 8.

4.2 Arithmetic Sequence

4.2.1 Define an arithmetic sequence

A sequence in which each term is formed by adding a fixed number to the one preceding it, is called an arithmetic sequence or an arithmetic progression (A.P.).

In A.P, the difference between the two consecutive terms is same. Here difference means that second term minus the first term or third term minus second term and so on. The difference is called the common difference denoted by 'd'. For instance, the sequence 5, 7, 9, 11, 13, 15...; is an arithmetic progression with common difference of 2.

4.2.2 Find the *n*th or general term of an arithmetic sequence

If the first term of an A.P. is a and the common difference d then by definition

a_1 = the first term $= a = a + (1-1)d$
a_2 = the second term $= a + d = a + (2-1)d$
a_3 = the third term $= a + 2d = a + (3-1)d$
a_4 = the fourth term $= a + 3d = a + (4-1)d$
a_5 = the fifth term $= a + 4d = a + (5-1)d$ and so on.

Hence, we conclude that

a_n = the nth term $= a + (n-1)d$

or $\quad a_n = a + (n-1)d$

This is the formula for finding the nth or general term of an arithmetic sequence whose first term is a and the common difference d, whereas

$$a, a + d, a + 2d, \ldots, a + (n-1)d \ldots$$

is known as the standard form of A.P.

4.2.3 Solve problems involving arithmetic sequence

The formula to find nth term of an A.P. is given by

$$a_n = a + (n-1)d$$

The formula involves four elements, namely; a_n, a, n, and d. We shall now consider various types of problems based on this formula.

Example 1. Find the 10th term of the A.P: $2, 5, 8, 11, \ldots$

Solution:

Here, $n = 10,\ a = 2,\ d = 5 - 2 = 8 - 5 = 3$

By using $a_n = a + (n-1)d$

We get, $a_{10} = 2 + (10-1)3 = 29$

Thus the 10th term of the given A.P. is 29.

Example 2. Find the number of terms in the A.P., if $a = 3, d = 7$ and $a_n = 59$.

Solution: Using $a_n = a + (n-1)d$,

We get, $59 = 3 + (n-1)7$ $(\because a_n = 59, a = 3 \text{ and } d = 7)$

$\Rightarrow \quad \frac{56}{7} = n - 1$

$\Rightarrow \quad n = 9$

Thus the number of terms in the A.P. is 9.

Example 3. Find the thirteenth term of the A.P. whose first term and the common difference are 3 and -4 respectively. Also write its first four terms.

Solution: Here, $a = 3$ and $d = -4$

We know that $a_n = a + (n-1)d$.

So, $a_n = 3 + (n-1)(-4) = 3 - 4n + 4$

or $a_n = 7 - 4n$ ...(i)

Thus, the general term of the A.P. is $7 - 4n$.

Taking $n = 13$ in (i), we have

$$a_{13} = 7 - 4(13) = 7 - 52 = -45$$

We can find a_2, a_3, a_4 by using $n = 2,3,4$ in (i), that is,

$$a_2 = 7 - 4(2) = -1$$
$$a_3 = 7 - 4(3) = -5$$
$$a_4 = 7 - 4(4) = -9$$

Hence, the first four terms of the sequence are $3, -1, -5, -9$.

Example 4. If $a_{n-2} = 4n - 13$, then find the nth term of the sequence.

Solution: For the first term we take

$$n - 2 = 1$$

or $$n = 3$$

So,

For $n = 3$, $a_1 = 4(3) - 13 = -1 = a$

For $n = 4$, $a_2 = 4(4) - 13 = 3$

For $n = 5$, $a_3 = 4(5) - 13 = 7$

So, we get an A.P $-1, 3, 7, \ldots$

Thus $a_n = a + (n-1)d = -1 + (n-1)4$ $(\because a = -1 \text{ and } d = 4)$

$$a_n = 4n - 5$$

Hence this is the nth term of the sequence.

Example 5. If the 5th term of an A.P.is 13 and 17th term is 49.
Find a_n and a_{13}.

Solution: Given that $a_5 = 13$ and $a_{17} = 49$.
Putting $n = 5$ in $a_n = a + (n-1)d$, we have

$a_5 = a + (5-1)d = a + 4d$

$\Rightarrow \quad 13 = a + 4d \quad \ldots$(i)

Also $\quad a_{17} = a + (17-1)d$

$\Rightarrow \quad 49 = a + 16d \quad \ldots$(ii)

Subtracting (i) from (ii), we get

$12d = 36 \Rightarrow d = 3$

By using $d = 3$ in (i), we get $13 = a + 4(3) \Rightarrow a = 13 - 12 = 1$

Thus, $a_{13} = 1 + (13-1)3 = 37$ and $a_n = 1 + (n-1)3 = 3n - 2$

Hence, $a_{13} = 37$ and $a_n = 3n - 2$

Example 6. An object falling from rest, falls 16 meters during the first second, 32 meters during the next second, 48 meters during the third second and so on. How much will it fall during the 9th second?

Solution: Fall of object for 1st second $= a_1 = 16m$
Fall of object for 2nd second $= a_2 = 32m$
Fall of object for 3rd second $= a_3 = 48m$
Fall of object for 9th second $= a_9 = ?$

So, $\quad 16, 32, 48, \ldots$ is an A.P.

Here, $\quad a = 16, d = 32 - 16 = 16$ and $n = 9$

We know that $\quad a_n = a + (n-1)d$

So, $\quad a_9 = 16 + (9-1)(16)$

$a_9 = 16 + (8)(16)$

$a_9 = 16 + 128 \Rightarrow a_9 = 144m$

Hence, the object falls $144m$ during the 9th second.

Exercise 4.2

1. Find the indicated term in each of the following A.P.
   (i) $1,5,9, \ldots ; a_{12}$ (ii) $-15, -9, -3, \ldots ; a_{10}$
   (iii) $2,6,10,14, \ldots ; a_7$ (iv) $-5, 4, 13, \ldots ; a_{30}$
   (v) $23, 26, 29, \ldots ; a_{14}$ (vi) $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots ;\ a_6$

2. Find the first five terms of the following arithmetic sequences, if
   (i) $a_n = 2n - 3$ (ii) $a_{n-2} = 3n - 5$
   (iii) $a_n = \frac{n}{2n+1}$ (iv) $a_n = (n+1)a_{n-1}, a_1 = 1$
   (v) $a_n - a_{n-1} = n + 2, a_1 = 2$
   (vi) $a = 5$ and other three consecutive terms are 23, 26, 29.

3. If $a_{n-3} = 2n - 5$, find nth term of the sequence.

4. Find 20th term of the sequence $1, 2 - x, 3 - 2x, \ldots$

5. Find the 21st term of the A.P if its 6th term is 11 and the 15th term is 47.

6. (a) Which term of the sequence $-15, -9, -3, \ldots$ is 75?

   (b) Which term of the sequence $5, 2, -1, \ldots$ is -85 ?

   (c) Which term of the sequence $-2, 4, 10, \ldots$ is 148?

7. Find the nth term of the sequence $\left(\frac{6}{7}\right)^2, \left(\frac{11}{7}\right)^2, \left(\frac{16}{7}\right)^2, \ldots$

8. If a, b, c are the lth, mth and nth terms of an A.P., show that:

   (i) $a(m-n) + b(n-l) + c(l-m) = 0$

   (ii) $l(b-c) + m(c-a) + n(a-b) = 0$

9. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$.

10. If $\frac{1}{a-b}, \frac{1}{b-c}, \frac{1}{c-a}$ are in A.P., then show that $\frac{a+c}{a-b} = \frac{a+b}{c-a}$.

11. An object falling from rest, falls 12 metres during the first second, 24 metres during the next second, 36 metres during the third second and so on. How much will it fall during the 8th second?

12. A man deposits Rs.13,000 in a bank in the first month; Rs. 14,500 in the second month; Rs. 16,000 in third month and so on. Find how much he has to deposit in the bank at the end of a year.

13. A boy saves Rs. 200 at the end of the first week and goes on increasing his saving for Rs. 25 weekly. After how many weeks, his weekly saving will be Rs. 2000.

4.3 Arithmetic Mean

4.3.1 Know arithmetic mean between two numbers

When three numbers are in arithmetic progression (A.P)., then the middle term is called their arithmetic mean i.e., if a, A, b are in A.P., then A is called the arithmetic mean (A.M.) of a and b, where a and b are called extremes. When more than three numbers are in A.P., then all the numbers between the extreme numbers (i.e. all the terms between the first and the last terms) are called arithmetic means. That is, if $a, A_1, A_2, A_3, \ldots A_n, b$ are in A.P, then there are $n+2$ terms in A.P., and $A_1, A_2, A_3, \ldots A_n$ are termed as the n arithmetic means between a and b. For example,

(i) As 2,5,8 are in A.P. So, 5 is the A.M. of 2 and 8

(ii) As 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 are in A.P. So, 5, 7, 9, 11, 13, 15, 17 and 19 are the eight A.Ms between 3 and 21.

4.3.2 Insert an arithmetic mean between two numbers

Let A be the A.M. between a and b. Then a, A, b are in arithmetic progression, so that the common difference is the same,

i.e., $A - a$ = Common difference = $b - A$

or $A - a = b - A$ or $A + A = a + b$ or $2A = a + b$

$$\Rightarrow \boxed{A = \frac{a+b}{2}}$$

Example: Find the A.M., between $\sqrt{2}$ and $3\sqrt{2}$.

Solution: Let, A be the A.M between $\sqrt{2}$ and $3\sqrt{2}$.

Here, $a = \sqrt{2}$ and $b = 3\sqrt{2}$

Now, $A = \frac{a+b}{2} = \frac{\sqrt{2}+3\sqrt{2}}{2} = 2\sqrt{2}$

4.3.3 Insert *n* arithmetic means between two numbers

Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between any two given numbers a and b. Then we have an A.P: $a, A_1, A_2, \ldots, A_n, b$ with a as its first term and b its $(n+2)th$ term. Suppose the common difference of this A.P. is d. Then, $a_{n+2} = b = a + \{(n+2) - 1\}d$,

or $b = a + (n+1)d$

or $b - a = (n+1)d$

or $d = \frac{b-a}{n+1}$

So, $A_1 =$ the second term $= a + d = a + \frac{b-a}{n+1} = \frac{na+b}{n+1}$,

$A_2 =$ the third term $= a + 2d = a + \frac{2(b-a)}{n+1} = \frac{(n-1)a+2b}{n+1}$

... ...

and $A_n =$ the $(n+1)th$ term $= a + nd = a + \frac{n(b-a)}{n+1} = \frac{a+nb}{n+1}$,

Hence, $\boxed{\frac{na+b}{n+1}, \frac{(n-1)a+2b}{n+1}, \ldots, \frac{a+nb}{n+1}}$

are the n arithmetic means between a and b.

Example: Find three arithmetic means between $\sqrt{3}$ and $3\sqrt{3}$.

Solution: Let, A_1, A_2, A_3 be three arithmetic means between $\sqrt{3}$ and $3\sqrt{3}$.

Then $\sqrt{3}, A_1, A_2, A_3, 3\sqrt{3}$ are in A.P.

Here, $a = \sqrt{3}$ and $a_5 = 3\sqrt{3}$

Using $a_n = a + (n-1)d$

We get, $a_5 = a + (5-1)d$ or $3\sqrt{3} = \sqrt{3} + 4d$ or $4d = 2\sqrt{3}$

Therefore, $d = \frac{\sqrt{3}}{2}$

Now, $A_1 = a + d = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$,

Similarly, $A_2 = A_1 + d = \frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$,

$A_3 = A_2 + d = 2\sqrt{3} + \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$,

Hence, the three arithmetic means between $\sqrt{3}$ and $3\sqrt{3}$ are $\frac{3\sqrt{3}}{2}$, $2\sqrt{3}$, $\frac{5\sqrt{3}}{2}$

Exercise 4.3

1. Find the A.M between.

(i) 18 and 26 (ii) $10 + 5\sqrt{3}$ and $4 - 5\sqrt{3}$

(iii) $3a + 2b$ and $5a - 6b$ (iv) $5 + 7\sqrt{5}i$ and $8 - 7\sqrt{5}i$

(v) $3a^2 - 5a + 6$ and $-a^2 + 7a - 4$ (vi) $3a - 5$ and $5a + 3$

2. Insert three A.Ms. between 3 and 11.

3. Insert four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.

4. Insert 5 A.Ms. between $\frac{\sqrt{5}}{2}$ and $4\sqrt{5}$.

5. Insert 6 A.Ms. between -30 and 30.

6. If 5 and 8 are two A.Ms. between c and d, find c and d.

7. Find n so that $\frac{a^{n+5}+b^{n+5}}{a^{n+4}+b^{n+4}}$ may be the A.M. between a and b.

4.4 Arithmetic Series

4.4.1 Define an arithmetic series

If $t_1, t_2, t_3, \ldots, t_n$ is an arithmetic sequence, then the expression

$$t_1 + t_2 + t_3 + \cdots + t_n$$

is called an arithmetic series. If an arithmetic series consists of a finite number of terms, it is called a finite arithmetic series; otherwise, it is called an infinite arithmetic series.

In general, an arithmetic series with a as its first term and d as common difference is

$$a + (a + d) + (a + 2d) + \cdots$$

The following are some examples of arithmetic series:

(i) $3 + 5 + 7 + \cdots + (2n + 1)$;

(ii) $-1 - 8 - 15 - \cdots$

4.4.2 Establish the formula to find the sum to n term of an arithmetic series

In general, an arithmetic series of n terms with a as its first term and d its common difference is:

$$a + (a + d) + (a + 2d) + \cdots + \{a + (n - 1)d\}$$

If S_n denotes the sum of the series to n terms, then

$$S_n = a + (a + d) + (a + 2d) + \cdots + \{a + (n - 2)d\} + \{a + (n - 1)d\} \ldots \text{(i)}$$

By writing the sum of the terms of the series in the reverse order,
we have

$$S_n = \{a + (n - 1)d\} + \{a + (n - 2)d\} + \cdots + (a + 2d) + (a + d) + a \qquad \ldots \text{(ii)}$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{2a + (n - 1)d\} + \{2a + (n - 1)d\} + \cdots + \{2a + (n - 1)d\}; \text{ (up to } n \text{ terms)}$$

$$\Rightarrow \quad \boxed{2S_n = n\{2a + (n - 1)d\}}$$

Hence, $$S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

Let l denotes the last term, i.e., $l = T_n = a + (n - 1)d$; then we have

$$S_n = \frac{n}{2}\{2a + (n - 1)d\} = \frac{n}{2}[a + \{a + (n - 1)d\}]$$

or $$\boxed{S_n = \frac{n}{2}(a + l)}$$

Thus, the sum to n terms of a series of A.P. is equal to n times the average of the first and the last term.

In each of the above two formulae for the sum of an arithmetic sequence, there are four elements, viz. S_n, n, a and d or l. Any three being known, the fourth one may be evaluated as illustrated by means of the examples given below.

Example 1. Find the sum of the first n terms of the arithmetic series $3 + 8 + 13 + \cdots$; also find the sum of the first 13 terms.

Solution: Given arithmetic series is: $3 + 8 + 13 + \cdots$

Here $a = 3, d = 8 - 3 = 5$

We have to find S_n and S_{13}

We know that $S_n = \frac{n}{2}[2a + (n - 1)d]$ ... (i)

B using $a = 3, d = 5$ in equation (i)

We get, $$S_n = \frac{n}{2}[2(3) + (n - 1)(5)] = \frac{n}{2}[6 + 5n - 5]$$

$$\Rightarrow \quad S_n = \frac{n}{2}[5n + 1] \qquad \ldots \text{(ii)}$$

By using $n = 13$ in equation (ii)

We get, $$S_{13} = \frac{13}{2}[5(13) + 1] = 429$$

Thus, the sum of first 13 terms is 429.

Example 2. Find the arithmetic series if $S_5 = 30$ and $S_9 = a_5 - 32$.

Solution: We know that $S_n = \frac{n}{2}[2a + (n-1)d]$ ... (i)

By using $n = 5$ in equation (i)

we get, $S_5 = \frac{5}{2}[2a + (5-1)d] = \frac{5}{2}(2a + 4d)$

$\Rightarrow \quad 30 = 5(a + 2d) \quad (\because S_5 = 30)$

$\Rightarrow \quad a + 2d = 6$ ... (ii)

Now, using $n = 9$ in equation (i)

$$S_9 = \frac{9}{2}[2a + (9-1)d] = \frac{9}{2}(2a + 8d)$$

$\Rightarrow \quad S_9 = 9(a + 4d)$

$\Rightarrow \quad S_9 = 9a + 36d$

We know that $a_n = a + (n-1)d$

and $a_5 = a + (5-1)d$

or $a_5 = a + 4d$

Now, we have, $S_9 = a_5 - 32$

i.e., $9a + 36d = a + 4d - 32$

$8(a + 4d) = -32$

$a + 4d = -4$ ... (iii)

Subtracting equation (iii) from (ii), we have

$(a + 2d) - (a + 4d) = 6 - (-4)$

$\Rightarrow \quad d = -5$

By using $d = -5$ in equation (ii) weget, $a + 2(-5) = 6 \Rightarrow \quad a = 16$

Now, $a_2 = a_1 + d = 16 + (-5) = 11$

$a_3 = a_2 + d = 11 + (-5) = 6$

$a_4 = a_3 + d = 6 + (-5) = 1$

Hence, the required arithmetic series is $16 + 11 + 6 + 1 + \cdots$

Example 3. The sum of three numbers in A.P. is 21 and their product is 231. Find the numbers.

Solution: Let the three numbers in A.P. are $a - d,\ a, a + d$

$\because$ Sum of three numbers = 21

$\therefore \quad (a - d) + a + (a + d) = 21$

$3a = 21$ or $a = 7$.

Now, product of three numbers = 231

i.e., $(a - d)(a)(a + d) = 231$ ....(1)

Put $a = 7$ in equation (1)

$(7 - d)(7)(7 + d) = 231$

$$(7-d)(7+d) = \frac{231}{7}$$

$$49 - d^2 = 33$$

$$49 - 33 = d^2$$

$$d^2 = 16 \text{ or } d = \pm 4.$$

If $d = 4$, then first number $= a - d = 7 - 4 = 3$, second number $= a = 7$

and third number $= a + d = 7 + 4 = 11$

Hence, three numbers are 3, 7 and 11.

If $d = -4$, then first number $= a - d = 7 + 4 = 11$, second number $= a = 7$

and third number $= a + d = 7 - 4 = 3$

Hence, three numbers are 11, 7 and 3.

Thus, required numbers are 3, 7, 11 or 11, 7, 3.

Example 4. Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

Solution: Let four numbers in A.P. are $a - 3d,\ a - d, a + d, a + 3d$

Sum of four numbers = 20

i.e., $a - 3d + a - d + a + d + a + 3d = 20$

$\Rightarrow \quad 4a = 20$ or $a = 5$.

Now, sum of squares of the numbers = 120

i.e., $(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 120$ ...(1)

Using $a = 5$ in equation (1)

We get $(5-3d)^2 + (5-d)^2 + (5+d)^2 + (5+3d)^2 = 120$

$\Rightarrow 25 - 30d + 9d^2 + 25 - 10d + d^2 + 25 + 10d + d^2 + 25 + 30d + 9d^2 = 120$

$\Rightarrow 20d^2 + 100 = 120$

$\Rightarrow 20d^2 = 20$

$\Rightarrow d^2 = 1$

$\Rightarrow d = \pm 1$

If $a = 5$ and $d = 1$, then

First number $= a - 3d = 5 - 3 = 2$

Second number $= a - d = 5 - 1 = 4$

Third number $= a + d = 5 + 1 = 6$

and Fourth number $= a + 3d = 5 + 3 = 8$

Hence, four numbers are 2, 4, 6 and 8.

If $a = 5$ and $d = -1$, then

First number $= a - 3d = 5 - (-3) = 8$

Second number $= a - d = 5 - (-1) = 6$

Third number $= a + d = 5 - 1 = 4$

Fourth number $= a + 3d = 5 + (-3) = 2$

Hence, four numbers are 8, 6, 4, 2.
Thus required numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

Note: For the sake of convenience, we take
two numbers in A.P as: $a-d, a+d$
four numbers in A.P as: $a-3d, a-d, a+d, a+3d$ and so on

Similarly,
three numbers in A.P as: $a-d,\ a,\ a+d$
five numbers in A.P as: $a-2d, a-d, a, a+d, a+2d$ and so on.

Example 5. The sums of the first n terms of two A.P.'s are in the ratio $5n-3:3n+31$. Show that their 9th terms are equal. Find also $a_n:a'_n$ and $a_9:a'_{11}$.

Solution: Let a, a' be the first terms and d, d' be the common differences of the two A.P.'s. If S_n and S'_n denotes the sum of these A.P.'s, then we are given

$$\frac{S_n}{S'_n}=\frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{2a'+(n-1)d'\}}=\frac{5n-3}{3n+31} \qquad \ldots \text{(i)}$$

Express R.H.S of equation (i) in the form of L.H.S

$$\therefore \qquad \frac{S_n}{S'_n}=\frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{2a'+(n-1)d'\}}=\frac{\frac{n}{2}\{2(1)+(n-1)(5)\}}{\frac{n}{2}\{2(17)+(n-1)(3)\}}$$

By comparing we have

$$a=1,\ d=5 \text{ and } a'=17, d'=3$$

Now, $\quad a_9=a+8d=1+8(5)=1+40=41$
and $\quad a'_9=a'+8d'=17+8(3)=17+24=41$
Thus, $a_9=a'_9=41$ showed.

$$\Rightarrow \qquad a_n:a'_n=\frac{a_n}{a'_n}=\frac{1+(n-1)(5)}{17+(n-1)(3)}=\frac{5n-4}{3n+14}$$

and $\quad a_9:a'_{11}=\frac{5(9)-4}{3(11)+14}=\frac{45-4}{33+14}=\frac{41}{47}=41:47.$

4.4.3 Show that sum of *n* arithmetic means between two numbers is equal to *n* times their arithmetic mean

Using the results from the article (4.3.3), we can find the sum of n arithmetic means as:

$$\text{Sum}=\{a+d\}+\{a+2d\}+\cdots+\{a+nd\}$$
$$=\left\{a+1\frac{(b-a)}{(n+1)}\right\}+\left\{a+2\frac{(b-a)}{(n+1)}\right\}+\cdots+\left\{a+\frac{n(b-a)}{(n+1)}\right\}$$
$$=(a+a+a\ldots\text{upto } n \text{ terms})+\left\{\frac{1(b-a)}{n+1}+\frac{2(b-a)}{n+1}+\cdots+\frac{n(b-a)}{n+1}\right\}$$
$$=na+\frac{(b-a)}{n+1}(1+2+\cdots+n)$$

$$= na + \frac{(b-a)}{(n+1)} \times \frac{n(n+1)}{2}$$
$$= na + \frac{n(b-a)}{2} = \frac{2na + nb - na}{2}$$
$$= n\left(\frac{a+b}{2}\right) = n \text{ times the A.M. of between } a \text{ and } b.$$

Hence, the sum of n A.M's between a and b is n times the single mean between them. Therefore, the sum of p A.M's = $S_p = p\left(\frac{a+b}{2}\right)$

Similarly, the sum of q A.M's = $S_q = q\left(\frac{a+b}{2}\right)$

Note: $\frac{S_p}{S_q} = \frac{p}{q}$ or $S_p : S_q = p : q$

4.4.4 Solve real life problems involving arithmetic series

Example: A grocery store, displays cans in such a way that 27 cans are in the bottom row, 24 cans in the next row and forming an arithmetic sequence. The top row has 3 cans. Find the total number of cans in the display.

Solution: Since the display of cans are in arithmetic sequence with

$$a = 27, a_n = 3 \text{ and } d = -3$$

Now, $a_n = a + (n-1)d \Rightarrow 3 = 27 - 3n + 3$

$$\Rightarrow n = 9$$

Now the total number of cans is given by

$$S_n = \frac{n}{2}(a + l) = \frac{9}{2}(27 + 3)$$
$$= 9(15) = 135 \text{ cans}$$

Exercise 4.4

1. Sum the series.

(i) $3 + 8 + 13 + \cdots$ to 16 terms.

(ii) $(-3) + (-1) + 1 + 3 + 5 + \cdots$ to 18 terms.

(iii) $4.57 + 4.87 + 5.17 + \cdots$ to 22 terms.

(iv) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \cdots$ to n terms.

(v) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \cdots$ to 15 terms.

2. Find the number of terms of the following.

(i) $(-7) + (-5) + (-3) + \cdots$; if sum = 65

(ii) $-7 + (-4) + (-1) + \cdots$; if sum = 114

(iii) $-9+(-6)+(-3)+\cdots$; if sum = 3225

(iv) $-9-6-3+0+\cdots$; if sum = 66

3. Sum the series.

(i) $3+5-7+9+11-13+15+17-19+\cdots$ to $3n$ terms.

(ii) $1+4-7+10+13-16+19+22-25+\cdots$ to $3n$ terms.

(iii) $9+12-15+18+21-24+27+30-33+\cdots$ to $3n$ terms.

4. If $S_n = n(2n-1)$, then find the series.

5. Find the sum of first 100 natural numbers which are neither exactly divisible by 3 nor by 7.

6. If S_2, S_3, S_5 are the sum of $2n, 3n$ and $5n$ terms of a_n A.P. show that $S_5 = 5(S_3 - S_2)$.

7. The sum of n terms of two arithmetic series are in the ratio of $3n+2 : n+1$. Find the ratio of their 8^{th} terms.

8. The sum of three numbers in A.P. is 27 and their product is 405. Find the numbers.

9. Find five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.

10. The sum of Rs. 42,000 is distributed among five persons so that each person after the first receives Rs. 80 less than the preceding person. How much does each person receive?

11. A well digging company charges Rs. 1,250 for the first meter, Rs.1500 for the second meter and Rs. 1,750 for the third meter and so on. What is the depth of a well that costs Rs. 50,000.

12. A man borrows Rs. 25,000 and agrees to repay with a total profit of Rs. 10,000 in 10 installments, each installment being less than the preceding by Rs. 200. What should be his first installment?

4.5 Geometric Sequence

4.5.1 Define a geometric sequence

A geometric sequence, also known as a geometric progression (G.P) is a sequence of numbers where each term after the first non-zero term is found by multiplying the previous one by a fixed, non-zero number called the common ratio denoted by 'r'. For example, the sequence 3, 6, 12, ...; is a geometric progression with common ratio 2.

Also, the sequence

$$2, 6, 18, 54, \ldots$$

is a G.P. because its common ratio is 3

i.e., $6 \div 2 = 3$, $18 \div 6 = 3$, and $54 \div 18 = 3$

4.5.2 Find the *n*th or general term of geometric sequence

If the first term of a G.P. is a and the common ratio is r, then by definition,

$$a = \text{the first term} = a = ar^{1-1};$$
$$a_2 = \text{the second term} = ar = ar^{2-1};$$
$$a_3 = \text{the third term} = ar^2 = ar^{3-1};$$
$$a_4 = \text{the fourth term} = ar^3 = ar^{4-1};$$

and so on.

Hence, we deduce that

$$\boxed{a_n = \text{the } n\text{th term} = ar^{n-1}} \quad \ldots\text{(i)}$$

which is the formula for finding the nth term of a geometric sequence whose first term is a and the common ratio is r.

The sequence

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

is known as the standard form of a G.P.

Example 1. Find 9th term of the G.P: 6, 12, 24, ...

Solution: Here, first term $= a = 6$

$$\text{Common ratio} = r = \frac{12}{6} = 2$$

$$9^{\text{th}} \text{ term} = a_9 = ?$$

We know that

$$a_n = ar^{n-1}$$

So, $a_9 = (6)(2)^{9-1} = (6)(2)^8 \Rightarrow a_9 = 1536$

Thus, the required 9th term of the given G.P is 1536.

Example 2. In a G.P., $a_3 = 12$ and $a_8 = 384$. Find the nth term.

Solution: We know that $a_n = ar^{n-1}$ ...(i)

Using $n = 3$ in equation (i), $a_3 = ar^{3-1} \Rightarrow ar^2 = 12$ ...(ii)

Now, using $n = 8$ in equation (i), $a_8 = ar^{8-1} \Rightarrow ar^7 = 384$ ...(iii)

From equation (ii) and (iii), by division

$$\frac{ar^7}{ar^2} = \frac{384}{12} \Rightarrow r^5 = 32 \Rightarrow r^5 = 2^5 \Rightarrow r = 2$$

Using $r = 2$ in equation (ii)

We get $a(2)^2 = 12$

$\Rightarrow 4a = 12$ or $a = 3$

Now, $a_n = ar^{n-1}$

Using $a = 3$ and $r = 2$

We get $a_n = 3(2)^{n-1}$

Hence, the required nth term $a_n = 3(2)^{n-1}$

Example 3. Suppose that the fourth term of a geometric sequence is 81 and the sixth term is 729. Find the first term and common ratio of the sequence.

Solution: Here, fourth term = $a_4 = 81$ and 6th term = $a_6 = 729$.

We have to find r and a.

We know that $a_n = ar^{n-1}$

$$a_4 = ar^{4-1} \Rightarrow 81 = ar^3 \quad \ldots\text{(i)}$$

and

$$a_6 = ar^{6-1} \Rightarrow 729 = ar^5 \quad \ldots\text{(ii)}$$

From equation (ii) and (i), by division we have

$$\frac{729}{81} = \frac{ar^5}{ar^3} \Rightarrow r = \pm 3$$

Taking $r = 3$ in eq. (i), we get

$$81 = a(3)^3 \Rightarrow a = 3$$

Here $a = 3$ and $r = 3$ the required geometric sequence will be $3, 9, 27, 81, \ldots$

Taking $r = -3$ in eq. (i), we get

$$81 = a(-3)^3 \Rightarrow a = -3$$

Here $a = -3$ and $r = -3$, the required geometric sequence will be

$$-3, 9, -27, 81, \ldots$$

Example 4. Find three consecutive numbers in G.P. whose sum is 14 and their product is 64.

Solution: Let the three numbers in G.P. are: a, ar and ar^2

As, sum of three numbers = 14

so, $a + ar + ar^2 = 14$

$$a(1 + r + r^2) = 14 \quad \ldots (1)$$

Now, product of three numbers = 64

$$(a) \times (ar) \times (ar^2) = 64$$

$$\Rightarrow (ar)^3 = (4)^3$$

$$\Rightarrow ar = 4 \quad \ldots(2)$$

From equation (1) by (2), by division

$$\frac{a(1 + r + r^2)}{ar} = \frac{14}{4}$$

$$\Rightarrow \frac{(1 + r + r^2)}{r} = \frac{7}{2}$$

$$\Rightarrow 2(1 + r + r^2) = 7(r)$$

$$\Rightarrow 2 + 2r + 2r^2 = 7r$$

$$\Rightarrow 2r^2 - 5r + 2 = 0$$

$$\Rightarrow (r - 2)(2r - 1) = 0$$

$$\Rightarrow r = 2 \quad \text{or} \quad r = \frac{1}{2}$$

By using $r = 2$ in equation (2), we get: $a(2) = 4$ or $a = 2$.

When $a = 2$ and $r = 2$ then

First number $= a = 2$

Second number $= ar = 2(2) = 4$

Third number $= ar^2 = 2(2)^2 = 8$

Hence the three numbers are: 2, 4 and 8.

By using $r = \frac{1}{2}$ in equation (2), we get: $a\left(\frac{1}{2}\right) = 4$ or $a = 8$.

When $a = 8$ and $r = \frac{1}{2}$, then

First number $= a = 8$

Second number $= ar = 8\left(\frac{1}{2}\right) = 4$

Third number $= ar^2 = 8\left(\frac{1}{2}\right)^2 = 2$

Hence the three numbers are: 8, 4 and 2.

Thus the required numbers are 2, 4, 8 or 8, 4, 2.

Note: For the sake of convenience, we take

two numbers in G.P as: $\frac{a}{r}$, ar

four numbers in G.P as: $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^2 and so on.

Similarly,

three numbers in G.P as: $\frac{a}{r}$, a, ar

five numbers in G.P as: $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2 and so on.

4.5.3 Solve problems involving geometric sequence

Example 1. If the population of a town increases geometrically at the rate of 7% annually and the present population is 70,000. What will be the population after 6 years from now?

Solution: Present population $= a = 70{,}000$

$\because$ Number of years $= n = 6$

$\because$ Number of terms $= n = 7$

Rate of increase of population $= r = 1 + 7\,\% = 1.07$

Population after 6 years $= a_7 = ?$

We know that $a_n = ar^{n-1}$

$a_7 = 70{,}000(1.07)^{7-1} = 70{,}000\,(1.07)^6$

$a_7 = 70{,}000(1.500730) = 105051$ approx.

Hence population after 7 years will be 105051.

Example 2. The number of bacteria in a culture increased in G.P from 1250 to 10,000 in 4 days. Find the daily rate of increase, assuming the rate of increase to be constant.

Solution: Here $a = 1250, a_n = 10{,}000, n = 4$ and $r = ?$

We know that

$$a_n = ar^{n-1}$$
$$10{,}000 = 1250r^{4-1}$$
$$\frac{10{,}000}{1250} = r^3$$
$$8 = r^3 \quad \text{or} \quad 2^3 = r^3 \quad \text{or} \quad r = 2$$

Hence the rate of increase of bacteria is 200%.

Exercise 4.5

1. Find the indicated terms of each of the following G.P.

(i) $3, 6, 12, \dots; 5th$ term (ii) $32, 16, 8, \dots;$ 9th term

(iii) $4, 2\sqrt{2}, 2 \dots,$ 12th term (iv) $i, 1, -i, \dots;$ 16th term

(v) $b^2 - c^2, b + c, \frac{b+c}{b-c}, \dots,$ 9th term

2. In a G.P, $a_1 = \frac{5}{9}, a_6 = \frac{15625}{9}$. Find its nth term.

3. Find the nth term of the geometric sequence if:

$\frac{a_5}{a_3} = \frac{4}{9}$ and $a_2 = \frac{4}{9}$

4. Show that the reciprocals of the terms of the G.P. $a_1, a_1r^2, a_1r^4, \dots$ form another G.P.

5. How many terms are in G.P. $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \dots, \frac{1}{196608}$.

6. Find three consecutive numbers in G.P. whose sum is 39 and their product is 729.

7. The number of bacteria in a culture increased in G.P from 515,000 to 15,45,000 in 7 days. Find the daily rate of increase, assuming the rate of increase to be constant.

8. Find the profit on Rs. 1000 for 5 years at 4% per annum compound profit.

4.6 Geometric Mean

4.6.1 Know geometric mean between two numbers

When three numbers are in G.P., the middle one is called their geometric mean, i.e., if numbers a, G, b are in G.P. then G is called a geometric mean of a and b where a and b are called extremes.

Similarly, if $a, G_1, G_2, G_3 \dots, G_n, b$ are in G.P, then $G_1, G_2, G_3 \dots, G_n$ are n geometric means between a and b.

Example: (i) As, 2, 6, 18 are in G.P, so 6 is a G.M between 2 and 18.
(ii) As, 1, 3, 9, 27, 81 are in G.P, so 3, 9, 27 are three geometric means between 1 and 81.

4.6.2 Insert a geometric mean between two numbers

Let G be a G.M. between a and b, then a, G, b are in G.P.

So that the common ratio is the same, i.e. $\frac{G}{a}$ = common ratio = $\frac{b}{G}$

Now, $G^2 = ab$ or $\boxed{G = \pm\sqrt{ab}}$

According to definition, $+\sqrt{ab}$ and $-\sqrt{ab}$ are two possible geometric means between a and b.

Thus, a geometric mean between two numbers is equal to the square root of their product.

Example: Find G.M between 3 and 6.

Solution: Here, $a = 3$ and $b = 6$

We know that $G = \pm\sqrt{ab} = \pm\sqrt{(3)(6)} = \pm 3\sqrt{2}$

Hence the required G.M between 3 and 6 is $\pm 3\sqrt{2}$.

4.6.3 Insert *n* geometric means between two numbers

Let $G_1, G_2, G_3, \ldots, G_n$ be n geometric means between any two given numbers a and b. Then $a, G_1, G_2, G_3, \ldots, G_n, b$ is a geometric progression with a as its first term and b its $(n+2)$th term.

Suppose the common ratio of this G.P. is r.

So, $a_{n+2} = b = ar^{(n+2)-1} \Rightarrow b = ar^{n+1} \Rightarrow \frac{b}{a} = r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

So, G_1 = the second term $= ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

G_2 = the third term $= ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

… … … … … … … … … … … …

and G_n = the $(n+1)$th term $= ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

Hence, $\boxed{a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}}$

are the required n G.M's between a and b.

Now, the product of all geometric means $= G_1 . G_2 . G_3 \dots G_n$

$$= a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} \cdot a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \cdot \dots \cdot a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{(1+2+3+\cdots+n)}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}} = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$$= (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n$$

Hence the product of n G.Ms between two numbers a and b is the n^{th} power of G.M of a and b.

Example: Insert three G.Ms between 3 and 243

Solution: Let, G_1, G_2, G_3 be three geometric means between 3 and 243 Then $3, G_1, G_2, G_3, 243$ is a G.P.

Here, $a = 3$ and $a_5 = 243$

Using $a_n = ar^{n-1}$,

$a_5 = ar^{5-1}$ or $243 = 3r^4$ or $r^4 = 81$

Therefore, $r = \pm 3$

If $r = 3$

then $G_1 = ar = 3(3) = 9$,

Similarly, $G_2 = G_1 r = 9(3) = 27$,

and $G_3 = G_2 r = 27(3) = 81$

If $r = -3$

$G_1 = ar = 3(-3) = -9$,

Similarly, $G_2 = G_1 r = -9(-3) = 27$,

$G_3 = G_2 r = 27(-3) = -81$

Hence, the three geometric means between 3 and 243 are 9, 27, 81 or -9, 27, -81.

Exercise 4.6

1. Find the G.M. between:

(i) -2 and -8 (ii) 3 and $\frac{1}{3}$

(iii) $8\sqrt{2}$ and $9\sqrt{2}$ (iv) $-2i$ and $8i$

(v) $7 + i$ and $7 - i$ (vi) $\frac{16}{9}$ and $\frac{9}{25}$

2. Insert:
   (i) Two G.Ms between 3 and 81.
   (ii) Three G.Ms between 2 and $\frac{1}{2}$.
   (iii) Four G.Ms between 3 and 96.
   (iv) Five G.Ms between 16 and 1024.

3. The A.M. between two numbers is 29 and the geometric mean is 21, find the numbers.
4. For what value of n, is $\frac{a^{n-2}+b^{n-2}}{a^{n-3}+b^{n-3}}$ the G.M. between a and b?
5. Show that the nth root of the product of n geometric means between x and y is the geometric mean between x and y.
6. The A.M. of two positive integral numbers exceeds their (positive) G.M by 2 and their sum is 20. Find the numbers.
7. The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

4.7 Geometric Series

4.7.1 Define a geometric series

If $a_1, a_2, a_3, \ldots, a_n$ is a geometric sequence, then the expression $a_1 + a_2 + a_3 + \cdots + a_n$ is called a geometric series. If the series consists of a finite number of terms, it is called a finite geometric series; otherwise it is called an infinite geometric series.

In general, a geometric series with a as its first term and r as its common ratio is: $a + ar + ar^2 + \cdots$

The following are some examples of geometric series:

(i) $2 + 4 + 8 + 16 + 32 + \cdots + 2^n$

(ii) $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots + \frac{2}{3^n}$

(iii) $\frac{1}{y} - \frac{1}{y^2} + \frac{1}{y^3} - \cdots + \frac{(-1)^{n+1}}{y^n}$

4.7.2 Find the sum of n terms of a geometric series

In general, a geometric series of n terms with a as its first term and r as its common ratio is: $a + ar + ar^2 + \cdots + ar^{n-1}$

If S_n denotes the sum to n terms of the series, we have

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \ldots. \text{(i)}$$

Multiplying each side by r, we have

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \text{.... (ii)}$$

Hence, from (i) and (ii), by subtraction:

$$S_n - rS_n = a - ar^n \quad \text{or} \quad S_n(1-r) = a(1-r^n).$$

If $r \neq 1$, dividing by $(1-r)$, we get $S_n = \frac{a(1-r^n)}{1-r}$ when $r < 1$ .... (iii)

or $S_n = \frac{a(r^n-1)}{r-1}$ when $r > 1$ ...(iv)

If l denotes the last term, i.e., if $l = a_n = ar^{n-1}$; then we have

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a - ar^n}{1-r} = \frac{\{a - r(ar^{n-1})\}}{1-r}$$

i.e., $S_n = \frac{a-rl}{1-r}$ when $r < 1$ .... (v)

or $S_n = \frac{rl-a}{r-1}$ when $r > 1$ .... (vi)

Thus, we can use the formula (v) and (vi) to find the sum of geometric series when its last term l is given.

Note: If $r = 1$, every term of the series is equal to a : hence the sum of the first n terms $= S_n = a + a + a + \cdots$ to n terms, i.e. $S_n = na$.

Example 1.

Find the sum of the first 7 terms of the G.P: $-4, 12, -36, \ldots$

Solution: Here $a = -4$ and $r = -3$. Using the formula $S_n = \frac{a(1-r^n)}{1-r}$,

$$S_7 = \frac{(-4)\{1-(-3)^7\}}{1-(-3)} = \frac{(-4)\{1-(-2187)\}}{4} = -2188$$

Thus the required sum of given G.P is -2188.

Example 2. Find the sum of n terms of the geometric series if $a_n = (-3)\left(\frac{2}{5}\right)^n$.

Solution: As we know

$$a_n = ar^{n-1} \quad \text{...(i)}$$

Given general term is: $a_n = (-3)\left(\frac{2}{5}\right)^n$

or $a_n = -3\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{n-1} = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$, that is $a_n = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$

Comparing it with eq. (i), we get $a = -\frac{6}{5}$ and $r = \frac{2}{5} < 1$

Thus, $S_n = \frac{a(1-r^n)}{1-r} = \frac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{1-\frac{2}{5}} = \left(-\frac{6}{5}\right)\left(\frac{5}{3}\right)\left[1-\left(\frac{2}{5}\right)^n\right]$

Thus, the sum of n terms of the geometric series is: $S_n = (-2)\left[1-\left(\frac{2}{5}\right)^n\right]$

4.7.3 Find the sum of an infinite geometric series

Consider the infinite geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \qquad \ldots\text{(i)}$$

We know that the sum S_n, of the first n terms of the infinite geometric series (i), is given by

$$S_n = a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

i.e.,

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \ldots\text{(ii)}$$

The following three cases arise for an infinite geometric series.

Case I: when $r = 1$

If $r = 1$, then sum of the first n terms of infinite geometric series is

$S_n = a + a + a + \cdots$ to n terms $= na$.

Since a is constant, as the number of terms tends to infinity, the sum also tends to infinity and we say that series is divergent. Thus, the sum of an infinite number of terms, S, of a G.P. is infinite, if $r = 1$.

For example, the sum of the infinite geometric series

$$2 + 2 + 2 + \cdots$$

with $r = 1$ is infinite or that the series is divergent.

Case II: when $|r| > 1$

In this case the sum of the series also tends to infinity as the number of terms tends to infinity. This result is otherwise also obvious, for if, the absolute value of r, $|r| > 1$, then r^n and consequently ar^n tends to infinity as n tends to infinity. Thus, the sum to an infinite number of terms, of a G.P. is infinite, if $|r| > 1$.

For example, the sum of the infinite geometric series $2 - 6 + 18 - 54 + \cdots$ with $|r| = |-3| > 1$, is infinite or that the series is divergent.

Case III: when $|r| < 1$

If $|r| < 1$, then the value of r^n and consequently of $\frac{ar^n}{1-r}$ tends to zero as n tends to infinity. But as n tends to infinity the first part of (ii); viz. $\frac{a}{1-r}$ remains unaffected.

Thus, the sum to an infinite number of terms of a G.P. is $S = \frac{a}{1-r}$, if $|r| < 1$

For example, the sum of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$

with $a = \frac{1}{2}$ and $|r| = \frac{1}{2} < 1$ is $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and we say that the series is convergent.

Note: (i) If the sum of a series is a definite finite value it, the series is said to be convergent and if its sum is infinity, it is said to be divergent.
(ii) Infinite geometric series is convergent if $|r| < 1$ and it is divergent if $|r| > 1$ or $r = 1$.

Example 1. Find the sum of the infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Solution:

Here, $a = 1$ and $|r| = \left|-\frac{1}{2}\right| < 1$

So, $S = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$

Example 2. If $a = 1 - x + x^2 - x^3 + \cdots$ where $|x| < 1$
$b = 1 + x + x^2 + x^3 + \cdots$ where $|x| < 1$

show that $2ab = a + b$

Solution: Here $a = \frac{1}{1-(-x)}$ $\because$ $(r = -x)$

$\Rightarrow$ $a = \frac{1}{1+x} \Rightarrow 1 + x = \frac{1}{a}$ ...(i)

Similarly, $b = \frac{1}{1-x}$ $(\because r = x)$

$\Rightarrow$ $1 - x = \frac{1}{b}$ ...(ii)

Adding (i) and (ii), we obtain $2 = \frac{1}{a} + \frac{1}{b}$ $\Rightarrow$ $2ab = a + b$

4.7.4 Convert recurring decimal fraction into an equivalent common fraction

A repeating or recurring decimal is the decimal representation of a number whose digits are periodic, (repeating its values at regular intervals). We convert recurring decimal into equivalent common fraction with the help of following example:

Example: Convert the recurring decimal $1.\dot{3}$ into an equivalent common fraction (Vulgar fraction).

Solution: $1.\dot{3} = 1.33333\ldots = 1 + 0.33333\ldots = 1 + 0.3 + 0.03 + 0.003 + \cdots$ ...(i)

Now, consider the series $0.3 + 0.03 + 0.003 + \cdots$

Here, $a = 0.3$ and $r = \frac{0.03}{0.3} = 0.1$

We know that $S = \frac{a}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$

Hence, $0.3 + 0.03 + 0.003 + \cdots = \frac{1}{3}$

Thus the given recurring decimal $1.\dot{3}$ is converted into an equivalent common fraction using equation (i)

i.e., $1.\dot{3} = 1 + \frac{1}{3} = \frac{3+1}{3} = \frac{4}{3}$

4.7.5 Solve real life problems involving geometric series

Example 1. A business man earned Rs. 10,000 in the first month, Rs. 15,000 in the second month, Rs. 22,500 in the third month and so on. Find the total amount he will earn in 12 months.

Solution:

Amount earned in first month $= a = Rs. 10{,}000$

Amount earned in second month $= a_2 = Rs. 15{,}000$

Amount earned in third month $= a_3 = Rs. 22{,}500$

Amount earned by 12 months $= S_{12} = ?$

We have geometric series:

$$10000 + 15000 + 22500 + \cdots \text{ to 12 terms.}$$

Here, $a = 10000$ and $r = \frac{15000}{10000} = \frac{3}{2} = 1.5 > 1$

We know that

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{10000\left[\left(\frac{3}{2}\right)^{12} - 1\right]}{\frac{1}{2}} = 20000\left(\frac{531441 - 4096}{4096}\right)$$

$$= 2574926.75$$

$$= 2{,}575{,}000 \text{ rupees (approximately)}$$

Hence, the total amount in 12 months is 2,575,000 rupees.

Example 2. The starting salary of a peon was Rs. 8000 and after each subsequent year his salary was increased by 15%. What total amount of salary he got for the first twelve years?

Solution: Initial salary of the peon $= a = \text{Rs. } 8000$

Rate of increase in salary $= r = 1 + 15\% = 1 + 0.15 = 1.15$

Number of years $= 12$

We know that

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{12} = \frac{8000(1.15^{12} - 1)}{1.15 - 1}$$

$$S_{12} = \frac{8000(5.35025 - 1)}{0.15}$$

$$S_{12} = \frac{34802.00084}{0.15} = 232013.339$$

Total amount of salary of the peon will be Rs. 232013 (approximately).

Exercise 4.7

1. Find the sum:

(i) $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{512}$ (ii) $2^6+2^7+2^8+\cdots+2^{13}$

(iii) $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots+\frac{1}{x^8}$

(iv) $0.9+0.99+0.999+\cdots$ to n terms

(v) $7+77+777+\cdots$ to n terms.

(vi) $0.2+0.22+0.222+\cdots$ to n terms

(vii) $3+33+333+\cdots$ to n terms

2. Find the sum to n terms of the following series:

(i) $1+(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+\cdots;$ where $a>1, b>1$ and $a>b$

(ii) $r+(1+k)r^2+(1+k+k^2)r^2+\cdots$, where k and r are proper fractions.

3. If $a_n=\left(\frac{1}{4}\right)^n$ then find its sum up to n terms.

4. Find the sum of the following infinite geometric series.

(i) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$ (ii) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

(iii) $\frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\cdots$ (iv) $2+\sqrt{2}+1+\cdots$

(v) $2+1+0.5+\cdots$ (vi) $4+2\sqrt{2}+2+\sqrt{2}+1+\cdots$

(vii) $0.1+0.05+0.025+\cdots$

5. Find the Vulgar fraction equivalent to the following recurring decimals.

(i) $0.\dot{7}$ (ii) $1.\dot{7}\dot{4}$ (iii) $0.\dot{2}5\dot{9}$

(iv) $1.1\dot{4}\dot{8}$ (v) $2.\dot{2}\dot{3}$

6. If $a=\frac{b}{2}+\frac{b^2}{4}+\frac{b^3}{8}+\cdots$ if $0<b<2$, then prove that $b=\frac{2a}{1+a}$.

7. The sum of an infinite geometric series is half the sum of the squares of its terms. If the sum of its first two terms is $4\frac{1}{2}$, find the series.

8. Joining the midpoints of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of nested equilateral triangles in this manner with the original triangle having perimeter $\frac{5}{2}$. What will be the total perimeter of all the triangles formed in this way?

4.8 Harmonic Sequence

4.8.1 Recognize a harmonic sequence

A sequence is said to be a harmonic sequence or a harmonic progression (H.P.) if the reciprocals of its terms are in arithmetic progression.

The harmonic progression derives its name from the fact that musical strings of equal thickness and tension will produce harmony if their lengths are to one another as the reciprocals of the natural numbers.

Examples of harmonic sequence

(i) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \dots$; is an H.P. Its corresponding A.P is: 5, 10, 15,...

(ii) $\frac{4}{5}, \frac{2}{5}, \frac{4}{15}, \dots$; is an H.P. Its corresponding A.P is: $\frac{5}{4}, \frac{5}{2}, \frac{15}{4}, \dots$

Since the general form of an A.P. is: $a, a+d, a+2d, \dots, a+(n-1)d$.

Therefore H.P. is a sequence of the form: $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots, \frac{1}{a+(n-1)d}$ provided that none of the denominators is zero.

Hence to every H.P, there is a corresponding A.P, the terms of which are the reciprocals of the corresponding terms of the H.P.

Most of the examples on an H.P. can be worked out by converting the given H.P into A.P and making use of the corresponding A.P.

Example: Find the *n*th term and 8th term of H.P: $\frac{1}{8}, \frac{1}{26}, \frac{1}{44}, \dots$

Solution: Corresponding A.P is 8,26,44, ...

Here, $a = 8$ and $d = 26 - 8 = 18$

Using these values in, $a_n = a + (n-1)d$,

we have $a_n = 8 + (n-1)18$

$\Rightarrow \quad a_n = 18n - 10$

Thus the *n*th term of the given H.P is $T_n = \frac{1}{a_n} = \frac{1}{18n-10}$

so, $T_8 = \frac{1}{18 \times 8 - 10} = \frac{1}{134}$.

4.8.2 Find *n*th term of harmonic sequence

Since the reciprocals of the terms of H.P form A.P. Therefore, the nth term of the harmonic progression is equal to the reciprocal of the nth term of the corresponding A.P. Thus, the formula to find the nth term of the harmonic progression sequence is given as:

The nth term of the Harmonic Progression (H.P) is: $T_n = \frac{1}{[a+(n-1)d]}$, where

"a" is the first term of corresponding A.P

"d" is the common difference of corresponding A.P

"n" is the number of terms of corresponding A.P

We will also find the general or nth term when the first two terms of an H.P. are given.

Let a and b respectively be the first and the second terms of an H.P. Then $\frac{1}{a}$ and $\frac{1}{b}$ are, respectively, the first and the second terms of the corresponding A.P. If d be the common difference of this A.P., then the common difference can be calculated as: $d = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}$.

Thus, the nth term of this A.P.

$$a_n = \frac{1}{a} + (n-1)\left\{\frac{a-b}{ab}\right\} = \frac{b + (n-1)\ \ (a-b)}{ab}$$

Hence the corresponding nth term of the H.P is:

$$\boxed{T_n = \frac{ab}{b + (n-1)(a-b)}}$$

Example 1. Find the 5th term of H.P: $\frac{1}{3}, \frac{1}{10}, \frac{1}{17}, \ldots$

Solution: The given harmonic sequence is: $\frac{1}{3}, \frac{1}{10}, \frac{1}{17}, \ldots$

Here , $a = 3,\quad n = 5,\quad d = 7$

We know that, $a_n = \frac{1}{a+(n-1)d}$

$\Rightarrow a_5 = \frac{1}{3+(5-1)7} = \frac{1}{31}$

Thus, the fifth term of the given H.P. is $\frac{1}{31}$.

Example 2. Find the 8th term of the H.P. $\frac{4}{5}, \frac{2}{5}, \frac{4}{15}, \ldots$

Solution:

Here H.P. is $\frac{4}{5}, \frac{2}{5}, \frac{4}{15}, \ldots$

We know that $T_n = \frac{ab}{b+(n-1)\ \ (a-b)}$ where, $a = \frac{4}{5}$, $b = \frac{2}{5}$ and $n = 8$

So, $$T_8 = \frac{\left(\frac{4}{5}\right)\left(\frac{2}{5}\right)}{\frac{2}{5}+(8-1)\left(\frac{4}{5}-\frac{2}{5}\right)} = \frac{\frac{8}{25}}{\frac{2}{5}+\frac{14}{5}} = \frac{\frac{8}{25}}{\frac{16}{5}}$$

$$= \frac{8}{25} \times \frac{5}{16} = \frac{1}{10}$$

Example 3. If the 3rd term and 7th term of an H.P. are $\frac{2}{9}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution: Since the 3rd term of the H.P. is $\frac{2}{9}$ and its 7th term is $\frac{2}{25}$, therefore the 3rd and 7th terms of the corresponding A.P. are $\frac{9}{2}$ and $\frac{25}{2}$ respectively.

Now taking a, the first term and d, the common difference of the corresponding A.P., we have

$$a + 2d = \frac{9}{2} \qquad \text{(3rd term)} \qquad \ldots\text{(i)}$$

and

$$a + 6d = \frac{25}{2} \qquad \text{(7th term)} \qquad \ldots\text{(ii)}$$

Subtracting (i) from (ii), gives: $4d = \frac{25}{2} - \frac{9}{2} \Rightarrow d = 2$

From (i), we get $a = \frac{9}{2} - 2d = \frac{9}{2} - 4 = \frac{1}{2}$

Thus a_2 of the A.P. $= a + d = \frac{1}{2} + 2 = \frac{5}{2}$

and a_4 of the A.P. $= a + 3d = \frac{1}{2} + 3(2) = \frac{13}{2}$

Hence the required H.P. is $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$

Exercise 4.8

1. Find the indicated terms in the following harmonic progressions.

(i) $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \ldots$; 9th term (ii) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$; 12th term

(iii) $\frac{9}{5}, \frac{9}{13}, \frac{9}{21}, \ldots$; 8th term (iv) $-\frac{1}{5}, -\frac{1}{3}, -1, \ldots$; 15th term

(v) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$; 8th term (vi) $\frac{1}{3}, \frac{1}{8}, \frac{1}{13}, \ldots$; 11th term

2. If the 7th and 10th terms of an H.P are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th and 20th terms.

3. If the sum of first and sixth terms of H.P. is $\frac{31}{116}$. Find the harmonic sequence if the first term is $\frac{1}{4}$.

4. The first term of H.P is $-\frac{1}{3}$ and fifth term is $\frac{1}{5}$. Find its 9th term.

5. If the pth term of an H.P. is q, the qth term is p: prove that the $(p+q)$th term is $\frac{pq}{(p+q)}$.

4.9 Harmonic Mean

4.9.1 Define a harmonic mean

When three numbers are in Harmonic Progression (H.P), then the middle one is called their harmonic mean i.e., if a,H,b are in H.P., then H is called the harmonic mean (H.M.) of a and b. When more than three numbers are in H.P., all the numbers between the extreme numbers are called Harmonic Means. That is, if

$$a,\ H_1, H_2,\ H_3, \ldots, H_n, b \text{ are in H.P.}$$

Then there are $n+2$ numbers in H.P., and $H_1,\ H_2,\ H_3, \ldots, H_n$ are termed as the n harmonic means between a and b.

For example,

(i) $\frac{2}{5}$ is the H.M. of 1 and $\frac{1}{4}$ because $1, \frac{2}{5}, \frac{1}{4}$ form an H.P.

(ii) $\frac{3}{5},\frac{3}{7},\frac{3}{9},\frac{3}{11},\frac{3}{13},\frac{3}{15},\frac{3}{17}$ and $\frac{3}{19}$ are the eight H.Ms between 1 and $\frac{3}{21}$ because

$$1, \frac{3}{5},\frac{3}{7},\frac{3}{9},\frac{3}{11},\frac{3}{13},\frac{3}{15}, \frac{3}{17}, \frac{3}{19}, \frac{3}{21} \text{ form an H.P}$$

4.9.2 Insert a harmonic mean between two numbers

Let H be the H.M. between a and b. Then a, H, b are in H.P.

So, $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in corresponding A.P. and $\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$

or $$\frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

Therefore, $$H = \frac{2ab}{a+b}$$

Thus, the harmonic mean of any two numbers is equal to twice their product divided by their sum.

Example: Find the harmonic mean of 2 and 5.

Solution: Here $a = 2$ and $b = 5$, using the formula of harmonic mean:

$$H = \frac{2ab}{a+b}$$

We get, $$H = \frac{2(2)(5)}{2+5} = \frac{20}{7}$$

Thus, the harmonic mean between 2 and 5 is $\frac{20}{7}$.

4.9.3 Insert n harmonic means between two numbers

Let, $H_1, H_2, \ldots, H_n$ be the n harmonic means between any two given numbers a and b. Then the corresponding A.P. will be:

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots, \frac{1}{H_n}, \frac{1}{b}$$

Let d be the common difference of this A.P.,

So, $\quad \frac{1}{b} = a_{n+2} = \frac{1}{a} + \{(n+2) - 1\}d$

i.e. $\quad \frac{1}{b} = \frac{1}{a} + (n+1)d$

or $\quad d = \frac{a-b}{(n+1)ab}$

Now, $\quad \frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{(a-b)}{(n+1)ab} = \frac{a+nb}{(n+1)ab}$

Therefore, $\quad H_1 = \frac{(n+1)ab}{a+nb}$

Now, $\quad \frac{1}{H_2} = \frac{1}{a} + 2d = \frac{1}{a} + 2\frac{(a-b)}{(n+1)ab} = \frac{2a+(n-1)b}{(n+1)ab}$

So, $\quad H_2 = \frac{(n+1)ab}{2a+(n-1)b}$

Similarly, $\quad H_3 = \frac{(n+1)ab}{3a+(n-2)b}$

and $\quad H_n = H_{n-1} + d = \frac{(n+1)ab}{na+b}$

Hence, $\quad \frac{(n+1)ab}{a+nb}, \frac{(n+1)ab}{2a+(n-1)b}, \frac{(n+1)ab}{3a+(n-2)b}, \ldots, \frac{(n+1)ab}{na+b}$

are n harmonic means between a and b.

Relation Between Arithmetic, Geometric and Harmonic Means

Following are two important relations involving the arithmetic, the geometric and the harmonic means between two given numbers.

(i) If A, G and H are respectively the arithmetic, the geometric and the harmonic means between any two given numbers, then A, G, H are in G.P.

$$\text{i.e.,} \quad \frac{G}{A} = \frac{H}{G} \quad \text{or} \quad G^2 = AH$$

(ii) If A, G and H are respectively the arithmetic, the positive geometric and the harmonic means between any two real, positive and unequal numbers then $A > G > H$.

Example 1. If A.M and H.M between two numbers are 5 and $\frac{21}{5}$ respectively. Find the numbers.

Solution: Let the numbers are a and b

$$\because \quad A.M = 5$$

$$\therefore \quad \frac{a+b}{2} = 5$$

$$a + b = 10 \quad \ldots \text{(i)}$$

and $$H.M = \frac{21}{5}$$

i.e., $$\frac{2ab}{a+b} = \frac{21}{5} \quad \ldots \text{(ii)}$$

From equation (i), using $a + b = 10$ in equation (ii), we get

$$\frac{2ab}{10} = \frac{21}{5}$$

$$\Rightarrow 2ab = \frac{21 \times 10}{5}$$

$$\Rightarrow ab = 21 \quad \ldots \text{(iii)}$$

From equation (i), we get $b = 10 - a$ ... (iv)

Using $b = 10 - a$ in equation (iii),

we get $$a(10 - a) = 21$$

$$a^2 - 10a + 21 = 0$$

$$\Rightarrow (a - 7)(a - 3) = 0$$

$$\Rightarrow a = 7 \text{ and } a = 3$$

Using $a = 7$ and $a = 3$ in equation (iv), we get

$b = 10 - 7 = 3$ and $b = 10 - 3 = 7$

Hence the required numbers are 3, 7 or 7, 3.

Example 2. If G.M and H.M between two numbers are 15 and $\frac{75}{13}$ respectively. Find the numbers.

Solution: Let the numbers be a and b

$$\because \quad G.M = 15$$

$$\therefore \quad \pm\sqrt{ab} = 15$$

Squaring both sides

$$\left(\pm\sqrt{ab}\right)^2 = (15)^2$$

$$ab = 225 \quad \ldots \text{(i)}$$

and $$H.M = \frac{75}{13}$$

$$\frac{2ab}{a+b} = \frac{75}{13} \quad \text{... (ii)}$$

From equation (i), using $ab = 225$ in equation (ii), we get

$$\frac{2(225)}{a+b} = \frac{75}{13}$$

$$a + b = \frac{26 \times 225}{75}$$

$$a + b = 78 \quad \text{...(iii)}$$

From equation (iii), we get $b = 78 - a$ ...(iv)

Using $b = 78 - a$ in equation (i),

$$a(78 - a) = 225$$

$\Rightarrow \quad a^2 - 78a + 225 = 0$

$\Rightarrow \quad (a - 75)(a - 3) = 0$

$\Rightarrow \quad a = 75$ and $a = 3$

Using $a = 75$ and $a = 3$ in equation (iv), we get

$$b = 78 - 75 = 3 \text{ and } b = 78 - 3 = 75$$

Hence the required numbers are 3, 75 or 75, 3.

Exercise 4.9

1. Insert the H.M between:

(i) $\frac{1}{64}$ and $\frac{1}{81}$ (ii) $\frac{1}{5}$ and $\frac{1}{17}$ (iii) $3 + \sqrt{2}$ and $3 - \sqrt{2}$

(iv) $2 + 3i$ and $2 - 3i$ (v) $\frac{1}{7}$ and $\frac{1}{11}$ (vi) 3 and 7

2. Insert four H.Ms between:

(i) 4 and 20 (ii) $\frac{1}{9}$ and $\frac{1}{81}$

3. Insert five harmonic means between $\frac{1}{4}$ and $\frac{1}{24}$

4. Prove that the square of the geometric mean of two numbers equals the product of the A.M and the H.M of the two numbers.

5. Find n so that $\frac{x^{n-5}+y^{n-5}}{x^{n-6}+y^{n-6}}$ may be H.M. between x and y.

6. If 5 is the harmonic mean between 2 and b, find b?

7. Find the arithmetic, harmonic and geometric means between 1 and 9. Also verify that $AH = G^2$.

8. The A.M of two numbers is 8 and H.M is 6. Find the numbers.

9. The H.M of two numbers is $\frac{24}{5}$ and G.M is 6. Find the numbers.

Review Exercise 4

1. Select correct option.

i. A sequence is a function whose domain is set of:

(a) integers (b) rational numbers
(c) natural numbers (d) real numbers

ii. If H is the harmonic mean between x and y then H is:

(a) $\frac{2(x+y)}{xy}$ (b) $\frac{x+y}{2xy}$ (c) $\frac{2xy}{x+y}$ (d) $\frac{xy}{x+y}$

iii. If $a_n - a_{n-1} = n+1$ and $a_4 = 14$ then $a_5 =$ -------------:

(a) 3 (b) 5 (c) 14 (d) 20

iv. A sequence $\{a_n\}$ in which $a_{n+1} - a_n$ is the same number for all $n \in \mathbb{N}$ is called:

(a) A.P (b) G.P (c) H.P (d) None of these

v. If a_{n-1}, a_n, a_{n+1} are in A.P, then a_n is called:

(a) A.M (b) G.M (c) H.M (d) Mid-point

vi. Arithmetic mean between c and d is:

(a) $\frac{c+d}{2}$ (b) $\frac{c+d}{2cd}$ (c) $\frac{2cd}{c+d}$ (d) $\frac{2}{c+d}$

vii. The harmonic mean between $\sqrt{2}$ and $3\sqrt{2}$ is:

(a) $4\sqrt{2}$ (b) $\frac{4}{\sqrt{2}}$ (c) $\frac{3}{2}\sqrt{2}$ (d) None of these

viii. For any G.P the common ratio r is equal to:

(a) $\frac{a_n}{a_{n+1}}$ (b) $\frac{a_{n-1}}{a_n}$ (c) $\frac{a_n}{a_{n-1}}$ (d) $a_{n+1} - a_n$ for $n \in \mathbb{N}, n > 1$

ix. No term of a G.P is:

(a) 0 (b) 1 (c) Negative (d) Imaginary number

x. The sum of infinite geometric series is a finite number if:

(a) $|r| > 1$ (b) $|r| = 1$ (c) $|r| \geq 1$ (d) $|r| < 1$

xi. If the reciprocals of the terms of a sequence form an A.P, then it is:

(a) Harmonic sequence (b) Arithmetic sequence
(c) Reciprocal sequence (d) Series

xii. If $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is A.M between a & b, then n is equal to:

(a) 0 (b) -1 (c) 1 (d) $\frac{1}{2}$

xiii. If $1, x-1, 3$ are in A.P., then $x =$ ------------

(a) 0 (b) 1 (c) 2 (d) 3

xiv. G.M between -2 and 8 is:

(a) $4i$ or $-4i$ (b) 4 or -4 (c) 16 or -16 (d) 3 or -5

xv. General term of a sequence is $(-1)^n n^2$. Its 4th term is:

(a) -4 (b) -16 (c) 16 (d) 4

xvi. If $|r| > 1$ then infinite geometric series is --------------:

(a) Convergent (b) Divergent (c) Undefined (d) both a and b

xvii. The harmonic mean of $\frac{1}{3}$ and $\frac{2}{5}$ is:

(a) $\frac{4}{11}$ (b) $\frac{3}{4}$ (c) $\frac{5}{11}$ (d) $\frac{11}{4}$

xviii. If $\frac{a-b}{b-c} = \frac{a}{b}$, then a, b and c are in:

(a) A.P (b) G.P (c) H.P (d) None of these

2. The 5th term of an arithmetic sequence is 60 and 8th term is 90. Find 12th term.

3. There are n A.Ms. between 8 and 32 such that the ratio of the third and 7th means is 3:5, find the value of n.

4. Find the sum of all the integers between 280 and 350 which are exactly divisible by 9.

5. How many terms are there in a G.P if $a = 8, a_n = \frac{1}{512}$ and $r = \frac{1}{2}$.

6. The yearly depreciation of a certain machine is 20% of the value at the beginning of the year. If the original cost of the machine is Rs. 50,000, find the value after 5 years.

7. Find r, if $S_{10} = 244S_5$ in a G.P.

8. If A.M and H.M between two numbers are 5 and $4\frac{1}{5}$ respectively. Find the numbers.

9. If G.M. and H.M. between two numbers are 15 and $5\frac{10}{13}$ respectively. Find the numbers.

10. The second term of an H.P is $\frac{1}{2}$ and the fifth term is $\frac{-1}{4}$. Find the 12th term.

Miscellaneous Series

Unit 5

5.1 Evaluation of Σn, Σn^2 and Σn^3

5.1.1 Recognize sigma (Σ) notation

Let $x_1 + x_2 + x_3 + \cdots + x_n$ be a series of first n terms, the sum of this series is denoted by

$$\sum_{i=1}^{n} x_i$$

where Σ is a Greek letter "Sigma" which is used for summation.

i.e., $$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$$

In summation notation $\sum_{i=1}^{n} x_i$, i is called index, x_i is the ith element of the series, whereas 1 and n are called lower and upper limits respectively.

For example, $\sum_{i=1}^{10} x_i$ means the sum of the values of x, starting with x_1 and ending at x_{10}.

i.e., $$\sum_{i=1}^{10} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

Similarly, the expression $\sum_{i=3}^{10} x_i$ means sum of the values of x, starting with x_3 and ending at x_{10}.

i.e., $$\sum_{i=3}^{10} x_i = x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

5.1.2 Find sum of

- **the first n natural numbers (Σn)**
- **the squares of the first n natural numbers (Σn^2)**
- **the cubes of the first n natural numbers (Σn^3)**

Let us find the sum of positive integral powers of natural numbers by using $\sum_{k=1}^{n} [k^m - (k-1)^m]$, where m, n and k are natural numbers.

Now, $\sum_{k=1}^{n}[k^m - (k-1)^m] = (1^m - 0^m) + (2^m - 1^m) + (3^m - 2^m) + (4^m - 3^m) + \cdots$

$$+[(n-1)^m - (n-2)^m] + [n^m - (n-1)^m] = n^m$$

i.e $$\sum_{k=1}^{n}[k^m - (k-1)^m] = n^m \quad \text{... (i)}$$

- **Sum of the first n natural numbers (Σn)**

Taking $m = 1$ in equation (i), we get

$$\sum_{k=1}^{n}[k - (k-1)] = n$$

i.e., $$\sum_{k=1}^{n} 1 = 1 + 1 + 1 + \cdots \text{ to } n \text{ terms} = n \quad \text{... (ii)}$$

Taking $m = 2$ in equation (i), we get

$$\sum_{k=1}^{n}[k^2 - (k-1)^2] = n^2$$

$$\Rightarrow \sum_{k=1}^{n}[k^2 - (k^2 - 2k + 1)] = n^2$$

$$\Rightarrow \sum_{k=1}^{n}(2k - 1) = n^2$$

$$\Rightarrow 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^2$$

$$\Rightarrow 2\sum_{k=1}^{n} k - n = n^2 \quad \text{[using (ii)]}$$

$$\Rightarrow \sum_{k=1}^{n} k = \frac{n^2+n}{2}$$

$$\text{or} \quad \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}} \quad \text{or} \quad \boxed{\sum n = \frac{n(n+1)}{2}} \qquad \ldots\text{(iii)}$$

This is the formula for getting the sum of first n natural numbers.

Example 1. Find the sum of

(i) first fifteen natural numbers (ii) first fifty-one natural numbers

Solution:

We know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(i) By using $n = 15$

$$\sum_{k=1}^{15} k = \frac{15(15+1)}{2}$$

$$= 120$$

Hence $1 + 2 + 3 + \cdots + 15 = 120$.

(ii) By using $n = 51$

$$\sum_{k=1}^{51} k = \frac{51(51+1)}{2}$$

$$= 51(26) = 1326$$

Hence $1 + 2 + 3 + \cdots + 51 = 1326$.

Example 2. (i) Find the sum of first fifteen natural numbers starting from 5.

(ii) Find the sum of first fifty one natural numbers starting from 19.

Solution:

By using the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(i) The sum of first fifteen terms of natural numbers starting from 5 is:

$$\sum_{k=5}^{19} k = \sum_{k=1}^{19} k - \sum_{k=1}^{4} k$$

$$= \frac{19(19+1)}{2} - \frac{4(4+1)}{2}$$

$$= 190 - 10 = 180$$

So, $$\sum_{k=5}^{19} k = 180$$

Hence, $5 + 6 + 7 + \cdots + 19 = 180$.

(ii) The sum of first fifty one natural numbers starting from 19 is:

$$\sum_{k=19}^{69} k = \sum_{k=1}^{69} k - \sum_{k=1}^{18} k$$

$$= \frac{69(69+1)}{2} - \frac{18(18+1)}{2}$$
$$= 69(35) - 9(19) = 2244$$

So, $$\sum_{k=19}^{69} k = 2244$$

Hence, $19 + 20 + 21 + \cdots + 69 = 2244$.

- **Sum of the squares of the first n natural numbers (Σn^2)**

Taking $m = 3$ in equation (i), we get

$$\sum_{k=1}^{n} [k^3 - (k-1)^3] = n^3$$

$$\Rightarrow \sum_{k=1}^{n} [k^3 - (k^3 - 3k^2 + 3k - 1)] = n^3$$

$$\Rightarrow \sum_{k=1}^{n} (k^3 - k^3 + 3k^2 - 3k + 1) = n^3$$

$$\Rightarrow \sum_{k=1}^{n} (3k^2 - 3k + 1) = n^3$$

$$\Rightarrow 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = n^3$$

$$\Rightarrow 3\sum_{k=1}^{n} k^2 = n^3 + \frac{3n(n+1)}{2} - n \quad \text{[using (ii) and (iii)]}$$

$$3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(2n^2 + 3n + 1)}{(2)(3)}$$

$$\sum_{k=1}^{n} k^2 = \frac{n[(2n(n+1) + 1(n+1)]}{6}$$

Thus, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ or $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ ...(iv)

This is the formula to find the sum of squares of first n natural numbers.

Example:

(i) Find the sum of squares of first six natural numbers.

(ii) Find the sum of squares of first thirty five natural numbers.

Solution:

By using formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(i) For $n = 6$

$$\sum_{k=1}^{6} k^2 = \frac{6(6+1)[2(6)+1]}{6} = \frac{6(7)(13)}{6} = 91$$

Hence, $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$

(ii) For $n = 35$

$$\sum_{k=1}^{35} k^2 = \frac{35(35+1)[2(35)+1]}{6} = \frac{35(36)(71)}{6} = 14910$$

Hence, $1^2 + 2^2 + 3^2 + \cdots + 35^2 = 14910$

- **Sum of the cubes of the first n natural numbers (Σn^3)**

Taking $m = 4$ in equation (i), we get

$$\sum_{k=1}^{n} [k^4 - (k-1)^4] = n^4$$

$$\Rightarrow \sum_{k=1}^{n} [k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1)] = n^4$$

$$\Rightarrow \sum_{k=1}^{n} (4k^3 - 6k^2 + 4k - 1) = n^4$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^4$$

$$4\sum_{k=1}^{n} k^3 - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n = n^4$$ [using (ii), (iii) and (iv)]

$$\Rightarrow 4\sum_{k=1}^{n} k^3 - n(n+1)(2n+1) + 2n(n+1) - n = n^4$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 = n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$

$$\Rightarrow 4\sum_{k=1}^{n} k^3 = n(n^3 + 2n^2 + n + 2n + 1 - 2n - 2 + 1) = n(n^3 + 2n^2 + n)$$

$$\Rightarrow \sum_{k=1}^{n} k^3 = \frac{n^2(n^2+2n+1)}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

Thus,

$$\boxed{\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2} \qquad \ldots(v)$$

This is the formula to find the sum of cubes of first n natural numbers.

Example: Find the sum of cubes of first seven natural numbers.

Solution: By using formula

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

For $n = 7$, $$\sum_{k=1}^{7} k^3 = \left[\frac{7(7+1)}{2}\right]^2 = \left[\frac{7(8)}{2}\right]^2 = (28)^2 = 784$$

Hence, $1^3 + 2^3 + 3^3 + \cdots + 7^3 = 784$

Exercise 5.1

Sum the following series up to indicated terms.

1. $1^2 + 2^2 + 3^2 + \cdots + 10^2$
2. $1^3 + 2^3 + 3^3 + \cdots + 15^3$
3. $1^2 + 3^2 + 5^2 + \cdots$ up to n terms
4. $2^2 + 4^2 + 6^2 + 8^2 + \cdots$ up to n terms
5. $1^3 + 3^3 + 5^3 + \cdots$ up to n terms
6. $2 \times 1^2 + 5 \times 2^2 + 8 \times 3^2 + \cdots$ up to n terms
7. $1 + (1+2) + (1+2+3) + \cdots$ up to n terms
8. $2 + (2+5) + (2+5+8) + \cdots$ up to n terms
9. Find the sum of n terms of the series whose nth term is:

   (i) $3n^2 + n + 1$ (ii) $n^2 + 4n + 1$

   (iii) $n^3 + 3n^2 + 2n + 1$

10. (i) Find the sum of first forty six natural numbers starting from 10.
(ii) Find the sum of last 12 terms of the series: $1 + 2 + 3 + \cdots + 39$.

11. Find the sum of 9^{th} to 21^{st} terms of the series of squares of first n natural numbers.

12. Find the sum of last 12 terms of the series of cubes of first 30 natural numbers.

5.2 Arithmetico-Geometric Series

5.2.1 Define arithmetico-geometric series

A series obtained by multiplying the corresponding terms of an A.P. and a G.P. is called an artithmetico-geometric (A.G.) series. Such a series will be of the form

$$a + (a + d)r + (a + 2d)r^2 + \cdots$$

This is the series which is obtained by multiplying the arithmetic series $a + (a + d) + (a + 2d) + \cdots$

and the geometric series $1 + r + r^2 + \cdots$

The n^{th} term of A.G series is, $T_n = \{a + (n - 1)d\}r^{n-1}$

5.2.2 Find sum to n terms of the arithmetico-geometric series

Sum of n terms S_n of the arithmetico-geometric series is obtained as under.

$$\text{Let } S_n = a + (a + d)r + (a + 2d)r^2 + \cdots + [a + (n - 1)d]r^{n-1} \quad \ldots\text{(i)}$$

Multiplying both sides by r

$$rS_n = ra + (a + d)r^2 + (a + 2d)r^3 + \cdots + [a + (n - 1)d]r^n \quad \ldots\text{(ii)}$$

Subtracting equation (ii) from (i), we get

$$S_n - rS_n = a + (dr + dr^2 + dr^3 + \cdots + dr^{n-1}) - [a + (n - 1)d]r^n$$

$$(1 - r)S_n = a + \left[\frac{dr(1 - r^{n-1})}{1 - r}\right] - [a + (n - 1)d]r^n$$

$$(1 - r)S_n = a + \frac{dr}{1 - r} - \frac{dr^n}{1 - r} - [a + (n - 1)d]r^n$$

Dividing both sides by $1 - r$

$$S_n = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} - \frac{dr^n}{(1 - r)^2} - \frac{[a + (n - 1)d]r^n}{1 - r} \quad \ldots\text{(iii)}$$

Example 1.

Find the sum of the series: $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 100 \cdot 2^{100}$.

Solution:

Let $S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 100 \cdot 2^{100}$ ...(i)

Multiplying both sides by 2

$$2S = 1 \cdot 2^2 + 2 \cdot 2^3 + \cdots + 99 \cdot 2^{100} + 100 \cdot 2^{101} \quad \text{...(ii)}$$

Subtracting eq. (ii) from eq. (i), we get

$$-S = 1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 \ldots + 1 \cdot 2^{100} - 100 \cdot 2^{101}$$

$\Rightarrow \quad -S = 2 + 2^2 + 2^3 + \cdots + 2^{100} - 100 \cdot 2^{101}$

$\Rightarrow \quad -S = 2(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{99}) - 100 \cdot 2^{101}$

$\Rightarrow \quad -S = \dfrac{2(2^{99} - 1)}{2 - 1} - 100 \cdot 2^{101}$

$\Rightarrow \quad -S = 2^{100} - 2 - 100 \cdot 2^{101}$

$\Rightarrow \quad -S = 2^{100}(1 - 100 \cdot 2^1) - 2$

$\Rightarrow \quad -S = 2^{100}(-199) - 2$

$\Rightarrow \quad S = 2^{100}(199) + 2$

or $\quad S = 199 \cdot 2^{100} + 2$

Example 2. Evaluate the sum of the series: $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$ to infinite terms.

Solution:

Let, $\quad S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$ ...(i)

Multiplying both sides by $\frac{1}{5}$

$$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots \quad \text{...(ii)}$$

Subtracting eq. (ii) from eq. (i), we get

$$\left(1 - \frac{1}{5}\right)S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \cdots$$

$\Rightarrow \quad \dfrac{4}{5}S = 1 + 3\left(\dfrac{\frac{1}{5}}{1 - \frac{1}{5}}\right) \qquad \left[\because S = \dfrac{a}{1 - r} \text{ where } |r| < 1\right]$

$\Rightarrow \quad \dfrac{4}{5}S = 1 + 3\left[\dfrac{1}{5} \cdot \dfrac{5}{4}\right] = 1 + \dfrac{3}{4}$

$\Rightarrow \quad S = \dfrac{7}{4} \cdot \dfrac{5}{4}$

Hence $\quad S = \dfrac{35}{16}$

Thus, the required sum of the given infinite series

$$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \quad \text{is } \frac{35}{16}$$

Example 3. Find the sum of the infinite series: $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+\cdots$

Solution: Let S denote the sum of the given AG series. Then

$$S = 1.\frac{1}{3}+3.\frac{1}{9}+5.\frac{1}{27}+7.\frac{1}{81}+\cdots \qquad \text{... (i)}$$

Multiplying both sides by $\frac{1}{3}$

$$\frac{1}{3}S = 1.\frac{1}{9}+3.\frac{1}{27}+5.\frac{1}{81}+\cdots \qquad \text{... (ii)}$$

Subtracting equation(ii) from equation (i)

$$S-\frac{1}{3}S = 1.\frac{1}{3}+2.\frac{1}{9}+2.\frac{1}{27}+2.\frac{1}{81}+\cdots$$

$$\Rightarrow \quad \frac{2}{3}S = \frac{1}{3}+2\left(\frac{\frac{1}{9}}{1-\frac{1}{3}}\right)$$

$$= \frac{1}{3}+2\left(\frac{1}{9}\times\frac{3}{2}\right)$$

$$\Rightarrow \quad \frac{2}{3}S = \frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

$$\Rightarrow \quad S = 1$$

Thus, the required sum of the given infinite series is 1,

i.e., $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+\cdots = 1$

Exercise 5.2

Sum the following series up to the indicated terms.

1. $1\cdot 2+2\cdot 2^2+3\cdot 2^3+4\cdot 2^4+\cdots$ to 50 terms

2. $1+2.5+3.5^2+4.5^3+\cdots$ to 30 terms

3. $1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\cdots$ to 25 terms

4. $1+\frac{8}{5}+\frac{15}{5^2}+\frac{22}{5^3}+\cdots$ to n terms

Find the sum to infinity of the following series:

5. $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$

6. $1+9a+25a^2+49a^3+\cdots$ where $|a|<1$,

7. $1+4b+7b^2+10b^3+\cdots$ where $|b|<|$

8. $1+\frac{4}{4}+\frac{7}{4^2}+\frac{10}{4^3}+\cdots$

9. $1+\frac{5}{3}+\frac{12}{3^2}+\frac{22}{3^3}+\frac{35}{3^4}+\cdots$

10. $1+5a+9a^2+13a^3+\cdots$ where $|a|<|$

5.3 Method of Differences

5.3.1 Define method of differences

The Method of Differences is the method in which differences of successive terms of the given series are used to find the sum of the series particularly when the general term of the series is that of an "Algebraic fraction".

Theorem:

If a series can be expressed in the form of $f(r+1)-f(r)$ or $f(r)-f(r+1)$ where $f(r)$ is the known function of r, then

$$\sum_{r=1}^{n}[f(r+1)-f(r)] = f(n+1)-f(1)$$

or

$$\sum_{r=1}^{n}[f(r)-f(r+1)] = f(1)-f(n+1)$$

Proof: Consider a series that is expressed in the form $f(r)-f(r+1)$

Now, $\sum_{r=1}^{n}[f(r)-f(r+1)] = \{f(1)-f(2)\}+\{f(2)-f(3)\}+\{f(3)-f(4)\}+\cdots+$
$\{f(n-1)-f(n)\}+\{f(n)-f(n+1)\}$

$= f(1)-f(2)+f(2)-f(3)+f(3)+\cdots+f(n-1)-f(n)+f(n)-f(n+1)$

$= f(1)-f(n+1)$

i.e,
$$\sum_{r=1}^{n}[f(r)-f(r+1)] = f(1)-f(n+1) \qquad \ldots \text{(i)}$$

Multiplying both sides by -1

$$(-1)\sum_{r=1}^{n}[f(r)-f(r+1)] = (-1)[f(1)-f(n+1)]$$

$$\Rightarrow \sum_{r=1}^{n}[f(r+1)-f(r)] = f(n+1)-f(1) \qquad \ldots \text{(ii)}$$

Hence, both relations (i) and (ii) are proved.

Example: Sum the series $\sum_{1}^{50} \frac{1}{(r+1)(r+2)}$ by the method of differences.

Solution:

Method 1:

The function can be written in the following form

$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2} \qquad \text{(by partial fractions)}$$

Taking summation on both sides up to fifty terms, we get

$$\sum_{r=1}^{50} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{50} \frac{1}{r+1} - \sum_{r=1}^{50} \frac{1}{r+2} \qquad \text{... (i)}$$

Now,

$$\sum_{r=1}^{50} \frac{1}{r+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50} + \frac{1}{51}$$

and

$$\sum_{r=1}^{50} \frac{1}{r+2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{50} + \frac{1}{51} + \frac{1}{52}$$

Substituting in equation (i), we get

$$\sum_{r=1}^{50} \frac{1}{(r+1)(r+2)} = \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50} + \frac{1}{51}\right] - \left[\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{50} + \frac{1}{51} + \frac{1}{52}\right]$$

After cancelling the values, we have

$$\sum_{r=1}^{50} \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{52} = \frac{25}{52}$$

Method 2: We know that

$$\sum_{r=1}^{n} [f(r) - f(r+1)] = f(1) - f(n+1) \qquad \text{... (i)}$$

Here $\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$ with $f(r) = \frac{1}{r+1}$ and $f(r+1) = \frac{1}{r+2}$

Now, $$\sum_{r=1}^{50} \left(\frac{1}{r+1} - \frac{1}{r+2}\right) = \frac{1}{1+1} - \frac{1}{51+1} \qquad [\text{using (i)}]$$

$$= \frac{1}{2} - \frac{1}{52} = \frac{25}{52}$$

5.3.2 Apply this method to find the sum of n terms of the series whose differences of the consecutive terms are either in arithmetic or in geometric sequence

We apply the method of differences to find the sum of a series which is not in any standard progression. However, differences of the consecutive terms of that series are either in arithmetic progression or in geometric progression. The following examples show the method of finding the sum of the series whose differences of the consecutive terms are either in A.P or in G.P respectively.

Example 1: Find sum of the series: $5 + 10 + 19 + 32 + 49 + \cdots$ to n terms

Solution: Here, differences of consecutive terms are in A.P.

i.e., $5, 9, 13, 17, \ldots$; is A.P.

So, we use method of differences

Let $\quad S_n = 5 + 10 + 19 + 32 + 49 + \cdots$ to n terms $\quad \ldots$ (i)

or $\quad S_n = \pm 5 \pm 10 \pm 19 \pm 32 \pm \cdots \pm a_n \quad \ldots$ (ii)

$$0 = 5 + \{5 + 9 + 13 + 17 + \cdots \text{ to } (n-1)\text{terms}\} - a_n$$

[by taking difference of equations (i) & (ii)]

$$\Rightarrow \quad a_n = 5 + \frac{(n-1)}{2}\{2(5) + (n-2)4\} \qquad \because S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= 5 + \frac{(n-1)}{2}(4n+2)$$

$$= 5 + (n-1)(2n+1)$$

$$a_n = 2n^2 - n + 4$$

For sum of the series,

$$S_n = 2\sum_1^n n^2 - \sum_1^n n + \sum_1^n 4$$

$$S_n = 2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n$$

$$= \frac{n\{2(n+1)(2n+1) - 3(n+1) + 24\}}{6}$$

$$= \frac{n\{2(2n^2+3n+1) - 3n - 3 + 24\}}{6}$$

$$= \frac{n(4n^2 + 6n + 2 - 3n + 21)}{6}$$

$$S_n = \frac{n(4n^2 + 3n + 23)}{6}$$

Hence, sum of the n terms of the given series is $S_n = \frac{n(4n^2+3n+23)}{6}$

Example 2: Find sum of the series: $3 + 7 + 15 + 31 + 63 + \cdots$ to n terms.

Solution: Here, differences of consecutive terms are in G.P.

i.e., $4, 8, 16, 32, \ldots$; is G.P.

So, we use method of differences

Let, $S_n = 3 + 7 + 15 + 31 + 63 + \cdots$ to n terms ... (i)

or $S_n = \quad \pm 3 \pm \; 7 \pm 15 \pm 31 \pm \cdots \pm a_n$ ... (ii)

$$0 = 3 + \{4 + 8 + 16 + 32 + \cdots \text{ to } (n-1) \text{ terms}\} - a_n$$

[by subtracting equation (ii) from (i)]

$$a_n = 3 + \frac{4(2^{n-1} - 1)}{2-1} \qquad \left[\because S_n = \frac{a(r^n - 1)}{r-1}\right]$$

$$= 3 + 4(2^{n-1} - 1)$$

$$= 3 + 2 \cdot 2^n - 4$$

$$a_n = 2 \cdot 2^n - 1$$

For sum of the series,

$$S_n = 2\sum_{1}^{n} 2^n - \sum_{1}^{n} 1$$

$$= 2\{2^1 + 2^2 + 2^3 + \cdots + 2^n\} - n$$

$$= 2\left[\frac{2(2^n - 1)}{2 - 1}\right] - n$$

$$S_n = 4(2^n - 1) - n$$

Hence, sum of the n terms of the given series is: $S_n = 4(2^n - 1) - n$

Exercise 5.3

Find the sum of the following series.

1. $3 + 6 + 12 + 21 + 33 + \cdots$ to n terms.
2. $5 + 10 + 17 + 26 + 37 + \cdots$ to n terms.
3. $7 + 14 + 26 + 43 + 65 + \cdots$ to n terms.
4. $3 + 6 + 15 + 42 + 123 + \cdots$ to n terms.
5. $7 + 14 + 45 + 100 + 179 + \cdots$ to n terms.
6. $3 + 6 + 21 + 96 + 471 + \cdots$ to n terms·

5.4 Summation of Series using Partial Fractions

Partial fractions also help in obtaining the sum of the series whose general terms are rational expressions which can be expressed as a sum of two or more fractions.

5.4.1 Use partial fractions to find the sum to n terms and to infinity the series of the type $\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \cdots$

Let us use partial fractions to find the sum to n terms and to infinity the series of the type

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \cdots$$

with general term $a_n = \frac{1}{[a+(n-1)d]\cdot[a+nd]}$

Example 1. By using partial fractions, evaluate the sum of the following series to n terms.

$$\frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \cdots$$

Solution: We have

$$\frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \cdots \text{to } n \text{ terms} = \sum_{1}^{n} \frac{1}{\{2+(n-1)3\}(2+3n)} = \sum_{1}^{n} \frac{1}{(3n-1)(2+3n)}$$

Now we find partial fractions of general term

Let, $$\frac{1}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2} \quad \ldots(i)$$

Multiplying both sides by $(3n-1)(3n+2)$

We get $$1 = A(3n+2) + B(3n-1) \quad \ldots(ii)$$

By using $n = \frac{1}{3}$ in (ii)

We get $1 = A(1+2) \Rightarrow 3A = 1$ or $A = \frac{1}{3}$

By using $n = -\frac{2}{3}$ in equation (ii)

We get $1 = B(-2-1) \Rightarrow -3B = 1$ or $B = -\frac{1}{3}$

By using values of A and B in equation (i)

We get $$\frac{1}{(3n-1)(2+3n)} = \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$$

Now, $$\sum_{1}^{n} \frac{1}{(3n-1)(3n+2)} = \sum_{1}^{n} \left\{\frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}\right\}$$

$$= f(1) - f(n+1) \qquad \left[\because f(r) = \frac{1}{3(3r-1)}\right]$$

$$= \frac{1}{3(3\times 1-1)} - \frac{1}{3\{3(n+1)-1\}}$$

$$= \frac{1}{6} - \frac{1}{3(3n+2)}$$

$$= \frac{3n+2-2}{6(3n+2)}$$
$$= \frac{n}{2(3n+2)}$$

So, $\frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \cdots$ to n terms $= \frac{n}{2(3n+2)}$.

Example 2. By using partial fractions, evaluate the sum of the following infinite series.

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots$$

Solution: We have

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

Now we find partial fractions of general term

Let $\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$ ... (i)

Multiplying both sides by $n(n+1)(n+2)$

We get $1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$ ... (ii)

For $n = 0$, (ii) becomes, $1 = 2A$ $\Rightarrow A = \frac{1}{2}$

For $n = -1$, (ii) becomes, $1 = B(-1)(-1+2)$ $\Rightarrow B = -1$

For $n = -2$, (ii) becomes, $1 = C(-2)(-2+1)$ $\Rightarrow C = \frac{1}{2}$

By using values of A, B and C in (i)

We get, $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} + \frac{-1}{n+1} + \frac{1}{2(n+2)}$

Now,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$
$$= \sum_{n=1}^{\infty} \left[\frac{1}{2n} + \frac{-1}{n+1} + \frac{1}{2(n+2)}\right]$$
$$= \sum_{n=1}^{\infty} \frac{1}{2n} + \sum_{n=1}^{\infty} \frac{-1}{n+1} + \sum_{n=1}^{\infty} \frac{1}{2(n+2)}$$
$$= \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right] + \left[\frac{-1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots\right] + \left[\frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots\right]$$

$$= \frac{1}{4} \qquad \text{(By cancellation of terms)}$$

Hence, $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots = \frac{1}{4}$.

Exercise 5.4

Find the sum of the following series:

1. $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots$ to n terms.

2. $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \frac{1}{9\cdot13} + \cdots$ to n terms.

3. $\frac{1}{1\cdot5\cdot9} + \frac{1}{5\cdot9\cdot13} + \frac{1}{9\cdot13\cdot17} + \cdots$ to infinity.

4. $\frac{1}{1\cdot4\cdot7} + \frac{1}{4\cdot7\cdot10} + \frac{1}{7\cdot10\cdot13} + \cdots$ to infinity.

Find sum of the series:

5. $\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$

6. $\sum_{k=1}^{n} \frac{1}{9k^2 + 3k - 2}$

7. $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$

8. $\sum_{k=2}^{n} \frac{1}{(k^2 - k)}$

Review Exercise 5

1. Select correct option.

i. If $a_n = 5 - 3n + 2n^2$, then $a_{2n} =$ --------------

(a) $5 - 6n + 2n^2$ (b) $5 - 6n + 4n^2$

(c) $5 + 6n + 4n^2$ (d) $5 - 6n + 8n^2$

ii. If $a_{n-2} = 3n - 11$, then $a_n =$ --------------

(a) $3n + 5$ (b) $3n - 5$ (c) $3n - 9$ (d) $3n - 13$

iii. $\sum_{1}^{50} n =$ -----------

(a) 1274 (b) 1275 (c) 1280 (d) 1285

iv. $\frac{1}{n(n+1)} =$ -------------

(a) $\frac{1}{n}+\frac{1}{n+1}$ (b) $\frac{1}{n}-\frac{1}{n+1}$ (c) $\frac{2}{n}+\frac{1}{n+1}$ (d) $\frac{1}{n}-\frac{3}{n+1}$

v. The n^{th} term of $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}$ ... is:

(a) $\frac{1}{3n-1}$ (b) $3n-1$ (c) $2n+1$ (d) $\frac{1}{3n+1}$

vi. $\sum n$ is equal to:

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$ (c) $\frac{n^2(n+1)^2}{2}$ (d) n^2

vii. $\sum n^2$ is equal to:

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$ (c) $\frac{n^2(n+1)^2}{2}$ (d) n^2

viii. $\sum n^3$ is equal to:

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$ (c) $\frac{n^2(n+1)^2}{4}$ (d) $\frac{n(n+1)^2}{4}$

ix. If $S_n = (n+1)^2$, then S_{2n} is equal to:

(a) $2n+1$ (b) $4n^2+4n+1$

(c) $(2n-1)^2$ (d) Cannot be determined

x. $\sum_{n=3}^{20} n^0 =$:-----------

(a) 1 (b) 19 (c) 20 (d) 18

2. Sum the following series up to n terms

$2\times 1^2 + 4\times 2^2 + 6\times 3^2 + \cdots$

3. Sum the series $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots$ to infinity.

Permutation, Combination and Probability

Unit 6

6.1 Factorial of a Natural Number

6.1.1 Know Kramp's factorial notation to express the product of first n natural numbers by n!

In mathematics, the factorial of a positive integer n, is the product of all positive integers (i.e., natural numbers) less than or equal to n. It may be noted that the factorial values for negative integers are not defined.

The product of first n natural numbers or factorial of n is denoted by n! and read as 'n factorial'. The factorial notation n! was introduced in 1808 AD by a French mathematician named Christian Kramp. The factorial notation is frequently used to write continued products in simplified form.

Now, we find some factorials by definition.

$$1! = 1$$
$$2! = 2\cdot 1 = 2$$
$$3! = 3\cdot 2\cdot 1 = 6$$
$$4! = 4\cdot 3\cdot 2\cdot 1 = 24$$

Thus, $n! = n(n-1)(n-2)\ldots.3\cdot 2\cdot 1$

or $n! = n(n-1)(n-2)!$ or $n! = n(n-1)!$

Also we define $0! = 1$.

Note: The symbol $\underline{|n}$ is also used for n!

Example 1. Evaluate: $\dfrac{15!}{2!10!}$

Solution: $\dfrac{15!}{2!10!} = \dfrac{15\cdot14\cdot13\cdot12\cdot11\cdot10!}{2!\,10!} = \dfrac{15\cdot14\cdot13\cdot12\cdot11}{2} = 180180$

Example 2. Write $\dfrac{n-4}{(n-2)(n-1)}$ in factorial form.

Solution: $\dfrac{n-4}{(n-2)(n-1)} = \dfrac{(n-4)}{(n-2)(n-1)} \times \dfrac{(n-1)!}{(n-1)!} \times \dfrac{(n-4)!}{(n-4)!}$

$$= \frac{(n-4)}{(n-2)(n-1)} \times \frac{(n-1)(n-2)(n-3)!}{(n-1)!} \times \frac{(n-4)!}{(n-4)(n-5)!}$$

$$= \frac{(n-3)!\,(n-4)!}{(n-1)!\,(n-5)!}$$

Example 3. Express $\frac{(n-1)!}{r!} - \frac{(n+1)!}{(r-1)!}$ as a single fraction:

Solution: $\frac{(n-1)!}{r!} - \frac{(n+1)!}{(r-1)!}$

$$= \frac{(n-1)!}{r(r-1)!} - \frac{(n+1)n(n-1)!}{(r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!}\left[\frac{1}{r} - n(n+1)\right]$$

$$= \frac{(n-1)!}{r(r-1)!}[1 - nr(n+1)]$$

$$= \frac{(1 - n^2r - nr)(n-1)!}{r!}$$

Exercise 6.1

1. Evaluate the following:

(i) $4!$ (ii) $6!$ (iii) $\frac{8!}{5!}$ (iv) $\frac{10!}{7!}$

(v) $5! \times 7! \times 3!$ (vi) $10! \div 8! + 5!$ (vii) $\frac{16!}{8!12!}$ (viii) $\frac{(9!)^2}{(5!)^3 \times 7!}$

(ix) $\frac{9!}{2! \times (9-2)!}$

2. Write the following in factorial form:

(i) $6 \cdot 5 \cdot 4$ (ii) $12 \cdot 11 \cdot 10$

(iii) $\frac{n(n+1)(n+2)}{6 \cdot 5}$ (iv) $20 \cdot 19 \cdot 18 \cdot 17$

3. Express the following as a single fraction:

(i) $\frac{(n+1)!}{(r+1)!} + \frac{n!}{r!}$ (ii) $\frac{(n+1)!}{r!} + \frac{n!}{(r+1)!}$

6.2 Permutation

6.2.1 Recognize the fundamental principle of counting and illustrate this principle using tree diagram

A process of determining the number of elements contained in a set is called counting. The symbol O(A) or n(A) or |A| is used to denote the number of elements in the set A.

For example, If $A = \{a, b, c\}$, then $O(A) = 3$.

Sum Principle of counting

If A and B are two sets, then

$$O(A \cup B) = O(A) + O(B) - O(A \cap B) \qquad \ldots \text{(i)}$$

This is known as the Sum Principle.

If A and B are two disjoint sets, then

$$O(A \cup B) = O(A) + O(B) \qquad \ldots. \text{(ii)}$$

This is known as the Sum Principle for disjoint sets.

This principle can be extended to any finite number of disjoint sets:

i.e., $O(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = O(A_1) + O(A_2) + O(A_3) + \cdots + O(A_n)$ ... (iii)

Fundamental Principle or Product Principle of counting

If A and B are any two sets and A×B is their Cartesian product, then

$$O(A \times B) = O(A) \cdot O(B) \qquad \ldots. \text{(iv)}$$

This is known as the fundamental principle or Product Principle.

The Product Principle can be extended to any finite number of finite sets:

i.e., $O(A_1 \times A_2 \times A_3 \times \ldots \times A_n) = O(A_1) \cdot O(A_2) \cdot O(A_3) \cdot \ldots \cdot O(A_n)$ ... (v)

Product principle can also be defined as:

If an operation performed in r distinct ways, and corresponding to each of these ways there are s distinct ways of performing a second operation, then the two consecutive operations can be performed together in rs distinct ways in that order.

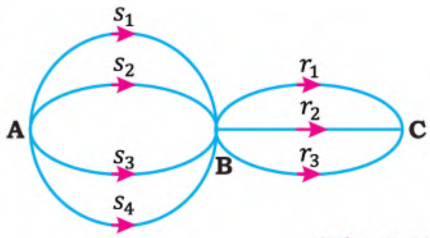
Example 1. There are 4 roads from village A to village B and 3 roads from village B to village C. By how many different routes may one can travel from A to C by way of B? Also enlist the various routes.

Solution: One can go from village A to village B by any one of the four paths s_1, s_2, s_3, s_4 and then he can go from village B to village C by any one of the three different paths r_1, r_2, r_3 (Fig. 6.1).

Note that by going from A to B by the path s_1, he can choose any of the three paths r_1, r_2, r_3 to go from B to C. Therefore, from A to C, he can go by any of the following three different routes:

$$s_1r_1 \quad s_1r_2 \quad s_1r_3$$

Thus, path s_1 from A to B is yielding three different paths to reach C. Similarly, each of the remaining three paths s_2, s_3 and s_4 from A to B, will also yield three different paths to reach at C. Hence, by the Fundamental Principle, he can go from A to C by 4 times 3, or 12, different routes.



(Fig. 6.1)

A list of various routes is as follows:

s_1r_1	s_1r_2	s_1r_3
s_2r_1	s_2r_2	s_2r_3
s_3r_1	s_3r_2	s_3r_3
s_4r_1	s_4r_2	s_4r_3

Example 2. How many positive odd numbers, each having three digits, can be formed from the digits 1,2,3,4,5,6 if no digit is to be repeated in a given number.

Solution: Since the required numbers are odd, the unit's place may be filled in one of three ways: by the 1, 3 or 5. In turn, the digit, in the tenth's place may be filled in any one of the five ways, and the digit in the hundredth's place may be filled in any one of the four ways. Hence, by the Fundamental Principle of counting, the three places can be filled in

$$(4)(5)(3) = 60 \text{ ways}$$

Hence, 60 positive odd numbers of different digits can be formed.

Example 3. How many two-digit whole numbers can be formed from the digits 3,4,5?

Solution: Any one of the three digits can be used in the tenth's place. Since the digits used need not to be different, so, the unit's place may also be filled in any one of the three different ways. Hence, by the Fundamental Principle, the two places can be filled in

$$3 \cdot 3 = 9 \text{ ways}$$

Hence, 9 whole numbers of two digits can be formed.

Tree Diagram

A tree diagram is a special type of diagram used in strategic decision making, valuation or probability calculations. The diagram starts at a single node, with branches emanating to additional nodes, which represent mutually exclusive decisions or events.

Tree diagrams are useful for organizing and visualizing the different possible outcomes of a sequence of events. For each possible outcome of the first event, we draw a branch where we write down the possibility of that outcome.

Then, for each possible outcome of the second event we do the same thing.

Below is an example of a simple tree diagram.

Example: Farhan has 3 books and Hashir has 2 books. In how many ways can they exchange a book? Also illustrate using tree diagram.

Solution: Farhan can give a book to Hashir in 3 ways (any one of his books F_1, F_2, or F_3 may be given); and then Hashir can give one of his 2 books H_1, H_2, to Farhan, so Hashir can give a book in exchange by 2 ways. Hence by the Fundamental Principle they can exchange a book in

$$(3)(2) = 6 \text{ ways.}$$

The various possibilities can be shown with the help of the following tree diagram (Fig. 6.2)

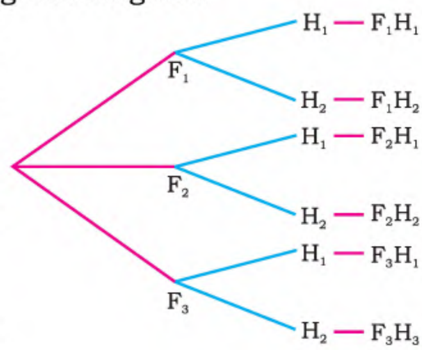


Fig. 6.2

6.2.2 Explain the meaning of permutation of '*n*' different objects taken *r* at a time and know the notation ^nP_r

Permutation is one of the methods of counting, very useful in the study of certain fields of mathematics.

We know that an ordered r-tuple (finite ordered list) whose first, second, ... , rth elements are respectively $t_1, t_2, ..., t_r$ is written as $(t_1, t_2, ..., t_r)$. When $r = 2$, we have the ordinary ordered pair; viz. (t_1, t_2).

An ordered r-tuple all of whose elements belong to a set S of n elements $(n \geq r)$, is called a permutation of r elements selected from the set S.

Permutation of r elements from a set of n elements $(n \geq r)$ are traditionally called "Permutations of n things taken r at a time" and denoted as ^nP_r.

The concept of a permutation is nothing more than an ordered set, with each ordering of the elements of the set being a different permutation.

From the above discussion, it is obvious that a permutation is an arrangement of all or part of a set of objects, with regard to the order of the arrangement. For example, suppose we have a set of three letters: A, B and C. If we arrange 2 letters from that set then each possible arrangement would be an example of a permutation. The complete list of possible permutations would be: AB, AC, BA, BC, CA, and CB. Thus, in this case $^3P_2 = 6$.

6.2.3 Prove that $^{n}P_r = n(n-1)(n-2)\cdots(n-r+1)$ and hence deduce that

(i) $^{n}P_r = \frac{n!}{(n-r)!}$ (ii) $^{n}P_n = n!$ (iii) $0! = 1$

We have already studied that if n distinct objects are given and we have to arrange r $(n \geq r)$ out of them at a time in which order is important, such an arrangement is called a permutation of n objects taken r at a time. The number of permutations is determined by the following theorem.

Theorem: The number of permutations of n elements of a set taken r at a time is given by $^{n}P_r = n(n-1)(n-2)\ldots(n-r+1)$, up to r factors

Proof:

Let $S = \{t_1, t_2, \ldots, t_n\}$ be a set of n elements. The number of permutations of n elements taken r at a time from this set are the ordered r-tuples which can be formed by using its elements. The number of permutations of r elements out of n is the same as the number of ways of filling up r blank places in an ordered r-tuple.

The first element of the ordered r-tuple can be chosen in n ways because there are n element of S for us to choose from. After the first element of the r-tuple is chosen in any one of these n ways, we are left with (n–1) elements of the set S. Thus the second element of the r-tuple can be chosen in (n–1) ways, so both 1st and 2nd elements can be chosen in $n(n-1)$ ways. After the first two elements of the r-tuple are chosen, we are left with (n–2) elements of the set S. Thus the third element of r-tuple can be chosen in (n–2) ways. Again, by the Fundamental Principle, the first three elements of the r-tuple can be chosen in n(n–1)(n–2) ways and so on. Continuing in this way, the number of ways in which the 'r' elements of the ordered r-tuple can be chosen is

$$n(n-1)(n-2)\ldots \text{ up to r factors.}$$

Therefore, the number of possible ordered r-tuples is:

$$^{n}P_r = n(n-1)(n-2)\ldots[n-(r-1)]$$
$$= n(n-1)(n-2)\ldots(n-r+1)$$

Hence, proved.

Deductions:

(i) $^{n}P_r = \frac{n!}{(n-r)!}$

Using the relation $^{n}P_r = n(n-1)(n-2)\ldots(n-r+1)$

Multiplying and dividing by $(n-r)!$ we get:

$$= \frac{[n(n-1)(n-2)\ldots(n-r+1)](n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}$$

Hence $^{n}P_r = \frac{n!}{(n-r)!}$

(ii) $^nP_n = n!$

If r = n, then

$$^nP_n = \frac{n!}{(n-n)!}$$
$$= \frac{n!}{0!} = \frac{n!}{1}$$
$$^nP_n = n!$$

(iii) $0! = 1$

We have, $^nP_n = 1$

By using $n = 1$

We get, $^1P_1 = 1!$

$$\Rightarrow \quad \frac{1!}{(1-1)!} = 1$$
$$\Rightarrow \quad \frac{1!}{0!} = 1$$
$$\Rightarrow \quad 1 = 0!$$

6.2.4 Apply ^nP_r to solve problems of finding the number of arrangements of *n* objects taken *r* at a time (when all *n* objects are different and when some of them are alike).

Example 1. How many different arrangements can be made by using all the letters of the word "DAUGHTER"?

Solution: There are 8 different letters in the word "DAUGHTER" and all the 8 letters are to be used. This is a case of permutations of 8 different things taking all of them at a time. Hence the total number of arrangements is:

$$^8P_8 = 8! = 40320.$$

Example 2. In how many ways can 6 books of Mathematics and 5 books of Biology be placed on a shelf so that the books on the same subject always remain together?

Solution: The 6 books of Mathematics can be arranged in $^6P_6$, i.e., 6! ways among themselves.

The 5 books of Biology can be arranged in $^5P_5$, i.e., 5! ways among themselves.

The 2 groupings of the books on the two subjects can be permutated in $^2P_2$, i.e., 2! ways. Hence by fundamental principle.

The total number of arrangements:

$$= 6!\ .5!\ .2!$$
$$= (720)(120)(2)$$
=1,72,800 arrangements.

Example 3. How many different three digit whole numbers can be formed with the numbers 9,8,7,5,3,2. Assuming that no digit can be repeated?

Solution: This is the case of permutations of 6 digits taking 3 at a time. Hence the total of three-digit numbers that can be formed is:

$$^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6.5.4.(3!)}{3!} = 120$$

Example 4. How many signals can be made with six flags of different colours, by hoisting 1 or 2 or 3 or 4 or 5 or all of them?

Solution: When only 1 flag is to be displayed at a time,

the number of signals = ${}^6P_1 = \frac{6!}{5!} = 6.$

When 2 flags are to be displayed at a time,

the number of signals = ${}^6P_2 = \frac{6!}{4!} = 30.$

When 3,4,5 and 6 flags are to be displayed at a time, the number of signals are ${}^6P_3$, ${}^6P_4$, ${}^6P_5$ and ${}^6P_6$ respectively,

Hence the total number of signals:

$$= {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6$$
$$= 6 + 30 + 120 + 360 + 720 + 720$$
$$= 1956 \text{ signals}$$

Example 5. If ${}^nP_3 = 12 . {}^{\frac{n}{2}}P_3$, find n.

Solution: We have, $\quad {}^nP_3 = 12 . {}^{\frac{n}{2}}P_3$

$$\Rightarrow \quad n(n-1)(n-2) = 12.\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)$$
$$\Rightarrow \quad 2n(n^2-3n+2) = 3n\,(n-2)(n-4)$$
$$\Rightarrow \quad 2(n^2-3n+2) = 3(n^2-6n+8)$$
$$\Rightarrow \quad n^2-12n+20 = 0$$
$$\Rightarrow \quad (n-10)(n-2) = 0$$
$$\Rightarrow \quad n = 10 \text{ or } 2.$$

But $\quad n = 2$ is inadmissible.

Hence, $\quad n = 10.$

The number of permutations can be determined by the following theorem if some objects are alike.

Theorem

The number of distinct permutations of n things taken all at a time, when r of them are alike of one kind, s of them alike of another kind, t of them are alike of a third kind and the rest, if any, all different is

$$\frac{n!}{r!\,s!\,t!}$$

Proof:

Let P be the required number of permutations. Take any one of these permutations and replace the r like things by r unlike things, different from each other and from all the rest, and permute them in all possible ways among themselves, leaving the others unchanged. We thus get $r!$ permutations out of the original one. Hence out of the P original permutations, we shall get $(r! \cdot \text{P})$ permutations.

Similarly, if we now replace the s like things by s unlike things, different from each other and from all the rest, and permute them in all possible ways among themselves, we obtain $(r!\cdot s!\cdot \mathrm{P})$ permutations out of the $(r!\cdot \mathrm{P})$ permutations previously obtained.

Repeat the process with the '*t*' like things of the third kind, replacing them by '*t*' unlike things different from each other and from all the rest, and permuting them in all possible ways among themselves. The number of permutations is thus increased $t!$ times. Hence we have $(r!\cdot s!\cdot t!\cdot \mathrm{P})$ permutations in the end. But the 'n' things are now all different, and the total number of their permutations must be $n!$. Therefore, $r!\cdot s!\cdot t!\cdot \mathrm{P} = n!$

or $$\mathrm{P} = \frac{n!}{r!\ s!\ t!}$$

This number of permutations is symbolically written as $\binom{n}{r,\ s,\ t}$

The above theorem can be easily extended to cases where there are more than three kinds of like things.

Note: If '*r*', '*s*' and '*t*' are such that none of the things are left over, then $r + s + t = n$.

Example 1. Find the number of permutations of 9 fruits, taken all at a time when 3 are apples, 2 are oranges and 2 are mangoes.

Solution: Here 3 apples are like things of the first kind, 2 oranges are like things of the second kind and 2 mangoes of the third kind and rest are the different. The total number of fruits is 9.

Hence the required number of permutations is:

$$\binom{9}{3,2,2} = \frac{9!}{3!\,2!\,2!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4(3!)}{(3!)2\cdot 1\cdot 2\cdot 1} = 15120$$

Example 2. How many whole numbers can be formed with the digits $1, 2, 3, 4, 3, 2, 1$ so that the even digits occupy the even places?

Solution: The arrangement $1, 2, 3, 4, 3, 2, 1$ is one of the type required. The even digits 2, 4, 2 can be permuted among themselves in their three places in:

$$\frac{3!}{2!} = 3 \text{ ways.}$$

The odd digits 1, 3, 3, 1 can be permuted among themselves in their four places in:

$$\frac{4!}{2!\ 2!} = 6 \text{ ways.}$$

By fundamental principle

The required number $= (3)(6) = 18$.

Example 3. A coin is tossed nine times repeatedly. In how many possible ways can we get 5 heads and 4 tails.

Solution: Out of 9 tosses, we are to have 5 heads and 4 tails. This can be done in:

$$\binom{9}{5,4} = \frac{9!}{5!\ \ 4!}$$
$$= \frac{9\cdot 8\cdot 7\cdot 6\cdot (5!)}{(5!).4\cdot 3\cdot 2\cdot 1}$$
$$= 126 \text{ ways.}$$

6.2.5 Find the arrangement of different objects around a circular permutation

Circular Permutation:

In an ordinary permutation of r things, the elements are arranged in a definite order and their places can be marked as the first, the second, the third... etc., so that every arrangement has a beginning and an end. In a circular permutation, where the elements are arranged round the circumference of a circle, there is neither a beginning nor an end, and the positions cannot be marked out absolutely as the first, the second, the third... etc. In a circular permutation therefore, it is the relative positions of elements that matter.

Example 1. Suppose a man a, a lady b, a boy c, and a girl d, are sitting around a round table as shown in Fig. 6.3.

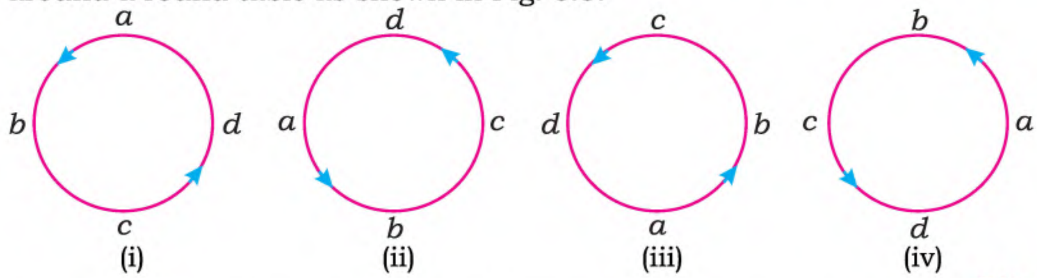


Fig. 6.3

Moving round the circumference in the anti-clockwise sense as indicated by the arrows, the four letters a, b, c and d occur in the same order and their relative positions are unchanged. Hence the various arrangements thus obtained are to be regarded as identical. The four figures show the four different forms in which the same arrangement appears to four different persons standing outside the circle opposite a, b, c and d respectively.

These identical arrangements round a circle correspond to four distinct ordinary permutations:

$$abcd\ ,\ dabc\ ,\ cdab\ ,\ bcda$$

obtained by taking in succession as first each of the four letters in the circular permutation. Hence the total number of circular permutations of four different things is one-fourth of the number of ordinary permutations. It is therefore equal to $\frac{1}{4}$. (4!) = 3!.

Since these identical arrangements arise because all the things are moved around without changing their relative positions, it follows that if we keep one of them fixed in position and permute all the rest among themselves, we shall get the required number of circular permutations.

In the above situation clockwise and anti-clockwise arrangements are distinct.

In the case of a necklace of beads (or a garland of flowers) the same necklace gives one order of beads if looked at from one side and the reverse order of looked at from the opposite side. If one of these arrangements is turned over, it becomes identical with the other.

Hence the same necklace is the result of both the orders of beads. In such cases, therefore, the clockwise and anti-clockwise arrangements give only one distinct permutation.

Theorem

The number of circular permutations of n elements of a set taken all at a time, is $(n-1)!$, if clockwise and anti-clockwise arrangements are regarded as distinct; if they are regarded as identical, the number is $\frac{1}{2}[(n-1)!]$

Proof:

The number of circular permutations of n elements can be obtained by fixing the position of any one of these elements wherever it may be and then by permuting the remaining $(n-1)$ elements.

This can be done in $(n-1)!$ ways.

Hence the required number of circular permutations is $(n-1)!$

If the clockwise and anti-clockwise arrangements are not regarded as distinct, the number is reduced by one-half.

Hence in such cases the required number is: $\frac{1}{2}[(n-1)!]$.

Example 1. In how many ways can 5 persons be seated at round table conference?

Solution: The number of circular permutations of n distinct objects is $(n-1)!$. So the number of ways in which 5 person be seated at round table conference is $(5-1)! = 4! = 24$ ways

Example 2. In how many ways, can 10 keys be arranged in a ring?

Solution: Since the clockwise and anticlockwise arrangements are identical.

So, the number of permutations $= \frac{(n-1)!}{2} = \frac{(10-1)!}{2} = \frac{9!}{2} = 181440$

6.2.6 Solve daily life problems involving permutation

Example 1. There are 9 competitors in a race for 3 prizes. In how many ways, can the prizes be given?

Solution:

Here $n = 9$ and $r = 3$

Now $\quad {}^9P_3 = \frac{9!}{(9-3)!}$

$= \frac{9!}{6!}$

$= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$

$= (9)(8)(7) = 504$

Hence in 504 ways 3 prizes can be given to 9 competitors.

Example 2. In how many ways, can Fiza take one Red ball, one Yellow ball and one Green ball from a box containing 4 Red, 3 Yellow and 5 Green balls?

Solution:

Fiza can take one red ball, one yellow ball and one green ball in

$P_1^4 \cdot P_1^3 \cdot P_1^5$ ways

$= 4 \cdot 3 \cdot 5$ ways.

$= 60$ ways.

Example 3. There are 9 different posts vacant of which 5 must be held by postgraduates and 3 by graduates, while the remaining 1 may be given to either postgraduates or graduates if 8 postgraduates and 6 graduates apply for these posts, in how many ways can the posts be filled?

Solution: There are 8 postgraduates and 5 different posts for postgraduates alone. These can, therefore, be filled in ${}^8P_5$ ways.

Similarly, the 6 graduates can fill the 3 posts for them alone in ${}^6P_3$ ways. When this has been done, 1 post still remains to be filled up and there remain 6 candidates for the same. Thus the remaining 1 post can be filled up by the remaining 6 candidates in ${}^6P_1$ ways.

Hence the total number of ways required to fill all the posts are:

$= {}^8P_5 \ . \ {}^6P_3 \ . \ {}^6P_1 = \frac{8!}{3!} \cdot \frac{6!}{3!} \cdot \frac{6!}{5!}$

$= (8)(7)(6)(5)(4)(6)(5)(4)(6)$

$= 4838400$ ways.

Exercise 6.2

1. Evaluate the following:

(i) $^{20}P_3$ (ii) $^{16}P_4$ (iii) $^{12}P_5$ (iv) $\binom{8}{3,2}$ (v) $\binom{9}{2,3,4}$

2. Find the value of n in each of the following:

(i) $^{n}P_2 = 30$ (ii) $^{n}P_3 = 60$ (iii) $^{8}P_n = 8.7.6$

(iv) $^{11}P_n = 11.10.9$ (v) $^{n+2}P_4 : {}^{n+1}P_3 = 5:1$ (vi) $^{n}P_4 : {}^{n-1}P_3 = 9:1$

3. In how many ways different batting orders are possible for a cricket team consisting of 11 players.

4. In how many ways can two English books, three Chemistry books and four Physics books be arranged on a shelf so that all the books of same subject are together?

5. How many different arrangement can be formed of the word "MATRIX" using all letters? Also find the number of arrangements if each of them start with "R".

6. How many signals can be given by 8 flags of different colours when 5 of them are used at a time?

7. A fair coin is tossed three times. How many outcomes are possible? Find by fundamental principle and illustrate using tree diagram.

8. How many 5-digit numbers can be formed, without repeating any digit:

(i) 5,6,7,8,9 (ii) 1,2,3,5,7,9 (iii) 2,4,6,8,0

9. Prove that:

(i) $^{n}p_r = {}^{n-1}p_r + r \cdot {}^{n-1}p_{r-1}$ (ii) $^{n}p_r = n \cdot {}^{n-1}p_{r-1}$

10. How many arrangements can be formed using all the letters of the following words.

(i) STATISTICS (ii) PLANE

(iii) OBJECT (iv) FASTENS

(v) MATHEMATICS (vi) ASSISTANCE

11. How many 8-digit numbers can be formed from the digits 2,2,2,3,3,3,5,6? How many among these are greater than 60,000,000?

12. How many distinct permutations of letters of the word "ESSENTIAL" are possible? How many will have the two S's (i) together (ii) separate.

13. In how many ways, can 8 persons be seated around a round table?

14. In how many ways, 7 keys be arranged in a circular key ring?

15. How many 3-digit numbers can be formed by using each one of the digits 2,3,5,7,9 only once?

16. How many numbers greater than 23,000 can be formed from the digits 1,2,3,5,6 if digits may repeat?

17. Find the number of 5-digit numbers that can be formed from the digits 1,2,4,6,8 (when no digit is repeated), but
    (i) the digits 2 and 8 are next to each other.
    (ii) the digits 2 and 8 are not next to each other.

18. How many 6-digit numbers can be formed without repeating any digit from the digits 0,1,2,3,4,5? In how many of them will 0 be at the tens place?

19. How many 5-digit multiplies of 5 can be formed from the digits 2,3,5,7,9 when digits may repreat?

6.3 Combination

Combination is another method of counting, like permutation. If we form subsets of a given set S of n elements, each one of which has a specified number of elements, say r $(r \leq n)$ without any regard to the order of the elements in any of these subsets, then each of these subsets is called a combination.

6.3.1 Define combination of n different objects taken r at a time

Combinations of r elements from a set of n elements $(n \geq r)$, are traditionally called "Combinations of n things taken r at a time".

The number of ways of selecting r things out of n different objects is denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$. In other words, the number of combinations of n objects taken r at a time $(r \leq n)$ is denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$. In combination order does not matter.

6.3.2 Prove the formula ${}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ and deduce that

- $\binom{n}{n} = \binom{n}{0} = 1$
- $\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{1} = \binom{n}{n-1} = n$,
- $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

The general formula for finding the number of combinations of n elements of a set taken r at a time is given by the following theorem.

Theorem: The number of combinations of n elements of a set taken r at a time is

$$\binom{n}{r} = \frac{{}^{n}P_{r}}{r!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!\,(n-r)!}$$
$$= \frac{n!}{r!\,(n-r)!}$$

Proof: We know that the number of permutations of the n elements of a set taken r at a time is given by

$${}^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1). \quad \ldots\text{(i)}$$

Each one of the combinations of r elements that can be formed provides $r!$ of these permutations. Thus by the Product Principle we have: $\binom{n}{r} r! = {}^{n}P_{r}$

Dividing each member by $r!$, we get:

$$\binom{n}{r} = \frac{{}^{n}P_{r}}{r!} \quad \ldots\text{(ii)}$$

Substituting the value of ${}^{n}P_{r}$ from (i) in (ii), we get:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

Multiplying the numerator and the denominator of the right member by $(n-r)!$, we have if $n \geq r$.

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r!\,(n-r)!}$$

or $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Deductions:

(i) $\binom{n}{n} = \binom{n}{0} = 1$

We have, $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

If $r = n$, then

We get, $$\binom{n}{n} = \frac{n!}{n!\,(n-n)!} = 1$$

i.e. the number of combinations of n things taken n (all) at a time is 1.

Now, if $r = 0$, then

We get, $$\binom{n}{0} = \frac{n!}{0\,!\,(n-0)!} = 1$$

(ii) $\binom{n}{r} = \binom{n}{n-r}$

Here we have $\binom{n}{r} = \dfrac{n!}{r\,!(n-r)!}$

If $r = n - r$, then $\binom{n}{n-r} = \dfrac{n!}{(n-r)![n-(n-r)]!}$

$$= \frac{n!}{(n-r)!\,r!}$$

$$= \binom{n}{r}$$

When we are selecting r elements out of n, in fact, we are leaving $(n-r)$ elements. Sometimes it is simpler to select $(n-r)$ elements which are left out. For example, if we have to select 90 students out of 100, it would be simpler to select 10 students whom we want to reject. Such combinations of r elements or $(n-r)$ elements, out of n elements are called complementary combinations.

(iii) $\binom{n}{r} = \binom{n}{n-1} = n$

We have, $\binom{n}{n-r} = \binom{n}{r}$

Taking $r = 1$

We get, $\binom{n}{n-1} = \binom{n}{1} = \dfrac{n!}{1!(n-1)!} = \dfrac{n!}{(n-1)!} = n$

Hence, $\binom{n}{1} = \binom{n}{n-1} = n.$

(iv) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

Proof: By using $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$,

we have, $\binom{n}{r-1} + \binom{n}{r} = \dfrac{n!}{(n-r+1)!(r-1)!} + \dfrac{n!}{(n-r)!r!}$

$$= \frac{n!\,r + n!\cdot(n-r+1)}{(n-r+1)!\,r!}$$

$$= \frac{n!\,(r+n-r+1)}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)n!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!}$$

$$= \binom{n+1}{r}$$

Hence, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

This rule is called Pascal's rule.

Example 1. Find the value of $\binom{11}{5}$.

Solution:

$$\binom{11}{5} = \frac{11!}{(11-5)!\cdot 5!}$$
$$= \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$
$$= 462$$

Example 2. If ${}^{20}C_r = {}^{20}C_{r+2}$, find ${}^{r}C_5$

Solution: Here $\quad {}^{20}C_r = {}^{20}C_{r+2}$

By using $\quad \binom{n}{r} = \frac{n!}{r!(n-r)!}$

We have, $\quad \frac{20!}{(20-r)!\cdot r!} = \frac{20!}{(20-r-2)!\cdot(r+2)!}$

$$\Rightarrow \quad \frac{1}{(20-r)!\cdot r!} = \frac{1}{(20-r-2)!\cdot(r+2)!}$$

$$\Rightarrow \quad \frac{1}{(20-r)!\cdot r!} = \frac{1}{(20-r-2)!\cdot(r+2)(r+1)!}$$

$$\Rightarrow \quad \frac{1}{(20-r)(20-r-1)(20-r-2)!\cdot r!} = \frac{1}{(20-r-2)!\cdot(r+2)(r+1)\cdot r!}$$

Cancelling out $[(20-r-2)!]\cdot r!$ from the denominators

We get, $\quad \frac{1}{(20-r)(20-r-1)} = \frac{1}{(r+2)(r+1)}$

$$\Rightarrow \quad (r+2)(r+1) = (20-r)(20-r-1)$$
$$\Rightarrow \quad (r+2)(r+1) = (20-r)(19-r)$$
$$\Rightarrow \quad r^2+3r+2 = 380-39r+r^2$$
$$\Rightarrow \quad 42r = 378$$
$$\Rightarrow \quad r = 9.$$

Hence, $\quad {}^{r}C_5 = {}^{9}C_5$

$$= \frac{9\cdot 8\cdot 7\cdot 6\cdot(5!)}{(9-5)!\cdot(5!)} = \frac{9\cdot 8\cdot 7\cdot 6}{4!} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 126.$$

Alternatively,

we know that $\quad {}^{n}C_r = {}^{n}C_{n-r}$

So, $\quad {}^{20}C_r = {}^{20}C_{20-r}$

But we are given that $\quad {}^{20}C_r = {}^{20}C_{r+2}$

Therefore by comparison,

We get, $20 - r = r + 2$

$\Rightarrow$ $r = 9$

Thus ${}^{r}C_5 = {}^{9}C_5 = 126.$

Example 3. Verify Pascal's rule when $n = 8$ and $r = 4$.

Solution: If $n = 8$ and $r = 4$ then, according to Pascal's rule $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$.

$$\text{L.H.S} = \binom{8}{3} + \binom{8}{4} = \frac{8!}{(8-3)! \cdot 3!} + \frac{8!}{(8-4)! \cdot 4!}$$
$$= \frac{8!}{5! \cdot 3!} + \frac{8!}{4! \cdot 4!} = \frac{8!}{5 \cdot 4! \cdot 3!} + \frac{8!}{4! \cdot 4 \cdot 3!}$$
$$= \frac{8!}{4! \cdot 3!}\left(\frac{1}{5} + \frac{1}{4}\right)$$
$$= \frac{9 \cdot 8!}{5 \cdot 4! \times 4 \cdot 3!} = \frac{9!}{5! \cdot 4!} = \frac{9!}{(9-4)!\,4!} = \binom{9}{4} = \text{R.H.S}$$

Hence, $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$

i.e., Pascal rule is verified.

6.3.3 Solve daily life problems involving combination

Example 1. How many matches can be played among five cricket teams, if each team has to play once against other.

Solution: This is the case of combination in which two teams are to be selected out of five teams.

So, required number of matches $= \binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \times 4 \times 3!}{2 \times 3!} = 10.$

Example 2. A department in a college consists of 3 professors and 6 students. A study tour is to be arranged. In how many ways can a party of five tourist be chosen so as to include at least one professor?

Solution: The party may consist of

(i) 1 professor and 4 students, or

(ii) 2 professors and 3 students, or

(iii) 3 professor and 2 students

These can be chosen in:

(i) $\binom{3}{1} \cdot \binom{6}{4}$ (ii) $\binom{3}{2} \cdot \binom{6}{3}$ (iii) $\binom{3}{3} \cdot \binom{6}{2}$ ways respectively.

All these choices are independent. Hence by the sum Principle, the required number of ways is:

$$\binom{3}{1} \cdot \binom{6}{4} + \binom{3}{2} \cdot \binom{6}{3} + \binom{3}{3} \cdot \binom{6}{2}$$
$$= (3)(15) + (3)(20) + 15 = 120.$$

Example 3. In how many ways can a party of 5 students and 4 teachers be formed out of 18 students and 6 teachers?

Solution: The students can be chosen in $\binom{18}{5}$ ways and the teachers in $\binom{6}{4}$ ways.

Hence, by the Product Principle, the required number of ways is:

$$\binom{18}{5}\cdot\binom{6}{4}$$
$$=\frac{18!}{(18-5)!\cdot 5!}\times\frac{6!}{(6-4)!\cdot 4!}$$
$$=\frac{18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13!}{13!\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\times\frac{6\cdot 5\cdot 4\cdot 3\cdot 2!}{2!\cdot 4!}=(8568)(15)$$
$$=128520 \text{ ways.}$$

Exercise 6.3

1. Evaluate the following:

(i) $^{12}C_{10}$ (ii) $^{12}C_3$ (iii) $\binom{20}{17}$

(iv) $\binom{7}{4}$

2. Find the value of n when:

(i) $^nC_2 = 21$ (ii) $^nC_3 = 4$ (iii) $^nC_{12} = {}^nC_6$

(iv) $^nC_{15} = \frac{17\times 16}{2!}$ (v) $^nC_5 = {}^nC_4$ (vi) $^nC_{10} = \frac{12\times 11}{2!}$

3. Find the values of n and r, when:

(i) $^nP_r = 210$ and $^nC_r = 35$ (ii) $^{n-1}C_{r-1} : {}^nC_r : {}^{n+1}C_{r+1} = 3:6:11$

4. If (i) $^{12}C_r = {}^{12}C_{r+4}$ then find ^6C_r.

(ii) $^nC_{12} = {}^nC_{10}$ then find $^{25}C_n$.

5. The members of a club are 12 boys and 8 girls. In how many ways,

(i) a committee of 5 members be formed?

(ii) a committee of 4 boys and 3 girls be formed?

6. A box contains 7 red balls and 6 black balls. In how many ways can 5 balls be selected such that exactly 3 are red?

7. How many (a) diagonals and (b) triangles can be drawn in a plane by joining the vertices of the polygon having:

(i) 6 sides (ii) 8 sides (iii) 10 sides

8. Show that: $^9C_6 + {}^9C_5 = {}^{10}C_6$.

9. There are 11 men and 9 women members of a club. How many committees of 8 members can be formed, having:

(i) exactly five men (ii) at most five women

(iii) at least five women.

10. Find the number of combinations of the letters of the word "Question" taken 5 at a time.

11. How many words can be formed by 3 vowels and 4 consonants out of 5 vowels and 7 consonants.

6.4 Probability

The word "Probability" is commonly used in everyday speech. We say: "It will probably rain tomorrow"; "He will probably pass the examination"; or "The black horse will probably win the race".

Probability has become a science that predicts the chances of success or failure of an untold number of occurrences for man's benefit. The numerical measure of uncertain statements is, infact, called probability.

The theory of probability is of great importance in a number of modern fields. It is particularly useful in solving problems related to mortality and insurance, biological, physical, medical, engineering and social phenomena. It is also useful in our daily life.

6.4.1 Define the following:

- **statistical experiment**
- **sample space and an event**
- **mutually exclusive events**
- **equally likely events**
- **dependent and independent events**
- **exhaustive events**
- **impossible event**
- **simple and compound events**

(i) Statistical Experiment

The process by which an observation is made is called an statistical experiment or a trial.

Examples:

(i) Rolling a dice.

(ii) Tossing a coin.

- **Outcome**

The results of an experiment are called outcomes or logical possibilities.

Examples:

(i) Outcomes of an experiment of tossing a coin are head and tail.

(ii) A possible outcome of an experiment of rolling a die is 1,2,3,4,5 or 6.

- **Sample Point**

Every possible outcome, no two of which may be outcomes at the same time, is called sample point or an element.

Example: Head is the sample point for the experiment of rolling a die.

(ii) Sample Space and an event

(a) Sample Space

A set of all sample points or outcomes of an experiment is called the sample space or an outcome set and is usually denoted by S.

The sample space for rolling a die is $S = \{1,2,3,4,5,6\}$.

(b) Event

Any subset of a sample space is called an event.

A subset of a sample space having no element at all is called a null space or empty space and is denoted by $\varnothing$.

(iii) Mutually Exclusive events

Two events are said to be mutually exclusive or incompatible if they cannot occur simultaneously in a single outcome. In other words, if one of the events excludes the occurrence of the other event in an outcome, then they are said to be mutually exclusive. Thus, two events A and B are said to be mutually exclusive if $A \cap B = \emptyset$, i.e. if A and B are disjoint sets. For example, in a single toss of a coin, the events A={head} and B={tail} are mutually exclusive, for if one occurs the other cannot happen.

(iv) Equally likely events

Outcomes of an experiment or a trial are said to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the other.

For example, in the tossing of an unbiased coin, both the faces are equally likely to come up; or in the rolling of a fair die, all the six faces are equally likely to occur. If each member of a set has an equal chance of being selected, we say that there is a random choice or a random selection.

(v) Dependent and Independent Events

(a) Dependent Events

Two events are dependent if the outcome or occurrence of the first affects the outcome or occurrence of the second.

Example: If we draw two colored balls from a bag, and the first ball is not replaced before you draw the second ball then the occurrence of the second drawn ball will be affected, hence both events are dependent events.

(b) Independent Events

Two events are independent if the outcome or occurrence of the first event does not affect the occurrence of the second event.

Some examples of independent events are:

(i) Getting a head in tossing a coin and getting a 5 in rolling a die.

(ii) Choosing a marble from a jar end getting tail in tossing a coin.

(vi) Exhaustive events

Let A and B be two events of a sample space S. Then A and B are said to be exhaustive, if $A \cup B = S$.

In the case of rolling a die, the two events,

$E = \{4\}$ and $F = \{3\}$ are not exhaustive for $E \cup F \neq S$ while the events $A = \{head\}$ and $B = \{tail\}$ in the tossing of a coin, are exhaustive as $A \cup B = S$ in this case.

(vii) Impossible event

An impossible event is an event that cannot happen.

Example: In flipping a coin once, an impossible event would be getting both a head and a tail.

(viii) Simple and Compound events

(a) Simple Event

An event containing only one element of the sample space, is called a simple or an elementary event.

For example, in throwing a dice, the event A of getting 4 is simple event. In this case sample space $= \{1, 2, 3, 4, 5, 6\}$ and event $A = \{4\}$.

(b) Compound Event

A compound event is one that can be expressed as the union of simple events.

The event B = A set of getting head or tail in tossing a coin. It is a compound event, since $B = \{head\} \cup \{tail\} = \{head, tail\}$

It may be noted that the union of simple events produces a compound event that is still a subset of the sample space.

Complementary Events

Two events A and B of sample space S are called complementary events if $A \cup B = S$ and $A \cap B = \emptyset$.

Note: The compelement event of the event A is denoted by A' or A^c

For example, the events A = $\{2, 4, 6\}$ and B = $\{1, 3, 5\}$ in the rolling of a die with S = {1, 2, 3, 4, 5, 6} are complementary events, because $A \cup B = S$ and $A \cap B = \emptyset$.

6.4.2 Recognize the formula for probability of occurrence of an event *E*, that is

$$P(E) = \frac{n(E)}{n(S)}, \qquad 0 \le P(E) \le 1$$

If S is a sample space and E is an event such that $E \subseteq S$, then the probability that the event E occurs is defined as

$$P(E) = \frac{n(E)}{n(S)}$$

Since $\quad E \subseteq S$

$\Rightarrow \quad n(E) \le n(S)$

$\Rightarrow \quad P(E) \le P(S)$

i.e. $\quad P(E) \le 1 \quad \ldots(i) \quad (\because P(S) = 1)$

$\because$ E is an event

$\therefore n(E) \ge 0$

$\Rightarrow \quad P(E) \ge 0 \quad \ldots(ii)$

Combining (i) and (ii), we get, $0 \le p(E) \le 1$.

Thus this result guarantees that no probability may be less than zero or greater than one. P(E) = 0 means that E cannot occur, i.e., when $E = \emptyset$ and P(E) = 1 means that E must occur, i.e. when E = S.

6.4.3 Apply the formula for finding probability in simple cases

Example 1. A bag contains a red, a yellow and a blue marble. What is the probability of picking a red marble?

Solution: There are three possible outcomes, picking a red marble (r), a yellow marble (y) and a blue marble (b).

Therefore, $\quad S = \{r, y, b\}$

Let, E be the event of picking a red marble, i.e, A = {r}.

Hence the probability of picking the red marble is:

$$P(E) = \frac{O(E)}{O(S)} = \frac{1}{3}$$

Example 2. A basket contains two white balls and two black balls. What is the probability of drawing two black balls?

Solution:

Here, the sample space $S = \{(w, w), (w, b), (b, w), (b, b)\}$.

There is only one favorable case for the event E.

E = An event set of drawing two black balls = $\{(b,b)\}$.

Hence the probability of drawing two black balls is: $P(A) = \frac{O(E)}{O(S)} = \frac{1}{4}$.

Example 3. Two coins are tossed together once. Find the probability of getting at least one head.

Solution: Here, sample space, $S = \{HH, HT, TH, TT\}$, where H denotes head and T denotes tail.

Let, E be the event of getting at least one head.

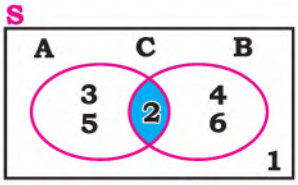
i.e., $= \{HH, HT, TH\}$

Therefore $O(S) = 4$ and $O(E) = 3$

Hence $P(E) = \frac{3}{4}$.

6.4.4 Use Venn diagrams and tree diagrams to find the probability for the occurrence of an event

Venn Diagram:



(Fig. 6.4)

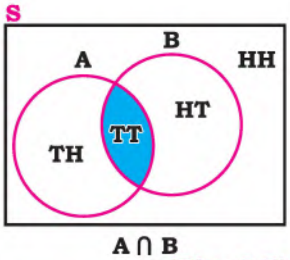
A Venn diagram can be used to represent the outcomes of an experiment and is very helpful to find probability. It generally consists of a rectangle that represents the sample space S together with circles or ovals. The circles or ovals represent events. For example, in throwing a dice, the events A, B and C along with the sample spaces are shown through Venn diagram (Fig 6.4) where A be the event of getting a prime number, B be the event of getting an even number and C be the event of getting both prime and even.

The use of Venn diagram in finding probability is explained with the help of the following examples.

Example 1.

Let A be the event of getting tail on the first coin. and B be the event of getting tail on the second coin when two coins are tossed.

Find P(A∩B) and P(AUB) with the help of Venn diagrams



(Fig. 6.5)

Solution:

Here, sample space is

S = {HH, HT, TH, TT}.

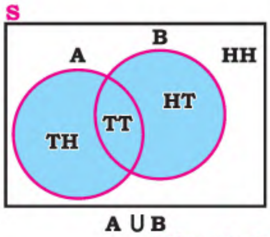
Event A ={TT, TH} and event B = {TT, HT}.

Therefore, A∩B = {TT} as shown in Fig 6.5.

Using Fig 6.5, P(A∩B) = $\frac{1}{4}$ = 0.25

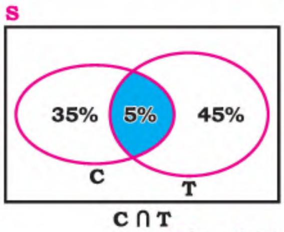
and AUB ={TH, TT, HT}. as shown in Fig 6.6.

Using Fig 6.6, P(AUB) = $\frac{3}{4}$ = 0.75



(Fig. 6.6)

Example 2. 40% of the students at a local college belong to a club and 50% work part time. 5% of the students work part time and belong to the club. Find the probability with the help of Venn diagram (i) that a student belongs to club and works part time.
(ii) that a student belongs to club or works part time.

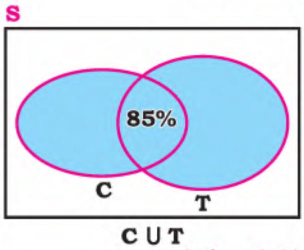


(Fig. 6.7)

Solution:

Let C be the event that student belongs to a club
and T be the event that student works part time.
By using Fig 6.7

- The probability that the student belongs to a club is: P(C) = 0.40
- The probability that the student works part time is: P(T) = 0.50



(Fig. 6.8)

Using Fig 6.7

- the probability that the student belongs to a club and works part time is: P(C ∩ T) = 0.05

Using Fig. 6.8,

- the probability that the student belongs to a club or works part time is:

$$P(C \cup T) = 0.85$$

Tree Diagram

We have already studied that a tree diagram is a special type of diagram used to determine the number of outcomes of an experiment. It consists of "branches" that are labelled with numbers of outcomes or probabilities. Tree diagrams can make some probability problems easier to visualize and solve. The following example illustrates how to use a tree diagram.

Example 1. There are 11 balls in a basket in which three balls are red (R) and eight balls are blue (B). Draw two balls, one at a time, with replacement. With the help of tree diagram, find the number of BR outcomes and

a. Calculate P(RR).
b. Calculate P(RB U BR).
c. Calculate P(R on 1st draw ∩ B on 2nd draw).
d. Calculate P(BB).

Solution:

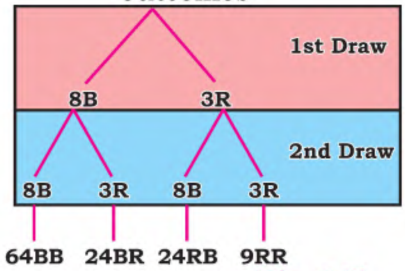
"With replacement" means that we put the first ball back in the basket before we select the second ball. The tree diagram shows all the possible outcomes (Fig. 6.9)

We have, Total numbers of outcomes

$$= 64 + 24 + 24 + 9 = 121$$

The first set of branches represents the first draw. The second set of branches represents the second draw. Each of the outcomes is distinct. We can list each draw containing red balls as R1, R2, and R3 and blue balls as B1, B2, B3, B4, B5, B6, B7, and B8. All BR outcomes can be listed as under.

Tree diagram showing outcomes



(Fig. 6.9)

B1R1	B1R2	B1R3	B2R1	B2R2	B2R3	B3R1	B3R2	B3R3	
B4R1	B4R2	B4R3	B5R1	B5R2	B5R3	B6R1	B6R2	B6R3	
B7R1	B7R2	B7R3	B8R1	B8R2	B8R3				(24 outcomes)

Also, with the help of tree diagram

Number of BR outcomes = 24

Using tree diagram (Fig. 6.9)

a: $P(RR) = \frac{9}{121}$

b: $P(RB \cup BR) = \frac{48}{121}$

c: $P(\text{R on 1st draw} \cap \text{B on 2nd draw}) = P(RB) = \frac{24}{121}$

d: $P(BB) = \frac{64}{121}$

Example 2. In a standard deck of 52 playing cards, twelve cards are face cards (F) and 40 cards are not face cards (N). Two cards are drawn one at a time, without replacement. Using tree diagram

a. Find P(FN or NF).

b. Find P(at least on face card).

Solution: The tree diagram is labeled with all possible probabilities. (Fig. 6.10)

a. $P(FN \text{ or } NF) = \frac{480}{2,652} + \frac{480}{2,652}$

$$= \frac{960}{2,652} = 0.362$$

Tree diagram showing probabilities

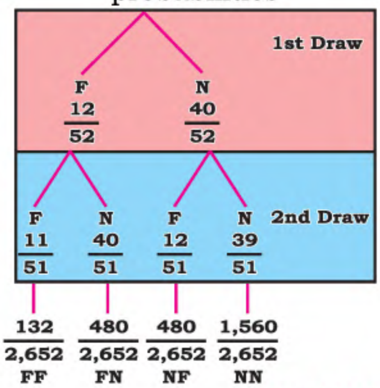


Fig (6.10)

b. P(at least one face card)

$$= \frac{(132 + 480 + 480)}{2,652} = 0.4118$$

6.4.5 Define the conditional probability

The conditional probability of an event B in relationship to an event A is the probability when event A has already occurred. The notation for conditional probability is P(B/A) [read as probability of event B given A] and is determined by:

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Example 1. A single six sided die is rolled once. Determine the probability that a 2 is rolled. Given that an even number has already been rolled.

Solution: The sample space for this experiment is $S = \{1,2,3,4,5,6\}$

A is an event that 2 is rolled i.e., $A = \{2\}$, B is an event that even number is rolled i.e., $B = \{2,4,6\}$. Now probability of A on the occurrence of B is:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

Example 2. A family has two children. Determine the probability that the family has; (i) one boy and one girl given that the first child is a boy.
(ii) two girls given that at least one is a girl.

Solution: Here, $S = \{BB, BG, GB, GG\}$

(i) Let A is an event of having one boy and one girl

i.e., $A = \{BG, GB\}$ and B is an event of having first child is a boy

i.e., $B = \{BB, BG\}$

We have, $A \cap B = \{B\,G\}$

Now, $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

(ii) Let A is an event of having both girls i.e., $A = \{G\,G\}$ and B is an event of having at least one a girl i.e., $B = \{BG, GB, GG\}$

Now, $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

6.4.6 Recognize the addition theorem (or law) of probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where, A and B are two events Deduce that $P(A \cup B) = P(A) + P(B)$ where A and B are mutually exclusive events

To write down the elements of a sample space S and to count the number of favourable cases often proves tedious in practical problems. To facilitate the computations of probabilities in such cases. We have the following theorems in which we shall denote

the probability of A or B by $P(A \cup B)$,

and the probability of A and B by $P(A \cap B)$.

Theorem 1: (Addition law of probability)

If A and B are two events of sample space S, then

$$P(A \cup B) + P(A \cap B) = P(A) + P(B).$$

This law is called addition law of probability.

This can also be written as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

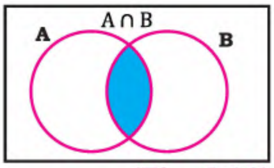
Proof: From Venn diagram (Fig. 6.11), it is clear that

$$O(A \cup B) + O(A \cap B) = O(A) + O(B)$$

Dividing both sides by O(S)

We get, $$\frac{O(A \cup B)}{O(S)} + \frac{O(A \cap B)}{O(S)} = \frac{O(A)}{O(S)} + \frac{O(B)}{O(S)}$$

i.e., $$P(A \cup B) + P(A \cap B) = P(A) + P(B)$$



(Fig. 6.11)

Corollary: If A and B be mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

This result can be extended to any finite number of mutually exclusive events $A_1, A_2, \ldots, A_n$. That is, if $A_1, A_2, \ldots, A_n$ are disjoint subsets of S, then the probability of at least one of them is given by

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Theorem 2:

If A and A′ are complementary events in a sample space S, then

$$P(A) + P(A') = 1$$

Proof: From Venn diagram (Fig 6.12)

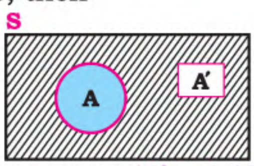
$$A \cup A' = S$$

$\Rightarrow$ $$P(A \cup A') = P(S)$$

$\Rightarrow$ $P(A) + P(A') = 1$, (By using addition law of probability)

$$P(A) = 1 - P(A')$$

and $$P(A') = 1 - P(A).$$



(Fig. 6.12)

Theorem 3:

$$P(\emptyset) = 0.$$

Proof: Since $\emptyset \cup \emptyset' = S$

Therefore, $P(S) = P(\emptyset \cup \emptyset')$

$\Rightarrow \quad 1 = P(\emptyset) + P(\emptyset')$, (By using addition law of probability)

$1 = P(\emptyset) + P(S)$, [Since $\emptyset' = S - \emptyset = S$]

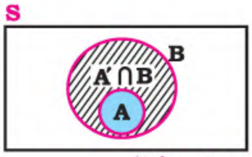
$\Rightarrow \quad 1 = P(\emptyset) + 1$, [Since $p(S)=1$]

$\Rightarrow \quad p(\emptyset) = 0.$

Theorem 4:

If $A \subseteq B \subseteq S$, then

$P(A) \leq P(B).$



(Fig. 6.13)

Proof: According to Venn diagram (Fig. 6.13)

$$B = A \cup (A' \cap B).$$

Also, $A \cup (A' \cap B) = (A \cup A') \cap (A \cup B)$ (distributive law)

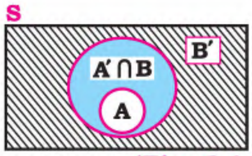
$= S \cap (A \cup B)$

$= A \cup B$

$= B$, $(\because A \subseteq B)$

From Venn diagram (Fig. 6.14)

$A \cap (A' \cap B) = \emptyset$



(Fig. 6.14)

i.e., A and $(A' \cap B)$ are disjoint sets.

Therefore $P(B) = P[A \cup (A' \cap B)]$

$= P(A) + P(A' \cap B)$

But $P(A' \cap B) \geq 0.$

Therefore $P(B) \geq P(A)$

or $P(A) \leq P(B).$

Hence proved.

6.4.7 Recognize the multiplication theorem (or law) of probability: $P(A \cap B) = P(A) \cdot P(B|A)$ or $P(A \cap B) = P(B) \cdot P(A|B)$ where $P(B|A)$ and $P(A|B)$ are conditional probabilities. Deduce that $P(A \cap B) = P(A) \cdot P(B)$ where A and B are independent events

Multiplication theorem of Probability

If A and B are events of sample space, then the probability that both A and B occur is equal to the probability of the event A times the probability of B given that A has occurred.

i.e., $P(A \cap B) = P(A) \cdot P(B/A)$

Also, $P(A \cap B) = P(B) \cdot P(A/B)$

where P(B/A) and P(A/B) are conditional probabilities. This is multiplication rule for two dependent events.

In the case where A and B are independent (where A has no effect on the probability of event B), the conditional Probability of event B given by event A, is simply the probability of event B, that is P(B).

So, $P(A \cap B) = P(A) \cdot P(B)$

Example 1. A box contain 5 black and 7 red balls. Two balls are drawn from the box one after the other without replacement, what is the probability that both balls are black balls.

Solution:

The total number of balls in the box is $5 + 7 = 12$. Let B_1 is the event of getting first black ball and B_2 is the event of getting second black ball.

Here, both events are dependent events.

So, $P(B_1 \cap B_2) = P(B_1)P(B_2/B_1)$

$$= \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$$

Example 2. A coin is tossed and a single 6-sided die is rolled. Find the probability of getting head of the coin and rolling a 3 on the die.

Solution:

Let A be the event of getting a head and B be the event of getting a 3 on the dice.

$\because$ Both events are independent

$\therefore$ $P(A \cap B) = P(A).P(B)$

$$= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

6.4.8 Use theorems of addition and multiplication of probability to solve related problems from daily life

(i) Problems of theorems of addition of probability.

Example 1. An integer is chosen at random from the first 200 positive integers. What is the probability that the chosen integer is divisible by 6 or 8?

Solution: Here $S = \{1, 2, 3, \ldots, 200\}$

Now the number of integers divisible by 6 in the first 200 positive integers = 33.

Again, the number of integers divisible by 8 in the first 200 positive integers = 25.

Also, the number of integers divisible by 24 (L.C.M. of 6 and 8) in the first 200 positive integers is 8.

Let A be the event that the chosen integer is divisible by 6.

i.e., $A = \{6, 12, 18, \ldots, 198\}$.

Let B be the event that the chosen integer is divisible by 8.

i.e., $B = \{8, 16, 24, \ldots, 200\}$.

$$A \cap B = \{24, 48, 72, 96, 120, 144, 168, 192\}.$$

Now,
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$
$$= \frac{50}{200} = \frac{1}{4}.$$

Example 2. If the probability of solving a problem by two students Ahsan and Umar are $\frac{1}{2}$ and $\frac{1}{3}$ respectively then what is the probability of the problem to be solved.

Solution: Let A and B be the events of solving the problem by Ahsan and Umar respectively. We have $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$.

The problem will be solved if it is solved by at least one of them.

So, we need to find $P(A \cup B)$.

∴ A and B are independent events

∴ $P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3}$

By addition theorem of probability, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

So,
$$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{2} \times \frac{1}{3} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{3 + 2 - 1}{6} = \frac{2}{3}$$

(ii) Problems of theorems of multiplication of probability

Example 1. Three cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing 3 aces?

Solution:
$$P(3 \text{ aces}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50}$$
$$= \frac{24}{1{,}32{,}600}$$
$$= \frac{1}{5{,}525}$$

Example 2. In a shipment of 20 computers, 3 are defective. Three computers are randomly selected and tested. What is the probability that all three are defective if the first and second ones are not replaced after being tested?

Solution: P(3 defectives) $= \frac{3}{20} \cdot \frac{2}{19} \cdot \frac{1}{18} = \frac{6}{6840} = \frac{1}{1140}$

Example 3. Suppose an individual applying to a college determines that he has an 80% chance of being accepted, and he knows that dormitory housing will only be provided for 60% of all of the accepted students. What is the chance of the student being accepted and receiving dormitory housing?

Solution: P (Accepted and Dormitory Housing)

$= \text{P(Dormitory Housing|Accepted)} \cdot \text{P(Accepted)}$

$= (0.60) \cdot (0.80) = 0.48.$

Example 4. A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?

Solution: P (green) $= \frac{5}{16}$

P (yellow) $= \frac{6}{16}$

$\because$ This is the case of independent events.

$\therefore$ P (green and yellow) = P (green) · P (yellow)

$= \frac{5}{16} \cdot \frac{6}{16} = \frac{30}{256}$

$= \frac{15}{128}$

Exercise 6.4

1. Two fair coins are tossed. Find the probability of getting:

(i) Same faces (ii) All heads (iii) At most one head

(iv) At least two tail (v) At least one tail

2. Two dice, one red and the other green, are rolled simultaneously. The numbers of dots on the tops are added. Find the probability of getting a sum of:

(i) 8 (ii) 10 (iii) 12

3. A bag contains 40 balls out of which 5 are green, 15 are red and the remaining are black. A ball is drawn out of the bag. Find the probability of getting:

(i) The ball is green (ii) The ball is black (iii) The ball is not green

4. A card drawn from a well shuffled deck of cards, find the probability of:

(i) getting a King (ii) getting a club (iii) getting a face card

5. A die is rolled. Find the probability using Venn diagram of getting:
(i) an even number (ii) a number greater than 4
(iii) a number which is even and greater than 4.

6. In a three child family, by using tree diagram. Find the probability of having:
(i) three boys (ii) exactly two boys (iii) at least one girl

7. In a single throw of two fair dice, find the probability that the product of the numbers on the dice is:
(i) between 2 and 10 (both inclusive) (ii) divisible by 5

8. A marble is drawn at random from a box containing 20 red, 10 white, 25 orange and 15 blue marbles. Find the probability that it is:
(i) orange or red (ii) not blue or red (iii) red, white or blue

9. The king, queen and jack of clubs are removed from a deck of 52 playing cards and then shuffled. A card is drawn from the remaining cards. Find the probability of getting:
(i) a heart (ii) a queen (iii) a club (iv) '9' of red color

10. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability that (i) the sum is 10, (ii) the sum is six or less.

11. Given that $P(A) = 0.3$, $P(B) = 0.7$, $P(A \cap B) = 0.21$ then find:
(i) $P(A/B)$ (ii) $P(B/A)$

12. If one card is selected at random from a deck of 52 playing cards, what is the probability that the card is a club or a face or both?

13. From two events A and B, $P(A) = 0.5$, $P(B) = 0.2$, $P(A \cup B) = 0.4$ then find $P(A \cap B)$?

14. A natural number is chosen out of first 35 natural numbers. What is the probability that the chosen number is divisible by 8 or 9?

15. A bag contains 15 black, 25 red and 10 white balls. A ball is drawn at random. Find the probability that it is either red or white?

16. Two events A and B are such that $P(A) = \frac{1}{4}$, $P(A/B) = \frac{1}{2}$, $P(B/A) = \frac{2}{3}$. Find $P(A \cap B)$ and $P(B)$?

17. A fair die is thrown twice. Find the probability that an even number of dots appear in first and the number of dots in the second throw is less than 4?

18. Three missiles are fired at a target. If the probability of hitting the target are 0.5, 0.3 and 0.6 respectively, and if the missiles are fired independently, what is the probability that all the missiles hit the target?

Review Exercise 6

1. Select correct answer.

i. If $n = 0$, then $\frac{(n+1)!}{n!} =$ ______________

(a) 0 (b) 1 (c) n (d) ∞

ii. Probability of getting 7 in throwing a dice is:

(a) 0 (b) 1 (c) -1 (d) Not defined

iii. The factorial form of $12 \cdot 11 \cdot 10 \cdot$ is:

(a) $\frac{12!}{9!}$ (b) 12! (c) $\left(\frac{12}{9}\right)!$ (d) $(12!) \cdot (9!)$

iv. If two independent events A and B occur in p and q ways respectively, then number of ways that both events can occur is:

(a) $p + q$ ways (b) $p.q$ ways (c) $(pq)^r$ ways (d) $rp + qr$ ways

v. An arrangement of n objects according to some definite order is called:

(a) Combination (b) Permutation

(c) Factorial (d) none of these

vi. An arrangement of n objects without any order is called:

(a) Combination (b) Permutation

(c) Factorial (d) none of these

vii. The number of permutation of the letters of the word COMMITTEE is:

(a) $\binom{9}{2,\ 2,\ 2}$ (b) $\binom{6}{1,\ 2,\ 2}$ (c) $\binom{9}{2,\ 2,\ 1}$ (d) $\binom{9}{2,\ 3,\ 1}$

viii. 8.7.6 is equal to:

(a) ${}^8P_3$ (b) ${}^8C_3$ (c) ${}^8P_5$ (d) ${}^8C_5$

ix. If $r = n$, then ${}^nP_r$ is equal to:

(a) $r!$ (b) $(n - r)!$ (c) 1 (d) 0

x. The number of ways that a necklace of n beads of different colours be made is:

(a) $(n - 1)!$ (b) $\frac{n!}{2}$ (c) $\frac{n!-1}{2}$ (d) $\frac{(n-1)!}{2}$

xi. Any subset of a sample space is called:

(a) Sample space (b) an event

(c) a Trial (d) Random variable

xii. For two events A and B if $A \cap B = \emptyset$, then events A and B are called:

(a) Mutually exclusive (b) Not mutually exclusive

(c) Overlapping (d) Dependent events

xiii. When a dice is rolled and coin is tossed, all possible outcome are:

(a) 6 (b) 12 (c)18 (d)24

xiv. If two events A and B have equal chance of occurrence, then the events are:

(a) Equally likely (b) Not equally likely
(b) Dependent (d) Not mutually exclusive

xv. If E be an event of a sample space S, then:

(a) $P(E) = \frac{n(S)}{n(E)}$ (b) $0 \leq P(E) \leq 1$
(c) $0 < P(E) < 1$ (d) all of these

xvi. The probability of getting the tail in a single toss of a coin is:

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

xvii. Three dice are rolled simultaneously, then $n(S)$ is equal to:

(a) 36 (b) 18 (c) 216 (d) 6

xviii. Two teams A and B are playing a match, the probability that team A does not lose is:

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 0

xix. If ${}^{n}C_6 = {}^{n}C_{12}$, then n equals:

(a) 18 (b) 12 (c) 6 (d) 20

2. Evaluate the following:

(i) $\frac{{}^{6}C_3 \times {}^{4}C_3}{{}^{10}C_4}$ (ii) $\frac{{}^{6}P_3 \times {}^{4}P_3}{{}^{10}P_4}$ (iii) $\sum_{r=1}^{3} {}^{3}C_r$

3. There are seven seats available in a compartment. In how many ways can seven persons be seated?

4. A room has 3 lamps. From a collection of 10 light bulbs of which 6 are no good, a person selects 3 at random and puts them in the sockets. What is the probability that he will have light?

5. If two dice are thrown simultaneously, what is the probability of obtaining a sum of 7 or a sum of 11?

6. A bag contains 8 white, 10 black and 12 red balls. 3 balls are drawn from the bag. What is the probability that the first ball is white, the second ball is black and the third ball is red, when every time the ball is replaced?

7. In how many ways can a football team of 11 players be selected out of 15 players? How many of them will include a particular player?

Mathematical Induction and Binomial Theorem

Unit 7

7.1 Mathematical Induction

A powerful method of proof, frequently used in mathematics is mathematical induction. This method is not to be confused with the method of inductive logic used in the experimental science in which generalization is formulated by observing many specific cases. In contrast, mathematical induction is a form of deductive reasoning in which conclusions are established beyond any doubt.

7.1.1 Describe principle of mathematical induction

Principle of Mathematical Induction

If P(n) is a proposition about a positive integer (natural number) n such that

- P (n) is true for $n = 1$, and
- P (n) is true for $n = k$ implies that it is also true for $n = k + 1$.

Then P (n) is true for all positive integers (natural numbers) n.

Principle of Mathematical Induction when Proposition P (n) is not true for the first few values of n

Sometimes a proposition P(n) is not true for the first few values of n and is true for all successive values after a certain stage.

For an application of the method of mathematical induction to such cases, the principle of mathematical induction is modified and restated as under.

If P(n) is a proposition about a positive integer n such that

- P(n) is true for $n = i$, where i is a positive integer, and
- P (n) is true for $n = k + 1$, whenever P(n) is true for any positive integer $n = k$.

7.1.2 Apply the principle to prove the statements, identities or formulae

Example 1. By using principle of mathematical induction, prove that the following formula is true for all positive integral values of n.

$$2 + 4 + 6 + \cdots + 2n = n(n+1).$$

Proof:

(i) For $n = 1$, we have $2 = 1(1+1)$ or $2 = 2$ i.e., $\mathrm{P}(n)$ is true for $n = 1$

(ii) Assume that the formula or $\mathrm{P}(n)$ is true for some positive integer $n = k$.

i.e., $2 + 4 + 6 + \cdots + 2k = k(k+1)$ ...(i)

Now we shall prove that the proposition is true for $n = k+1$, that is, we shall prove that: $2+4+6+\cdots+2k+2(k+1) = (k+1)\{(k+1)+1)\}$.

By our hypothesis, we have

$$2+4+6+\cdots+2k = k(k+1)$$

Adding $2(k+1)$ to both the sides of equation (i), we get

$$\begin{aligned} 2+4+6+\cdots+2k+2(k+1) &= k(k+1)+2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)\{(k+1)+1\}. \end{aligned}$$

i.e., $\mathrm{P}(n)$ is true for $n = k+1$ whenever it is true for $n = k$. Hence by principle of mathematical induction the proposition is true for all positive integral values of n.

Example 2. Prove that $2^n > (2n+1)$ for all integral values of $n \geq 3$.

Solution:

It can well be seen that for $n = 1$ and 2, the proposition gives $2 > 3$ and $4 > 5$ which are false.

We therefore, apply the modified form of mathematical induction.

(i) Here, $i = 3$ and so $\mathrm{P}(n)$ for $n = 3$ gives

$2^3 > (2 \cdot 3 + 1)$

or $8 > 7$ which is true. So, $\mathrm{P}(n)$ is true for $i = 3$.

(ii) Assume $\mathrm{P}(n)$ to be true for $n = k$, i.e.

$$2^k > (2k+1)$$

Now, multiplying both sides of the above inequality by 2, we get

$2.2^k > 2(2k+1)$,

$\Rightarrow \quad 2^{k+1} > 4k+2$,

$\Rightarrow \quad 2^{k+1} > 2k+2k+2$.

Deducting $2k$ from right side and keeping 1 instead, the inequality statement holds and we have,

$2^{k+1} > 2k+1+2$

or $\quad 2^{k+1} > 2k + 2 + 1.$

i.e., $\quad 2^{k+1} > 2(k+1) + 1.$ Therefore, P (n) is also true for $n = k + 1$.

Hence, by the principle of mathematical induction P(n) is true for all integral values of $n \geq 3$.

Example 3. Using the principle of mathematical induction prove that:

$$1 + 2 + 3 + \cdots n = \frac{n(n+1)}{2}, \text{ for all natural numbers } n.$$

Proof:

(i) For $n = 1$, P(n) becomes $1 = \frac{1(1+1)}{2}$

or $\quad 1 = 1$

So, P(n) is true for $n = 1$.

(ii) Assuming the result to be true for $n = k$, we get the hypothesis:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Now we have to prove that the result is also true for $n = k + 1$. To do so, we add $(k + 1)$ to both sides of the above hypothesis and get

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\ (\frac{k}{2} + 1) = (k+1) \cdot \frac{(k+2)}{2}$$

$$= \frac{(k+1)\{(k+1)+1\}}{2}$$

i.e., P(n) is true for $n = k + 1$.

Thus, by the principle of mathematical induction P$(n)$is true for all natural numbers n.

Example 4. Prove that:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all natural numbers } n.$$

Proof:

If P (n) represents the above proposition, then

(i) for $n = 1$, P(n) becomes $1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$

$\Rightarrow \quad 1 = \frac{1(2)(3)}{6}$

or $\quad 1 = 1$ which is true.

i.e., P(n) is true for $n = 1$.

(ii) Suppose that P(n) is true for $n = k$, i.e.,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \ldots(i)$$

To prove P(n) to be true for $n = k + 1$ we add $(k+1)^2$ to both sides of the above equation (i).

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{(k+1)}{6}\{k(2k+1) + 6(k+1)\}$$
$$= \frac{(k+1)}{6}\{2k^2 + 7k + 6\}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6}$$

This, in fact, is the same as the given proposition P(n) for $n = k + 1$.
Thus, the truth of P(n) for $n = k$ implies its truth for $n = k + 1$.
Hence, by the principle of mathematical induction, P(n) is true for all natural numbers n.

Example 5. Prove that: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, for all natural numbers n.

Proof:

i. For $n = 1$, P(n) becomes

$$1^3 = \left[\frac{1.(1+1)}{2}\right]^2$$
$$1 = 1, \text{ which is true.}$$

i.e., P(n) is true for $n = 1$

ii. Assume that P(n) is true for $n = k$.

i.e., $$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Adding $(k+1)^3$ to both the sides, we have

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$
$$= \frac{(k+1)^2}{4}\{k^2 + 4(k+1)\}$$
$$= \frac{(k+1)^2}{4}\{k^2 + 4k + 4\}$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$
$$= \left[\frac{(k+1)\{(k+1)+1\}}{2}\right]^2,$$

Thus, it is true for $n = k + 1$.

Hence, by mathematical induction P(n) is true for all natural numbers.

Example 6. Prove that $2^{3n} - 7n - 1$, is divisible by 49 where n is any positive integer.

Proof:

(i) For $n = 1$, the given expression becomes

$$2^{3n} - 7n - 1 = 2^{3\cdot 1} - 7\cdot 1 - 1$$
$$= 8 - 7 - 1 = 0$$

Since zero is divisible by 49, the given statement is true for $n = 1$.

(ii) Assume that the statement is true for $n = k$, i.e.,

$$2^{3k} - 7k - 1 \text{ is divisible by } 49$$

Now for, $n = k + 1$, we have

$$2^{3(k+1)} - 7(k+1) - 1 = 2^{3k+3} - 7k - 7 - 1$$
$$= 8.2^{3k} - 7k - 8$$
$$= 8.2^{3k} - 8(7k) - 8 + 7(7k)$$
$$= 8(2^{3k} - 7k - 1) + 49k$$

By our hypothesis $(2^{3k} - 7k - 1)$ is divisible by 49 and $49k$ is obviously divisible by 49. Therefore, the given expression is also divisible by 49 for $n = k + 1$.

Hence by mathematical induction the given statement is true for all positive integral values of n.

Exercise 7.1

1. Prove the following propositions by mathematical induction for every positive integer n.

(i) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

(ii) $3 + 6 + 9 + \cdots + 3n = \frac{3}{2}n(n + 1)$

(iii) $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$

(iv) $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2(1 - \frac{1}{2^n})$

(v) $2 + 6 + 18 + \cdots + 2\cdot 3^{n-1} = 3^n - 1$

(vi) $2 + 6 + 12 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$.

(vii) $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2}{3}n(n + 1)\ (2n + 1)$.

(viii) $1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$.

(ix) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

(x) $1.2+2.2^2+3.2^2+4.2^2+\cdots+n.2^n=(n-1).2^{n+1}+2$.

(xi) $1!\cdot 1+2!\cdot 2+3!\cdot 3+\cdots+n!\cdot n=(n+1)!-1$.

(xii) $\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+\cdots+\frac{1}{(a+n-1)(a+n)}=\frac{n}{a(a+n)}$

(xiii) $\frac{1^2}{1.3}+\frac{2^2}{3.5}+\cdots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$.

(xiv) (i) $a+ar+ar^2+\cdots+a\,r^{n-1}=\frac{a(a-r^n)}{1-r},(r\neq 1)$.

(ii) $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$.

(iii) $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r};\ n\geq r$.

2. Prove the following statement by mathematical induction.

(i) $2^{3n+2}-28n-4$ is divisible by 49,$\forall\ n\in\mathbb{N}$.

(ii) $3^{2n+2}-8n-9$ is divisible by 64,$\forall\ n\in\mathbb{N}$.

(iii) 7^n-4^n is divisible by 3

3. Prove that:

(i) $2^{n+1}>(2n+3)$, for all integral values of $n\geq 2$,

(ii) $3^{n-1}>2^n$, for all integral values of $n\geq 3$,

(iii) $n!>3^{n-1}$, for all integral values of $n\geq 5$.

7.2 Binomial Theorem

7.2.1 Use Pascal's triangle to find the expansion of $(x+y)^n$ where n is a small positive integer

An expression consisting of two terms connected by +ve or –ve sign is called a binomial expression or simply a binomial.

For example, $a+b$, $2x-3y$, z^3-2z, a^2+b^2 are all binomial expressions.

For $(x+y)^n$ where $(x+y)$ is a binomial and natural number n is its exponent or index. The following products can be verified by actual multiplication.

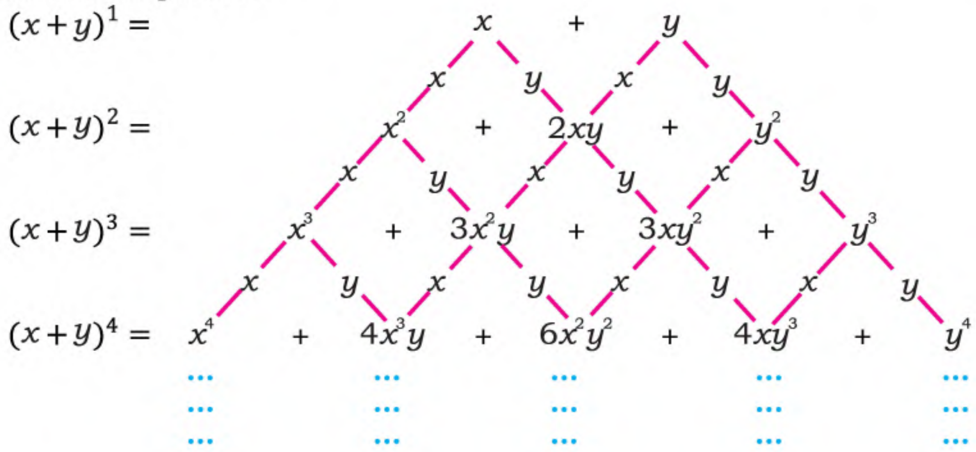
$$(x+y)^1=x+y$$

$$(x+y)^2=x^2+2xy+y^2$$

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

In the binomial theorem we wish to state a general law of formation that lies behind these expansions; i.e., we wish to state a general law for the expansion of $(x+y)^n$, where n is a natural number. The following diagram illustrates the process.



(Fig. 7.1)

The coefficients in the above expansions may be shown in the following scheme, which is called Pascal's triangle.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

... ...

... ...

... ...

(Fig. 7.2)

A very definite pattern is evident in this array. The first and the last number in each row is 1. Each of the other elements is the sum of the two elements to its left and right in the row immediately preceding. Thus we would expect the coefficients of the expansion of $(x+y)^5$ to be

1 5 10 10 5 1

Direct multiplication will verify this result. Similarly, for $(x+y)^6$, we have 1 6 15 20 15 6 1

It may also be noted that there is a connection between the coefficients in these expansions and the possible number of combinations of x and y. Thus, Pascal's triangle can also be given in the following form.

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

(Fig. 7.3)

Each element in a row, other than the 1's is the sum of the two elements to its left and right in the row immediately preceding it, that is, in general

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

This is called Pascal's rule. Apart from coefficients, a definite pattern about the terms of the expansion of $(x+y)^n$ is also evident from (Fig. 7.1). Each term contains product of powers of x and y. From first term to the last term the products are:

$$x^n y^0, x^{n-1}y, x^{n-2}y^2, \dots, x^0 y^n$$

Example: Using Pascal's triangle, expand $(x+y)^5$.

Solution: According to Pascal's triangle, the row related to $n = 5$, is: $1, 5, 10, 10, 5, 1$

So, $\quad (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

7.2.2 State and prove binomial theorem for positive integral index

It is difficult to find the co-efficients in the expansion of $(x+y)^n$ from the scheme given above when the exponent n is large. In this section we state and prove a theorem, known as the Binomial Theorem, due to UMER KHYAM (1074 A.D).

Statement:

If n is a positive integer, then

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

Proof: We prove that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r-1}a^{n-r+1}b^{r-1} + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

with the help of the Principle of Mathematical Induction.

(i) For $n = 1$, we have

$$(a+b)^1 = a^1 + \binom{1}{1} a^{1-1} b^1 = a + b$$

Thus, the theorem is true for $n = 1$

(ii) Assume the theorem to be true for $n = k$, i.e., let the hypothesis be

$$(a+b)^k = a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} + \binom{k}{r} a^{k-r} b^r + \cdots + b^k$$

Multiplying each term of the formula in the above hypothesis by $a + b$, we have

$$(a+b)^{k+1} = a\left\{a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} + \binom{k}{r} a^{k-r} b^r + \cdots + b^k\right\}$$

$$+b\left\{a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \cdots + \binom{k}{r-1} a^{k-r+1} b^{r-1} + \binom{k}{r} a^{k-r} b^r + \cdots + b^k\right\}$$

$$= \left\{a^{k+1} + \binom{k}{1} a^k b + \binom{k}{2} a^{k-1} b^2 + \cdots + \binom{k}{r-1} a^{k-r+2} b^{r-1} + \binom{k}{r} a^{k-r+1} b^r + \cdots + ab^k\right\}$$

$$+\left\{a^k b + \binom{k}{1} a^{k-1} b^2 + \binom{k}{2} a^{k-2} b^3 + \cdots + \binom{k}{r-1} a^{k-r+1} b^r + \binom{k}{r} a^{k-r} b^{r+1} + \cdots + b^{k+1}\right\}$$

$$= a^{k+1} + \left\{\binom{k}{1} + \binom{k}{0}\right\} a^k b + \left\{\binom{k}{2} + \binom{k}{1}\right\} a^{k-1} b^2 + \cdots + \left\{\binom{k}{r} + \binom{k}{r-1}\right\} a^{k-r+1} b^r + \cdots + b^{k+1}$$

[By grouping the like terms]

But by Pascal's rule

$$\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$$

Therefore, $\binom{k}{1} + \binom{k}{0} = \binom{k+1}{1}$; $\binom{k}{2} + \binom{k}{1} = \binom{k+1}{2}$; ... and so on

Hence,

$$(a+b)^{k+1} = a^{k+1} + \binom{k+1}{1} a^k b + \binom{k+1}{2} a^{k-1} b^2 + \cdots + \binom{k+1}{r} a^{k+1-r} b^r + \cdots + b^{k+1}$$

Therefore, the theorem is also true for $n = k + 1$. Hence by the Principle of Mathematical Induction the theorem is true for all positive integral exponents.

Characteristics of Binomial Theorem

We may notice the following points in connection with the binomial formula for the positive integral index.

(i) Using the other notation for combinations the formula can also be written in the form as:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2}.b^2 + \cdots + {}^nC_n.b^n.$$

(ii) The binomial formula is frequently written as:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!} a^{n-r} b^r + \cdots + b^n$$

(iii) The number of terms in the expansion of $(a + b)^n$ is $n + 1$.

(iv) In the successive terms index of "a" decreases by one and index of "b"

increases by one so that the sum of two indices is always n.

(v) The coefficients of successive terms are 1, ${}^nC_1$, ${}^nC_2, \ldots,$ ${}^nC_{n-1}$, 1. These are known as binomial coefficients.

(vi) Since the expansion of $(a+b)^n$ is symmetrical w.r.t. a and b, it follows that the coefficients of $a^{n-r}b^r$ and $a^r b^{n-r}$ are equal, i.e., the coefficients of terms equidistant from the beginning and the end are equal.

(vii) since $\binom{n}{0} = 1 = \binom{n}{n}$, we may, for the sake of uniformity write $\binom{n}{0}$ as the coefficient of a^n and $\binom{n}{n}$ as the coefficient of b^n in the expansion.

(viii) If we put $a = b = 1$ in the binomial formula, then

$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{r} + \cdots + \binom{n}{n}$

This formula also determines the total number of subsets of a set S consisting of n elements. Whereas ${}^nC_r$ is the total number of subsets of S each consisting of r elements.

(ix) Since $(a-b) = \{a + (-b)\}$, we have

$$\{a+(-b)\}^n = a^n + \binom{n}{1}a^{n-1}(-b) + \binom{n}{2}a^{n-2}(-b)^2 + \cdots + \binom{n}{r}a^{n-r}(-b)^r + \cdots + (-b)^n;$$

i.e., $(a-b)^n = a^n - \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 - \cdots + (-1)^r\binom{n}{r}a^{n-r}(b)^r + \cdots + (-1)^n b^n$

Thus, the terms in the expansion of $(a-b)^n$ are alternatively positive and negative, the last term being $+b^n$ or $-b^n$ according as n is even and odd respectively.

(x) Putting $a = 1$ and $b = x$ in the expansion of $(a-b)^n$, we have

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \cdots + (-1)^r\binom{n}{r}x^r + \cdots + (-1)^n x^n$$

Similarly, $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + x^n.$

Example 1. Expand $(x+y)^7$ by the binomial theorem.

Solution:

$$(x+y)^7 = x^7 + 7x^6y + \frac{7.6}{2!}x^5y^2 + \frac{7.6.5}{3!}x^4y^3 + \frac{7.6.5.4}{4!}x^3y^4 + \frac{7.6.5.4.3}{5!}x^2y^5 + \frac{7.6.5.4.3.2}{6!}xy^6 + y^7$$

$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Example 2. Expand $(ax - \frac{b}{x})^6$ by the binomial theorem.

$$(ax - \frac{b}{x})^6 = (ax)^6 - 6(ax)^5\left(\frac{b}{x}\right) + \frac{6.5}{2!}(ax)^4\left(\frac{b}{x}\right)^2 - \frac{6.5.4}{3!}(ax)^3\left(\frac{b}{x}\right)^3 + \frac{6.5.4.3}{4!}(ax)^2\left(\frac{b}{x}\right)^4$$

$$-\frac{6.5.4.3.2}{5!}(ax)\left(\frac{b}{x}\right)^5 + \left(\frac{b}{x}\right)^6$$

$$= a^6x^6 - 6a^5bx^4 + 15a^4b^2x^2 - 20a^3b^3 + \frac{15a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$$

Example 3. Compute $(1.01)^9$ by means of the binomial theorem correct to three decimal places.

Solution: $(1.01)^9 = (1+0.01)^9$

$$= 1 + \frac{9}{1}\cdot 1^9 \cdot (0.01) + \frac{9.8}{1.2}\cdot 1^8 \cdot (0.01)^2 + \frac{9.8.7}{1.2.3}\cdot 1^7 \cdot (0.01)^3 + \text{up to last term}$$

$$= 1 + 0.09 + 0.0036 + 0.000084 + \cdots \text{ up to the last term}$$

$= 1.093684$ approx: $= 1.094$ correct to three decimal places.

7.2.3 Expand $(x+y)^n$ using binomial theorem and find its general term

If n is a positive integer, then

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

which is called the binomial expansion.

The term $T_{r+1} = \binom{n}{r}x^{n-r}\cdot y^r$ is called the general term in the expansion of $(x+y)^n$ where n is a +ve integer, we observe that

$$T_1 = \binom{n}{0}x^n = x^n$$

$$T_2 = \binom{n}{1}x^{n-1}y = nx^{n-1}y,$$

$$T_3 = \binom{n}{2}x^{n-2}y^2 = \frac{n(n-1)}{2!}x^{n-2}y^2, \quad \text{and so on.}$$

Example 1. Find the sixth term in the expansion of

$$\left(\frac{2x}{3} - \frac{3}{2x}\right)^{10}$$

Solution: $T_6 = T_{5+1}$

$$= \binom{10}{5}\cdot\left(\frac{2x}{3}\right)^{10-5}\cdot\left(-\frac{3}{2x}\right)^5$$

$$= \frac{10!}{5!\,(10-5)!}\cdot\frac{2^5x^5}{3^5}\cdot\left(-\frac{3^5}{2^5x^5}\right)$$

$$= -\frac{10.9.8.7.6(5!)}{5.4.3.2.1(5!)} = -252$$

Example 2. Find the coefficient of x^6 in $(a^3 + 3bx^2)^6$

Solution: Here, $T_{r+1} = \binom{6}{r}(a^3)^{6-r}(3bx^2)^r$

$$= \binom{6}{r}(a^3)^{6-r}(3b)^r x^{2r}$$

Now, T_{r+1} will contain x^6, if $2r = 6$ or $r = 3$.

So, T_4 contains x^6.

i.e., $T_4 = \binom{6}{3}(a^3)^{6-3}(3bx^2)^3$

$= \frac{6!}{3!\,(6-3)!}a^9(27b^3x^6)$

$= 540a^9b^3x^6$

So, the coefficient of x^6 is $540a^9b^3$.

7.2.4 Find the specified term in the expansion of $(x + y)^n$

In the expansion of $(x + y)^n$, where n being a positive integral index. We can find specified terms like middle terms, the term involving particular power of x, the term independent of x etc.

The term involving particular power of x and independent of x

Example 1. Write in the simplified form the term involving x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution: Suppose x^{-17} occurs in T_{r+1}.

Now, $T_{r+1} = \binom{15}{r}\cdot(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r$

$= (-1)^r\binom{15}{r}x^{60-4r}\cdot x^{-3r}$

$= (-1)^r\cdot\binom{15}{r}\cdot x^{60-7r}$

Thus $60 - 7r = -17$ or $r = 11$.

So, $T_{r+1} = T_{11+1}$

$= (-1)^{11}\cdot\binom{15}{11}\cdot x^{60-7(11)} = -\frac{15.14.13.12.\,(11\,!)}{(11)!\,.4.3.2.1}x^{-17} = -1365x^{-17}$.

Hence the term involving x^{-17} is $-1365\ x^{-17}$.

Example 2. Find the term independent of x in $\left(2x + \frac{1}{3x^2}\right)^9$

Solution: Let $(r + 1)$th term be independent of x.

Now, $T_{r+1} = \binom{9}{r}\cdot(2x)^{9-r}\cdot\left(\frac{1}{3x^2}\right)^r = \binom{9}{r}\cdot\frac{2^{9-r}}{3^r}\cdot\frac{x^{9-r}}{x^{2r}} = \binom{9}{r}\cdot\frac{2^{9-r}}{3^r}\cdot x^{9-3r}$

Since this term is supposed to be independent of x, we must have $9 - 3r = 0$. or $r = 3$.

Thus, the required term, $T_{r+1} = T_{3+1}$.

Now, $T_{3+1} = \binom{9}{3}\cdot\frac{2^{9-3}}{3^3}x^{9-3(3)} = \frac{9.8.7.6!}{3.2.1.6\,!}\cdot\frac{2^6}{3^3} = 3.4.7\cdot\frac{2^6}{3^3} = \frac{1792}{9}$

Hence, the term independent of x is $\frac{1792}{9}$.

Example 3. Does the expansion of $\left(x + \frac{1}{x}\right)^7$ contain a term

(i) independent of x, (ii) involving x^6

Solution:

(i) Suppose the term independent of x (i.e. the constant term) is $(r + 1)$th term.

Now, $T_{r+1} = \binom{7}{r} x^{7-r} . \left(\frac{1}{x}\right)^r = \binom{7}{r} x^{7-2r}$

Since, this term is supposed to be independent of x, we must have

$7 - 2r = 0$ or $r = \frac{7}{2}$ Then $r + 1 = \frac{7}{2} + 1 = \frac{9}{2}$

Since the position of a term cannot be fractional.

Therefore, there does not exist a term independent of x in the given expansion.

(ii) Let x^6 occurs in $(r + 1)$th term of the expansion.

Now $T_{r+1} = {}^7C_r \ x^{7-r} \frac{1}{x^r} = {}^7C_r \ x^{7-2r}$,

According to the supposition $7 - 2r = 6$

$\Rightarrow$ $r = \frac{1}{2}$, which is a fraction.

So there does not exist a term involving x^6 in the given expansion.

Middle Term:

We shall find the middle term or terms in the expansion of $(x + y)^n$, n being a positive integral index.

The number of terms in the expansion of $(x + y)^n$ is $n + 1$. If n is even then there is one middle term and if n is odd, then there are two middle terms.

If n is even, say $n = 2k$, then the number of terms is $(2k + 1)$. Hence only one term, i.e., $(k + 1)th = \left(\frac{n+2}{2}\right)$th term is the middle term. On the other hand. If n is odd, say $n = 2k + 1$, then in the expansion there are $n + 1 = 2k + 2$ terms, i.e., the number of terms is even. In this case there are two, $(k + 1)$th and $(k + 2)$th, middle terms. Thus the required middle terms are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

Example 1. Find the middle term in the expansion of $\left(x - \frac{2y}{3}\right)^{10}$.

Solution: Here, $n = 10$ is an even. So, there is only one middle term that is $\left(\frac{10+2}{2}\right)$th term = 6^{th} term.

So, $$T_6 = T_{5+1} = \binom{10}{5}(x)^{10-5}\left(\frac{-2y}{3}\right)^5$$
$$= (-1)^5 \frac{10!}{5! \times 5!} \times x^5 \times \frac{2^5 \cdot y^5}{3^5}$$
$$= \frac{-252x^5 \cdot 32 \cdot y^5}{243}$$
$$= \frac{-896x^5y^5}{27}$$

So, the required middle term is $\frac{-896x^5y^5}{27}$.

Example 2. Find the middle terms of $\left(x^3 + \frac{1}{x^2}\right)^7$.

Solution: Here, $n = 7$ is an odd. So, there are two middle terms, that is $\left(\frac{7+1}{2}\right)$th and $\left(\frac{7+3}{2}\right)$th the terms. Hence T_4 and T_5 are two middle terms.

Now, $$T_4 = \binom{7}{3}(x^3)^{7-3}\left(\frac{1}{x^2}\right)^3$$
$$= 35\, x^{12} \times \frac{1}{x^6}$$
$$= 35x^6$$

and $$T_5 = \binom{7}{4}(x^3)^{7-4}\left(\frac{1}{x^2}\right)^4$$
$$= 35\, x^9 \times \frac{1}{x^8}$$
$$= 35x$$

Hence, $35x^6$ and $35x$ are the two required middle terms.

Exercise 7.2

1. Expand by means of the binomial theorem:

(i) $(a+b)^8$ (ii) $(2x-3y)^4$ (iii) $\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^4$

(iv) $\left(\frac{x^2}{2} - \frac{2}{x}\right)^6$ (v) $(3x^2 - 2y^3)^5$

2. (a) Use the binomial theorem, to find the value of:

(i) $(11)^5$ (ii) $(19)^6$ (iii) $(99)^4$

(b) Use the binomial theorem to compute the values of the following correct to four places of decimal.

(i) $(1.01)^6$ (ii) $(1.02)^7$ (iii) $(2.03)^5$

3. Evaluate:

(i) $(1+2\sqrt{a})^4-(1-2\sqrt{a})^4$ (ii) $(2-\sqrt{1-a})^5+(2+\sqrt{1-a})^5$

(iii) $(2+\sqrt{3})^5-(2-\sqrt{3})^5$

4. Find the indicated term in each of the following expansions.

(i) $\left(\frac{2}{3}x-\frac{3}{2}\right)^{10}$; the 8th term (ii) $\left(\frac{a}{b}-\frac{b}{a}\right)^{8}$; the 7th term

(iii) $\left(2x-\frac{1}{y}\right)^{10}$ the last term

5. Find the middle term in the expansion of:

(i) $\left(b-\frac{2c}{3}\right)^{10}$ (ii) $\left(1-\frac{1}{2}a^2\right)^{14}$ (iii) $\left(\frac{a}{b}-\frac{b}{a}\right)^{18}$ (iv) $\left(a-\frac{1}{a}\right)^{2n}$

6. Find the two middle terms of:

(i) $\left(x^3+\frac{1}{x^2}\right)^7$ (ii) $\left(2b-\frac{b^2}{4}\right)^9$

7. Write, in the simplified form, the term independent of x in

(i) $\left(2x+\frac{1}{x^2}\right)^9$ (ii) $\left(2x+\frac{5}{x}\right)^6$

8. Obtain in the simplified form:

(i) the term involving x^6 in the expansion of $\left(2x^3-\frac{1}{x^2}\right)^7$.

(ii) the term involving a^8 in the expansion of $\left(a^2-\frac{1}{a^2}\right)^{12}$.

(iii) the coefficient of x in the expansion of $\left(x^2+\frac{a^2}{x}\right)^5$.

9. Does the expansion of $\left(\frac{3a}{2}-\frac{1}{3a^3}\right)^9$ contain a term.

(i) independent of a, (ii) involving a^{10}

10. The coefficients of the fifth, sixth and seventh terms of the expansion of $(1+x)^n$ form an A.P. Find n.

7.3 Binomial Series

7.3.1 Expand $(1+x)^n$ where n is a positive integer and extend this result for all rational values of n

We know that

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n \quad \ldots\text{(i)}$$

By putting $x=1$ and $y=x$ in equation (i)

We get $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \quad \ldots\text{(ii)}$$

When n is a positive integer then in the expansion of $(x+y)^n$ or $(1+x)^n$, there are $(n+1)$ terms or finite number of terms and is valid for any real value of x.

But if n is negative or rational (fraction), then the above expansion never ends or have infinite number of terms and is valid only for $-1 < x < 1$ or $|x| < 1$ thus in such a case the expansion is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)x^r}{r!} + \cdots$$

This series is called binomial series and its general term is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Example 1. Expand $(2-x)^{-3}$ to four terms, where $|x| < 1$.

Solution:

$(2-x)^{-3} = 2^{-3}(1-\frac{x}{2})^{-3}$, making the first term of the binomial as 1.

So, $(2-x)^{-3} = 2^{-3}\left(1-\frac{x}{2}\right)^{-3} = 2^{-3}\{1+\left(-\frac{x}{2}\right)\}^{-3}$

$$= 2^{-3}\left\{1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\cdot\left(-\frac{x}{2}\right)^2+\frac{(-3)(-3-1)(-3-2)}{3!}\cdot\left(-\frac{x}{2}\right)^3+\cdots\right\};\ |x|<1$$

$$= \frac{1}{8}\left(1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{5x^3}{4}+\cdots\right)$$

Example 2. Expand $\left(1+\frac{2x}{3}\right)^{-\frac{3}{2}}$ to four terms, when $|x| < 1$

Solution:

$$\left(1+\frac{2x}{3}\right)^{-\frac{3}{2}} = 1+\left(-\frac{3}{2}\right)\left(\frac{2x}{3}\right)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}\cdot\left(\frac{2x}{3}\right)^2+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{3!}\cdot\left(\frac{2x}{3}\right)^3+\cdots;\ |x|<1$$

$$= 1 - x + \frac{15}{18}x^2 - \frac{105}{162}x^3 + \cdots$$

$$= 1 - x + \frac{5}{6}x^2 - \frac{35}{54}x^3 + \cdots$$

Example 3. Find the first negative term in the expansion of $(1+x)^{\frac{7}{2}}$

Solution: By using the general term fromula

$$T_{r+1} = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\ldots\left(\frac{7}{2}-r+1\right)}{r!}x^r$$

This will be the first negative term when $\frac{7}{2} - r + 1$ is negative,

i.e., $\frac{9}{2} - r < 0$ or $r > \frac{9}{2}$,

i.e., when $r > 4\frac{1}{2}$, so $r = 5$.

Hence, when $r = 5$, we get the first negative term,

$$T_6 = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\left(\frac{7}{2}-3\right)\left(\frac{7}{2}-4\right)}{5\,!}x^5 = -\frac{7}{256}x^5.$$

7.3.2 Expand $(1+x)^n$ in ascending power of x and explain its validity/ convergence for $|x| < 1$ where n is a rational number

We know that if n is negative or rational (fraction), then the expansion of $(1+x)^n$ has infinite number of terms and is valid only for $-1 < x < 1$ or $|x| < 1$ and the expansion is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad ; |x| < 1$$

As we see in the expansion, we get ascending power of x and the terms progressively get smaller and smaller for $-1 < x < 1$ or $|x| < 1$, so the series will be valid or convergent.

Let us explain the validity or convergence of this series

We have, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r\,!}x^r + \cdots,$ ... (i)

Replacing x by $-x$ in (i), we have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2\,!}x^2 - \frac{n(n-1)(n-2)}{3\,!}x^3 + \cdots + (-1)^r \cdot \frac{n(n-1)\ldots(n-r+1)}{r\,!}x^r + \cdots ; |x| < 1 \quad \ldots \text{(ii)}$$

Changing the sign of n in (i) and (ii), we get

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2\,!}x^2 - \frac{n(n+1)(n+2)}{3\,!}x^3 + \cdots + (-1)^r \cdot \frac{n(n+1)\ldots(n+r-1)}{r\,!}x^r + \ldots ; |x| < 1 \quad \ldots \text{(iii)}$$

and

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2\,!}x^2 + \frac{n(n+1)(n+2)}{3\,!}x^3 + \cdots + \cdot \frac{n(n+1)\ldots(n+r-1)}{r\,!}x^r + \cdots ; |x| < 1 \quad \ldots \text{(iv)}$$

Hence, for $|x| < 1$ and $n = 1$, we have from (iii) and (iv)

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots \quad \ldots \text{(v)}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots \quad \ldots \text{(vi)}$$

Similarly for $n = 2$, we have

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^r \cdot (r+1)x^r + \cdots \quad \ldots \text{(vii)}$$

and $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (r+1)x^r + \cdots$ ...(viii)

We see that the expansion (vi) does not hold for $x = 2$. In this case the equation (vi) becomes

$$(-1)^{-1} = -1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^r + \cdots, \text{ which is absurd.}$$

Summing the infinite given series on the right-hand side of the expansion (vi), we have

$$1 + x + x^2 + x^3 + \cdots + x^r + \cdots = \frac{1}{1-x} = (1-x)^{-1}$$

which is true only when $|x| < 1$.

Thus expansion (i) is valid or convergent for $|x| < |$ where n is a rational number.

Example 1. If $y = 3x + 6x^2 + 10x^3 + \cdots,$

then prove that, $x = \frac{y}{3} - \frac{1.4}{3^2.2!}y^2 + \frac{1.4.7}{3^3.3!}y^3 - \cdots$

Solution: Since $y = 3x + 6x^2 + 10x^3 + \cdots$

So, $1 + y = 1 + 3x + 6x^2 + 10x^3 + \cdots$

i.e., $1 + y = (1 - x)^{-3}$

Therefore, $(1 - x) = (1 + y)^{-\frac{1}{3}}$

i.e., $1 - x = 1 + \left(-\frac{1}{3}y\right) + \frac{(-\frac{1}{3})(-\frac{1}{3} - 1)}{2!}y^2 + \frac{(-\frac{1}{3})(-\frac{1}{3} - 1)(-\frac{1}{3} - 2)}{3!}y^3 + \cdots$

or $1 - x = 1 - \frac{1}{3}y + \frac{1.4}{3^2.2!}y^2 - \frac{1.4.7}{3^3.3!}y^3 + \cdots$

Hence, $x = \frac{1}{3}y - \frac{1.4}{3^2.2!}y^2 + \frac{1.4.7}{3^3.3!}y^3 - \cdots$

Example 2. Evaluate $\sqrt[5]{31}$ to five places of decimals.

Solution:

$$\sqrt[5]{31} = (31)^{\frac{1}{5}} = (32 - 1)^{\frac{1}{5}} = \left\{32\left(1 - \frac{1}{32}\right)\right\}^{\frac{1}{5}}$$

$$= (32)^{\frac{1}{5}}\left(1 - \frac{1}{32}\right)^{\frac{1}{5}} = 2(1 - \frac{1}{32})^{\frac{1}{5}}$$

$$= 2\left\{1 + \frac{1}{5}\cdot\left(\frac{-1}{32}\right) + \left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)\cdot\frac{1}{2!}(-\frac{1}{32})^2 + \frac{1}{5}\cdot\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)\cdot\frac{1}{3!}(-\frac{1}{32})^3 + \cdots\right\}$$

$$= 2(1 - 0.00625 - 0.000078125 - 0.000001464 + \cdots)$$

$$= 2 \times 0.99367 = 1.98734.$$

7.3.3 Determine the approximate values of the binomial expansions having indices as – ve integers or fractions

We know that

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad \text{when } |x| < 1.$$

where n is negative integer or fraction.

We see that the terms progressively get smaller and smaller. In such a case, to get an approximate value of the expansion, we may omit the terms containing squares and higher powers of x.

Thus, we can get approximation of this expression.

$(1 + x)^n = 1 + nx$ approximately and this approximation is called 1st approximation.

Similarly, up to a second approximation, we have

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2\,!}x^2$$

and so on.

Example 1. If $|x| < 1$, find the first approximation of

$$\frac{\sqrt[3]{1-7x}}{\sqrt[4]{(1+2x)^3}}$$

Solution: $\dfrac{\sqrt[3]{1-7x}}{\sqrt[4]{(1+2x)^3}}$

$$= (1-7x)^{\frac{1}{3}}(1+2x)^{-\frac{3}{4}}$$

$$= \left\{1 - \frac{1}{3}(7x) + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2\,!}(7x)^2 - \cdots\right\}\left\{1 + \left(-\frac{3}{4}\right)(2x) + \frac{\left(-\frac{3}{4}\right)\left(-\frac{3}{4}-1\right)}{2\,!}(2x)^2 + \cdots\right\}$$

$= \left(1 - \frac{7}{3}x\right)\left(1 - \frac{3}{2}x\right)$, neglecting squares and higher powers of x

$$= \left(1 - \frac{7}{3}x - \frac{3}{2}x + \frac{7}{2}x^2\right)$$

$= \left(1 - \frac{23}{6}x\right)$ approximately (neglecting the term containing the square of x)

Example 2. If x be so small that its squares and higher powers can be neglected, prove that

$$\frac{\sqrt{1+2x} + (16+3x)^{\frac{1}{4}}}{(1-x)^2} = 3 + \frac{227}{32}x$$

Solution: The given expression

$$= \frac{(1+2x)^{\frac{1}{2}} + (16)^{\frac{1}{4}}\left(1+\frac{3}{16}x\right)^{\frac{1}{4}}}{(1-x)^2}$$

$= \dfrac{\left(1+\frac{1}{2}2x\right) + 2\left(1+\frac{1}{4}\cdot\frac{3}{16}x\right)}{(1-2x)}$, approximating

$$= \frac{(1+x) + 2\left(1+\frac{3}{64}x\right)}{(1-2x)}$$

$$= \left(3 + \frac{35}{32}x\right)(1-2x)^{-1}$$

$= \left(3 + \frac{35}{32}x\right)(1+2x)$, approximating again

$= 3 + \frac{227}{32}x$, approximately.

Example 3. If $(a-b)$ be small as compared with a or b then show that

$$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}=\left(\frac{a}{b}\right)^{\frac{1}{n}}$$

Solution: Let $a=b+h$ where h is a small quantity.

So, $$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}=\frac{(n+1)(b+h)+(n-1)b}{(n-1)(b+h)+(n+1)b}$$

$$=\frac{2nb+(n+1)h}{2nb+(n-1)h}$$

$$=\frac{1+\frac{n+1}{2n}\cdot\frac{h}{b}}{1+\frac{n-1}{2n}\cdot\frac{h}{b}}$$

$$=\left(1+\frac{n+1}{2n}\cdot\frac{h}{b}\right)\left(1+\frac{n-1}{2n}\cdot\frac{h}{b}\right)^{-1}$$

$$=\left(1+\frac{n+1}{2n}\cdot\frac{h}{b}\right)\left(1-\frac{n-1}{2n}\cdot\frac{h}{b}\right), \text{ approximating}$$

$$=1+\frac{h}{b}\cdot\frac{2}{2n} \quad \text{(Again approximating)}$$

$$=1+\frac{h}{nb} \qquad \ldots\text{(i)}$$

Also $$\left(\frac{a}{b}\right)^{\frac{1}{n}}=\left(\frac{b+h}{b}\right)^{\frac{1}{n}}=\left(1+\frac{h}{b}\right)^{\frac{1}{n}} \qquad [\text{Since, } a=b+h]$$

$$=1+\frac{h}{nb}, \text{ approximating} \qquad \ldots\text{(ii)}$$

Hence, from (i) and (ii), we have

$$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}=\left(\frac{a}{b}\right)^{\frac{1}{n}}$$

7.3.4 Application of summation of series

In general, most of the infinite series can be summed up very quickly by identifying them with some binomial expansion, as is shown in the following examples.

Example 1. Sum to infinity the series:

$$1+\frac{1}{3^2}+\frac{1.4}{1.2}\cdot\frac{1}{3^4}+\frac{1.4.7}{1.2.3}\cdot\frac{1}{3^6}+\cdots$$

Solution: Identifying the given series with

$$(1+x)^n=1+nx\frac{n(n-1)}{2\,!}x^2+\cdots$$

By comparing,

we get $nx = \frac{1}{9}$ ...(i)

$\Rightarrow \quad n^2x^2 = \frac{1}{81}$ ... (ii)

and $\frac{n(n-1)x^2}{2!} = \frac{2}{81}$ ...(iii)

From (ii) and (iii), by division, we have $\frac{n-1}{2n} = 2$ or $n - 1 = 4n$

Therefore $3n = -1$ or $n = -\frac{1}{3}$

By using $n = -\frac{1}{3}$ in equation (i), we get $x = -\frac{1}{3}$

Using $x = -\frac{1}{3}$, $n = -\frac{1}{3}$

we have $(1 + x)^n = \left(1 - \frac{1}{3}\right)^{-\frac{1}{3}}$

$$= \left(\frac{2}{3}\right)^{-\frac{1}{3}} = \sqrt[3]{\frac{3}{2}}.$$

Hence, the sum of the given series is $\sqrt[3]{\frac{3}{2}}$.

Example 2. If $x = \frac{1}{3} + \frac{1.3}{3.6}\frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \cdots$, prove that $x^2 + 2x - 2 = 0$.

Solution: Adding 1 to both the sides of the given series,

We have $x + 1 = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots$ ...(i)

Let the series on the right hand side of (i) be identified with

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \cdots$

By comparing

we get, $nx = \frac{1}{3}$ ...(ii)

$\Rightarrow \quad n^2x^2 = \frac{1}{9}$ ...(iii)

and $\frac{1}{2}n(n-1)x^2 = \frac{1.3}{3.6}$ ...(iv)

From (iii) and (iv) by division, we get

$$\frac{n-1}{2n} = \frac{3}{2} \text{ or } 2(n-1) = 6n \text{ or } 2n - 2 = 6n \text{ or } 4n = -2$$

Therefore, $n = -\frac{1}{2}$.

Hence, from (ii), $x = -\frac{2}{3}$

By substituting $x = -\frac{2}{3}$ and $n = -\frac{1}{2}$ in $(1 + x)^n$ we have

Sum of the given series is

$$= \left(1-\frac{2}{3}\right)^{-\frac{1}{2}} = \left(\frac{1}{3}\right)^{-\frac{1}{2}} = \sqrt{3} \qquad \ldots\text{(iv)}$$

Thus, from (i) and (iv) give we get $\quad x+1=\sqrt{3}$

Squaring both sides, we have

$$x^2+2x+1=3 \quad \text{or} \quad x^2+2x-2=0$$

Exercise 7.3

1. Find the first four terms in the following expansions:

(i) $\sqrt[3]{8-16x}$ (ii) $\frac{3}{3+x}$ (iii) $(1+3x)^{\frac{-1}{3}}$ (iv) $\frac{\sqrt{1+2x}}{1-x}$

2. Find first negative term in the expansion of

(i) $(1+y)^{\frac{4}{3}}$ (ii) $(1+2x)^{\frac{5}{2}}$

3. If x is so small that its square and higher powers may be neglected, then show that:

(i) $\sqrt{1+\frac{1}{4}x} \approx 1+\frac{1}{8}x$ (ii) $\sqrt[4]{1+8x} \approx 1+2x$

4. Using binomial series, find the value of the following up to three places of decimals:

(i) $\sqrt{24}$ (ii) $\sqrt[3]{28}$ (iii) $\sqrt[5]{241}$

(iv) $(1280)^{\frac{1}{4}}$ (v) $\frac{1}{\sqrt[5]{252}}$

5. Identify the following series as binomial expansion and find the sum in each case.

(i) $1+\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\cdots$ (ii) $1+\frac{1}{2}\cdot\frac{1}{5}-\frac{1}{2!}\cdot\frac{1}{4}\cdot\frac{1}{5^2}+\cdots$

(iii) $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1\cdot3}{2!4}+\left(\frac{1}{4}\right)^2-\frac{1\cdot3\cdot5}{3!8}\left(\frac{1}{4}\right)^3+\cdots$

(iv) $1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{2}\right)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{2}\right)^3+\cdots$

(v) $1-\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{3}\right)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{3}\right)^3+\cdots$

6. Use binomial theorem to show that:

(i) $1+\frac{1}{4}+\frac{1\cdot3}{4\cdot8}+\frac{1\cdot3\cdot5}{4\cdot8\cdot12}\ldots=\sqrt{2}$ (ii) $1+\frac{1}{4}+\frac{1\cdot3}{4\cdot6}+\frac{1\cdot3\cdot5}{4\cdot6\cdot8}+\cdots=\sqrt[3]{4}$

7. If $y=\frac{1}{2}\cdot\frac{1}{16}-\frac{1}{2!\cdot4}\cdot\frac{1}{16^2}+\frac{1}{3!}\cdot\frac{1\cdot3}{8}\cdot\frac{1}{16^3}\ldots$

then show that $16y(y+2)=1$.

8. If $\frac{1}{x}=\frac{2}{5}+\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{2}{5}\right)^3+\cdots,$

then show that $4x^2 - 2x - 1 = 0$.

9. If $\frac{y}{2} = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1\cdot3}{2^2\cdot2!}\left(\frac{4}{9}\right)^2 + \frac{1\cdot3\cdot5}{2^3 3!}\left(\frac{4}{9}\right)^3 + \cdots$,
then show that $5y^2 + 20y - 16 = 0$.

Review Exercise 7

1. Select correct answer.

i. The general term of the binomial expansion $(a+x)^n$ is where $n \in N$

(a) $\binom{n}{r} a^n x$ (b) $\binom{n}{r} a^{n-r} x^r$ (c) $\binom{n}{r} a^r x^{n-r}$ (d) $\binom{n}{r} (ax)^{n-r}$

ii. The number of terms in the expansion of $(a+b)^{2n}$ are:

(a) n (b) $2n+1$ (c) 2^n (d) 2^{n-1}

iii. In the expansion $(a+x)^n$, the exponent of 'x':

(a) decreases from n to 0 (b) increases from 0 to n
(c) remains n every where (d) becomes 0 at the end

iv. Middle term in the expansion of $(a+b)^{2n}$ is:

(a) nth term (b) $(n+1)$th term
(c) $(2n+1)$th term (d) $(2n-1)$th term

v. The term independent of x in the expansion of $(a+2x)^n$ is:

(a) First term (b) Middle term
(c) Last term (d) 2nd last term

vi. The coefficient of the last term in the expansion of $(2-x)^7$ is/are:

(a) 1 (b) -1 (c) 7 (d) -7

vii. In the expansion $\left(a+\frac{1}{2}\right)^7$, the number of middle terms is/are:

(a) one (b) two (c) three (d) four

viii. Sum of odd binomial coefficients in the expansion of $(a+x)^n$ is:

(a) 2^n (b) 2^{n-1} (c) 2^{n+1} (d) $n+1$

ix. $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n+1}$ is equal to:

(a) 2^n (b) 2^{n+1} (c) 2^{n-1} (d)Cannot be determined

x. The number of terms in the expansion of $(1+x)^{\frac{1}{3}}$ is __________.

(a) $\frac{4}{3}$ (b) 4 (c) ∞ (d) 2

xi. $1 - x + x^2 - x^3 + \cdots$ is equal to:

(a) $(1+x)^{-1}$ (b) $(1-x)^{-1}$ (c) $(1+x)^{-2}$ (d) $(1-x)^{-2}$

xii. The expansion of $(1-2x)^{-2}$ is valid if:

(a) $|x| < 0$ (b) $|x| < \frac{1}{2}$ (c) $|x| < 2$ (d) $|x| < 1$

xiii. The middle term in the expansion of $(a+b)^n$ is $\left(\frac{n}{2}+1\right)$ th term; then n is:

(a) Odd (b) Even

(c) Prime (d) None of these

xiv. In $\left(a+\frac{1}{a}\right)^8$, the sum of the binomial coefficients is:

(a) 64 (b)128 (c)256 (d) 512

2. Prove by principle of mathematical induction $\forall n \in \mathrm{N}$ that $8 \times 10^n - 2$ is divisible by 6.

3. Prove by mathematical induction

$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

4. Given that the coefficient of y^2 in the expansion $\frac{1}{(1+my)^2}$ is 96, find m.

5. Given that the terms involving m^4 and higher powers may be neglected and that $\frac{1}{(a+bm)^3} - \frac{1}{(1+3m)^4} = cm^2 + dm^3$, find the values of a, b, c and d.

Functions and Graph

Unit 8

8.1 Function

8.1.1 Recall

- **function as a rule or correspondence**
- **domain, co-domain and range of a function**
- **one to one and onto functions**

a. Function as a rule or correspondence

A function f from a set X to a set Y is a rule or correspondence that assigns each element of X to, one and only one element of Y. The elements of X are called pre-images of the function and the corresponding elements of Y are called the images of the function.

Symbolically, we write it as, a function $f: X \rightarrow Y$, where $y = f(x)$, $\forall x \in X$ and $y \in Y$.

The variable x is called the independent variable and y is called the dependent variable.

Example 1. Let $X = \{a, b, c\}$ and $Y = \{4,5,6\}$. State whether or not the relations indicated by the following figures are functions from X to Y.

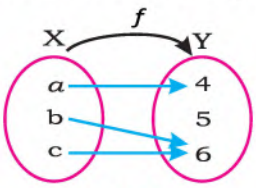


Fig. 8.1

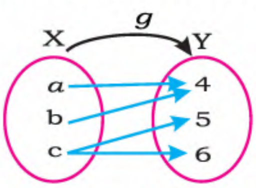


Fig. 8.2

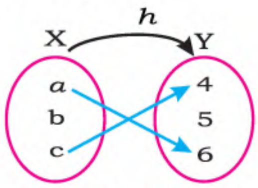


Fig. 8.3

Solution:

f is a function, because each element of X has a unique image in Y. (Fig. 8.1)
g is not a function, because the element c of X has two images in Y. (Fig. 8.2)
h is not a function, because the element b of X has no image in Y. (Fig. 8.3)

Example 2. If function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x) = 3x + 5$, then find

(i) $f(2)$ (ii) $f(-3)$ (iii) $f\left(\frac{2}{3}\right)$

Solution: (i) $f(2) = 3(2) + 5 = 11$

(ii) $f(-3) = 3(-3) + 5 = -4$

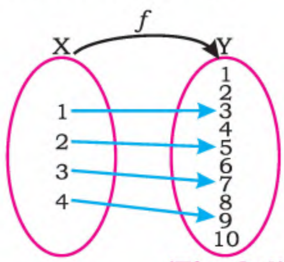
(iii) $f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) + 5 = 7$

b. Domain, co-domain and range of a function

Let $f : X\to Y$ be a function from a set X to a set Y, then X is called the domain and Y is called co-domain of the function f.Whereas the range is the set of all images of the function.

Example 1.

If $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the function f is defined by $f(x) = 2x + 1$, $\forall\, x \in X$, then find range of f.



(Fig. 8.4)

Solution:

Here $f(x) = 2x + 1$

for $x = 1$, we get $f(1) = 3$

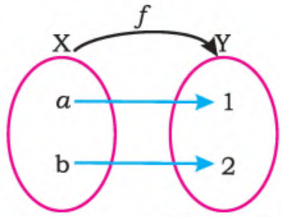
for $x = 2$, we get $f(2) = 5$

for $x = 3$, we get $f(3) = 7$

for $x = 4$, we get $f(4) = 9$,

Thus, Range of $f = \{3,5,7,9\}$ as shown in Fig. 8.4.

Example 2. If function $f: X \to Y$ is a function as shown in Fig. 8.5, where $X = \{a, b\}$ and $Y = \{1,2\}$, then write domain of f and show that Co-domain of f =Range of f.



(Fig. 8.5)

Solution: Here, from Fig. 8.5, we have

Domain of $f = \{a, b\}$

Co-domain of $f = \{1,2\}$ and Range of $f = \{1,2\}$

Hence, Co-domain of f =Range of f.

Example 3. Find the domain of $f(x) = \frac{(x-2)(x-4)}{(x-1)(x-3)}$

Solution: We have to find those values of x for which $f(x)$ is undefined, so that these values may be excluded from $\mathbb{R}$.

The function is undefined when the denominator is zero.

Let $(x - 1)(x - 3) = 0 \implies x = 1$ or $x = 3$

So, $f(x)$ is undefined for $x = 1$ or $x = 3$

i.e., $f(1) = \frac{(1-2)(1-4)}{(1-1)(1-3)} = \frac{(-1)(-3)}{(0)(-2)} = \frac{6}{0}$ (Undefined),

$$f(3) = \frac{(3-2)(3-4)}{(3-1)(3-3)} = \frac{(1)(-1)}{(2)(0)} = \frac{-1}{0} \text{ (Undefined)}$$

Thus, the domain of $f = \{x | x \in \mathbb{R} \text{ and } x \neq 1 \text{ or } 3\}$

Example 4. If $f(x) = x^2$ then find range of f.

Solution: Let $y = x^2$

$\because \quad x^2 \geq 0 \quad \forall x \in \mathbb{R}$

$\therefore \quad \text{Range} = \{y | y \in \mathbb{R} \wedge y \geq 0\}$

$\qquad = \mathbb{R}^+ \cup \{0\}.$

Example 5. Find domain of the function $f(x) = \sqrt{x+9}$

Solution: For real values, we take radicand as greater than or equal to zero

i.e., $\quad x + 9 \geq 0$

$\Rightarrow \quad x \geq -9$

Hence the domain of $f(x) = \{x | x \in \mathbb{R} \text{ and } x \geq -9\} = [-9, \infty)$

Example 6. Find the domain and range of the function $f(x) = \sqrt{x^2 - 4}$

Solution: For real values, we take radicand as greater than or equal to zero

i.e., $\quad x^2 - 4 \geq 0$

$\Rightarrow \quad x^2 \geq 4$

or $\quad x \geq 2 \text{ or } x \leq -2$

Hence domain of the function $f(x) = \{x | x \in \mathbb{R} \text{ and } x \geq 2 \text{ } or \text{ } x \leq -2\}$

Let $\quad y = f(x) = \sqrt{x^2 - 4}$

Here $\quad f(-2) = f(2) = 0$

and $\quad f(x) > 0 \ \forall x < -2 \text{ or } x > 2$

Hence Range $= \{y | y \in \mathbb{R} \wedge y \geq 0\} = \mathbb{R}^+ \cup \{0\}$

Example 7. Find the domain of the function $f(x) = \frac{1}{\sqrt{x-1}}$

Solution:

We have to find those values of x for which $f(x)$ is undefined or non-real so that these values may be excluded from $\mathbb{R}$.

The function is undefined or non-real when $x - 1$ is less than or equal to 0.

i.e., $\quad x - 1 \leq 0$

$\Rightarrow \quad x \leq 1$

So, the domain of $\quad f = \mathbb{R} - \{x | x \in \mathbb{R} \wedge x \leq 1\}$

or domain of $f = \{x | x \in \mathbb{R} \text{ and } x > 1\}$

c. One to one and onto Functions

(i) One – to – one function (Injective function)

A function $f: X \longrightarrow Y$ is one-to-one (injective) if distinct elements of set X have distinct images in set Y.

i.e., if $x_1, x_2 \in X$ and $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$

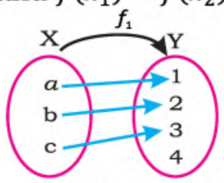


Fig. 8.6

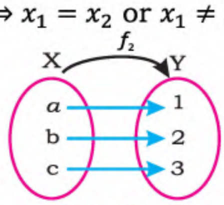


Fig. 8.7

Mapping diagrams of Fig.8.6 and Fig.8.7 represent that f_1 and f_2 are one-to-one functions.

(ii) Onto function (Surjective function)

A function $f: X \longrightarrow Y$ is onto function (Surjective), if the range of f is Y, that is co-domain is equal to range. In other words, if each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$ then f is onto function.

Let $X = \{a, b, c\}$ and $Y = \{1,2,3\}$ then

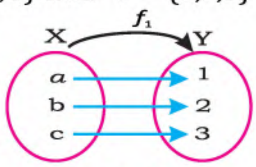


Fig. 8.8

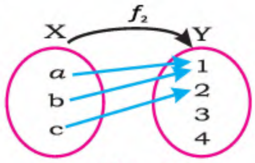
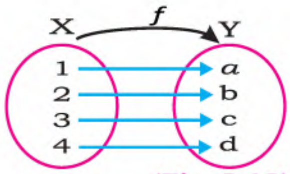


Fig. 8.9

Mapping diagram of Fig.8.8 represents a function f_1 which is onto.
Mapping diagram of Fig.8.9 represents a function f_2 which is not onto.

(iii) One to one and onto (Bijective Function)

A function $f: X \longrightarrow Y$ is called one-to-one and onto or bijective function if each element of Y has one and only one pre-image in X. Functions that are both one-to-one and onto are referred to as Bijective functions.



(Fig. 8.10)

Let $X = \{1,2,3,4\}$ and $Y = \{a, b, c, d\}$, then f is bijective function as shown in mapping diagram of Fig. 8.10.

8.1.2 Know linear, quadratic and square root functions

In this section we are concerned with the definitions of linear, quadratic and square root functions, however, we will first define an important function, which is polynomial function.

(i) Polynomial function

A function $p: \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

for all $x \in \mathbb{R}$ where the co-efficient $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are all real numbers and n is non-negative integer, is called a polynomial function.

If $a_n \neq 0$, then $p(x)$ is called a polynomial function of degree n and a_n is the leading co-efficient of $p(x)$.

For example, $p(x) = 4x^3 - 3x^2 + 2x + 1$,

$q(x) = x^2 - 4x + 2$,

$r(x) = 2x + 3$

and so on are polynomial functions. The degree of $p(x)$ is 3, $q(x)$ is 2 and $r(x)$ is 1 with leading co-efficients as 4, 1 and 2 respectively.

(ii) Linear Function

A polynomial function of degree 1 is called a linear function. A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined as a linear function of the form $f(x) = ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$

For example, $f(x) = 2x + 3$ and $g(x) = -5x + 7$ are linear functions.

(iii) Quadratic Function

A polynomial function of degree 2 is called a quadratic function. A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined as a quadratic function of the form

$f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

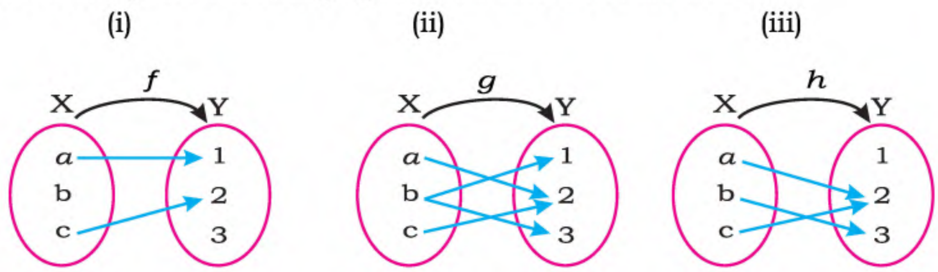
For example, $f(x) = 2x^2 + 5x + 6$, and $g(x) = 3x^2 - 2x - 5$ are quadratic functions.

(iv) Square Root Function

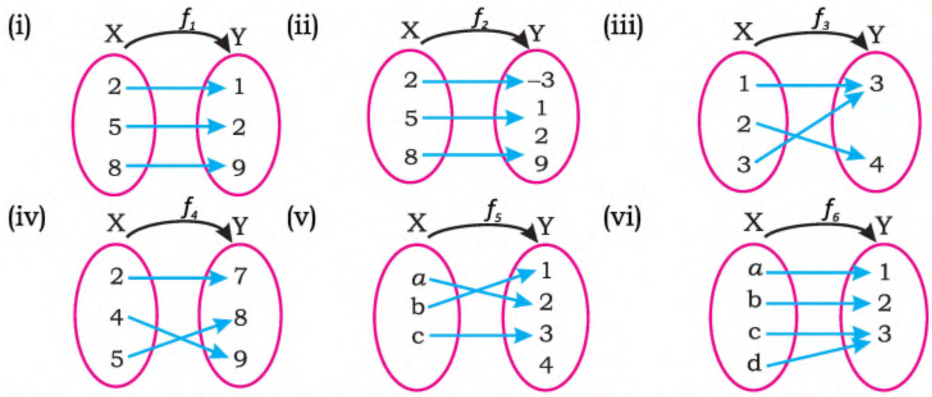
A function $f: \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ defined as $f(x) = \sqrt{x}$, where $x \geq 0$, is called a square root function.

Exercise 8.1

1. Let $X = \{a, b, c\}$ and $Y = \{1,2,3\}$. State whether or not the relations indicated by the following figures are functions from X to Y.



2. Which of the following functions are injective, surjective and bijective. Give reason.



3. If function $f: Z \longrightarrow \mathbb{R}$ defined by $f(x) = x^3 + 1$, find the following if possible:

(i) $f(0)$ (ii) $f(5)$ (iii) $f\left(\frac{3}{4}\right)$ (iv) $f(-2)$

4. If $f(x) = x^3$, find the values of :

(i) $f(2)$ (ii) $f(-10)$ (iii) $f\left(\frac{1}{2}\right)$ (iv) $f(5a)$ (v) $f\left(\frac{a}{3}\right)$

(vi) $f(a+h)$ (vii) $\frac{f(a+h)-f(a-h)}{2h}, (h \neq 0)$ (viii) $f(a+h) - f(a-h)$

5. Given that $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$, find:

(i) $\frac{f(5+h)-f(5)}{h}, (h \neq 0)$ (ii) $\frac{f(a+h)-f(a)}{h}, (h \neq 0)$

6. The domain of the function $f(x) = 5x + 1$ is $\{0,1,2,3,4,5\}$. Find its range.

7. Find the domain of the following functions:

(i) $f(x) = x^2 + 3$ (ii) $f(x) = \sqrt{x+3}$ (iii) $f(x) = \frac{1}{x}$

(iv) $f(x) = \sqrt{x-5}$ (v) $f(x) = \frac{x+1}{x-2}$ (iv) $f(x) = \frac{x^2}{x^2-9}$

8. The domain of the function $f(x)$ is $\{1,2,3,4,5\}$. Find the range of:

(i) $f(x) = 5x^2 + 3$ (ii) $f(x) = \frac{x}{x+1}$

9. Find the range of the function $f(x) = \sqrt{x^2+1}$.

10. Find the domain and range of the function $f(x) = \frac{1}{1+x^2}$.

11. Find the domain and range of $f(x) = \frac{1}{\sqrt{25-x}}$

12. A function is defined by $f(x) = ax + b$. The images of 1 and 5 are -2 and 10 respectively.

(i) Find the value of a and b.

(ii) The domain of $f = \{1,3,5\}$, find the range of f.

13. If $f(x) = x^3 - ax^2 + bx + 1$. Find the values of a and b, where $f(2) = -3$ and $f(-1) = 0$.

8.2 Inverse Function

8.2.1 Define inverse functions and demonstrate their domain and range with examples

If $f: X \longrightarrow Y$ be a one to one and onto (Bijective) function, then inverse of f, i.e., $f^{-1}: Y \longrightarrow X$ is inverse function of f and is defined as

$$x = f^{-1}(y), \ \forall\, y \in Y$$

if and only if $y = f(x), \forall x \in X$. It is evident that f and f^{-1} are inverses of each other. The Fig 8.11 illustrates the concept of inverse function.

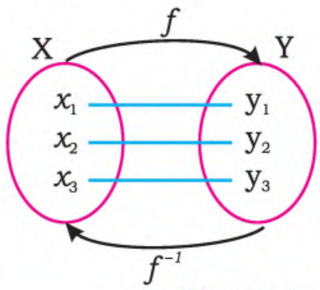


Fig. 8.11

It is clear from the definition of inverse function that

domain of f^{-1} = range of f

and range of f^{-1} = domain of f.

Note: (i) Inverse of f is not always a function.

(ii) Inverse of f is inverse function if f is bijective.

Example: If a function $f: X \longrightarrow Y$, is given by Fig. 8.12. Find f^{-1} and decide whether it is inverse function or not.

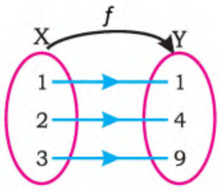


Fig. 8.12

Solution: From Fig 8.12

$f: X \longrightarrow Y$ is defined as $f = \{(1,1),(2,4),(3,9)\}$ or $y = x^2$

where $y = f(x), \forall x \in X$,

with domain of $f = \{1,2,3\}$

and range of $f = \{1,4,9\}$

Now $f^{-1}: Y \longrightarrow X$ is inverse function of f and will be as under

$f^{-1} = \{(1,1),(4,2),(9,3)\}$ as shown in Fig. 8.13 or $x = \sqrt{y}$

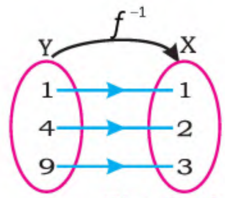


Fig. 8.13

where $x = f^{-1}(y), \ \forall\, y \in Y$

with, domain of $f^{-1} = \{1,4,9\}$

and range of $f^{-1} = \{1,2,3\}$

Since f is both one–to–one and onto, the function is bijective. Therefore f^{-1} is inverse function.

Method for Finding the Inverse of a Function

Inverse of a function $f(x)$ can be found by the following steps:

Step 1. Write $y = f(x)$

Step 2. Express x in terms of y

Step 3. In the resulting equation in step 2, replace x by $f^{-1}(y)$

Step 4. Replace each y by x in the result of step 3 to get $f^{-1}(x)$

Step 5. Verify that $f^{-1}(f(x)) = x$

Example 1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = 5x + 7$. Find $f^{-1}(x)$.

Solution: We have $f(x) = 5x + 7$;

Step1. Let $f(x) = 5x + 7 = y$

Step 2. $\Rightarrow \quad x = \frac{y-7}{5}$.

Step 3. Replacing x by $f^{-1}(y)$, we get $f^{-1}(y) = \frac{y-7}{5}$

Step 4. To find $f^{-1}(x)$, replace y by x, we have $f^{-1}(x) = \frac{x-7}{5}$

Step 5. Verification: $f^{-1}(f(x)) = f^{-1}(5x + 7) = \frac{5x+7-7}{5} = x$.

Example 2. Let $f(x) = \frac{x-3}{x-7}, x \neq 7$. Find f^{-1} and also find the domain and range of f^{-1}.

Solution: Since, $f(x) = \frac{x-3}{x-7}$ is not defined for $x = 7$,

$\therefore$ Domain of $f = \mathbb{R} - \{7\}$

Let $f(x) = \frac{x-3}{x-7} = y$,

$\Rightarrow \quad (x - 7)y = x - 3$

$\Rightarrow \quad xy - 7y = x - 3$

$\Rightarrow \quad xy - x = 7y - 3$

$\Rightarrow \quad x = \frac{7y-3}{y-1}$

Replacing x by $f^{-1}(y)$, we get $f^{-1}(y) = \frac{7y-3}{y-1}$

To find $f^{-1}(x)$, replace y by x, we have $f^{-1}(x) = \frac{7x-3}{x-1}$

Verification

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x-3}{x-7}\right) = \frac{7\left(\frac{x-3}{x-7}\right) - 3}{\frac{x-3}{x-7} - 1}$$

$$= \frac{\frac{(7x-21)-(3x-21)}{x-7}}{\frac{(x-3)-(x-7)}{x-7}}$$

$$= \frac{(7x-21)-(3x-21)}{(x-3)-(x-7)} = \frac{4x}{4} = x$$

We see that $f^{-1}(x)$ is not defined at $x = 1$, so it is not in the domain of f^{-1}.

So, domain of $f^{-1} = \mathbb{R} - \{1\}$

we know that Range of f^{-1} = Domain of $f = \mathbb{R} - \{7\}$.

Exercise 8.2

1. If $f(x) = -5x + 1$, find the values of :

(i) $f^{-1}(36)$ (ii) $f^{-1}(-14)$ (iii) $f^{-1}(0)$ (iv) $f^{-1}(a)$

2. Given that $g(t) = \frac{1}{t-5}$, $(t \neq 5)$, find the values of:

(i) $g^{-1}\left(\frac{1}{2}\right)$ (ii) $g^{-1}(2)$ (iii) $g^{-1}(-1)$ (iv) $g^{-1}(a)$

3. Find the inverse of the following functions:

(i) $f(x) = 12 - \frac{1}{2}x$ (ii) $f(x) = \frac{1}{2}(x-3)$ (iii) $f(x) = \frac{2x+1}{5}$

(iv) $f(x) = \frac{5}{9}(x-32)$ (v) $g(x) = 180(x-2)$ (vi) $h(x) = 2\pi x$

(vii) $f(t) = t^2 + 5$ $(t \geq 0)$ (viii) $f(t) = 5\sqrt{t}, (t \geq 0)$ (ix) $f(t) = (t-5)^3$

(x) $f(t) = \sqrt[3]{t+1}$ (xi) $g(x) = \frac{1}{x-3}, (x \neq 3)$

(xii) $g(x) = \frac{1}{2x+1}, \left(x \neq \frac{1}{2}\right)$

4. Find inverse of f and determine the domain and range of f^{-1} for the real valued function f defined by

i) $f(x) = \frac{x-1}{x-3}$, $x \neq 3$ ii) $f(x) = 5x + 7$

iii) $f(x) = \frac{1}{x+5}$, $x \neq -5$ iv) $f(x) = (x-5)^2, x \geq 5$

v) $f(x) = \sqrt{x+7}, x \geq -7$ vi) $f(x) = \frac{x-1}{x-4}$, $x \neq 4$

8.3 Graphical Representation of Functions

If $y = f(x)$ is a function, then the set of all points (x, y) such that x is in the domain of f and y is in the range of f, is called graph of function $f(x)$.

8.3.1 Sketch graphs of

- **linear function (e.g. $y = ax + b$)**
- **non-linear function (e.g. $y = x^2$)**
- **square root functions (e.g. obtained from $x^2 + y^2 = a^2$)**

- **linear function (e.g. $y = ax + b$)**

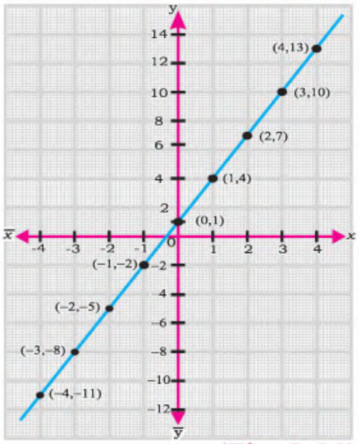
We sketch the graph of a linear function of the form $y = ax + b$, $a, b, x \in \mathbb{R}$, $a \neq 0$ with the help of following example.

Example: Sketch the graph of the function $f(x) = 3x + 1, \forall x \in \mathbb{R}$

Solution: By putting some values of x in the given function $y = f(x) = 3x + 1$, we get corresponding values of y, as shown in the following table.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 3x + 1$	-11	-8	-5	-2	1	4	7	10	13

The points are plotted and the graph is obtained which represents a line as shown in Fig. 8.14.



(Fig. 8.14)

- **non-linear functions (e.g., $y = x^2$)**

We sketch the graph of non-linear function with the help of the following example.

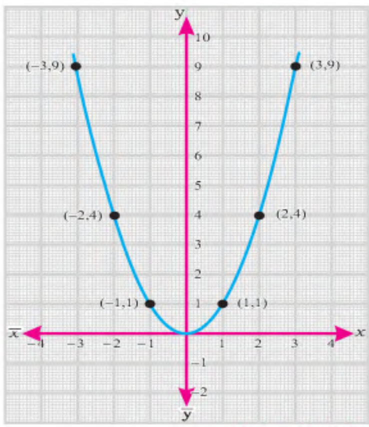
Example: Sketch the graph of the non-linear function $y = f(x) = x^2; \forall x, y \in \mathbb{R}$.

Solution:

We find some values of y by putting the values of x in the function as shown in the following table.

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

The graph of $y = x^2$ represents a parabola as shown in fig. 8.15.



(Fig. 8.15)

- **Square root functions (e.g. obtained from $x^2 + y^2 = a^2$)**

The square root function is the function $f: \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}^+ \cup \{0\}$ defined as $f(x) = \sqrt{x}; \ \forall x \in \mathbb{R}^+ \cup \{0\}$. We know that the equation $x^2 + y^2 = a^2$ represents a circle of radius a, where $x \leq a$.

Now, from $\quad x^2 + y^2 = a^2; \quad \forall x, y \in \mathbb{R}$

We get, $\quad y^2 = a^2 - x^2$

$$\Rightarrow \quad y = f(x) = \pm\sqrt{a^2 - x^2}, \quad \forall x \leq a$$

Here f is not a function as each element of domain has not unique image

But $y = f(x) = \sqrt{a^2 - x^2}$, $\forall x \le a$ and $y = f(x) = -\sqrt{a^2 - x^2}$, $\forall x \le a$ represent square root functions.

Example:

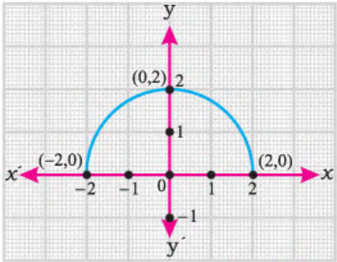
Sketch the graphs of the functions $y = \pm\sqrt{4 - x^2}$ $\forall x \le 2$,

Solution:

a) Consider the function $y = \sqrt{4 - x^2}$, $\forall x \le 2$

Some values of x and the corresponding values of y are mentioned in the table.

x	-2	-1	0	1	2
$y = \sqrt{4 - x^2}$	0	$\sqrt{3}$	2	$\sqrt{3}$	0



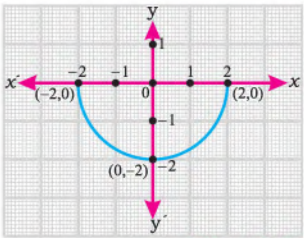
(Fig. 8.16)

Graph of the square root function shows that it is a half circle which opens downward meeting x-axis at (–2,0), (2,0) and y–axis at (0,2). Its center is at origin with radius 2 units as shown in Fig. 8.16.

b) Consider the function $y = -\sqrt{4 - x^2}$, $\forall x \le 2$

Some values of x and the corresponding values of y are mentioned in the table.

x	-2	-1	0	1	2
$y = -\sqrt{4 - x^2}$	0	$-\sqrt{3}$	-2	$-\sqrt{3}$	0



(Fig. 8.17)

Graph of the square root function shows that it is a half circle which opens upward meeting x-axis at (–2,0), (2,0) and y–axis at (0,-2). Its center is at origin with radius 2 units as shown in Fig. 8.17.

8.3.2 Sketch the graph of the function $y = x^n$ where

(a) n is a +ve integer

(b) n is a −ve integer $(x \neq 0)$,

(c) n is a rational number for $x > 0$

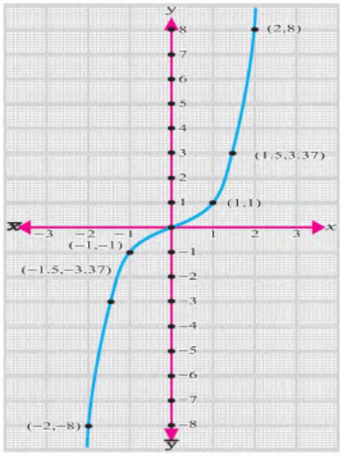
(a) n is a +ve integer

We sketch the graph of the function $y = x^n$, where n is a +ve integer, by taking $n = 3$ and $n = 4$.

(i) For $n = 3$, we have $y = x^3$, $\forall x \in \mathbb{R}$. Some corresponding values of x and y are given in the following table.

x	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5
$y = x^3$	-15.62	-8	-3.37	-1	-0.12	0	0.12	1	3.37	8	15.62

The graph of $y = x^3$ is a curve shown in Fig. 8.18.

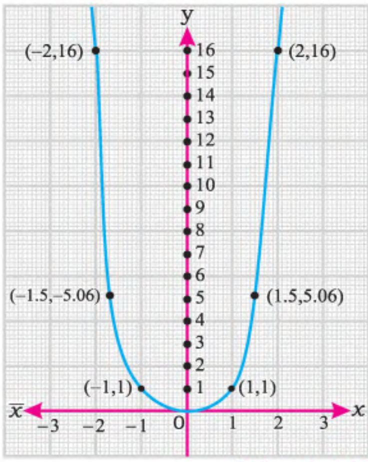


(Fig. 8.18)

(ii) For $n = 4$, we have $y = x^4, \forall x \in \mathbb{R}$. Some corresponding values of x and y are given in the following table.

x	-2	-1.5	-1	0	1	1.5	2
$y = x^4$	16	5.06	1	0	1	5.06	16

The graph of $y = x^4$ is shown in fig. 8.19.



(Fig. 8.19)

(b) *n* is a $-$ve integer $(x \neq 0)$,

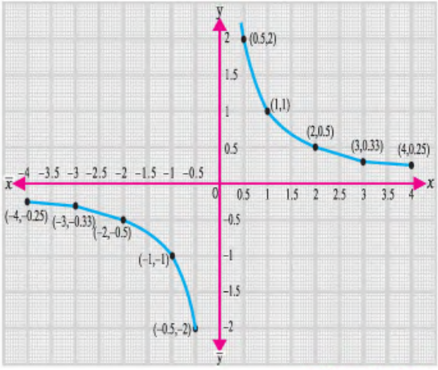
We sketch the graph of $y = x^n$, where n is a negative integer, by taking $-1, -2, -3$ and -4 as values of n.

(i) When $n = -1$, then $y = x^{-1} = \frac{1}{x}$, $x \neq 0$

Some values of x and their corresponding values of y are mentioned in the table.

x	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
$\frac{1}{x}$	−0.25	−0.33	−0.5	−1	−2	2	1	0.5	0.33	0.25

The graph $y = \frac{1}{x}$ is a curve as shown in Fig. 8.20.



(Fig. 8.20)

(ii) When n = −2 , then $y = x^{-2} = \frac{1}{x^2}$, $x \neq 0$

Following table shows some corresponding values of x and y of the function $y = \frac{1}{x^2}$

x	-2.5	−2	−1.5	−1	−0.5	0.5	1	1.5	2	2.5
$\frac{1}{x^2}$	0.16	0.25	0.44	1	4	4	1	0.44	0.25	0.16

The graph $y = \frac{1}{x^2}$ is a curve as shown in Fig.8.21

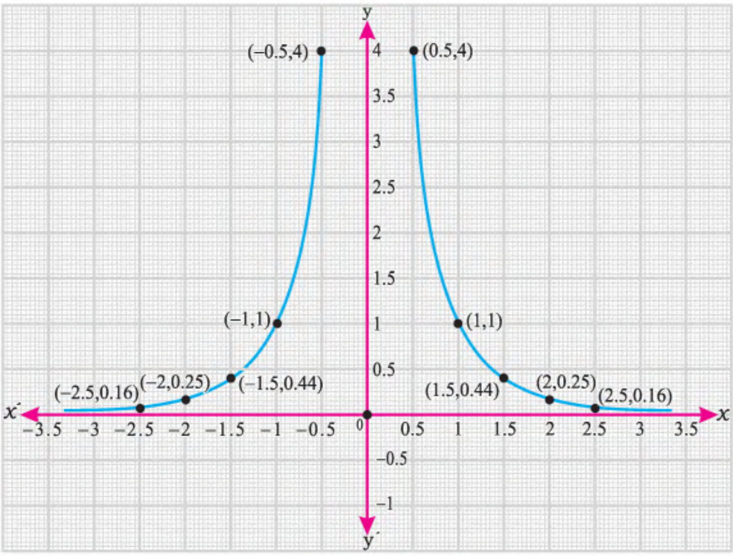


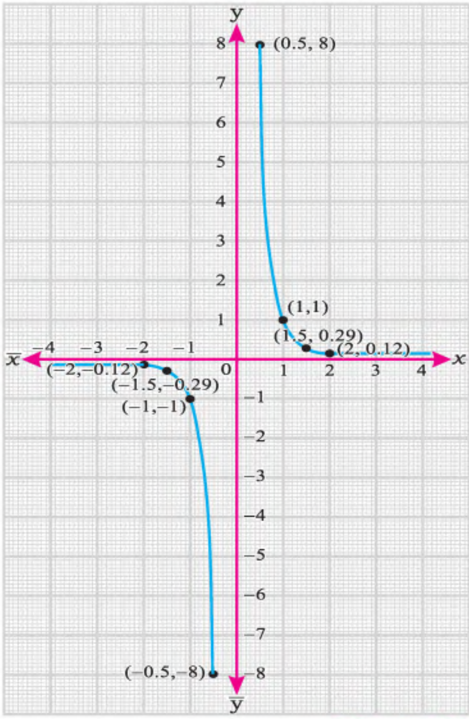
Fig. 8.21

(iii) When $n = -3$, then $y = x^{-3} = \frac{1}{x^3}$, $x \neq 0$

Following table shows some corresponding values of x and y of the function $y = \frac{1}{x^3}$

x	-2	-1.5	-1	-0.5	0.5	1	1.5	2
$y = \frac{1}{x^3}$	-0.12	-0.29	-1	-8	8	1	0.29	0.12

The graph $y = \frac{1}{x^3}$ is a curve as shown in Fig.8.22.



(Fig. 8.22)

(iii) When n= -4 , then $= x^{-4} = \frac{1}{x^4}$, $x \neq 0$

Following table shows some corresponding values of x and y of the function $y = \frac{1}{x^4}$

x	±0.7	±0.8	±0.9	±1	±1.5	±2	±2.5
$y = \frac{1}{x^4}$	4.16	2.44	1.52	1	0.19	0.06	0.02

The graph $y = \frac{1}{x^4}$ is a curve as shown in Fig.8.23.

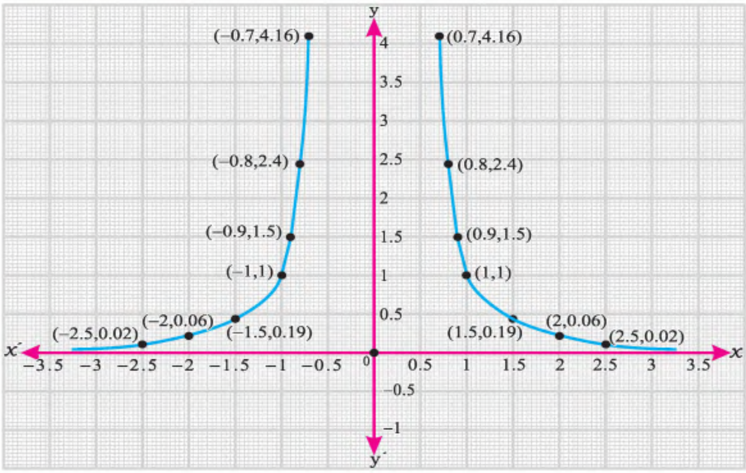


Fig. 8.23

Note: (i) When the value of n is negative odd then the graphs of $y = x^n$ are symmetric about origin. In this case, all the graphs have the same general pattern as of $\frac{1}{x}$.

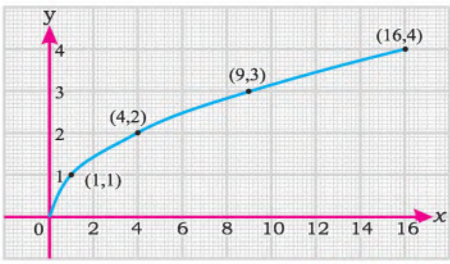
(ii) When the value of n is negative even then the graphs of $y = x^n$ are symmetric about y axis. In this case, all the graphs have the same general pattern as of $\frac{1}{x^2}$.

(iii) When the value of n is positive odd then the graphs of $y = x^n$ are symmetric about origin. In this case, all the graphs have the same general pattern as $y = x^3$

(iv) When the value of n is positive even then the graphs of $y = x^n$ are symmetric about y-axis. In this case, all the graphs have the same general pattern as $y = x^2$

(c) n is a rational number for $x > 0$

For n is a rational number, we sketch the graph of the function $y = f(x) = x^n$, $x > 0$ by taking $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ as values of n.



(Fig. 8.24)

(1) When $n = \frac{1}{2}$ then $y = x^{\frac{1}{2}} = \sqrt{x}$, $x \geq 0$. Following table shows some corresponding values of x and y of the function $y = \sqrt{x}$

x	0	1	4	9	16
$y=\sqrt{x}$	0	1	2	3	4

The graph of the function is a curve in xy –plane as shown in Fig. 8.24.

(2) When $n = \frac{1}{3}$ then $y = x^{\frac{1}{3}} = \sqrt[3]{x}$.

Following table shows some corresponding values of x and y of the function $y = \sqrt[3]{x}$

x	0	1	8	27
$y=\sqrt[3]{x}$	0	1	2	3

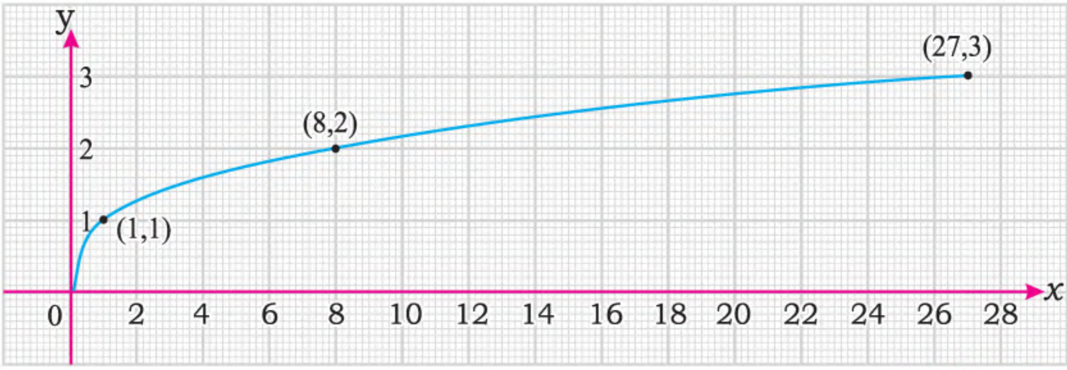


Fig. 8.25

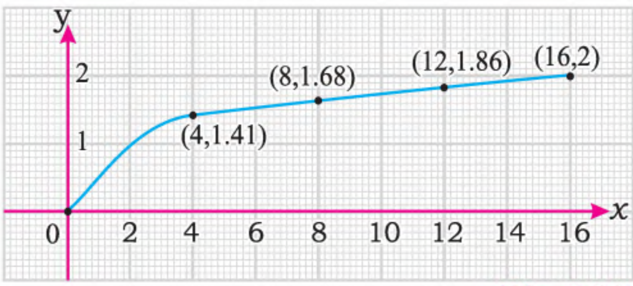
The graph of the function is a curve in xy –plane as shown in Fig. 8.25.

(3) When $n = \frac{1}{4}$ then $y = x^{\frac{1}{4}} = \sqrt[4]{x}$.

Following table shows some corresponding values of x and y of the function $y = \sqrt[4]{x}$

x	0	4	8	12	16
$y=\sqrt[4]{x}$	0	1.41	1.68	1.86	2

The graph of the function is a curve in xy-plane as shown in the Fig. 8.26.



(Fig. 8.26)

8.3.3 Sketch graph of quadratic function of the form $y = ax^2 + bx + c, (a \neq 0), a, b, c$ are integers

A nonlinear function that can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$ and a, b, c are integers, is called a quadratic function. Graph of every quadratic function is a parabola and the parent quadratic function is $y = x^2$.

Example 1. Sketch the graph of the function $y = f(x) = x^2 + 4x + 5$.

Solution: $y = x^2 + 4x + 5$

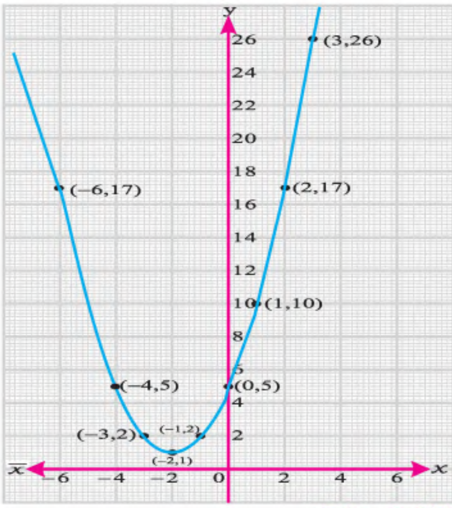
$= x^2 + 4x + 4 + 1$

$= (x+2)^2 + 1$

Following table shows some corresponding values of x and y of the given function

x	-6	-4	−3	−2	−1	0	1	2	3
y	17	5	2	1	2	5	10	17	26

The graph of the given quadratic function is a parabola. Its vertex is at a point $(-2, 1)$ and it opens upward as shown in Fig. 8.27.



(Fig. 8.27)

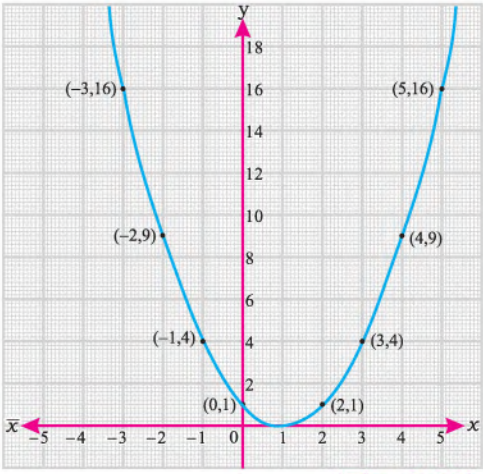
Example 2. Sketch the graph of the function $f(x) = x^2 - 2x + 1$.

Solution: $y = x^2 - 2x + 1 \Rightarrow y = (x-1)^2$

Some corresponding values x and y of the function $y = x^2 - 2x + 1$ are given in the following table.

x	−3	−2	−1	0	1	2	3	4	5
y	16	9	4	1	0	1	4	9	16

The graph of the given function is a parabola. Its vertex is at point (1,0) and it opens upward as shown in Fig. 8.28.



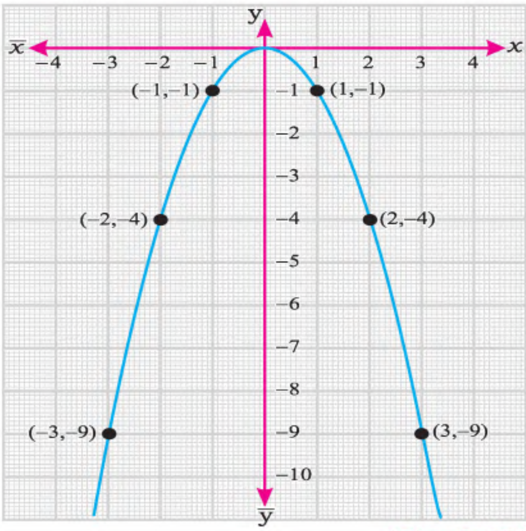
(Fig. 8.28)

Example 3. Sketch the graph of function $y = f(x) = -x^2$.

Solution: Some corresponding values of x and y of the function $y = -x^2$ are given in the following table.

x	-3	-2	-1	0	1	2	3
y	-9	-4	-1	0	-1	-4	-9

The graph of the given function is a parabola. Its vertex is at point (0,0) and it opens downward as shown in Fig. 8.29.



(Fig. 8.29)

Note: (i) When the coefficient of x^2 is positive in quadratic function $y = ax^2 + bx + c, (a \neq 0)$ i.e., $a > 0$, the graph of the function opens upward.

(ii) When the coefficient of x^2 is negative in quadratic function $y = ax^2 + bx + c, (a \neq 0)$ i.e., $a < 0$, the graph of the function opens downward.

8.3.4 Sketch graph using factors

We draw the graph of quadratic function in the form of factors $y = f(x) = a(x - p)(x - q)$ by using the following steps:

1. Identify the points $(p, 0)$ and $(q, 0)$ where the graph of the function cuts x- axis.
2. Take $x = 0$ in the function to identify the point $(0, y)$ where the graph cuts y – axis.
3. The sign of the constant 'a' indicates the shape of the graph opens upward or downward.
4. To draw the graph, we get some additional points of the graph.
5. Graph of each quadratic functions is a parabola. So, we draw parabola through the points.
6. Locate the correct point where the graph is turning.

The method of sketching the graph of a quadratic function using factors is illustrated through the following examples.

Example 1. Sketch the graph of the function $y = f(x) = -2x^2 + 6x$.

Solution: $y = -2x^2 + 6x \implies y = -2x(x - 3)$

In order to draw the graph, we proceed as follows

i. To identify the points where the graph of function cuts x- axis.
we compare with $y = f(x) = a(x - p)(x - q)$
and get $a = -2$; $p = 0$; $q = 3$.
or we put $y = 0$ in the given equation, we get
$x = 0$ and $x = 3$
so the points are (0,0) and (3,0) where the graph cuts the x- axis

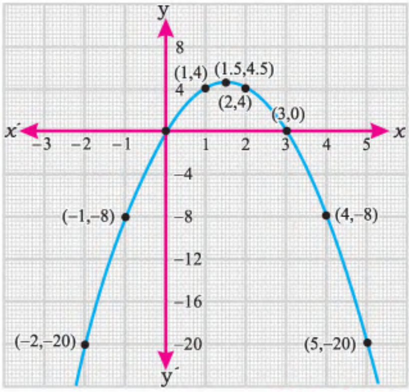
ii. To identify the points where the graph cuts y – axis.
We put $x = 0$ in $y = -2x^2 + 6x$, and get
$y = 0$. So the point is (0,0) and the graph cuts y-axis at (0,0).

iii. We check the sign of the constant 'a' which is negative in this case, therefore the graph opens downward.

iv. For plotting the graph, some additional points will be obtained from the function $y = -2x^2 + 6x$ as shown in the table.

x	-2	-1	0	1	1.5	2	3	4	5
y	-20	-8	0	4	4.5	4	0	-8	-20

v. We draw the graph of the function which is a parabola that opens downward with vertex at $(1.5, 4.5)$ as shown in Fig. 8.30.



(Fig. 8.30)

Example 2. Sketch the graph of the function $y = f(x) = x^2 - x - 6$.

Solution: $y = x^2 - x - 6 = (x+2)(x-3)$

In order to draw the graph, we proceed as follows:

i. To identify the points where the graph of function cuts x- axis.
we compare with $y = f(x) = a(x-p)(x-q)$.
and get $a = +1$; $p = -2$; $q = 3$.
or we put $y = 0$ in the given equation and get $x = -2$ and $x = 3$
So, the points are $(-2,0)$ and $(3,0)$ where the graph cuts the x- axis

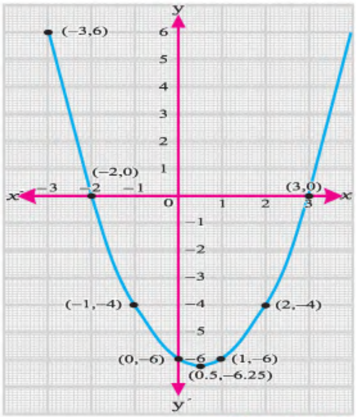
ii. To identify the points where the graph cuts y – axis.
We put $x = 0$ in $y = x^2 - x - 6$ and get $y = -6$
So the point is $(0,-6)$ and the graph cuts y-axis. at $(0,-6)$.

iii. Check the sign of the constant 'a' which is positive in this case, therefore the graph opens upward.

iv. For plotting the graph, some additional points will be obtained from the function
$y = x^2 - x - 6$ as under:

x	-3	-2	-1	0	0.5	1	2	3
y	6	0	-4	-6	-6.25	-6	-4	0

v. We draw the graph of the function which is a parabola that opens upward with vertex at $(0.5, -6.25)$ as shown in Fig. 8.31.



(Fig. 8.31)

8.3.5 Predict functions from their graphs (use the factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c$, if two points where the graph crosses x-axis and third point on the curve' are given)

We use factor form of quadratic equation to predict the equation of a given graph whose function is of the form $f(x) = ax^2 + bx + c$ provided that two points on the graph which cuts the x - axis and third point on the curve are given.

The equation of the curve which passes through $x - axis$ at the points $(p, 0)$ and $(q, 0)$ in the factor form of quadratic equation is

$$y = a(x - p)(x - q) \qquad \ldots(i)$$

By taking all three points from the graph and substituting in equation (i) we get the value of a. and by the value of p, q and a we can form the desired equation of that curve.

The method is illustrated in the following example.

Example 1: Predict the equation of the function from the given graph of the type $y = ax^2 + bx + c$

Solution: The equation of the curve which passes through $x - axis$ at the points$(p, 0)$ and $(q, 0)$ has the form $y = a(x - p)(x - q)$ ...(i)

Here the curve which passes through the points $(-2,0)$ and $(2,0)$ is shown in figure (8.32). So, $p = -2$ and $q = 2$. By substituting these values in eq.(i)

We get $y = a(x + 2)(x - 2)$ ...(ii)

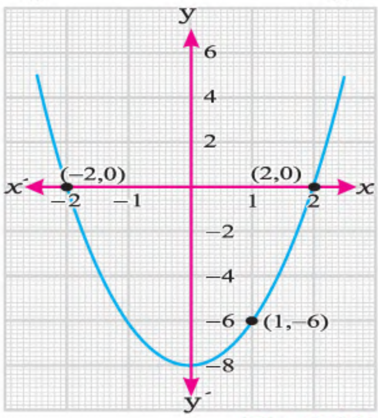
The point $(1, -6)$ lies on the curve, so it must satisfy equation (ii)

so, $-6 = a(1 + 2)(1 - 2) \Rightarrow a = 2$

Therefore equation (ii) becomes

$y = 2(x + 2)(x - 2)$

$y = 2x^2 - 8$, is the required equation of given graph of the parabola.



(Fig. 8.32)

Example 2: Predict equation of the graph of the function of the type $y = ax^2 + bx + c$ which cuts the x − axis at the points (4,0) and (−2,0) and also passes through the point (0,8) as shown in Fig. 8.33.

Solution: The equation of the curve which passes through x − axis at the points$(p, 0)$ and $(q, 0)$ has the form $y = a(x - p)(x - q)$ ...(i)

The curve which passes through the

points (4,0) and (−2,0) is shown in figure (8.33). Here, $p = 4$ and $q = -2$. By substituting these values in equation (i)

We have $\quad y = a(x-4)(x+2) \quad$ ...(ii)

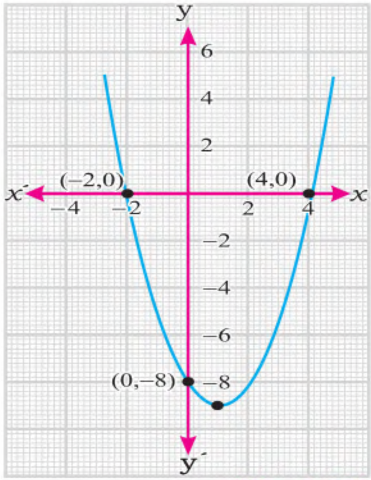
The point (0,8) lies on the curve, so it must satisfy equation (ii)

So, $\quad -8 = a(0-4)(0+2) \Rightarrow a = 1$

Therefore equation (ii) becomes

$$y = (x-4)(x+2)$$

$y = x^2 - 2x - 8$, is the required equation of the graph of parabola.



(Fig. 8.33)

Exercise 8.3

1. Sketch the graph of the following functions:
(i) $y = 5x + 2$ (ii) $y = x^2 + 5$ (iii) $y = x^5$

2. Sketch the graphs of the square root functions obtained from $x^2 + y^2 = 25$.

3. Sketch the graph of the following quadratic functions:
(i) $y = x^2 - 2x + 1$ (ii) $y = -3x^2 + 6x$

4. Sketch the graph of the following functions using factors:
(i) $y = x^2 + 3x + 2$ (ii) $y = -x^2 - 5x$

5. Find the equation of the function of the type $y = f(x) = ax^2 + bx + c$ which cuts the x-axis at the points (−4,0) and (3,0) also passes through the point (2, −4).

6. Find the equation of the graph of the function of the type $y = ax^2 + bx + c$ which crosses the x-axis at the point (−8,0) and (9,0) and also passes through (5,10)

7. Find the equation of the function of the type $y = 3x^2 + bx + c$ of the parabola which cuts x-axis at the points (−3,0) and (5,0).

8. Find the equation of the graph of the function of the type $y = 2x^2 + bx + c$ which crosses the x-axis at the points (−4,0) and (5,0).

8.4 Intersecting Graphs

8.4.1 Find the Intersecting point graphically when the intersection occurs between;

- a linear function and coordinate axes,
- two linear functions,
- a linear and quadratic function.

- **Points of intersection of a linear function and co-ordinate axes**

When we draw the graph of a linear function $y = f(x) = ax + b\,;\ \forall\ a, b, x \in \mathbb{R},\ a \neq 0$ and $b \neq 0$ we get a straight line that intersects x-axis and y-axis at points $P(a, 0)$ and $Q(0, b)$ respectively as shown in Fig.8.34(i) where a is called x-intercept and b is called y-intercept of the line l on the coordinate axes.

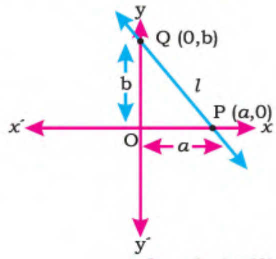


Fig. 8.34(i)

In case, when $b = 0$ in linear function
$y = f(x) = ax + b\,;\ \forall\ a, b, x \in \mathbb{R},\ a \neq 0$
then the line l will pass through origin as shown in Fig. 8.34(ii). So, point of intersection between linear function and coordinate axes is $(0, 0)$.
Method of finding points of intersection between linear function and coordinate axes graphically is explained in the following example.

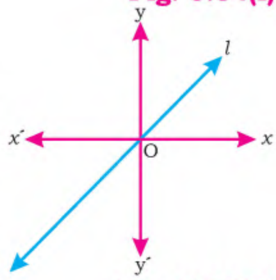
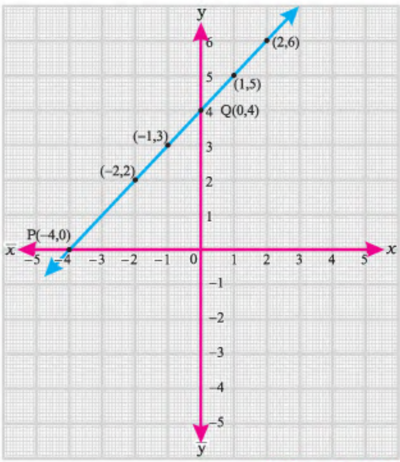


Fig. 8.34(ii)

Example: Find the points of intersection of the linear function $y = f(x) = x + 4$ with co-ordinate axes graphically.
Solution: Here, $y = f(x) = x + 4$
Some corresponding values of x and y of the function are given in the following table.

x	-2	-1	1	2
y	2	3	5	6

From the graph, we see, that x-intercept and y-intercept are -4 and 4 respectively as shown in Fig. 8.35.
Therefore, points of intersections with coordinate axes are $P(-4,0)$ and $Q(0, 4)$.



(Fig. 8.35)

(b) Point of intersection of two linear functions

In order to find the point of intersection of linear functions. we draw the graphs of the functions on the same graph paper and then with the help of graphs we locate the point of intersection of these functions.

If we have two different linear functions described by lines l_1 and l_2

l_1: $y = f_1(x) = a_1x + b_1$; ...(i)

l_2: $y = f_2(x) = a_2x + b_2$; ...(ii) $\forall\ a_1, b_1, a_2, b_2, x \in \mathbb{R}$,

then these lines l_1 and l_2 may or may not intersect. If they intersect, then the point of intersection is unique. In case they do not intersect, the lines will be parallel having no common point.

The method is illustrated in the following example.

Example: Find the point of intersection of the functions $f(x) = x - 3$ and $g(x) = 2x - 5$ graphically

Solution:

Let $f(x) = y$ then $y = x - 3$.

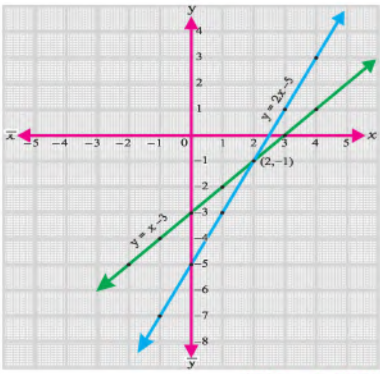
Some corresponding values of x and y are shown in the table.

x	-2	-1	0	1	2	3	4
$y = x - 3$	-5	-4	-3	-2	-1	0	1

Now, let $g(x) = y$ then $y = 2x - 5$.

Some corresponding values of x and y are shown in the table.

x	-1	0	1	2	3	4	5
$y = 2x - 5$	-7	-5	-3	-1	1	3	5



(Fig. 8.36)

From the graph, we find that the two linear functions intersect each other at the point $(2, -1)$ as shown in Fig 8.36.

(c) Points of intersection of a linear function and quadratic function

In order to find the points of intersection, we draw the graph of linear function and quadratic function on the same graph paper. The points of intersection will be located using this graph.

We know that the graphs of linear and quadratic functions are line and parabola respectively. These curves may or may not intersect each other. If they intersect then they will intersect at one or two points.

The method is illustrated by the following example.

Example: Find the points of intersection of functions

$$y = f(x) = 2x + 1 \quad \text{and} \quad y = g(x) = x^2 - 4x + 6,\ \forall\ x \in \mathbb{R}$$

Solution:

Some corresponding values of x and y of the function $y = 2x + 1$ are given in the following table.

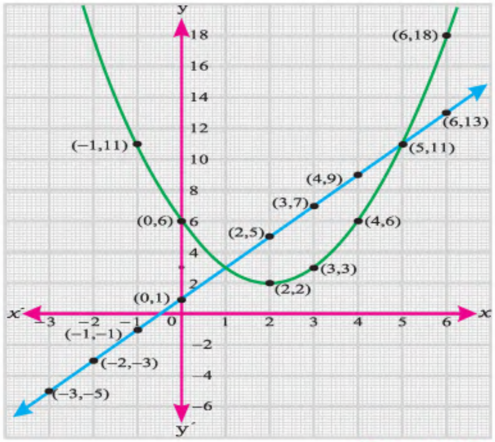
x	-3	-2	-1	0	1	2	3	4	5	6
$y = 2x + 1$	-5	-3	-1	1	3	5	7	9	11	13

Some corresponding values of x and y of function $y = x^2 - 4x + 6$ are given in the following table.

x	-1	0	1	2	3	4	5	6
$y = x^2 - 4x + 6$	11	6	3	2	3	6	11	18

Graphs of the above two functions are shown in figure (8.37)

From graph, the points of intersection of linear function and quadratic function are (1,3) and (5,11).



(Fig. 8.37)

8.4.2 Solve, graphically, appropriate problems from daily life

Example 1. A group of 45 school children visited a zoo on special occasion and at discount, paid Rs 60 altogether for entry tickets. The entry ticket of class 1 was Rs. 2 per child where as that of class KG Rs.1 per child, how many children were in the group from each class.

Solution:

Let x be the number of children from class 1

y be the number of children from class KG

According to the given condition

Total number of children are:

$$x + y = 45 \quad \ldots(i)$$

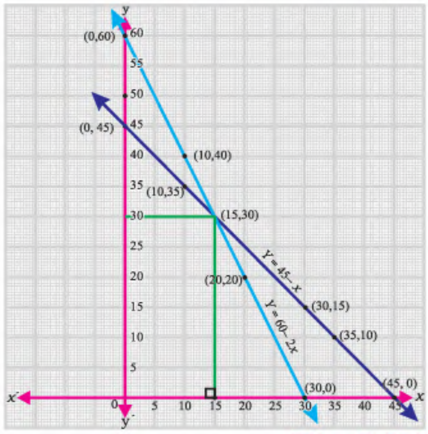
Amount paid for the entry tickets of class 1 and KG children, at the rate of Rs. 2 and Rs.1 respectively is:

$$2x + y = 60 \ldots(ii)$$

Some corresponding values of x and y of eq. (i) and (ii) are given in the following tables.

x	0	15	35	45
$y = 45 - x$	45	30	10	0

x	0	10	20	30
$y = 60 - 2x$	60	40	20	0



(Fig. 8.38)

In order to find children in each group, we plot the graphs of eq. (i) and (ii) as shown in figure 8.38.

From the graph, the intersection point is (15,30)

Hence,

x = number of children from class 1 = 15

and y = number of children from class KG=30

Example 2. The difference of the ages of Waqar and Saqib is 9 years while the age of Waqar is 3 years more than the square of the age of Saqib. Find their ages.

Solution:

Let the age of Saqib be x years.

and age of Waqar be y years.

According to the given condition

Difference of the ages of Waqar and Saqib = 9 years

i.e., $y - x = 9$

$\Rightarrow$ $y = x + 9$ ...(i)

Now, square of the age of Saqib $= x^2$

$\because$ Age of Waqar is 3 years more than the square of the age of Saqib

$\therefore$, $y = x^2 + 3$ ...(ii)

From equation (i)

$$y = x + 9$$

Some corresponding values of x and y of function $y = x + 9$ are given in the following table.

From equation (i)

x	-3	-2	-1	0	1	2	3	4
$y = x + 9$	6	7	8	9	10	11	12	13

From equation (ii)

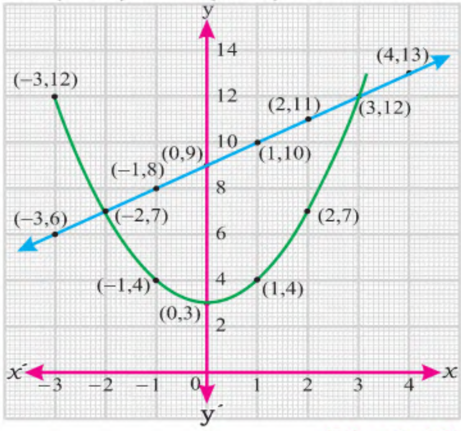
x	-3	-2	-1	0	1	2	3
$y = x^2 + 3$	12	7	4	3	4	7	12

From the graph, we get the intersection points $(-2,7)$ and $(3,12)$ as shown in Fig 8.39.

Since, age cannot be negative, so the required point of intersection is (3, 12).

Thus, age of Saqib = 3 years

and age of Waqar = 12 years



(Fig. 8.39)

Exercise 8.4

1. Graphically find the point of intersection of the following linear functions with co-ordinate axes.

(i) $f(x) = x + 7$ (ii) $f(x) = -x + 5$

(iii) $f(x) = 3x + 2$ (iv) $f(x) = -2x + 4$

2. Find the point of intersection of the following functions graphically where y is function of x.

(i) $2x + y = 5$ and $x + y = 4$ (ii) $3x - 2y = 4$ and $x + 4y = 6$

3. Find the point of intersection of the following functions graphically:

(i) $f(x) = x + 2$ and $g(x) = x^2 - 4x + 6$

(ii) $f(x) = x + 4$ and $g(x) = x^2 - 6x + 10$

4. The paths of two aeroplanes A and B in the plane are determined by the straight lines $2x - y = 6$ and $3x + y = 4$ respectively. Graphically find the point where the two paths cross each other.

5. If the sum of two numbers is 6 and the square of the first number is greater than 6 by the second number. Solve graphically and find the numbers.

Review Exercise 8

1. Select correct answer.

i. If $f: A \to B$ be a function, then it is an onto function if:
(a) Range = B (b) Range $\subset$ B
(c) Image is not repeated (d) Domain $\neq$ A

ii. The graph of $y = x^6$ is symmetric to ____________.
(a) x − axis (b) y − axis (c) origin (d) none

iii. An one to one function is also called __________ function:
(a) Injective (b) Surjective (c) Bijective (d) Inverse

iv. Inverse of a function exists only if it is:
(a) Injective (b) Bijective (c) Surjective (d) all of these

v. The function $f = \{(x, y) \mid y = mx + c\}, m$ and c are real numbers is:
(a) Linear (b) Quadratic (c) A circle (d) A point

vi. The function $f = \{(x, y) \mid y = ax^2 + bx + c, a \neq 0\}$ is:
(a) Linear (b) Quadratic (c) A circle (d) A point

vii. The range of $f(x) = x^2 + 3\ \forall x \in \mathbb{R}$ is __________.
(a) $\mathbb{R}$ (b) $\mathbb{R}^-$ (c) $(3, \infty)$ (d) $[3, \infty]$

viii. Graph of $y = x^n$ is symmetric to _________ if n is an odd integer.
(a) x − axis (b) y − axis (c) origin (d) none

ix. The graph of linear function is:
(a) Circle (b) Straight line
(c) Parabola (d) Triangle

x. If f is function from A to B. Domain of f is equal to:
(a) Any subset of A (b) A × B (c) A (d) B

xi. Every function is a:
(a) Relation (b) Inverse function
(c) One to one (d) None of these

xii. If f and g are equal where $f(x) = 7x - 4$ and $g(x) = x$ then x =:
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{4}{3}$

xiii. Domain of the function $f(x) = \frac{3x+1}{x-1}$ is set of all:
(a) real numbers (b) rational numbers
(c) complex numbers (d) real numbers except 1

xiv. A function $f(x) = |x| - x^2$ is:
(a) odd (b) Linear
(c) Even (d) Neither even nor odd

2. Find domain of the following functions.

(i) $f(x) = \frac{2x-1}{x+4}$ (ii) $f(x) = \sqrt{x+3}$

3. Graphically, find the point of intersection of the function $f(x) = x$ and $g(x) = x^2 - x$.

4. Find the equation, in the form $y = ax^2 + bx + c$ of the parabola which cuts x-axis at (1,0) and (5,0) and cuts y-axis at (0,15).

5. The function f is defined by $f(x) = ax^2 + bx + c$. Given that $f(0) = 5$, $f(-1) = 15$ and $f(1) = 1$, find the values of a, b and c.

6. The function f is defined by $f(x) = x^2 - 3x + 5$. Find the values of x for which $f(x) = f(3)$.

Linear Programming (LP)

Unit 9

9.1 Introduction

Linear programming was developed as a discipline in 1940s motivated initially by the need to solve complex planning problems in war times operations. Its development accelerated rapidly in the postwar period as many industries found valuable uses for linear programming. During World War-II, linear programming was used extensively to deal with transportation, scheduling, and allocation of resources subject to certain restrictions such as costs and operations.

9.1.1 Define linear programming (LP) as planning of allocation of limited resources to obtain an optimal result

Linear Programming (LP) is a mathematical technique for allocating limited resources in optimum manner.

If we have limited resources at our disposal then we intend to seek optimal utilization of those resources. The resources may be time, money, space etc. For example, we have 50 square feet of office space to use for storage and we have a budget of Rs 20,000 and there are variety of cabinet types and sizes from which to choose. So how best the available space, we should utilize and stay within the allocated budget. In another example a company manufactures three products using the basic raw material, some are more expensive to produce than others and few of them are perishable, and need to be used quickly. How much of each product should the company manufacture to minimize the cost and which combination produces the least waste.

In above examples the situations are complex, as so many variables and constraints are involved for consideration. In order to handle such type of problems, we take the help of linear programming. Linear Programming is a mathematical technique that determines the best way to use available resources. Managers use the process to help make decisions about the most efficient use of limited resources.

9.2 Linear Inequalities

In mathematics, a linear inequality involves a linear expression in two or more variables by using any of the relational symbols such as $<, >, \leq$ or $\geq$.

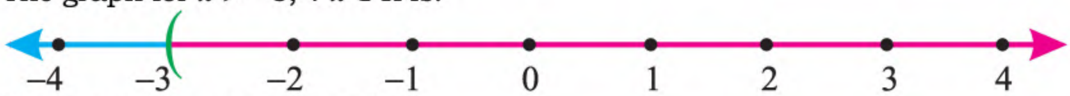
9.2.1 Find algebraic solutions of linear inequalities in one variable and represent them on number line

Inequalities of the form $ax < b, ax \leq b, ax > b$ or $ax \geq b$, where a and b are constants, known as linear inequalities in one variable.

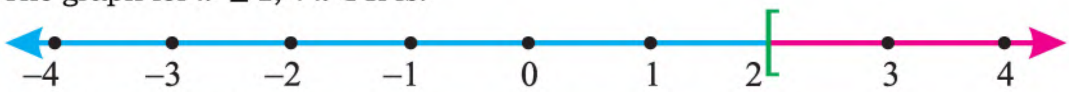
$x < -5$, $5x \leq 10$, $-2x - 6 > 10$ and $-2(x + 2) \geq 4 - x$ are few examples of linear inequalities in one variable. Solutions of a linear inequality in one variable are the values of the variable which satisfy the linear inequality.

The graphic solution of a linear inequality in one variable is represented by a number line. We use the left parenthesis symbol "(" and right parenthesis symbol ")" for ">" and "<" respectively. We also use the left square bracket symbol "[" and right square bracket symbol "]" for "≥" and "≤" respectively for graphic solutions of linear inequalities as shown below.

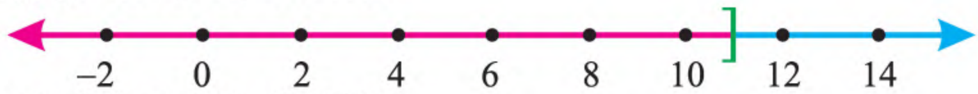
The graph for $x > -3$, $\forall x \in \mathbb{R}$ is:



The graph for $x \geq 2$, $\forall x \in \mathbb{R}$ is:



The graph for $x \leq 11$, $\forall x \in \mathbb{R}$ is:



The graph for $x < 11$, $\forall x \in \mathbb{R}$ is:

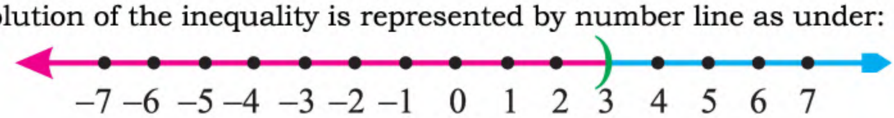
Example 1. Solve the inequality $x - 3 < 0$ and represent the solution on number line, where $x \in \mathbb{R}$.

Solution: We have $\quad x - 3 < 0$

Adding 3 to both sides $\quad x - 3 + 3 < 0 + 3 \Rightarrow x < 3$

Thus, the solution of the inequality is the set of all real values of x that are less than 3, i.e., solution set $= \{x | x \in \mathbb{R} \wedge \; x < 3\}$

The solution of the inequality is represented by number line as under:



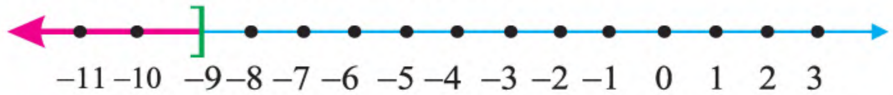
Example 2. Solve the inequality: $2(x+2) \leq x-5, \quad \forall\, x \in \mathbb{R}$ and represent the solution on number line.

Solution: We have $2(x+2) \leq x-5$

$\Rightarrow \quad 2x+4 \leq x-5 \Rightarrow \; x \leq -9$

The solution set $= \{x | x \in \mathbb{R} \wedge \quad x \leq -9\}$

Representation of solution on number line:



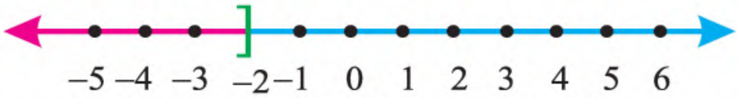
Example 3. Solve the inequality $2(x-3) \geq 3x-4$, $\forall\, x \in \mathbb{R}$ and represent the solution on number line.

Solution: We have $\quad 2(x-3) \geq 3x-4$

$\Rightarrow \quad 2x-6 \geq 3x-4 \;\Rightarrow\; -x \geq 2 \Rightarrow \; x \leq -2$

Thus, solution set $= \{x | x \in \mathbb{R} \wedge \quad x \leq -2\}$

Representation of solution on number line:



Note that the graph has an arrow indicating that the line continues without end to the left i.e., $-\infty$.

The solution can be expressed in Interval notation as $(-\infty, -2]$, i.e., all real values of x less than or equal to -2.

> **Note:** Interval notations: $[a, \infty) = \{x \mid x \in \mathbb{R} \wedge x \geq a\}$. Also $(a, \infty) = \{x | x \in \mathbb{R} \wedge x > a\}$
> $(-\infty, a] = \{x \mid x \in \mathbb{R} \wedge x \leq a\}$. Also $(-\infty, a) = \{x | x \in \mathbb{R} \wedge x < a\}$

Example 4. Solve the inequality $19 < 3x+7 \leq 28$, $\forall\, x \in \mathbb{R}$ and represent solution on number line.

Solution: We have $19 < 3x+7 \leq 28$, $\forall\, x \in \mathbb{R}$.

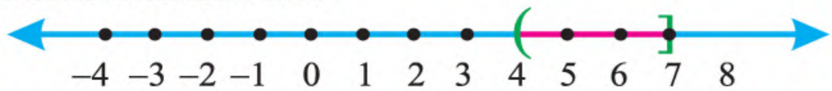
$\Rightarrow \quad 19 < 3x+7$ and $3x+7 \leq 28$

$\Rightarrow \quad 12 < 3x \qquad \Rightarrow \quad 3x \leq 21$

$\Rightarrow \quad 4 < x \qquad \Rightarrow \quad x \leq 7$

Thus, solution set $= \{x | x \in \mathbb{R} \wedge \quad 4 < x \leq 7\}$

Representation on number line:



9.2.2 Interpret graphically the linear inequalities in two variables.

The general form of inequalities: $ax+by < c$, $ax+by > c$, $ax+by \leq c$ and $ax+by \geq c$ are known as linear inequalities in two variables, where $a \neq 0$, $b \neq 0$ and c are constants, x and y are variables. The solution of a linear inequality in two variables like $ax + by > c$ is an ordered pair (x, y) that produces a true statement when the values of x and y are substituted into the inequality. The graph of linear inequalities in two variables is a set of all solutions that constitutes a region representing half portion of the plane.

Graphical solution of linear inequalities in two variables

Consider the linear inequality

$$ax + by \leq c, \quad a \neq 0,\ b \neq 0 \text{ and } c \neq 0 \qquad \text{.... (1)}$$

Following are the steps to graph the solution region of above inequality.

Step 1. Consider the inequality as an equation

$$ax + by = c, \qquad \text{.... (2)}$$

Step 2. Find x-intercept, and y-intercept

For x-intercept, we put $y = 0$ in (2)

We get $ax + b(0) = c \Rightarrow ax = c \Rightarrow x = \frac{c}{a}$ so, x-intercept is $\frac{c}{a}$ and $\left(\frac{c}{a}, 0\right)$ is the point where line cuts x-axis.

Similarly, for y-intercept, put $x = 0$ in (2)

We get $a(0) + by = c \Rightarrow by = c \Rightarrow y = \frac{c}{b}$ so, y-intercept is $\frac{c}{b}$ and $\left(0, \frac{c}{b}\right)$ is the point where line cuts y-axis.

Step 3. Plot the points in a graph and draw the straight line.

Step 4. Substitute (0, 0) in the inequality (1),

(a) if inequality is true then origin is a part of solution and shade the region involving origin.

(b) if inequality is false then origin is not a part of solution and shade the region which does not involve origin.

Example 1. Graph the solution of the inequality: $3y + 2x \leq 6$.

Solution: We have $3y + 2x \leq 6$ ...(1)

(i) Consider the inequality as an equation

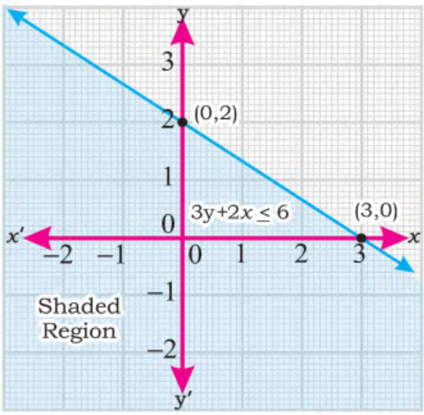
i.e., $3y + 2x = 6$ ...(2)

(ii) For y-intercept, we put $x = 0$ in (2), and get $3y + 2(0) = 6 \Rightarrow y = 2$

So, intersection point of line and y-axis is (0, 2)

For x-intercept, we put $y = 0$ in (2), and get $3(0) + 2x = 6 \Rightarrow x = 3$

So, intersection point of line and x-axis is (3, 0)



(Fig. 9.1)

(iii) We plot the above intersection points (0, 2) and (3, 0) in the graph.

(iv) We put $x = 0$ and $y = 0$ in (1), we get $3(0) + 2(0) \leq 6 \Rightarrow 0 < 6$,

which is true. So, origin is a part of solution of inequality (1).
Now shade the region in the graph as shown in Fig. 9.1 which represents the solution set.

Example 2. Graph, the solution of the inequality $3x+5y \geq 30$

Solution: We have $3x + 5y \geq 30$ ...(1)

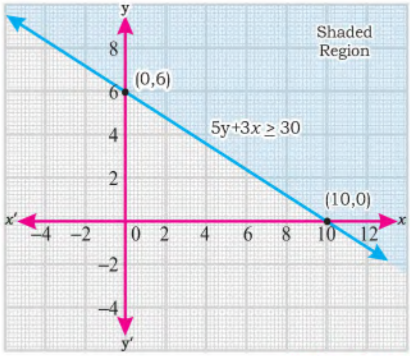
(i) Consider the inequality as an equation

i.e., $3x + 5y = 30$ ...(2)

(ii) For y-intercept, we put $x = 0$ in (2),
and get $3(0) + 5y = 30 \implies y = 6$
Intersection point of line and y-axis is (0,6)
For x-intercept, we put $y = 0$ in (2),
and get $3x + 5(0) = 30 \implies x = 10$
Intersection point of line and x-axis is (10,0)

(iii) We plot the above points (0, 6) and (10, 0) in the graph.

(iv) We put $x = 0$ and $y = 0$ in (1), and get
$3(0) + 2(0) \geq 30 \implies 0 \geq 30$ which is not true.
So, origin is not a part of solution of inequality (1).



(Fig. 9.2)

Now shade the region as shown in the Fig. 9.2 which represents the solution set.

9.2.3 Determine graphically the region bounded by up to 3 simultaneous linear inequalities of non-negative variables and shade the region bounded by them

Two or more linear inequalities form a system of linear inequalities. The solution of the system of linear inequalities in two variables x and y can be obtained by graphing each inequality and then taking intersection of their regions. The common region is the solution region of the system of linear inequalities. In case of '≤', solution region is below the line and in case of '≥', solution region is above the line.

Example 1. Solve graphically

$$5x + y \geq 10 \quad \text{...(i)}$$
$$x + y \geq 6 \quad \text{...(ii)}$$
$$x + 4y \geq 12 \quad \text{...(iii)}$$
$$x, y \geq 0$$

Solution: Let the corresponding equations of above inequalities are

$$5x + y = 10 \quad \text{...(iv)}$$
$$x + y = 6 \quad \text{...(v)}$$
$$x + 4y = 12 \quad \text{...(vi)}$$
$$x, y \geq 0$$

Inequality (i)

For y-intercept, we put $x = 0$ in (iv), and get $5(0) + y = 10 \Rightarrow y = 10$

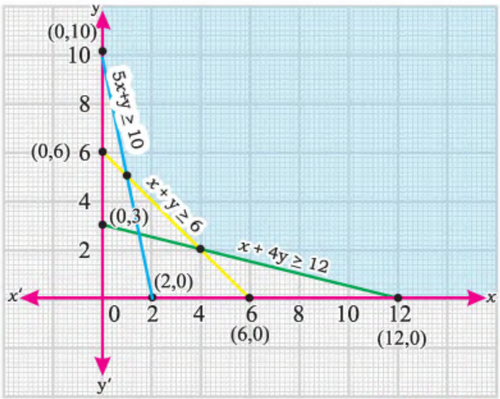
Intersection point of line and y-axis is (0,10)

For x-intercept, we put $y = 0$ in (iv), and get $5x + (0) = 10 \Rightarrow x = 2$

Intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (i) we get , $0 \geq 10$ which is not true

So, origin is not a part of solution of inequality (i)



(Fig. 9.3)

Inequality (ii)

For y-intercept, we put $x = 0$ in (v), and get $0 + y = 6 \Rightarrow y = 6$

Intersection point of line and y-axis is (0,6)

For x-intercept, we put $y = 0$ in (v), and get $x + 0 = 6 \Rightarrow x = 6$

Intersection point of line and x-axis is (6, 0)

For origin, we put $x = 0$ and $y = 0$ in (ii) we get $0 \geq 6$

which is not true. So, origin is a not a part of solution of inequality (ii)

Inequality (iii)

For y-intercept, we put $x = 0$ in (vi), and get $0 + 4y = 12 \Rightarrow y = 3$

Intersection point of line and y-axis is (0,3)

For x-intercept, we put $y = 0$ in (vi), and get $x + 4(0) = 12 \Rightarrow x = 3$

Intersection point of line and x-axis is (12, 0)

For origin, we put $x = 0$ and $y = 0$ in (iii) we get $0 \geq 12$

which is not true. So, origin is a not a part of solution of inequality (iii),

We draw the lines by plotting the above intersection points on the graph. The region bounded by the intersection of three inequalities is shaded (Fig. 9.3) which is the required solution satisfied by all the inequalities.

Note: Since all the inequalities are of type "≥" and coefficients of y are positive. Therefore, the solution region is above all the lines.

Example 2. Solve the following system graphically

$$5x + 10y \leq 50 \quad \ldots(1)$$

$$8x + 2y \geq 16 \quad \ldots(2)$$

$$3x - 2y \leq 6 \quad \ldots(3)$$

$$x, y \geq 0$$

Solution:

Let the corresponding equations of above inequalities are

$$5x + 10y = 50 \qquad ...(4)$$
$$8x + 2y = 16 \qquad ...(5)$$
$$3x - 2y = 6 \qquad ...(6)$$
$$x, y \geq 0$$

Inequality (1)

For y-intercept, we put $x = 0$ in (4), and get $5(0) + 10y = 50 \Rightarrow y = 5$

Intersection point of line and y-axis is (0,5)

For x-intercept, we put $y = 0$, and get $5x + 10(0) = 50 \Rightarrow x = 10$

Intersection point of line and x-axis is (10, 0)

For origin, we put $x = 0$ and $y = 0$ in (1)

we get , $0 \leq 50$ which is true

So, origin is a part of solution of inequality (1)

Inequality (2)

For y-intercept, put $x = 0$ in (5)

we get $y = 8$

Intersection point of line and y-axis is (0,8)

For x-intercept, we put $y = 0$ in (5)

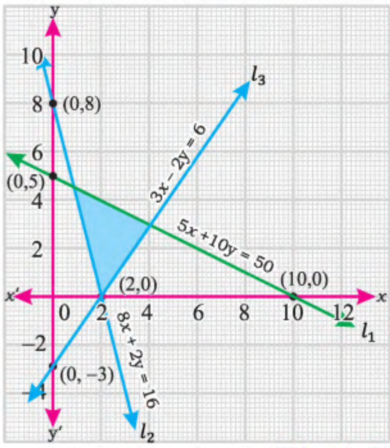
and get $x = 2$

Intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (2)

We get , $0 \geq 16$ which is not true

So, origin is not a part of solution of inequality (2)



(Fig. 9.4)

Inequality (3)

For y-intercept, we put $x = 0$ in (6) and get $3(0) - 2y = 6 \Rightarrow y = -3$

Intersection point of line and y-axis is $(0, -3)$

For x-intercept, we put $y = 0$ in (6) and get $3x - 2(0) = 6 \Rightarrow x = 2$

Intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (3)

We get , $0 \leq 6$ which is true.

So, origin is a part of solution of inequality (3)

We draw the lines by plotting the above intersection points on the graph. The region bounded by the intersection of three inequalities is shaded (Fig. 9.4) which is the required solution satisfied by all the inequalities.

Since coefficient y are positive. Therefore

Note: First inequality is of type (≤) so, solution region is below the line l_1.

Second inequality is of type (≥) so, solution region is above the line l_2.

Third inequality is of type (≤) so, solution region is below the line l_3.

Exercise 9.1

1. Solve the following inequalities and represent solution on number line in each case.

(i) $2x + 5 < 9,\ \forall\, x \in \mathbb{R}$ (ii) $3x - 1 > 10, \forall\, x \in \mathbb{R}$

(iii) $4x - 5 \leq 3 + 2x, \forall\, x \in \mathbb{R}$ (iv) $6 < 3(x + 2) < 21, \forall\, x \in \mathbb{R}$

2. Draw graph of the following linear inequalities.

(i) $2x - y \leq 7$ (ii) $3x + 4y \geq 10$

(iii) $x - 2y \geq 5$ (iv) $2x + y \leq 4$

3. Solve the following system of linear inequalities graphically.

(i) $3x - y \geq 8$, $2x + 3y \geq 5$

(ii) $3x - 2y \leq 6$, $x - y \leq 4$

4. Draw graph of the solution region of the following system of linear inequalities. Also check whether the graph is bounded or not.

(i) $2x + y \leq 5$, $3x - 2y \leq 7$, $x \geq 0$

(ii) $x + 2y \leq 6$, $2x - 3y \geq 8$, $x + 2y \leq 4$

9.3 Feasible Region

In linear programming problems, a feasible region is the set of all possible points that satisfy the problem's constraints, including inequalities and equalities.

9.3.1 Define

- **linear programming problem,**
- **objective function,**
- **problem constraints,**
- **decision variables.**

(i) Linear programming problem

A linear programming problem (LP problem) is one that is concerned with finding the optimal value (maximum or minimum) of a linear function (called objective function) of several variables, subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).

The term "linear" indicates that all the mathematical relations used in the problem are linear relations while the term "programming" refers to the method of determining a particular programme or plan of action.

(ii) Objective function

An objective function expresses the main aim of the LP problem which is either to be minimized or maximized. The objective function evaluates some quantitative criterion of immediate importance such as cost, profit, utility, or yield. In an LP problem, an objective function $Z = ax + by$, where a, b are constants, has to be maximized or minimized subject to the given linear inequalities. To find the Maximum Z or Minimum Z, we convert the inequalities into equations.

(iii) Problem Constraints

The linear inequalities or equations or restrictions on the variables of a linear programming problem are called problem constraints. The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions.

In LP problem, the constraints are equal or unequal mathematical sentences defining limitations on decisions. Constraints arise from a variety of sources such as limited resources, contractual obligations, or physical laws etc.

(iv) Decision variables

Decision variables describe the quantities that the decision makers would like to determine. They are the unknowns of a mathematical programming problem. In general, decision variables are represented by x, y, z etc.,. These variables are required to be non-negative.

9.3.2 Define and show graphically the feasible region (or solution space) of an LP problem

The common region determined by all the constraints including non-negative constraints $(x \geq 0, y \geq 0)$ of a linear programming problem is called the feasible region (or solution space) for the problem. Points within and on the boundary of the feasible region represent feasible solutions of the constraints. The feasible solution region on the graph is the one which is satisfied by all the constraints.

The method of showing the feasible region of an LP problem is illustrated by the following example.

Example:

Find the feasible region or solution space of the following inequalities

$5x + y \leq 100$; $x + y \leq 60$; $x \geq 0,\ y \geq 0$

Solution:

First, we convert the inequalities into equations as follows:

$$5x + y = 100 \qquad \text{...(i)}$$

$$x + y = 60 \qquad \text{...(ii)}$$

$$x \geq 0, y \geq 0$$

We consider equation (i)

For y-intercept, we put $x = 0$ and get $5(0) + y = 100 \Rightarrow y = 100$

Therefore, intersection point of line and y-axis is (0,100)

For x-intercept, we put $y = 0$ and get $5x + 0 = 100 \Rightarrow x = 20$

Therefore, intersection point of line and x-axis is (20, 0)

For origin, we put $x = 0$ and $y = 0$ in $5x + y \leq 100$

We get , $0 \leq 100$ which is true.

So, origin is a part of solution of inequality $5x + y \leq 100$

Now, we consider equation (ii)

For y-intercept, we put $x = 0$ and get

$0 + y = 60 \Rightarrow y = 60$

Therefore, intersection point of line and y-axis is (0, 60)

For x-intercept, we put $y = 0$ and get

$x + 0 = 60 \Rightarrow x = 60$

Therefore, intersection point of line and x-axis is (60,0)

For origin, we put $x = 0$ and $y = 0$ in $x + y \leq 60$

We get , $0 \leq 60$ which is true.

(Fig. 9.5)

So, origin is a part of solution of inequality $x + y \leq 60$

We draw the lines by plotting the above points in the graph. The shaded region OABC formed by the intersection of these inequalities is the feasible region (or solution space) for the given inequalities as shown in Fig. 9.5.

9.3.3 Identify the feasible region of simple LP problems

The method of identification of feasible region of simple LP problems is explained by the following examples.

Example 1. Find the feasible region of the following LP problem

Maximize $Z = 2x + 3y$

subject to $2x + y \leq 100; \quad x + y \leq 80; \quad x \geq 0, y \geq 0$

Solution:

We first, consider the constraints as equations as follows:

$$2x + y = 100 \quad \ldots(i)$$
$$x + y = 80 \quad \ldots(ii)$$
$$x \geq 0, y \geq 0$$

To get the intersection points with the coordinate axes.

We consider equation (i)

For y-intercept, we put $x = 0$ and get $2(0) + y = 100 \Rightarrow y = 100$

Therefore, intersection point of line and y-axis is (0,100)

For x-intercept, we put $y = 0$ and get

$2x + 0 = 100 \Rightarrow x = 50$

Therefore, intersection point of line and x-axis is (50, 0)

For origin, we put $x = 0$ and $y = 0$ in

$2x + y \leq 100$

We get , $0 \leq 100$ which is true.

So, origin is a part of solution of inequality $2x + y \leq 100$

Now, we consider equation (ii)

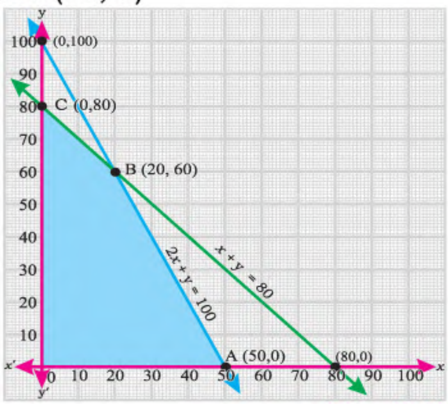
For y-intercept, we put $x = 0$ and get

$0 + y = 80 \Rightarrow y = 80$

Intersection point of line and y-axis is (0,80)

For x-intercept, we put $y = 0$ and get

$x + 0 = 80 \Rightarrow x = 80$



(Fig. 9.6)

So, the intersection point of line and x-axis is $(80, 0)$

For origin, we put $x = 0$ and $y = 0$ in $x + y \leq 80$

We get , $0 \leq 80$ which is true.

So, origin is a part of solution of inequality $x + y \leq 80$

By plotting the above points and drawing the lines in the graph, the shaded region OABC formed by the intersection of these inequalities, is the feasible region for the given inequalities as shown in Fig. 9.6.

Example 2. Find the feasible region of the following LP problem

Maximize Z= 7x +3y

subject to $6x + 2y \geq 12$; $2x + 2y \geq 8$; $4x + 12y \geq 24$; $x, y \geq 0$

Solution:

Inequalities can be written in simplified form as:

$$3x + y \geq 6, \qquad \ldots(i)$$

$$x + y \geq 4 \qquad \ldots(ii)$$

and

$$x + 3y \geq 6 \qquad \ldots(iii)$$

First, we consider the inequalities (i),(ii) and (iii) as equations

$$3x + y = 6 \qquad \ldots(iv)$$

$$x + y = 4 \qquad \ldots(v)$$

$$x + 3y = 6 \qquad \ldots(vi)$$

$$x \geq 0, y \geq 0$$

To get the intersection points with the coordinate axes:

Now, we consider equation (iv)

For y-intercept, we put $x = 0$ and get $3(0) + y = 6 \Rightarrow y = 6$

Therefore, intersection point of line and y-axis is (0,6)

For x-intercept, we put $y = 0$,

we get $3x + (0) = 6 \quad \Rightarrow x = 2$

Therefore, intersection point of line and x-axis is (2, 0)

For origin, we put $x = 0$ and $y = 0$ in (i), we get $0 \geq 6$ which is not true.

So, origin is not a part of solution of inequality (i)

Now, we consider equation (v)

For y-intercept, we put $x = 0$ and get

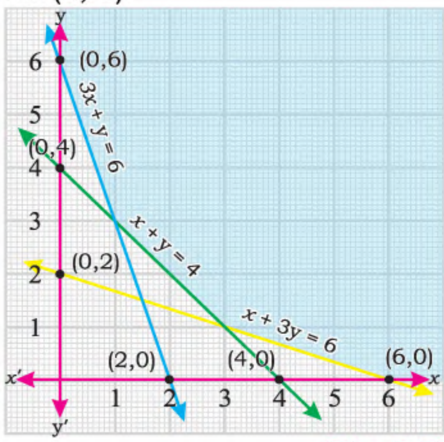
$0 + y = 4 \quad \Rightarrow y = 4$

Therefore, intersection point of line and y-axis is (0,4)

For x-intercept, we put $y = 0$ and get

$x + 0 = 4 \quad \Rightarrow x = 4$

Therefore, intersection point of line and x-axis is (4,0)



(Fig. 9.7)

For origin, we put $x = 0$ and $y = 0$ in (ii), and get $0 \geq 4$ which is not true.

So, origin is not a part of solution of inequality (ii)

Now, we consider equation (vi)

For y-intercept, put $x = 0$, we get $3y = 6 \quad \Rightarrow y = 2$

Therefore, intersection point of line and y-axis is (0,2)

For x-intercept, we put $y = 0$ and get $x + 3(0) = 6 \quad \Rightarrow x = 6$

Therefore, intersection point of line and x-axis is (6,0)

For origin, we put $x = 0$ and $y = 0$ in (iii), we get $0 \geq 6$ which is not true.

So, origin is not a part of solution of inequality (iii)

By plotting the above points and drawing the lines in the graph, the shaded region formed by the intersection of these inequalities is the feasible region (or solution space) for the given inequalities as shown in Fig. 9.7.

Example 3. Find the feasible region and also find its corner points for the following LP problem

Minimize $Z = x - 9y$

subject to $2x + 3y \leq 48$; $x \leq 15$; $y \leq 10$; $x, y \geq 0$

Solution:

First, we consider the constraints as equations and find intersection points with coordinate axes as under.

Constraint 1: $2x + 3y = 48$

When $x = 0$, we get $y = 16$

and $y = 0$, we get $x = 24$

The points of intersection of line with axes are (0, 16) and (24, 0).

For origin, we put $x = 0$ and $y = 0$ in $2x + 3y \le 48$

We get, $0 \le 48$ which is true.

So, origin is a part of solution of inequality $2x + 3y \le 48$

We draw the line through $(0, 16)$ and $(24, 0)$.

Constraint 2: $x = 15$

The point of intersection of line with x-axis is $(15, 0)$

For origin, we put $x = 0$, in $x \le 15$ we get,

$0 \le 15$ which is true. So, origin is a part of solution of inequality $x \le 15$

We draw the line: $x = 15$ through $(15, 0)$.

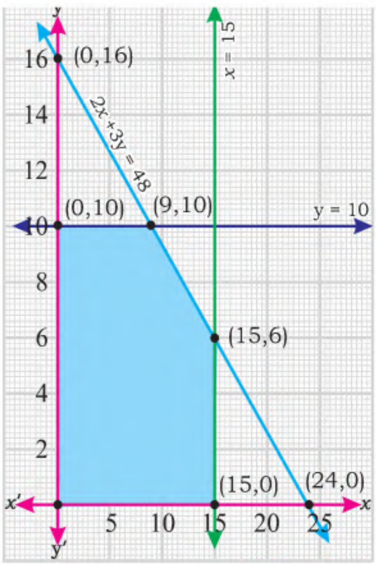
Constraint 3: $y = 10$

The point of intersection of line with y-axis is $(0, 10)$

For origin, we put $y = 0$, in $y \le 10$ we get,

$0 \le 10$ which is true. So, origin is a part of solution of inequality $y \le 10$

We draw the line: $y = 10$ through $(0, 10)$



(Fig. 9.8)

The shaded region as shown in Fig. 9.8 is the feasible region for the given inequalities, with the corner points (0,0), (15,0) (15,6), (9,10) and (0,10).

Exercise 9.2

1. Draw graph of the following system of linear inequalities and identify the feasible region and the corner points.

(i) $2x - 3y \le 8$, $3x + 2y \le 10$, $x \ge 0, y \ge 0$

(ii) $-5x + 4y \ge 10$, $2x + y \le 8$, $x \ge 0, y \ge 0$

(iii) $4x + 3y \le 12$, $3x - 2y \le 9$, $x \ge 0, y \ge 0$

2. Draw graph of the following inequalities and also identify the solution space and the corner points.

(i) $5x - y \le 16$, $3x + 2y \le 6$, $x + 2y \le 5$, $x \ge 0, y \ge 0$

(ii) $5x + 3y \le 12$, $2x - 3y \le 9$, $x + 2y \le 4$, $x \ge 0, y \ge 0$

(iii) $6x + 5y \ge 14$, $4x + 2y \le 10$, $3x - 2y \le 8$, $x \ge 0, y \ge 0$

9.4 Optimal Solution

9.4.1 Define optimal solution of an LP problem

Feasible solution: A feasible solution to a linear programming problem is a solution that satisfies all constraints.

Optimal solution: An optimal solution to a linear programming problem is the feasible solution with the largest objective function value for a maximization problem and the smallest objective function value for a minimization problem.

9.4.2 Find optimal solution (graphical) through the following systematic procedure:

- establish the mathematical formulation of LP problem,
- construct the graph,
- identify the feasible region,
- locate the solution points,
- evaluate the objective function,
- select the optimal solution,
- verify the optimal solution by actually substituting values of variables from the feasible region.

In order to find the optimal solution of Linear Programming problem the above mentioned steps are to be followed as explained in the following examples.

Example 1. Find an optimal solution of the following LP problem.

Maximize the objective function $z = f(x, y) = 5x + 7y$,

Subject to the constraints $x \leq 6$; $2x + 3y \leq 19$; $x + y \leq 8$; and $x, y \geq 0$

Solution:

Step 1. Mathematical formulation of LP problem

The problem is already in the form of mathematical formulation

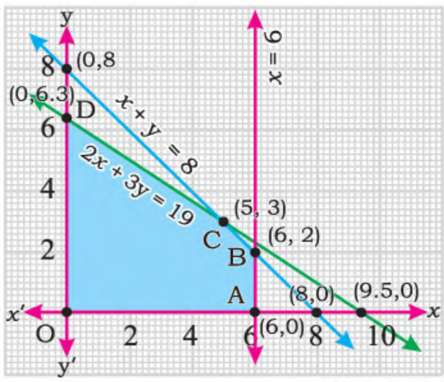
i.e., Maximize $z = 5x + 7y$

Subject to the constraints $x \leq 6$;

$2x + 3y \leq 19$; $x + y \leq 8$; and $x, y \geq 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes and draw lines.



(Fig. 9.9)

Constraint 1: $x \leq 6$

Consider the constraint as an equation i.e., $x = 6$, Draw the line for $x = 6$. The line is parallel to y-axis through $(6, 0)$.

Constraint 2: $\mathbf{2x + 3y \leq 19}$

Consider the constraint as an equation

$2x + 3y = 19$

When $x = 0$, we get $y = 6.33$. and

when $y = 0$, we get $x = 9.5$.

The intersection points are (0, 6.33) and (9.5, 0). We draw line through these points.

Constraint 3: $x + y \leq 8$

Consider the constraint as an equation $x + y = 8$

When $x = 0$, we get $y = 8$.

and for $y = 0$, we get $x = 8$.

The points of intersection are

(0, 8) and (8, 0).

We draw line through these points.

Step 3. Identify the feasible region

The intersection of three linear inequalities is the required feasible region OABCD which is the shaded area in the graph as shown in Fig. 9.9.

Step 4. Locate the solution points

The solution points or corner points of region OABCD are O(0,0), A (6,0), B (6,2), C(5,3) and D (0,6.33) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x, y) = 5x + 7y$
O (0, 0)	$f(0,0) = 5(0) + 7(0) = 0 + 0 = 0$
A (6, 0)	$f(6,0) = 5(6) + 7(0) = 30 + 0 = 30$
B (6, 2)	$f(6,2) = 5(6) + 7(2) = 30 + 14 = 44$
C (5, 3)	$f(5,3) = 5(5) + 7(3) = 25 + 21 = 46$
D (0, 6.33)	$f(0,6.33) = 5(0) + 7(6.33) = 0 + 44.31 = 44.31$

Step 6. Select the Optimal Solution

From the above table, as the maximum value of the objective function is 46 at the point (5, 3). Therefore, the optimal solution to the given LP problem is: $f_{maximum} = 46$; $x = 5,\ y = 3$

Step 7. Verify the optimal solution

For the optimal solution $(5, 3)$,

1st constraint $x \le 6$ becomes, $5 \le 6$ which is true.

2nd constraint $2x + 3y \le 19$ becomes, $19 \le 19$ which is true.

3rd constraint $x + y \le 8$ becomes, $8 \le 8$ which is true.

∵ All the constraints are satisfied by the optimal solution.

∴ it is verified.

Example 2. Find the optimal solution of the following LP problem:

Maximize $z = 6x - 8y$

Subject to $30x + 20y \le 300$; $5x + 10y \le 110$; and $x, y \ge 0$

Solution:

Step 1. Mathematical formulation of LP problem

The mathematical formulation is already given as

Maximize $z = 6x - 8y$

Subject to $30x + 20y \le 300$; $5x + 10y \le 110$; and $x, y \ge 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes.

Constraint 1. $30x + 20y \le 300$

We consider it as an equation $30x + 20y = 300$

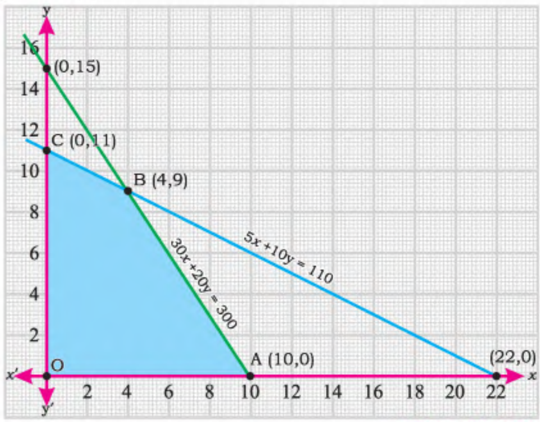
When $y = 0$, we get $x = 10$

The intersection point of line with x-axis is (10, 0)

When $x = 0$, we get $y = 15$

The intersection point of line with y-axis is (0, 15)

We draw line through these points.



(Fig. 9.10)

Constraint 2. $5x + 10y \le 110$

We consider it as an equation

$5x + 10y = 110$

When $y = 0$, we get $x = 22$

The intersection point of line with x-axis is (22, 0)

When $x = 0$, we get $y = 11$

The intersection point of line with y-axis is (0, 11)

We draw line through these points.

Step 3. Identify the feasible region

The intersection of two linear inequalities is the required feasible region OABC which is the shaded area in the graph as shown in Fig. 9.10.

Step 4. Locate the solution points

The solution points or corner points of the region OABC are O (0,0), A (10,0), B (4,9) and C (0,11) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x, y) = 6x - 8y$
O (0, 0)	$f(0,0) = 6(0) - 8(0) = 0 - 0 = 0$
A (10, 0)	$f(10,0) = 6(10) - 8(0) = 60 - 0 = 60$
B (4, 9)	$f(4,9) = 6(4) - 8(9) = 24 - 72 = -48$
C (0, 11)	$f(0,11) = 6(0) - 8(11) = 0 - 88 = -88$

Step 6. Select the Optimal Solution

From the above table, as the maximum value of the objective function is 60 at the point (10, 0). Therefore the optimal solution to the given LP problem is: $f_{maximum} = 60$; $x = 10, \ y = 0$

Step 7. Verify the optimal solution

For the optimal solution $(10, 0)$,

1st constraint $30x + 20y \leq 300$ becomes, $300 \leq 300$ which is true.

2nd constraint $5x + 10y \leq 110$ becomes, $50 \leq 110$ which is true.

$\because$ All the constraints are satisfied by the optimal solution.

$\therefore$ it is verified.

Example 3. Find the optimal solution of the following LP problem:

Minimize $z = 3x + 4y$

Subject to the constraints

$2x + 3y \geq 6$; $x + y \leq 8$ and $x \geq 0, \ y \geq 0$

Solution:

Step 1. Mathematical formulation of LP problem

The mathematical formulation is as already given

Minimize $Z = 3x + 4y$

Subject to $2x + 3y \geq 6$; $x + y \leq 8$ and $x \geq 0, \ y \geq 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes.

Constraint 1. $2x + 3y \geq 6$

We consider it as an equation $2x + 3y = 6$

When $y = 0$, we get $x = 3$

The intersection point of line with x-axis is (3, 0)

When $x = 0$, we get $y = 2$

The intersection point of line with y-axis is (0, 2)

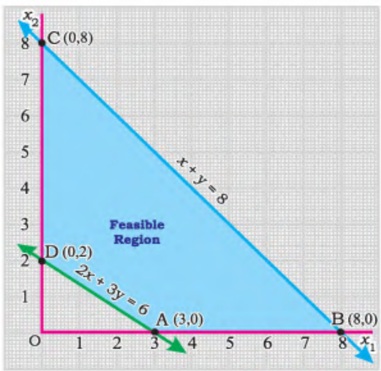
We draw line through these points.

Constraint 2. $x + y \leq 8$

We consider it as an equation $x + y = 8$

When $y = 0$, we get $x = 8$

The intersection point of line with x-axis is (8, 0)



(Fig. 9.11)

When $x = 0$, we get $y = 8$
The intersection point of line with y-axis is (0, 8)
We draw line through these points.

Step 3. Identify the feasible region

The intersection of two linear inequalities is the required feasible region ABCD which is the shaded area in the graph as shown in Fig. 9.11.

Step 4. Locate the solution points

The solution points or corner points of the region ABCD are, A (3,0), B (8,0), C (0,8) and D (0,2) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $Z = f(x, y) = 3x + 4y$
A (3, 0)	$f(3,0) = 3(3) + 4(0) = 9 + 0 = 9$
B (8, 0)	$f(8,0) = 3(8) + 4(0) = 24 + 0 = 24$
C (0, 8)	$f(0,8) = 3(0) + 4(8) = 0 + 32 = 32$
D (0, 2)	$f(0,2) = 3(0) + 4(2) = 0 + 8 = 8$

Step 6. Select the Optimal Solution

From the above table, as the minimum value of the objective function is 8 at the point (0, 2). Therefore the optimal solution to the given LP problem is: $f_{minimum} = 8$; $x = 0, \ y = 2$

Step 7. Verify the optimal solution

For the optimal solution $(0, 2)$,
1st constraint $2x + 3y \geq 6$ becomes, $6 \geq 6$ which is true.
2nd constraint $x + y \leq 8$ becomes, $2 \leq 8$ which is true.
$\because$ All the constraints are satisfied with the optimal solution.
$\therefore$ it is verified.

9.4.3 Solve real life simple LP problems

Following are some examples of real-life simple LP problems which are solved through graphical method.

Example 1. A company has two flour mills, A and B, which have different capacities for high, medium and low-grade flour. This company has to supply flour to a firm every week 12, 8 and 24 quintals. (1 quintal = 100 kg) of high, medium and low grade respectively. It costs the company Rs.1000 and Rs. 800 per day to run mill A and mill B respectively.

On a day, mill A produces 6, 2 and 4 quintals and mill B produces 2, 2 and 12 quintals of high, medium and low-grade flour respectively.

How many days per week each mill is operated in order to supply the flour to the firm most economically?

Solution:

Step 1. Mathematical formulation of LP problem

The problem can be formulated in the following tabular form:

Flour Production	Capacity		Requirement
	Mill A	Mill B	
High grade	6	2	12
Medium grade	2	2	8
Low grade	4	12	24
Cost (Rs)/day	1000	800	

Let x be the number of days per week mill A operates.

Let y be the number of days per week mill B operates.

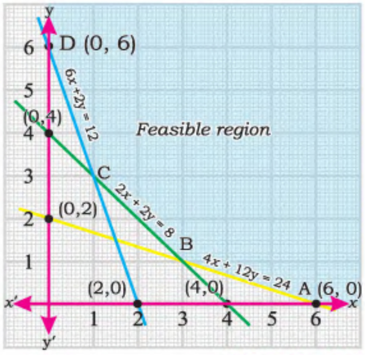
The objective is to minimize the total cost of operation and to find the values of x and y.

Now, we can write mathematically LP problem as follows:

Minimize $f = 1000\,x + 800\,y$

Subject to $6x + 2y \geq 12$; $2x + 2y \geq 8$; $4x + 12y \geq 24$; $x, y \geq 0$

Step 2. Construct the graph



(Fig. 9.12)

For the graph, first we find intersection points of the constraints with axes.

Constraint 1. $6x + 2y \geq 12$

We consider it as an equation $6x + 2y = 12$

when $x = 0$, we get $y = 6$ and

when $y = 0$, we get $x = 2$

The intersection points of line with axes are (0, 6) and (2, 0).

We draw line through these points.

Constraint 2. $2x + 2y \geq 8$

We consider it as an equation $2x + 2y = 8$

when $x = 0$, we get $y = 4$ and

when $y = 0$, we get $x = 4$

The intersection points of line with axes are (0, 4) and (4, 0).

We draw line through these points.

Constraint 3. $4x + 12y \geq 24$

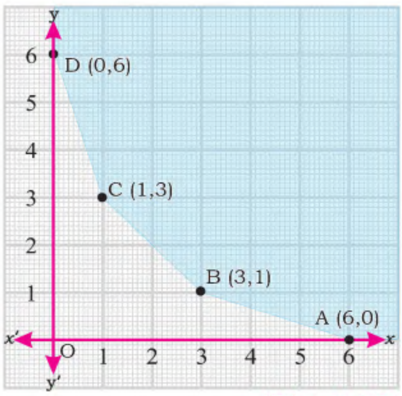
We consider it as an equation $4x + 12y = 24$

when $x = 0$, we get $y = 2$ and

when $y = 0$, we get $x = 6$

The intersection points of line with axes are (0, 2) and (6, 0).

We draw line through these points.



(Fig. 9.13)

Step 3. Identify the feasible region

The intersection of three linear inequalities is the required feasible region ABCD which is the shaded area in the graph as shown in Fig. 9.12 and Fig. 9.13.

Step 4. Locate the solution points

The solution points or corner points ABCD are A (6, 0), B (3,1), C (1,3) and D (0, 6) in the graph.

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x, y) = 1000x + 800y$
A (6, 0)	$f(6,0) = 1000(6) + 800(0) = 6000 + 0 = 6000$
B (3, 1)	$f(3,1) = 1000(3) + 800(1) = 3000 + 800 = 3800$
C (1, 3)	$f(1,3) = 1000(1) + 800(3) = 1000 + 2400 = 3400$
D (0, 6)	$f(0,6) = 1000(0) + 800(6) = 0 + 4800 = 4800$

Step 6. Select the Optimal Solution

We note that the minimum cost at point C = (1, 3) is Rs. 3400. The company operates mill A for one day per week and mill B for three days per week to supply flour to a firm most economically by paying Rs. 3400.

Therefore, the optimal solution to the given LP problem is

$f_{minimum} = 3400$ Rupees; $x = 1,\ y = 3$

Step 7. Verify the optimal solution

For the optimal solution $(1, 3)$,

1st constraint $6x + 2y \geq 12$ becomes, $12 \geq 12$ which is true.

2nd constraint $2x + 2y \geq 8$ becomes, $8 \geq 8$ which is true.

3rd constraint $4x + 12y \geq 24$ becomes, $40 \geq 24$ which is true.

$\because$ All the constraints are satisfied by the optimal solution.

$\therefore$ it is verified.

Example 2. A workshop has three types of machines A, B and C; it can manufacture two products 1 and 2, and all products have to go to each machine and each one goes in the same order; First to the machine A, then to B and then to C. The following table shows:

- The hours needed at each machine, per product unit
- The total available hours for each machine, per week
- The cost of each product per unit sold

Find maximum profit under given constraints.

Type of Machine	Product 1	Product 2	Available hours per week
A	2	2	16
B	1	2	12
C	4	2	18
Profit per unit	Rs. 1	Rs. 1.5	

Solution:

Step 1. Mathematical formulation of LP problem

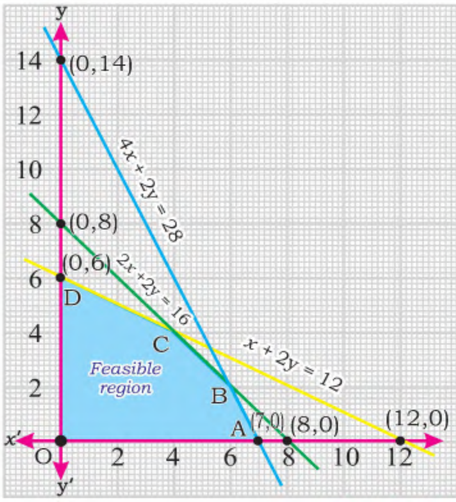
Let x be the units produced weekly of the Product 1
and y be the units produced weekly of the Product 2
Now, we can write mathematically LP problem as follows:

Minimize $f = x + 1.5\,y$

Subject to $2x + 2y \leq 16$; $x + 2y \leq 12$; $4x + 2y \leq 28$; $x, y \geq 0$

Step 2. Construct the graph

For the graph, first we find intersection points of the constraints with axes.



(Fig. 9.14)

Constraint 1. $2x + 2y \leq 16$

We consider it as an equation $2x + 2y = 16$
when $x = 0$, we get $y = 8$ and
when $y = 0$, we get $x = 8$
The points of intersection of line with axes are (0, 8) and (8, 0).
We draw line through these points.

Constraint 2. $x + 2y \leq 12$

We consider it as an equation $x + 2y = 12$
when $x = 0$, we get $y = 6$ and
when $y = 0$, we get $x = 12$
The intersection points of line with axes are (0, 6) and (12, 0).
We draw line through these points.

Constraint 3. $4x + 2y \leq 28$

We consider it as an equation $4x + 2y = 28$
when $x = 0$, we get $y = 14$ and
when $y = 0$, we get $x = 7$
The points of intersection of line with axes are (0,14) and (7,0). We draw line trough these points.

Step 3. Identify the feasible region

The intersection of three linear inequalities is the required feasible region OABCD which is the shaded area in the graph as shown in Fig. 9.14 and Fig. 9.15.

Step 4. Locate the solution points

In the graph, the solution points or corner points of the region OABCD are O(0,0), A (7, 0), B (6, 2), C (4, 4) and D (0, 6).

Step 5. Evaluate the objective function

Solution points or Corner points	Objective function $f(x,y) = 1x + 1.5y$
O (0, 0)	$f(0,0) = 1(0) + 1.5(0) = 0 + 0 = 0$
A (7, 0)	$f(7,0) = 1(7) + 1.5(0) = 7 + 0 = 7$
B (0, 8)	$f(6,2) = 1(6) + 1.5(2) = 6 + 3 = 9$
C (4, 4)	$f(4,4) = 1(4) + 1.5(4) = 4 + 6 = 10$
D (0, 6)	$f(0,6) = 1(0) + 1.5(6) = 0 + 9 = 9$

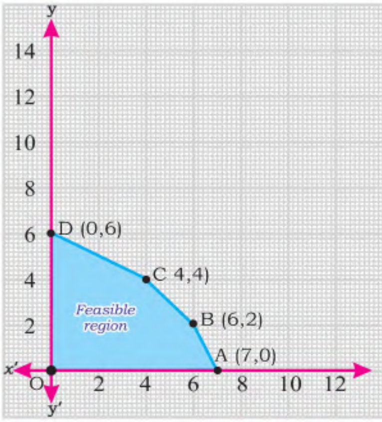
Step 6. Select the Optimal Solution

We note that the maximum profit at the point C = (4,4) is Rs. 10

Therefore, the optimal solution to the given LP problem is

$f_{maximum} = 10$ rupees $x = 4, \ y = 4$

$x = 4$ units of product 1 and $y = 4$ units of product 2.



(Fig. 9.15)

Step 7. Verify the optimal solution

For the optimal solution (4,4),

1st constraint $2x + 2y \leq 16$ becomes $16 \leq 16$ which is true.

2nd constraint $x + 2y \leq 12$ becomes, $12 \leq 12$ which is true.

3rd constraint $4x + 2y \leq 28$ becomes, $24 \leq 28$ which is true.

$\because$ All the constraints are satisfied by the optimal solution.

$\therefore$ it is verified.

Exercise 9.3

1. Solve the following linear programming problems by graphical method when $x \geq 0, y \geq 0$

(i) Maximize $Z(x,y) = 10x + 11y$

Subject to: $2x + 3y \leq 8$; $6x + 3y \leq 10$

(ii) Maximize $Z(x,y) = 30x + 36y$

Subject to: $4x + 2y \leq 12$; $6x + 5y \leq 20$

(iii) Maximize $Z(x,y) = 41x + 38y$

Subject to: $4x + 5y \leq 26$; $8x + 5y \leq 22$; $5x + 2y \leq 10$

(iii) Maximize $Z(x,y) = 41x + 38y$

Subject to: $4x + 5y \leq 26$; $8x + 5y \leq 22$; $5x + 2y \leq 10$

(iv) Maximize $Z = 4x + y$

Subject to: $x + y \leq 50$; $3x + y \leq 90$; $x - y \leq 40$

(v) Minimize $Z = 200x + 500y$

Subject to the constraints: $x + 2y \geq 10$; $3x + 4y \leq 24$

(vi) Minimize $Z = 3x + 2y$

Subject to the constraints: $2x + y \geq 6$; $x + y \geq 4$

(vii) Minimize $Z = 0.12x + 0.15y$

Subject to the constraints: $x + y \geq 5$; $2x + y \geq 6$; $x + 3y \geq 9$

2. A landlord has 50 hectares of land to grow two crops X and Y with profits per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide no more than 800 litres, is to be used for crops X and Y at rates of 20 liters and 10 liters per hectare. How much land should be allocated to each crop so as to maximize the total profit.

3. A toy factory manufactures two types of toys A and B and sells for Rs.25 and Rs.20 respectively. The toys A and B respectively, require 20 units and 12 units daily out of 2000 available resource units. Both require a production time of 5 minutes and total working hours are 9 per day. What should be the quantity for each of the toys to maximize the selling amount?

4. At a university, Professor Abdul Sattar wishes to employ two people, Farhan and Sarfaraz, to grade papers for his classes. Farhan is a graduate student and can grade 20 papers per hour; Farhan earns $15 per hour for grading papers. Sarfaraz is a post-doctoral associate and can grade 30 papers per hour; Sarfaraz earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment. If Prof. Abdul Sattar has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Review Exercise 9

1. **Select correct answer.**

i. If $2x - 5 \geq 3$ then its solution set is __________, where $x \in \mathbb{Z}$.

(a) $\{2, 3, 4, ...\}$ (b) $\{4, 5, ...\}$ (c) $\{2, 1, 0, ...\}$ (d) $\{1, 0, -1, ...\}$

ii. If $4x - 3y = 8$ is the corresponding equation of an inequality $4x - 3y \leq 8$ then its x-intercept is ________.

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 2 (d) $\frac{1}{2}$

iii. $\{x | x \in \mathbb{R} \wedge x > 5\} =$ ___________.

(a) $[5, \infty)$ (b) $(-\infty, 5)$ (c) $(5, \infty)$ (d) $(-\infty, 5]$

iv. Linear Programming (LP) is used to obtain _______solution.

(a) Feasible (b) Trivial (c) Optimal (d) Infeasible

v. Solution space of the linear inequality $2x + 3y \leq 6, \forall x, y \in \mathbb{R}$ includes ____.

(a) All points above the line (b) All points on and below the line

(c) All points below the line (d) All points on and above the line

vi. If $f(x, y) = 5x + 4y$ is an objective function and the corner points of the feasible region are $(5, 4), (0, 0)$, $(4, 0)$ and $(0, 6)$ then the function is maximum at:

(a) (4,0) (b) (0,6) (c) $(0, 0)$ (d) $(5, 4)$

vii. _____ are the entities whose values are to be determined from the solution of the LP problem.

(a) Objective function (b) Decision Variables

(c) Constraints (d) Opportunity cost

viii. __________ specifies the objective or goal of solving the LP problem.

(a) Objective function (b) Decision Variables

(c) Constraints (d) Opportunity cost

ix. _______ are the restrictions or limitations imposed on the LP problem:

(a) Variables (b)Costs (c) Profits (d)Constraints

x. The region of solution in LP problem is called ____.

(a) Infeasible region (b) Unbounded region

(c) Infinite region (d) Feasible region

xi. In case of an '______' constraint, the feasible region is a straight line.

(a) Less than or equal to (b) Greater than or equal to

(c) Mixed (d) Equal to

xii. $ax + b \leq 0,\ a \neq 0,\ a, b \in \mathbb{R}$, is called ____________.

(a) Non-linear inequality (b) Linear inequality

(c) Linear equality (d) Complex inequality

xiii. A point of a solution region where two of its boundary lines intersect, is called ____________.

(a) Middle point (b) Origin

(c) Corner point (d) Feasible point

xiv. The solution region of an inequality restricted to the first quadrant is called __________ region.

(a) Combined (b) Unbounded

(c) Infeasible (d) Feasible

xv. A function which is to be maximized or minimized is called ________.

(a)Linear function (b) Equal function

(c) Objective function (d) Non linear function

xvi. The feasible solution which maximizes or minimizes the objective function is called the __________.

(a) Optimal solution (b) Corner solution

(c) Initial solution (d) Complex solution

2. Draw graph of the following system of linear inequalities and find feasible region.

$$2x + 3y \geq 10; \quad 3x + 2y \geq 12; \quad x \geq 0, y \geq 0$$

3. Find the feasible region graphically subject to the following constraints and also find its corner points.

$$4x + y \geq 9; \quad 2x + 3y \leq 14; \quad x - y \leq 5; \quad x \geq 0, y \geq 0$$

4. Find the maximum and minimum values of the function $f(x,y) = 4x - 3y$ subject to the constraints

$$2x + 3y \geq 5; \quad 3x + 2y \leq 12; \quad x \geq 0, y \geq 0$$

5. A dealer wishes to purchase some of fans and sewing machines. He has only ¥ 5760 to invest and space of at most 20 items. The costs of a fan and a sewing machine are ¥ 360 and ¥ 240 respectively. He wants profit on a fan and a sewing machine ¥ 22 and ¥ 18 respectively. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

TRIGONOMETRIC IDENTITIES OF SUM AND DIFFERENCE OF ANGLES

Unit 10

10.1 Fundamental Law of Trigonometry

10.1.1 Recall trigonometric ratios

We have already studied trigonometric ratios in pervious classes. Let us recall. "The ratios of the lengths of sides of a right-angled triangle, are called trigonometric ratios."

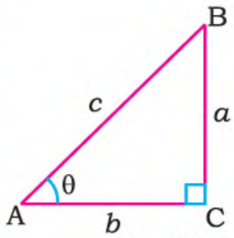


Fig. 10.1

There are six trigonometric ratios, namely sine, cosine, tangent, cotangent, secant and cosecant. These six trigonometric ratios are abbreviated as sin, cos, tan, cot, sec, and cosec.

Consider a right-angled triangle ABC in which $m < C = 90°$ and $m < A = \theta$ as shown in Fig. 10.1.

Trigonometric ratios for acute angle θ, are defined as under.

1. $\sin\theta = \dfrac{\text{length of opposite side of }\theta}{\text{length of hypotenuse}} = \dfrac{a}{c}$
2. $\cos\theta = \dfrac{\text{length of adjacent side of }\theta}{\text{length of hypotenuse}} = \dfrac{b}{c}$
3. $\tan\theta = \dfrac{\text{length of opposite side of }\theta}{\text{length of adjacent side of }\theta} = \dfrac{a}{b}$
4. $\cot\theta = \dfrac{\text{length of adjacent side of }\theta}{\text{length of opposite side of }\theta} = \dfrac{b}{a}$
5. $\sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side of }\theta} = \dfrac{c}{b}$
6. $\operatorname{cosec}\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side of }\theta} = \dfrac{c}{a}$

We have also studied in previous class that these ratios become trigonometric functions when θ is any real number representing measure (in radians) of an angle in standard position in unit circle and $P(x, y)$ be any point on the circle as shown in Fig. 10.2.

Trigonometric functions for any angle θ are defined as under.

$\sin\theta = y$, $\cos\theta = x$,

$\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$,

$\sec\theta = \frac{1}{x}$, $\operatorname{cosec}\theta = \frac{1}{y}$.

Where $m\overline{OA} = x$ and $m\overline{AP} = y$ in right angled ΔOAP.

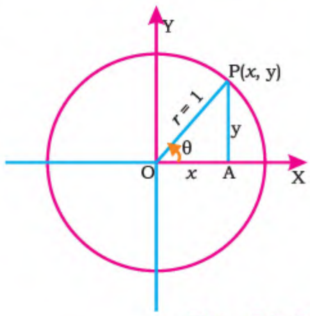


Fig. 10.2

10.1.2 Use distance formula to establish fundamental law of trigonometry

- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and deduce that
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Recall that the distance between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is found by the formula known as distance formula:

$$|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Fundamental Law $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Consider a unit circle with centre at O(0,0) as shown in Fig 10.3.

Let $P(\cos\beta, \sin\beta)$ and $Q(\cos\alpha, \sin\alpha)$ be any two points on the unit circle.

Then by using distance formula

$$|\overline{PQ}| = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \quad \ldots\text{(i)}$$

Now rotate the axes so that the positive direction of the x-axis passes through the point P. Then with respect to this coordinate system, the coordinates of P and Q, respectively become $(1,0)$ and $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ as shown in Fig 10.4.

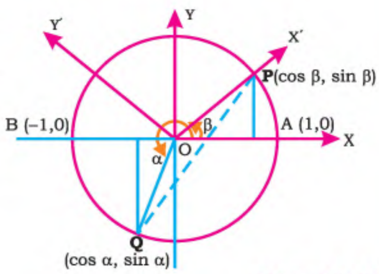


Fig. 10.3

So,

$|\overline{PQ}| = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$ ...(ii)

Hence, by comparing equation (i) and (ii)

we get, $\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$

$= \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$

Squaring both sides

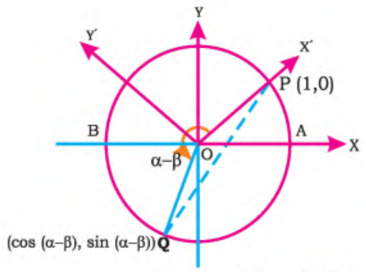


Fig. 10.4

$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= [\cos(\alpha - \beta) - 1]^2 + \sin^2(\alpha - \beta)$

or $(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) =$

$[\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)] - 2\cos(\alpha - \beta) + 1$

$\Rightarrow \quad 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha - \beta)$

So, $\boxed{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$ ... **I**

This law is called fundamental law of trigonometry.

Example: If $\cos\alpha = \frac{1}{2}, \cos\beta = \frac{\sqrt{3}}{2}$ and α, β are in the first quadrant. Find $\cos(\alpha - \beta)$.

Solution: We know $\cos^2\alpha + \sin^2\alpha = 1$

$$\Rightarrow \quad \sin\alpha = \pm\sqrt{1 - \cos^2\alpha}$$

$$= \pm\sqrt{1 - \left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$$

$\sin\alpha = \frac{\sqrt{3}}{2}$ (since α is in the first quadrant).

Again from $\cos^2\beta + \sin^2\beta = 1$,

we have $\sin\beta = \pm\sqrt{1 - \cos^2\beta}$

$$= \pm\sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} = \pm\frac{1}{2}.$$

$\sin\beta = \frac{1}{2}$. (since β is in the first quadrant)

Now, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$$

Deductions from the Fundamental Law

Using the above identity (I), we deduce some other identities.

Taking $\alpha = 0$ in (I)

We get, $\cos(0 - \beta) = \cos 0\cos\beta + \sin 0\sin\beta$

$$\Rightarrow \qquad \boxed{\cos(-\beta) = \cos\beta} \qquad [\because \cos 0 = 1, \sin 0 = 0] \qquad \ldots \textbf{i}$$

Again, putting $\alpha = \frac{\pi}{2}$, we have

$$\cos\left(\frac{\pi}{2} - \beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$

So, $$\boxed{\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta} \qquad \left[\because \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1\right] \qquad \ldots \textbf{ii}$$

Taking $-\beta$ for β in (ii), we have

$$\cos\left(\frac{\pi}{2} - (-\beta)\right) = \sin(-\beta)$$

$$\Rightarrow \qquad \cos\left(\beta - \left(-\frac{\pi}{2}\right)\right) = \sin(-\beta)$$

$$\Rightarrow \qquad \cos\beta\cos\left(-\frac{\pi}{2}\right) + \sin\beta\sin\left(-\frac{\pi}{2}\right) = \sin(-\beta)$$

$$\Rightarrow \qquad \cos\beta\,(0) + \sin\beta\,(-1) = \sin(-\beta)$$

So, $$\boxed{\sin(-\beta) = -\sin\beta} \qquad \ldots \textbf{iii}$$

Now, $$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

So, $$\boxed{\tan(-\theta) = -\tan\theta} \qquad \ldots \textbf{iv}$$

Again, $$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

So, $$\boxed{\cot(-\theta) = -\cot\theta} \qquad \ldots \textbf{v}$$

We know that $$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

So, $$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \qquad \boxed{\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta} \qquad \ldots \textbf{vi}$$

Now, $$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

So, $$\boxed{\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta} \qquad \ldots \textbf{vii}$$

Also, $$\cot\left(\frac{\pi}{2} - \theta\right) = \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

So, $$\boxed{\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta} \qquad \ldots \textbf{viii}$$

Similarly, $\text{cosec}\,(-\theta) = -\text{cosec}\,\theta$

and $\sec(-\theta) = \sec\theta$

Also $\text{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

and $\sec\left(\frac{\pi}{2} - \theta\right) = \text{cosec}\,\theta$

Deduction of $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

By Fundamental law

We have $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

and by putting $-\beta$ for β, we have

$$\cos(\alpha+\beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

But $\sin(-\beta) = -\sin\beta$ and $\cos(-\beta) = \cos\beta$.

So, $\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$ ...**II**

Deduction of $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

Since $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, therefore by putting $(\alpha+\beta)$ for θ,

we have $\sin(\alpha+\beta) = \cos\left[\frac{\pi}{2}-(\alpha+\beta)\right] = \cos\left\{\left(\frac{\pi}{2}-\alpha\right)-\beta\right\}$

$$= \cos\left(\frac{\pi}{2}-\alpha\right).\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right).\sin\beta$$

So, $\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$ ... **III**

By putting $-\beta$ for β in (III),

we have $\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$

i.e., $\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$ ...**IV**

Example 1. Find the value of $\sin\frac{7}{12}\pi$ without using tables or calculator.

Solution: $\sin\frac{7}{12}\pi = \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

By using values,

We get,

$$\sin\frac{7}{12}\pi = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 2. Without using tables or calculator, find $\sin 15°$.

Solution: $\sin 15° = \sin(60°-45°) = \sin 60°\cos 45° - \cos 60°\sin 45°$

By using values, $\sin 15° = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example 3. Prove that: $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$; $(\cos\alpha \neq 0, \cos\beta \neq 0)$.

Proof:

$$\text{L.H.S} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}$$

$$= \tan\alpha + \tan\beta$$

$$= \text{R.H.S}$$

$\because$ L.H.S = R.H.S

$$\therefore \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$$

Hence proved.

Example 4. Show that:

$$\sin(180° + \theta) = -\sin\theta.$$

Solution:

$$\text{L.H.S} = \sin(180° + \theta)$$

$$= \sin 180° \cos\theta + \cos 180° \sin\theta$$

$$= (0)\cos\theta + (-1)\sin\theta$$

$$= -\sin\theta = \text{R.H.S}$$

So, L.H.S = R.H.S, hence shown.

Deduction of $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \pm \tan\alpha\tan\beta}$

We know that $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Dividing the numerator and the denominator by $\cos\alpha\cos\beta \neq 0$,

we have $\boxed{\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$ ... **V**

Again putting $-\beta$ for β in (V),

We get, $\tan(\alpha-\beta) = \dfrac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

So, $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$ ...**VI**

Now, $$\cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}$$

$$=\frac{1}{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

$$=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

$$=\frac{1-\frac{1}{\cot\alpha}\cdot\frac{1}{\cot\beta}}{\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}}=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

So, $$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$ ... **VII**

Again putting $-\beta$ for β in (VII), we have

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot(-\beta)-1}{\cot\alpha+\cot(-\beta)}$$

$$=\frac{-\cot\alpha\cot\beta-1}{\cot\alpha-\cot\beta}$$

So, $$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$$ ...**VIII**

Example 1. Prove that $\tan(270°-\theta)=\cot\theta$

Proof: $$\text{L.H.S}=\tan(270°-\theta)$$

$$=\frac{\sin(270°-\theta)}{\cos(270°-\theta)}$$

$$=\frac{\sin 270°\cos\theta-\cos 270°\sin\theta}{\cos 270°\cos\theta+\sin 270°\sin\theta}$$

$$=\frac{(-1)\cos\theta-(0)\sin\theta}{(0)\cos\theta+(-1)\sin\theta}$$

$$=\frac{-\cos\theta}{-\sin\theta}$$

$$=\cot\theta$$

$$=\text{R.H.S}$$

Example 2. Without using tables or calculator, find the value of tan75°.

Solution:

Here, $\sin 75°=\sin(30°+45°)=\sin 30°\cos 45°+\cos 30°\sin 45°$

By using values

We get, $\sin 75° = \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

and $\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$

By using values,

We get, $\cos 75° = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Now, $\tan 75° = \frac{\sin 75°}{\cos 75°}$

$$= \frac{\frac{1+\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{\sqrt{3}+1}{2\sqrt{2}}\cdot\frac{2\sqrt{2}}{\sqrt{3}-1}$$

So, $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

Example 3. Find the value of $\cot\frac{1}{12}\pi$ without using tables or calculator.

Solution: $\cot\frac{1}{12}\pi = \cot\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\cot\frac{\pi}{3}\cot\frac{\pi}{4}+1}{\cot\frac{\pi}{4} - \cot\frac{\pi}{3}}$

By using values, $\cot\frac{\pi}{12} = \frac{\frac{1}{\sqrt{3}}.1+1}{1-\frac{1}{\sqrt{3}}}$

$$= \frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}$$

So, $\cot\frac{1}{12}\pi = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

Exercise 10.1

1. Prove that:

(i) $\cos(360° + \theta) = \cos\theta$

(ii) $\sin(180° + \theta) = -\sin\theta$

(iii) $\tan(180° - \theta) = -\tan\theta$

(iv) $\sin(270° - \theta) = -\cos\theta$

(v) $\cot(270° + \theta) - \tan\theta$ (vi) $\cot(270° - \theta) = \tan\theta$

(vii) $\text{cosec}(360° - \theta) = -\text{cosec}\,\theta$ (viii) $\sec(360° - \theta) = \sec\theta$

2. Evaluate the following.

(i) $\sin 150° \cos 300° + \sin 300° \cos 150°$

(ii) $\cos 19° \cos 11°$- $\sin 19° \sin 11$

3. Verify that:

(i) $\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\cos(180° + 45°)\cos(180° - 45°) = \frac{1}{2}$

4. Without using tables or calculator. Find the values of:

(i) $\sin 75°$ (ii) $\tan 105°$ (iii) $\cos 165°$ (iv) $\sin 255°$

(v) $\tan\frac{5\pi}{12}$ (vi) $\cos\frac{13\pi}{12}$ (vii) $\sin\frac{23\pi}{12}$ (viii) $\cos\frac{25\pi}{12}$

5. Prove that:

(i) $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$ (ii) $\frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{\sin(\alpha+\beta)+\sin(\alpha-\beta)} = \cot\alpha$

6. Prove that:

(i) $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

(ii) $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$

7. Prove that: $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

8. If $\sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$ then find:

(i) $\cos(\alpha + \beta)$ (ii) $\tan(\alpha + \beta)$ (iii) $\cot(\alpha - \beta)$ (iv) $\cos(\alpha - \beta)$

(v) $\sin(\alpha + \beta)$ (vi) $\sin(\alpha - \beta)$ (vii) $\cot(\alpha + \beta)$ (viii) $\tan(\alpha - \beta)$

Also find quadrants of angle $(\alpha + \beta)$ and $(\alpha - \beta)$

10.2 Trigonometric Ratios of Allied Angles

10.2.1 Define allied angles

A group of angles is said to be allied angles of basic agnel θ, if sum or difference of any two of them gives the integral multiple of $90°$ or $\frac{\pi}{2}$ radian.

So, the angles of measure of $90° \pm \theta, 180° \pm \theta, 270° \pm \theta, 360° \pm \theta$ are allied angles to basic angle θ.

Note: The General formula to produce allied angles is $n(90°) \pm \theta°$ or in radians $n\left(\frac{\pi}{2}\right) \pm \theta$ where $n \in \mathbb{Z}$ and $0 < \theta < \frac{\pi}{2}$

Example: Generate all allied angles if $\theta = 30°$.

Solution: We use $n(90°) \pm \theta$, to generate allied angles for all integers.

If n is non-negative integer

For $n = 0$, we have $0 \pm 30° = \pm 30°$
For $n = 1$, we have $90° \pm 30° = 60°, 120°$
For $n = 2$, we have $180° \pm 30° = 150°, 210°$ and so on
If n is negative integer.
For $n = -1$, we have $-90° \pm 30° = -60°, -120°$
For $n = -2$, we have $-180° \pm 30° = -150°, -210°$ and so on.
Hence all allied angles are $\pm 30°, \pm 60°, \pm 120°, \pm 150°, \pm 210°, ...$

10.2.2 Use fundamental law and its deductions to derive trigonometric ratios of allied angles

Express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta + \emptyset)$ where $a = r\cos\emptyset$ and $b = r\sin\emptyset$

Trigonometric ratios of allied angles in all four quadrants are given in the following table. Some of which are obtained from section 10.1.2 and the following solved examples. Remaining proofs are left as an exercise.

Example 1. Prove that $\sin(90° + \theta) = \cos\theta$

Proof: L.H.S $= \sin(90° + \theta)$
$= \sin 90° \cos\theta + \cos 90° \sin\theta$
$= (1)\cos\theta + (0)\sin\theta$
$= \cos\theta$
$=$ R. H. S

Example 2. Prove that $\cos(180° - \theta) = -\cos\theta$

Proof: L.H.S $= \cos(180° - \theta)$
$= \cos 180° \cos\theta + \sin 180° \sin\theta$
$= (-1)\cos\theta + (0)\sin\theta$
$= -\cos\theta$
$=$ R. H. S

Example 3. Prove that $\tan(360° - \theta) = -\tan\theta$

Proof: L.H.S $= \tan(360° - \theta)$
$$= \frac{\tan 360° - \tan\theta}{1 + \tan 360° \tan\theta}$$
$$= \frac{0 - \tan\theta}{1 + 0 \times \tan\theta}$$
$= -\tan\theta$
$=$ R. H. S

Trigonometric ratios of Allied angles in all the Four Quadrants

−θ (4th Quadrant)		90° − θ (1st Quadrant)		90° + θ (2nd Quadrant)	
sin (−θ)	−sinθ	sin (90° − θ)	cosθ	sin (90° + θ)	cosθ
cos (−θ)	cosθ	cos (90° − θ)	sinθ	cos (90° + θ)	−sinθ
tan (−θ)	−tanθ	tan (90° − θ)	cotθ	tan (90° + θ)	−cotθ
cosec (−θ)	−cosecθ	cosec (90° − θ)	secθ	cosec (90° + θ)	secθ
sec (−θ)	secθ	sec (90° − θ)	cosecθ	sec (90° + θ)	−cosecθ
cot (−θ)	−cotθ	cot (90° − θ)	tanθ	cot (90° + θ)	−tanθ

180° − θ (2nd Quadrant)		180° + θ (3rd Quadrant)		270° − θ (3rd Quadrant)	
sin (180° − θ)	sinθ	sin (180° + θ)	−sinθ	sin (270° − θ)	−cosθ
cos (180° − θ)	−cosθ	cos (180° + θ)	−cosθ	cos (270° − θ)	−sinθ
tan (180° − θ)	−tanθ	tan (180° + θ)	tanθ	tan (270° − θ)	cotθ
cosec (180° − θ)	cosecθ	cosec (180° + θ)	−cosecθ	cosec (270° − θ)	−secθ
sec (180° − θ)	−secθ	sec (180° + θ)	−secθ	sec (270° − θ)	−cosecθ
cot (180° − θ)	−cotθ	cot (180° + θ)	cotθ	cot (270° − θ)	tanθ

270° + θ (4th Quadrant)		360° − θ (4th Quadrant)		360° + θ (1st Quadrant)	
sin (270° + θ)	−cosθ	sin (360° − θ)	−sinθ	sin (360° + θ)	sinθ
cos (270° + θ)	sinθ	cos (360° − θ)	cosθ	cos (360° + θ)	cosθ
tan (270° + θ)	−cotθ	tan (360° − θ)	−tanθ	tan (360° + θ)	tanθ
cosec (270° + θ)	−secθ	cosec (360° − θ)	−cosecθ	cosec (360° + θ)	cosecθ
sec (270° + θ)	cosecθ	sec (360° − θ)	secθ	sec (360° + θ)	secθ
cot (270° + θ)	−tanθ	cot (360° − θ)	−cotθ	cot (360° + θ)	cotθ

Example 1. Find the values of

(i) cos 495° (ii) sin 1230° (iii) tan (−1590°).

Solution:

(i) $\cos 495° = \cos(360° + 135°)$

$= \cos 135°$

$= \cos(90° + 45°)$

$= -\sin 45°$

$= -\frac{1}{\sqrt{2}}$

(ii) $\sin 1230° = \sin(3x\,360 + 150)°$

$= \sin 150°$

$= \sin(180° - 30°)$

$= \sin 30° = \frac{1}{2}$

(iii) $\tan(-1590°) = -\tan 1590°$

$= -\tan(4 \times 360 + 150)°$

$= -\tan 150°$

$$= -\tan(180° - 30°)$$
$$= -(-\tan 30°)$$
$$= \tan 30° = \frac{1}{\sqrt{3}}$$

Example 2. Without using the tables or calculator, find the values of:

(i) $\text{cosec}(-870°)$ (ii) $\frac{6\cot 62°}{\tan 28°} + \frac{\tan 70°}{2\cot 20°}$

Solution:

(i) $\text{cosec}(-870°) = \frac{1}{\sin(-870°)} = \frac{1}{-\sin 870°}$

$$= \frac{1}{-\sin(720°+150°)}$$
$$= \frac{1}{-\sin 150°}$$
$$= \frac{1}{-\sin(90°+60°)}$$
$$= \frac{1}{-\cos 60°}$$
$$= -2$$

(ii) $\frac{6\cot 62°}{\tan 28°} + \frac{\tan 70°}{2\cot 20°} = \frac{6\cot(90°-28°)}{\tan 28°} + \frac{\tan(90°-20°)}{2\cot 20°}$

$$= 6\frac{\tan 28°}{\tan 28°} + \frac{\cot 20°}{2\cot 20°} \quad [\because \tan(90° - \theta) = \cot\theta]$$
$$= 6 + \frac{1}{2} = \frac{13}{2} \quad [\because \cot(90° - \theta) = \tan\theta]$$

Example 3. Prove that: $\frac{3\cot(90°-\theta)}{\tan\theta} - \frac{2\sin\theta}{\cos(90°-\theta)} = 1$

Solution:

$$\text{L.H.S} = \frac{3\cot(90°-\theta)}{\tan\theta} - \frac{2\sin\theta}{\cos(90°-\theta)} = 3\frac{\tan\theta}{\tan\theta} - 2\frac{\sin\theta}{\sin\theta} \quad [\because \cos(90° - \theta) = \sin\theta]$$
$$= 3 - 2 = 1 = \text{R.H.S} \quad [\because \cot(90° - \theta) = \tan\theta]$$

Express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta + \emptyset)$ where $a = r\cos\emptyset$ and $b = r\sin\emptyset$

Let $a = r\cos\emptyset$ and $b = r\sin\emptyset$.

Now by putting values of a and b in $a\sin\theta + b\cos\theta$

we have, $a\sin\theta + b\cos\theta = r\cos\emptyset.\sin\theta + r\sin\emptyset.\cos\theta$

$$= r(\cos\emptyset.\sin\theta + \sin\emptyset.\cos\theta)$$
$$= r\sin(\theta + \emptyset) \quad \text{[By using identity]}$$

Thus $a\sin\theta + b\cos\theta = r\sin(\theta + \emptyset)$

Example: Express $4\sin\theta + 3\cos\theta$ in the form $r\sin(\theta + \emptyset)$, where θ and $\emptyset$ are in the first quadrant.

Solution:

Since, θ and $\emptyset$ are in the first quadrant, therefore, all trigonometric functions have positive values.

Let $\quad 4 = r\cos\emptyset \quad \ldots(1)$

and $\quad 3 = r\sin\emptyset \quad \ldots(2)$

Squaring both sides of equations (1) and (2), and then adding

We get $\quad 4^2 + 3^2 = r^2(\cos^2\emptyset + \sin^2\emptyset)$

$\Rightarrow \quad 25 = r^2 \times 1$

$\Rightarrow \quad r = 5$

Therefore, $\quad \cos\emptyset = \frac{4}{r} = \frac{4}{5}$ and $\sin\emptyset = \frac{3}{r} = \frac{3}{5}$

Now, $\quad 4\sin\theta + 3\cos\theta = r\cos\emptyset\sin\theta + r\sin\emptyset\cos\theta$

$= r(\sin\theta\cos\emptyset + \cos\theta\sin\theta)$

$= 5\sin(\theta + \emptyset);$ $\quad$ where, $\cos\emptyset = \frac{4}{5}$ and $\sin\emptyset = \frac{3}{5}$

Exercise 10.2

1. Generate all allied angles if

(i) $\theta = 60°$ $\quad$ (ii) $\theta = 50°$

2. Show that: $\dfrac{\sin(90°-\theta)}{\text{cosec}(90°-\theta)} + \dfrac{\cos(90°-\theta)}{\sec(90°-\theta)} = 1$

3. Evaluate the following without using calculator.

(i) $\sin(675°)$ $\quad$ (ii) $\tan\left(\frac{25\pi}{6}\right)$ $\quad$ (iii) $\text{cosec}(2130°)$

(iv) $\dfrac{\sec 55°}{\text{cosec } 35°} + \dfrac{\sin 48°}{\cos 42°}$ $\quad$ (v) $2\dfrac{\sin 56°}{\cos 34°} - \dfrac{\cot 37°}{\tan 53°} - \sqrt{2}\sin 45°$

4. Show that:

$\cos(45° - \theta) = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$

5. Express the following in the form $r\sin(\theta + \emptyset)$ where θ and $\emptyset$ are in the first quadrant.

(i) $15\sin\theta + 8\cos\theta$ $\quad$ (ii) $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$

(iii) $\sin\theta + \cos\theta$

6. Prove that: $\dfrac{\tan\alpha}{\cot(90°-\alpha)} + \dfrac{\cos(90°-\alpha)}{\sin\alpha} = 2$

7. If $A + B + C = 180°$ then prove that

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

10.3 Double, Half and Triple Angle Identities

10.3.1 Derive double angle, half angle and triple angle identities from fundamental law and its deductions

(a) Double angle identities for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Taking $\alpha = \beta = \theta$, we get

$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$

i.e.,
$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \quad \dots (1)$$

We know that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Taking $\alpha = \beta = \theta$

We get, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

By using $\sin^2\theta + \cos^2\theta = 1$, we get

$$\boxed{\begin{array}{l}\cos 2\theta = 2\cos^2\theta - 1 \\ \cos 2\theta = 1 - 2\sin^2\theta\end{array}} \quad \dots (2)$$

or

Again, we know that $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Taking $\alpha = \beta = \theta$, we have

$$\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

i.e.,
$$\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}} \quad \dots (3)$$

Example 1. Prove that: $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

Solution:
$$\text{L.H.S} = \frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta}$$
$$= \frac{2\sin\theta\cos\theta}{\sin\theta} - \frac{2\cos^2\theta - 1}{\cos\theta} \quad \text{[By using (1) and (2)]}$$
$$= 2\cos\theta - \frac{2\cos^2\theta - 1}{\cos\theta}$$
$$= \frac{2\cos^2\theta - 2\cos^2\theta + 1}{\cos\theta}$$
$$= \frac{1}{\cos\theta} = \sec\theta = \text{R.H.S}$$

Example 2. Prove that: $\sec 4\theta = \frac{\tan 2\theta + \cot 2\theta}{\cot 2\theta - \tan 2\theta}$

Solution: R.H.S

$$= \frac{\tan 2\theta + \cot 2\theta}{\cot 2\theta - \tan 2\theta}$$

$$= \frac{\frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}}{\frac{\cos 2\theta}{\sin 2\theta} - \frac{\sin 2\theta}{\cos 2\theta}}$$

$$= \frac{\frac{\sin^2 2\theta + \cos^2 2\theta}{\sin 2\theta \cos 2\theta}}{\frac{\cos^2 2\theta - \sin^2 2\theta}{\sin 2\theta \cos 2\theta}}$$

$$= \frac{1}{\cos^2 2\theta - \sin^2 \theta}$$

$$= \frac{1}{\cos^2 2\theta - \sin^2 2\theta}$$

$$= \frac{1}{\cos 4\theta} = \sec 4\theta = \text{L.H.S}$$

(b) Half angle identities for $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$ and $\tan \frac{\theta}{2}$

We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= 1 - 2\sin^2\theta$$

Putting $\frac{\theta}{2}$ for θ, we have,

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$\Rightarrow \quad 2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

$$\Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\Rightarrow \quad \boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}} \quad \ldots (1)$$

We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= 2\cos^2\theta - 1$$

Putting $\frac{\theta}{2}$ for θ, we have,

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow \quad 2\cos^2\frac{\theta}{2} = 1 + \cos\theta$$

$$\Rightarrow \quad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

So, $$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \dots (2)$$

From (1) and (2), by division, we have, $\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

$$\text{or} \qquad \tan\frac{\theta}{2} = \left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\left(\sqrt{\frac{1+\cos\theta}{1+\cos\theta}}\right)$$
$$= \frac{\sqrt{1-\cos^2\theta}}{1+\cos\theta}$$
$$= \frac{\sin\theta}{1+\cos\theta}$$

So,
$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \qquad (\text{where } \cos\theta \neq -1) \qquad \dots(3a)$$

$$\text{or} \qquad \tan\frac{\theta}{2} = \frac{\sin^2\theta}{\sin\theta\,(1+\cos\theta)}$$
$$= \frac{1-\cos^2\theta}{\sin\theta\,(1+\cos\theta)}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$

So, $$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} \qquad (\text{where } \sin\theta \neq 0) \qquad \dots (3b)$$

Example 1. Show that $\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = 1 - \sin\theta$

Solution:

$$\text{L.H.S} = \left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= 1 - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= 1 - \sin\theta = \text{R.H.S}$$

Example 2. If $\sin\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$ then, find the values of

(i) $\cos\frac{\theta}{2}$ (ii) $\sin\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$

Solution:

$\because$ θ is in 1st quadrant

$\therefore \cos\theta = \sqrt{1-\sin^2\theta}$

$$\Rightarrow \cos\theta = \sqrt{1-\frac{9}{25}}$$

$$= \sqrt{\frac{16}{25}}$$

$$= \frac{4}{5}$$

(i) $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$ $\left(\because 0 < \theta < \frac{\pi}{2}\right)$

$$= \sqrt{\frac{1+\frac{4}{5}}{2}}$$

$$= \sqrt{\frac{9}{10}}$$

$$= \frac{3}{\sqrt{10}}$$

(ii) $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ $\left(\because 0 < \theta < \frac{\pi}{2}\right)$

$$= \sqrt{\frac{1-\frac{4}{5}}{2}}$$

$$= \sqrt{\frac{1}{10}}$$

$$= \frac{1}{\sqrt{10}}$$

(iii) $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

$$= \frac{\frac{3}{5}}{1+\frac{4}{5}}$$

$$= \frac{3}{5}\times\frac{5}{9}$$

$$= \frac{1}{3}$$

Example 3. Find the value of $\sin 15°$.

Solution: $\sin 15° = \sin\frac{30°}{2}$

$$= \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

(c) Triple angle identities $\sin 3\theta$, $\cos 3\theta$ and $\tan 3\theta$

We know that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

By using $\alpha = 2\theta$ and $\beta = \theta$, we get

$$\sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$
$$= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$

$$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta} \quad \ldots (1)$$

Similarly, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

By using, $\alpha = 2\theta$ and $\beta = \theta$, we get

$$\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta} \quad \ldots (2)$$

We also know that, $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

By using, $\alpha = 2\theta$ and $\beta = \theta$, we get

$$\tan 3\theta = \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$$
$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$$

$$= \frac{\frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{1-\tan^2\theta}}{\frac{(1-\tan^2\theta)-2\tan^2\theta}{1-\tan^2\theta}}$$

$$\boxed{\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}} \quad \ldots.(3)$$

Example: Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Solution:

L.H.S $= \cos 5\theta = \cos(2\theta + 3\theta)$

$= \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta$

$= (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - (2\sin\theta\cos\theta)(3\sin\theta - 4\sin^3\theta)$

$= (8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta) - \cos\theta(6\sin^2\theta - 8\sin^4\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta\sin^2\theta(3 - 4\sin^2\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta(1-\cos^2\theta)\{3 - 4(1-\cos^2\theta)\}$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - (2\cos\theta - 2\cos^3\theta)(3 - 4 + 4\cos^2\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - (2\cos\theta - 2\cos^3\theta)(4\cos^2\theta - 1)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - (8\cos^3\theta - 2\cos\theta - 8\cos^5\theta + 2\cos^3\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 10\cos^3\theta + 2\cos\theta + 8\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ = R.H.S

Exercise 10.3

Prove the following identities:

1. $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

2. $\cot 2\theta = \frac{\cot\theta - \tan\theta}{2}$

3. $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

4. $\tan^2\frac{\theta}{2} = \frac{\tan\theta - \sin\theta}{\tan\theta + \sin\theta}$

5. $\operatorname{cosec}^2\frac{\theta}{2} = \frac{2\sec\theta}{\sec\theta - 1}$

6. $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$

7. $\cos 2\theta = \frac{\cot^2\theta - 1}{\cot^2\theta + 1}$

8. $\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2 = 1 - \sin\theta$

9. $\cos 4\theta = 8\cos^4\theta - 8\cos\theta^2 + 1$

10. $\tan\frac{\alpha}{2} = \operatorname{cosec}\alpha - \cot\alpha$

11. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

12. $\frac{\sin\theta + \sin 2\theta}{1+\cos\theta + \cos 2\theta} = \tan\theta$

13. $\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{\tan^4\theta - 6\tan^2\theta + 1}$

14. $\sec\theta = \frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}$

15. $\dfrac{\sin 3\alpha}{\sin\alpha} - \dfrac{\cos 3\alpha}{\cos\alpha} = 2$

16. $\dfrac{\sin 4\theta}{\sin 2\theta} - \dfrac{\cos 4\theta}{\cos 2\theta} = \sec 2\theta$

17. Evaluate the following

(i) $\tan 22.5°$ (ii) $\cos 15°$

18. If $\cos\theta = \frac{-4}{5}$ and $\frac{\pi}{2} < \theta < \pi$ then find the following:

(i) $\sin\frac{\theta}{2}$ (ii) $\cos\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$ (iv) $\sin 2\theta$ (v) $\cos 2\theta$ (vi) $\tan 2\theta$

19. If $\sin\theta = \frac{12}{13}$, where $0 < \theta < \frac{\pi}{2}$ then find the following:

(i) $\sin\frac{\theta}{2}$ (ii) $\cos\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$ (iv) $\sin 2\theta$ (v) $\cos 2\theta$ (vi) $\tan 2\theta$

20. If $\cos\theta = \frac{3}{5}$, where $0 < \theta < \frac{\pi}{2}$, then find the following:

(i) $\sin 3\theta$ (ii) $\cos 3\theta$ (iii) $\tan 3\theta$

10.4 Sum, Difference and Product of sine and cosine

10.4.1 Express the product (of sines and cosines) as sums or differences (of sines and cosines)

We know that
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
Then by adding and subtracting, we get
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \quad \ldots(i)$$
and
$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \quad \ldots(ii)$$
Similarly, from
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$
by adding and subtracting, we get
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \quad \ldots(iii)$$
and
$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)] \quad \ldots(iv)$$

Example 1. Express $2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$ as sum or difference.

Solution:

Here $\alpha = \frac{3\theta}{2}$ and $\beta = \frac{\theta}{2}$

Using $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$,

so we get, $2\sin\frac{3\theta}{2}\cos\frac{\theta}{2} = \sin\left(\frac{3\theta}{2}+\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}-\frac{\theta}{2}\right)$

$$= \sin\frac{4\theta}{2} + \sin\frac{2\theta}{2}$$
$$= \sin 2\theta + \sin\theta$$

Example 2. Express $\cos(y+z)\cos(y-z)$ as sum or difference.

Solution:

Here, $\alpha = y + z$ and $\beta = y - z$

Using $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$,

We get, $\cos(y+z)\cos(y-z) = \frac{1}{2}[\cos(y+z+y-z) + \cos(y+z-y+z)]$

$$= \frac{1}{2}(\cos 2y + \cos 2z)$$

Example 3. Verify that: $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$

Solution: L.H.S $= \sin 10° \sin 30° \sin 50° \sin 70°$

$$= \sin 10° \sin 70° \sin 30° \sin 50°$$
$$= \frac{-1}{4}\sin 10°(-2\sin 50° \sin 70°)$$
$$= \frac{-1}{4}\sin 10° [\cos(50°+70°) - \cos(50°-70°)]$$
$$= \frac{-1}{4}\sin 10° [\cos 120° - \cos(-20°)]$$
$$= \frac{-1}{4}\sin 10° [\cos(90+30)° - \cos 20°] \qquad [\because \cos(-\theta) = \cos\theta]$$
$$= \frac{-1}{4}\sin 10° [-\sin 30° - \cos 20°] \qquad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$
$$= \frac{-1}{4}\sin 10° \left[-\frac{1}{2} - \cos 20°\right]$$
$$= \frac{1}{8}\sin 10° + \frac{1}{4}[\sin 10° \cos 20°]$$
$$= \frac{1}{8}\sin 10° + \frac{1}{8}[2\sin 10° \cos 20°]$$
$$= \frac{1}{8}\sin 10° + \frac{1}{8}[\sin(10°+20°) + \sin(10°-20°)]$$
$$= \frac{1}{8}\sin 10° + \frac{1}{8}[\sin 30° + \sin(-10°)]$$
$$= \frac{1}{8}\sin 10° + \frac{1}{8}\left[\frac{1}{2} - \sin 10°\right]$$
$$= \frac{1}{8}\sin 10° + \frac{1}{16} - \frac{1}{8}\sin 10°$$
$$= \frac{1}{16} = \text{R.H.S}$$

10.4.2 Express the sums or differences (of sines and cosines) as products (of sines and cosines)

From (i), we have

$$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

or $$2\sin\alpha \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

Putting $\alpha+\beta = u$ and $\alpha-\beta = v$, also $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$.

we have $$2\sin\frac{u+v}{2}\cos\frac{u-v}{2} = \sin u + \sin v$$

i.e., $$\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2} \quad \ldots (v)$$

Similarly, from (ii), we have

$$\sin u - \sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2} \quad \ldots (vi)$$

Again from (iii), we have $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

or $$2\cos\alpha \cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

Putting $\alpha+\beta = u$ and $\alpha-\beta = v$, also $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$.

we have $$2\cos\frac{u+v}{2}\cos\frac{u-v}{2} = \cos u + \cos v$$

i.e., $$\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2} \quad \ldots (vii)$$

Similarly from (iv), we have

$$\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2} \quad \ldots (viii)$$

Example 1. Express the sum $\cos 20° + \cos 10°$ in the product form.

Solution: We know that $\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}$

So, $$\cos 20° + \cos 10° = 2\cos\frac{20°+10°}{2}\cos\frac{20°-10°}{2}$$

$$= 2\cos 15° \cos 5°$$

Example 2. Prove that $\frac{\cos 6\theta - \cos 2\theta}{\sin 3\theta + \sin\theta} = \frac{-\sin 4\theta}{\cos\theta}$

Solution: By using $$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

and $$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

So, $$\text{L.H.S} = \frac{\cos 6\theta - \cos 2\theta}{\sin 3\theta + \sin\theta} = \frac{-2\sin\left(\frac{6\theta+2\theta}{2}\right)\sin\left(\frac{6\theta-2\theta}{2}\right)}{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}$$

$$= \frac{-\sin\frac{8\theta}{2}\ \sin\frac{4\theta}{2}}{\sin\frac{4\theta}{2}\ \cos\frac{2\theta}{2}}$$

$$= -\frac{\sin 4\theta\ \sin 2\theta}{\sin 2\theta \cos\theta}$$

$$= \frac{-\sin 4\theta}{\cos\theta}$$

$$= \text{R.H.S}$$

Exercise 10.4

1. Express the following products as sums or differences:

(i) $2\sin 6\theta\cos 3\theta$ (ii) $\cos 3\theta \sin 6\theta$

(iii) $2\sin(\alpha-\beta)\cos(\alpha+\beta)$ (iv) $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

(v) $\sin(\theta+45^\circ)\sin(\theta-45^\circ)$ (vi) $2\cos(2\theta+60^\circ)\cos(2\theta-60^\circ)$

2. Express the following sums or differences as products.

(i) $\sin 2\alpha - \sin 2\beta$ (ii) $\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2}$

(iii) $\cos 25^\circ + \cos 65^\circ$ (iv) $\cos(\theta+30^\circ) - \cos(\theta-30^\circ)$

(v) $\sin\frac{\pi}{2} - \sin\frac{\pi}{4}$ (vi) $\sin 2(\theta+40^\circ) + \sin 2(\theta-40^\circ)$

3. Prove the following identities.

(i) $\frac{\sin\alpha-\sin\beta}{\cos\alpha+\cos\beta} = \tan\left(\frac{\alpha-\beta}{2}\right)$

(ii) $\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

(iii) $\frac{\sin 6\theta+\sin 4\theta}{\cos 6\theta+\cos 4\theta} = \tan 5\theta$

(iv) $\frac{\sin 2\theta+\sin 4\theta+\sin 6\theta+\sin 8\theta}{\cos 2\theta+\cos 4\theta+\cos 6\theta+\cos 8\theta} = \tan 5\theta$

4. Evaluate:

(i) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$ (ii) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

5. Show that:

(i) $\frac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ} = \tan 37^\circ$ (ii) $\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ} = \tan 56^\circ$

Review Exercise 10

1. Select the correct option.

(i) $\cos(\alpha-\beta)$ is equal to:

(a) $\cos\alpha\cos\beta+\sin\alpha\sin\beta$ (b) $\cos\alpha\cos\beta-\sin\alpha\sin\beta$
(c) $\sin\alpha\cos\beta+\cos\alpha\sin\beta$ (d) $\sin\alpha\cos\beta-\cos\alpha\sin\beta$

(ii) Angles associated with basic angles of measure θ to a right angle or its multiple are called:

(a) Coterminal angles (b) Angles in standard positions
(c) Allied angles (d) Obtuse angles

(iii) $\sin\left(\frac{\pi}{2}+\theta\right)$ is equal to:

(a) $\cos\theta$ (b) $\sin\theta$ (c) $-\cos\theta$ (d) $-\sin\theta$

(iv) $\cos(\pi-\theta)$ is equal to:

(a) $\sin\theta$ (b) $\cos\theta$ (c) $-\sin\theta$ (d) $-\cos\theta$

(v) $\tan(\pi+\theta)$ is equal to:

(a) $\tan\theta$ (b) $-\cot\theta$ (c) $-\tan\theta$ (d) $\cot\theta$

(vi) $\cos(2\pi+\theta)$ is equal to:

(a) $\sin\theta$ (b) $\cos\theta$ (c) $-\sin\theta$ (d) $-\cos\theta$

(vii) $\sin 540^\circ$ is equal to:

(a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

(viii) $\sin(-300^\circ)$ is equal to:

(a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) 0 (d) -1

(ix) If α,β and γ are the angles of Δ ABC, then $\sin(\alpha+\beta)$ is equal to:

(a) $\sin\gamma$ (b) $-\sin\gamma$ (c) $\cos\gamma$ (d) $-\cos\gamma$

(x) $\cos 2\alpha$ is equal to:

(a) $\cos^2\alpha-\sin^2\alpha$ (b) $2\cos^2\alpha-1$ (c) $1-2\sin^2\alpha$ (d) all of these

(xi) $\sin\frac{\alpha}{2}$ is equal to:

(a) $\pm\sqrt{\frac{1+\sin\alpha}{2}}$ (b) $\pm\sqrt{\frac{1-\cos\alpha}{2}}$ (c) $\pm\sqrt{\frac{1+\cos\alpha}{2}}$ (d) $\pm\sqrt{\frac{1-\sin\alpha}{2}}$

(xii) $\cos 3\alpha$ is equal to: (a) $3\cos\alpha-4\cos\alpha$ (b) $3\cos^3\alpha+4\cos\alpha$
(c) $4\cos^3\alpha-3\cos\alpha$ (d) $4\cos^3\alpha+4\cos\alpha$

(xiii) $\cos\alpha-\cos\beta=$ (a) $2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ (b) $2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
(c) $2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ (d) $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

(xiv) $2\sin 7\theta \sin 2\theta$ is equal to: (a) $\cos 5\theta - \cos 9\theta$ (b) $\cos 9\theta - \cos 5\theta$ (c) $\sin 9\theta + \sin 5\theta$ (d) $\sin 9\theta - \sin 5\theta$

(xv) An allied angle to θ is ________.
(a) $270° + \theta$ (b) $60° + \theta$ (c) $45° + \theta$ (d) $30° + \theta$

(xvi) The value of $\cos(\alpha - 2\pi)$ is equal to:
(a) $-\cos\alpha$ (b) $-\sin\alpha$ (c) $\cos\alpha$ (d) $\sin\alpha$

(xvii) The value of $\sin 7\pi$ is equal to:
(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

2. Prove the following:

(i) $\cos\left(\frac{\pi}{4} - \alpha\right)\cos\left(\frac{\pi}{4} - \beta\right) - \sin\left(\frac{\pi}{4} - \alpha\right)\sin\left(\frac{\pi}{4} - \beta\right) = \sin(\alpha + \beta)$

(ii) $\cos\left(\frac{3\pi}{4} + \theta\right) - \cos\left(\frac{3\pi}{4} - \theta\right) = -\sqrt{2}\sin\theta$

(iii) $\sin^2 6\theta - \sin^2 4\theta = 4\sin 5\theta \cos 5\theta \sin\theta \cos\theta$

(iv) $\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan 4\theta$

(v) $\frac{\cos 4\theta + \cos 3\theta + \cos 2\theta}{\sin 4\theta + \sin 3\theta + \sin 2\theta} = \cot 3\theta$

(vi) $\sin 5\theta - \sin 3\theta + \sin 2\theta = 4\sin\theta\cos\frac{3\theta}{2}\cos\frac{5\theta}{2}$

Application of Trigonometry

Unit 11

11.1 Solving Triangles

We know that a triangle has six elements which are three sides and three angles. If measures of any three of them including the measure of at least one side are known, then we can find the measures of other sides and angles.

11.1.1 Solve right angled triangle when measures of:
(i) two sides are given
(ii) one side and one angle are given

(i) When two sides are given

The following example is suitable to understand the method of solving a right-angled triangle, when measures of two sides are given.

Example:

Solve the right triangle ABC with $\beta = 90°$, $a = 8.6$ cm and $b = 11.4$ cm.

Solution: Given and unknown elements are mentioned in Fig. 11.1.

We have to find α, γ and c.

Here, $\sin\alpha = \frac{a}{b} = \frac{8.6}{11.4} = 0.754$

$\Rightarrow \quad \alpha = \sin^{-1} 0.754$

$\Rightarrow \quad \alpha = 49°$

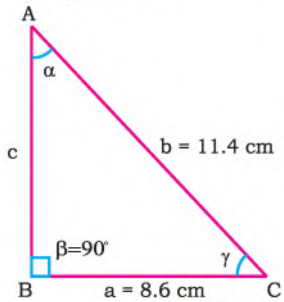
and $\quad \gamma = 180° - \alpha - \beta = 180° - 49° - 90° = 41°$

By Pythagoras theorem,

$b^2 = a^2 + c^2$

$\Rightarrow \quad c^2 = b^2 - a^2 = (11.4)^2 - (8.6)^2$

$\Rightarrow \quad c = 7.5$ cm



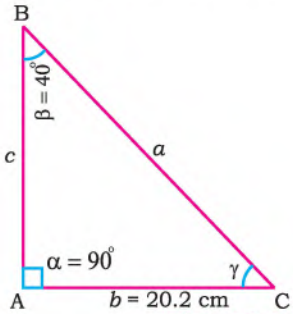
(Fig. 11.1)

(ii) When one side and one angle are given

If we are given a right-angled triangle, in which one side and one angle are given, we can find measures of its unknown sides and angles as explained in the following examples.

Example 1. Solve the right-angled triangle ABC with $\alpha = 90°, \beta = 40°$ and $b = 20.2$ cm.

Solution: Given and unknown elements are mentioned in Fig. 11.2, we have to find γ, c and a.



(Fig. 11.2)

As $\alpha + \beta + \gamma = 180°$

so, $\gamma = 180° - \alpha - \beta = 180° - 90° - 40°$

$\gamma = 50°$.

Now, $\tan\beta = \frac{b}{c}$

$\Rightarrow \tan 40° = \frac{20.2}{c}$

$\Rightarrow c = \frac{20.2}{\tan 40°} \Rightarrow c = 24.0734$ cm

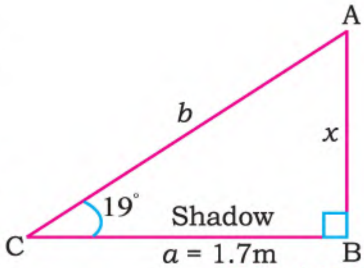
By Pythagoras theorem

$$a^2 = b^2 + c^2 = (20.2)^2 + (24.0762)^2 = 987.7$$

Therefore, $a = 31.42$ cm.

Example 2. Find the height of an object if the angle of elevation of the sun is 19° and the length of the shadow of the object is 1.7 meters.

Solution: Let x be the height of the object. Its shadow has length $a = 1.7$ m as shown in Fig. 11.3.



(Fig. 11.3)

Then, in ΔABC, $\tan 19° = \frac{x}{a}$

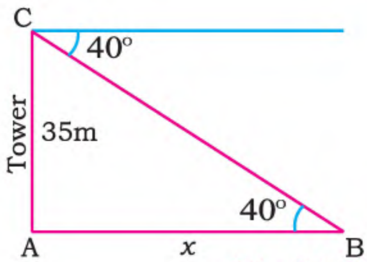
$\Rightarrow x = 1.7 \tan 19°$

$\Rightarrow x = 0.585$

So, the height of the object is 0.585 meters

Example 3. From the top of a tower, the angle of depression to the ship at its waterline is 40°. If the height of the tower is 35m, find the distance between the ship and the foot of the tower.

Solution: Let x be the distance of the ship B from the foot A of the tower $\overline{AC}$ as shown in Fig. 11.4.



(Fig. 11.4)

Then, in ΔABC, $\tan 40° = \frac{35}{x}$

$x = \frac{35}{\tan 40°}$

$x = 41.711$

Thus, the distance of ship from the foot of the tower is 41.711 meters.

Exercise 11.1

1. Solve the right angled triangle ABC in which $\gamma = 90°$

(i) $a = 4\text{m}$, $b = 3\text{m}$ (ii) $a = 12\text{m}$, $b = 5\text{m}$

(iii) $a = 8\text{m}$, $b = 6\text{m}$ (iv) $c = 8\text{m}$, $\beta = 42° 30'$

(v) $a = 140\text{m}$, $\alpha = 38° 20'$ (vi) $b = 30.8m$, $\alpha = 41° 50'$

2. A vertical stick 16m long, casts a 12cm long shadow. Find the angle of elevation of the sun.

3. Find the distance of a man from the foot a tower 169m high. if the angle of depression of the man from its top is $48° 30'$.

4. A string of a flying kite is 200 meters long and its angle of elevation is $58° 30'$. Find height of the kite when the string is fully stretched.

5. A man 1.8m tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is $31°45'$. What is the height of tree?

6. From the top of a cliff 80m high, the angle of depression of a boat is $12°30'$. How far is the boat from the cliff?

7. Two masts are 20m and 12m high. If the line joining their tops makes an angle of 35° with the horizontal; find distance between them.

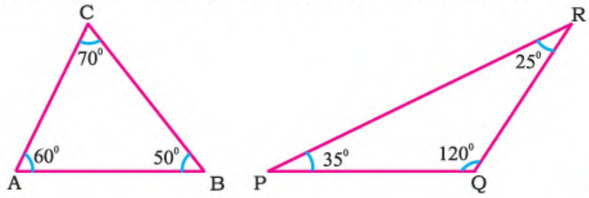
8. A window washer is working in a hotel building. An observer at a distance of 20 m from the building finds the angle of elevation of the worker to be of 30°. The worker climbs up 12m and the observer moves 4m further away from the building. Find the new angle of elevation of the worker.

11.1.2 Define an oblique triangle and prove

- **The laws of sines**
- **The laws of cosines**
- **The laws of tangents**

and deduce respective half angle formulae

An oblique triangle is a triangle with no right angle. It has either three acute angles, or one obtuse and two acute angles. In Fig 11.5, ΔABC and ΔPQR are oblique triangles. An oblique triangle can be solved, if a side and any two other elements are known by using the following laws.



(Fig. 11.5)

(i) The law of sines

Consider a triangle ABC, in which a, b, c are the measures of sides opposite to the angles α, β and γ respectively as shown in Fig.11.6. Take a rectangular coordinate system, in order to place the point C in the standard position. The coordinates of point A will be (b cosγ , b sinγ).

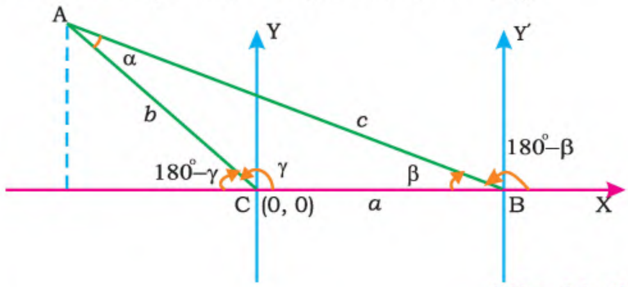
If the point B is taken as the origin and measure of the angle XBA $= 180° - \beta$ then the point A will have the coordinates $(c\cos(180°-\beta), c\sin(180°-\beta))$.

Since y – coordinate is the same in both the cases; we have

$$b\sin\gamma = c\sin(180°-\beta)$$

$$\Rightarrow \quad b\sin\gamma = c\sin\beta$$

$$\Rightarrow \quad \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad \ldots \text{(i)}$$



(Fig. 11.6)

Similarly, we have

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \quad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}} \quad \ldots \text{(iii)}$$

This is called the law of sines, which states that the measures of the sides of the triangle are proportional to the sines of the measures of the opposite angles. This rule is due to a muslim mathematician Al-Beruni.

(ii) The law of cosines

Consider Fig.11.6, the vertices of the triangle ABC are $A(b\cos\gamma, b\sin\gamma)$, $B(a,0)$ and C(0,0). By using distance formula,

$$|\overline{AB}| = \sqrt{(b\cos\gamma - a)^2 + (b\sin\gamma - 0)^2}.$$

So,

$$c = \sqrt{(b^2\cos^2\gamma - 2ab\cos\gamma + a^2 + b^2\sin^2\gamma}$$

$$\Rightarrow \quad c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma} \quad \ldots (1)$$

Similarly, we can prove that

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha} \quad \ldots (2)$$

and

$$\boxed{b^2 = a^2 + c^2 - 2ac\cos\beta} \quad \ldots (3)$$

The above rules (1), (2) and (3) are called the laws of cosines.
The laws can also be written as:

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc},\ \cos\beta = \frac{a^2+c^2-b^2}{2ac},\ \cos\gamma = \frac{a^2+b^2-c^2}{2ab}.$$

(iii) The laws of tangents

By the law of sines, $\frac{\sin\alpha}{\sin\beta} = \frac{a}{b} \Rightarrow \frac{\sin\alpha}{\sin\beta} - 1 = \frac{a}{b} - 1$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\sin\beta} = \frac{a-b}{b} \quad \dots (1)$$

Again $\frac{\sin\alpha}{\sin\beta} + 1 = \frac{a}{b} + 1$

$$\Rightarrow \quad \frac{\sin\alpha + \sin\beta}{\sin\beta} = \frac{a+b}{b} \quad \dots (2)$$

From (1) and (2), by division, we get

$$\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{a-b}{a+b}$$

$$\Rightarrow \quad \frac{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}} = \frac{a-b}{a+b}$$

$$\Rightarrow \quad \cot\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2} = \frac{a-b}{a+b}$$

$$\Rightarrow \quad \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} = \frac{a-b}{a+b} \quad \dots (3)$$

Similarly,

$$\frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} = \frac{b-c}{b+c} \quad \dots (4)$$

and

$$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} = \frac{c-a}{c+a} \quad \dots (5)$$

The above relations (3), (4) and (5) are called the laws of tangents.

(iv) Deduction of half angle formulae of Sine, Cosine and Tangent in terms of lengths of the sides of a triangle:

By the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$$\text{or} \qquad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \quad 1 - \cos\alpha = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \quad 1 - \cos\alpha = \frac{a^2 - b^2 - c^2 + 2bc}{2bc}$$

$$\Rightarrow \quad 2\sin^2\frac{\alpha}{2} = \frac{a^2 - b^2 - c^2 + 2bc}{2bc}$$

$$= \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

So, $$2\sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a-b+c)}{2bc} \qquad \text{... (i)}$$

Again, we take $$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow 1 + \cos\alpha = 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \quad 2\cos^2\frac{\alpha}{2} = \frac{b^2 + c^2 - a^2 + 2bc}{2bc}$$

$$= \frac{(b^2 + c^2 + 2bc) - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

So, $$2\cos^2\frac{\alpha}{2} = \frac{(a+b+c)(b+c-a)}{2bc} \qquad \text{... (ii)}$$

By dividing equation (i) by equation (ii),

We get, $$\frac{2\sin^2\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}$$

or $$\tan^2\frac{\alpha}{2} = \frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)} \qquad \text{... (iii)}$$

If we set, $s = \frac{1}{2}(a + b + c)$.

then we have, $a + b = 2s - c$

or $a + b - c = 2s - 2c$

Similarly, $a + c - b = 2s - 2b$

and $b + c - a = 2s - 2a$

Half angle formulae of sine in terms of sides of triangle:

From equation (i), $2\sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a-b+c)}{2bc}$

or $2\sin^2\frac{\alpha}{2} = \frac{(2s-2c)(2s-2b)}{2bc}$

$\sin^2\frac{\alpha}{2} = \frac{(s-b)(s-c)}{b\,c}$

Thus, $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{b\,c}}$

Similarly, $\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{a\,c}}$

and $\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Half angle formulae of cosine in terms of sides of triangle:

From equation (ii),

$2\cos^2\frac{\alpha}{2} = \frac{(a+b+c)(b+c-a)}{2bc}$

or $2\cos^2\frac{\alpha}{2} = \frac{2s(2s-2a)}{2bc}$

$\Rightarrow$ $\cos^2\frac{\alpha}{2} = \frac{s(s-a)}{b\,c}$

$\Rightarrow$ $\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{b\,c}}$

Similarly, $\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{a\,c}}$

and $\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Half angle formulae of tangent in terms of sides of triangle:

From equation (iii), $\tan^2\frac{\alpha}{2} = \frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)} = \frac{(s-c)(s-b)}{s(s-a)}$

$\Rightarrow$ $\tan\frac{\alpha}{2} = \sqrt{\frac{(s-c)(s-b)}{s(s-a)}}$

Similarly, $\tan\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

and $\tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

If we put $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$,

then, $\boxed{\tan\frac{\alpha}{2} = \frac{r}{s-a}}$

Similarly, $\tan\frac{\beta}{2} = \frac{r}{s-b}$

and $\tan\frac{\gamma}{2} = \frac{r}{s-c}$

11.1.3 Apply above laws to solve oblique triangles

If we are given any type of triangle, we can find its unknown sides and angles using sine, cosine and tangent laws and also with the help of their half angle formulas.

Example 1. Solve the triangle ABC in which $\alpha = 49°, \beta = 60°$ and $c = 39\ cm$.

Solution: Here, we have to find, γ, a and b.

Since $\alpha + \beta + \gamma = 180°$

$$\gamma = 180° - \alpha - \beta$$
$$= 180° - 49 - 60$$
$$= 71°$$

By the law of sines, $\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$

$$\Rightarrow \frac{a}{\sin 49°} = \frac{39}{\sin 71°}$$
$$\Rightarrow a = \frac{39\sin 49°}{\sin 71°}$$
$$\Rightarrow a = 31.13\text{cm}$$

Again by the law of sines, $\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

$$\Rightarrow \frac{b}{\sin 60°} = \frac{39}{\sin 71°}$$
$$\Rightarrow \text{b} = \frac{39\sin 60°}{\sin 71°}$$
$$\Rightarrow \text{b} = 35.72\text{ cm.}$$

Example 2. Solve the triangle ABC in which $a = 70$ cm, $c = 58$ cm and $\beta = 55°$

Solution: Here, we have to find b, α and γ

By using the law of tangents.

$$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} = \frac{c-a}{c+a} = \frac{58-70}{58+70} = \frac{-12}{128} = -0.09375 \qquad \ldots(i)$$

Now, $\gamma + \alpha = 180° - \beta = 180° - 55° = 125°$

$$\Rightarrow \quad \frac{\gamma+\alpha}{2} = \frac{125°}{2} = 62.5°$$

So, from equation (i),

we have
$$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan 62.5°} = -0.09375$$

$$\tan\frac{\gamma-\alpha}{2} = -0.09375 \tan 62.5°$$

$$\Rightarrow \quad \tan\frac{\gamma-\alpha}{2} = -0.1801$$

$$\Rightarrow \quad \frac{\gamma-\alpha}{2} = -10.21°$$

or $\qquad \gamma-\alpha = -20.42° \quad$ But, $\quad \gamma+\alpha = 125°$

So, $\qquad 2\gamma = 104.58°$

or $\qquad \gamma = 52.29°$

Now, $\quad \alpha = 180° - \beta - \gamma = 180° - 55° - 52.29° = 72.71°$

Now, by the law of sines, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$$\frac{70}{\sin 72.71°} = \frac{b}{\sin 55°} \quad \Rightarrow \quad b = \frac{70\sin 55°}{\sin 72.71°}$$

$$\Rightarrow \quad b = 60.05 \text{ cm.}$$

Note: The above example may also be solved by using laws of cosines.

Example 3. Find the measure of the largest angle in the triangle ABC with $a = 10\text{cm}$, $b = 20\text{cm}$ and $c = 26\text{cm}$.

Solution: Here $\qquad s = \frac{1}{2}(a+b+c) = \frac{1}{2}(10+20+26) = 28$

The side c is greater than the sides a and b, so γ will be the largest angle in the given triangle.

By using, $\qquad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

$$= \sqrt{\frac{(18)(8)}{(10)(20)}}$$

$$= 0.8485$$

$$\Rightarrow \quad \frac{\gamma}{2} = \sin^{-1}(0.8485)$$

$$\Rightarrow \quad \frac{\gamma}{2} = 58°$$

$\Rightarrow \quad \gamma = 116°$ is the largest angle of the triangle

Exercise 11.2

1. Solve the following triangles by using law of sines.

(i) $a = 70,\ b = 34,\ \alpha = 79°$ (ii) $b = 136,\ \gamma = 104.2°,\ \alpha = 43.1°$

(iii) $\alpha = 48°45',\ \beta = 69°15',\ c = 40.5$

2. Solve the following triangles by using laws of cosines.

(i) $\alpha = 78.2°,\ b = 62.4,\ c = 150$ (ii) $a = 48,\ b = 33.3,\ \gamma = 41°30'$

(iii) $a = 73,\ c = 40,\ \beta = 76°45'$

3. Solve by using laws of tangents.

(i) $a = 432,\ c = 325,\ \beta = 42°$ (ii) $b = 39,\ c = 35,\ \alpha = 75°$

(iii) $a = 39.14,\ b = 34.21,\ \gamma = 78°10'$

4. Solve the triangle by using suitable laws.

(i) $a = 9,\ b = 7,\ c = 5$ (ii) $b = 35,\ a = 37, \alpha = 23°25'$

5. Solve the following triangles by using half angle formula of sines.

(i) $a = 55,\ c = 65,\ b = 85$

(ii) $b = 15,\ a = 20,\ c = 14$

(iii) $c = 1.3,\ b = 2.3,\ a = 2.7$

6. Solve the following triangles by using half angle formula of cosines.

(i) $a = 95,\ b = 76,\ c = 85$

(ii) $a = 10,\ b = 7,\ c = 5$

(iii) $c = 10,\ a = 15,\ b = 7$

7. Solve the following by using half angle formula of tangent.

(i) $c = 23,\ a = 13,\ b = 16$

(ii) $a = 25,\ b = 20,\ c = 18$

(iii) $a = 16,\ b = 11,\ c = 13$

8. Find the largest angle of ΔABC, when

$a = 6\text{cm},\ b = 8\text{ cm}$ and $c = 9.4\text{ cm}$

9. Find the smallest angle in ΔABC, when

$a = 25\text{ cm},\ b = 18\text{ cm}$ and $c = 21\text{ cm}$

10. Find the length of the third side of a triangular building that faces 13.6 meters along one street and 13 meters along another street. The angle of intersection between the streets measures72°.

11. If one side of a triangle is y units long, another side is 3 times as long and the angle between the two sides measure 35°, find the measures of other two angles and the third side.

12. If the length of larger side of a parallelogram is 55 cm and one diagonal of the parallelogram makes angles of measure 30° and 50° with a pair of adjacent sides, find the length of the diagonal.

13. The sides of a parallelogram are 25cm and 35cm long and one of its angles is 36°. Find the lengths of its diagonals.

14. Two hikers start from the same point; one walks 9 km heading east, the other one 10 km heading 55° north east. How far apart are they at the end of their walks?

15. Two planes start from Karachi International Airport at the same time and fly in directions that make an angle of 127° with each other. Their speeds are 525km/h. How far apart they are at the end of 2 hours of flying time?

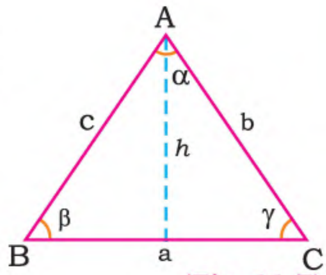
11.2 Area of a Triangle

11.2.1 Derive the formulae to find the area of a triangle in terms of the measures of

- two sides and their included angle,
- one side and two angles,
- three sides (Hero's formula)

(i) Two sides and their included angle

Consider a triangle ABC, in which h is its altitude as shown in Fig. 11.7. We know, from elementary geometry, that



(Fig. 11.7)

Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{altitude}$

or $\quad \blacktriangle = \frac{1}{2}\, a\, h$

i.e., $\quad \boxed{\blacktriangle = \frac{1}{2} ab \sin\gamma} \quad (\because h = b\sin\gamma)$

where ▲ denotes the area of the triangle.

Similarly, $\quad \blacktriangle = \frac{1}{2} ac\sin\beta$ and $\blacktriangle = \frac{1}{2} bc\sin\alpha$

Example: Find the area of triangle ABC, in which $a = 5.34$ cm, $b = 9.3$ cm and $\gamma = 53^\circ\, 34'$.

Solution: We know that, $\quad \blacktriangle = \frac{1}{2} ab\sin\gamma$

So, $\quad \blacktriangle = \frac{1}{2}(5.34)(9.3)\sin 53^\circ\, 34'$

$\Rightarrow \quad \blacktriangle = \frac{39.918}{2} = 19.98$ sq. cm

(ii) One side and two angles

By the law of sines, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$\Rightarrow \quad a = \dfrac{c\sin\alpha}{\sin\gamma}$ and $b = \dfrac{c\sin\beta}{\sin\gamma}$ ...(i)

We know that

Area of the triangle $\quad \blacktriangle = \dfrac{1}{2}ab\sin\gamma$

So, $\quad \blacktriangle = \dfrac{1}{2}\dfrac{c\sin\alpha}{\sin\gamma}\cdot\dfrac{c\sin\beta}{\sin\gamma}\sin\gamma \quad$ (using equation (i))

or $\quad \boxed{\blacktriangle = \dfrac{1}{2}c^2\dfrac{\sin\alpha\sin\beta}{\sin\gamma}}$

Similarly, $\quad \blacktriangle = \dfrac{1}{2}b^2\dfrac{\sin\alpha\sin\gamma}{\sin\beta}$ and $\blacktriangle = \dfrac{1}{2}a^2\dfrac{\sin\gamma\sin\beta}{\sin\alpha}$

Example: Find the area of triangle ABC, with a =36cm, β =46° and γ = 66°.

Solution: Here $\quad \alpha = 180° - \beta - \gamma = 180° - 46° - 66° = 68°$

Now $\quad \blacktriangle = \dfrac{1}{2}a^2\dfrac{\sin\gamma\,\sin\beta}{\sin\alpha}$

$= \dfrac{1}{2}\dfrac{(36)^2\times\sin 66°\times\sin 46°}{\sin 68°} = \dfrac{1}{2}\dfrac{1296\times 0\cdot 913\times 0\cdot 719}{0\cdot 927}$

$= 459.27$ sq. cm

(iii) Three sides (Hero's formula)

As Area of triangle $\quad \blacktriangle = \dfrac{1}{2}ac\sin\beta,$

So, $\quad \blacktriangle = ac\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2} \quad \left[\because \sin\beta = 2\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2}\right]$

or $\blacktriangle = ac\sqrt{\dfrac{(s-a)(s-c)}{ac}\cdot\dfrac{s(s-b)}{ac}}, \quad \left[\because \sin\dfrac{\beta}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}} \text{ and } \cos\dfrac{\beta}{2} = \sqrt{\dfrac{s(s-b)}{ac}}\right]$

i.e., $\quad \boxed{\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}}$

Example: Find the area of triangle ABC with a = 120 cm, b = 200cm, c = 98 cm.

Solution: Here, $s = \dfrac{1}{2}(a+b+c) = \dfrac{1}{2}(120+200+98) = 209$

Then, $s-a = 209-120 = 89, \quad s-b = 9, \quad s-c = 111$

By Hero's formula, $\quad \blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$

or $\quad \blacktriangle = \sqrt{(209)(89)(9)(111)} = \sqrt{18582399}$

or $\quad \blacktriangle = 4310.7$ sq. cm.

Exercise 11.3

1. Find the area of triangle ABC when

   i. $a = 8.6,\ b = 11.4,\ \gamma = 90°$ ii. $b = 63,\ c = 17,\ \alpha = 120°$

   iii. $c = 5.34,\ a = 9.30,\ \beta = 53°34'$ iv. $c = 36,\ \alpha = 46°,\ \beta = 66°$

   v. $a = 54,\ \beta = 37°,\ \gamma = 54°$ vi. $b = 4.8,\ \alpha = 35°9',\ \gamma = 85°31'$

   vii. $a = 37,\ b = 41,\ c = 37$ viii. $a = 41.34,\ b = 35.65,\ c = 56.81$

   ix. $a = 98,\ b = 120,\ c = 200$

2. The area of triangle is 3.346 square unit. If $\beta = 20.9°, \gamma = 117.2°$. Find a and angle α.
3. The measures of the two sides of a triangle are 4 and 5 units. Find the third side so that area of the triangle is 6 square units.
4. Find area of the equilateral triangle whose each side is x units long.

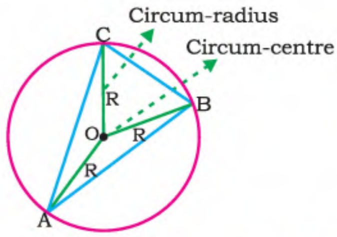
11.3 Circles connected with triangle

There are three types of circles connected with the triangle namely circum-circle, in-circle and escribed-circle.

11.3.1 Define circum-circle, in-circle and escribed-circle

(i) Circum-circle

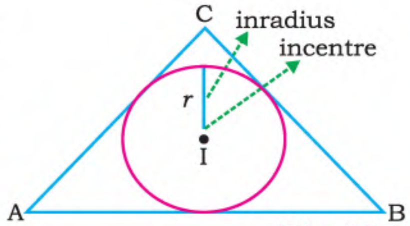
Consider a triangle ABC, (Fig. 11.8). The circle which passes through vertices A, B and C of the triangle is called the *circum-circle* of the triangle. The centre of the circle is called *circum-centre* and the radius is called *circum-radius* and is denoted by R.



(Fig. 11.8)

(ii) in-circle

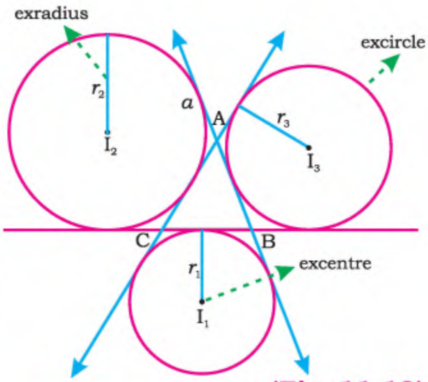
The circle inscribed within a triangle so that it touches all the sides of the triangle is called the in-circle of the triangle (Fig. 11.9). The centre and the radius of this circle are called in-centre and in-radius respectively. Incentre is denoted by I and in-radius by r.



(Fig. 11.9)

(iii) Escribed circle or excircle

A circle which touches one side of the triangle externally and two extended sides internally is called *escribed-circle* or ex-circle or *e-circle*. The centre and radius of excircle opposite to vertex A are denoted by I_1 and r_1 respectively. Similarly the centers and radii of the e-circles opposite to the vertices B and C are denoted by I_2, I_3 and r_2, r_3 respectively as shown in Fig. 11.10.



(Fig. 11.10)

11.3.2 Derive the formulae to find
- **circum-radius,**
- **in-radius,**
- **escribed-radii,**

and apply them to deduce different identities.

(i) Circum-radius of a triangle

Let O be the circum-centre of the circum-circle of triangle ABC. Join O and B. Produce $\overline{BO}$ to D and join C with D, then $m\overline{BD} = 2R$. If the triangle ABC is acute angled triangle (Fig 11.11) then $m\angle A = \alpha = m\angle D$. If the triangle ABC is obtuse angled triangle (Fig 11.12) then $m\angle D = \pi - \alpha$.

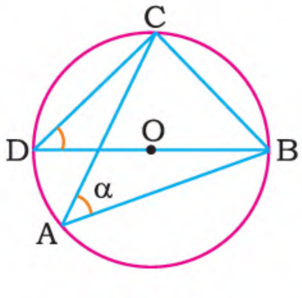


Fig. 11.11

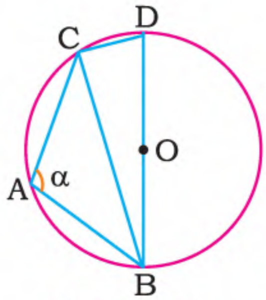


Fig. 11.12

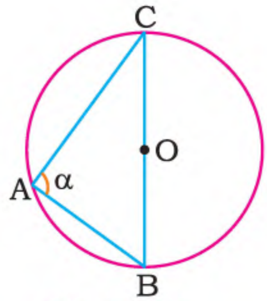


Fig. 11.13

Since, $\sin(\pi - \alpha) = \sin\alpha$,

Therefore, in both cases

$$\sin(m < D) = \sin\alpha.$$

But $\sin(m < D) = \sin\alpha = \frac{m\overline{BC}}{m\overline{BD}} = \frac{a}{2R}$,

so $R = \frac{a}{2\sin\alpha}$

If the triangle is a right triangle (Fig 11.13), then $\sin\alpha = \frac{a}{2R} = 1$ $(\because\ \alpha = 90°)$

Thus $R = \frac{a}{2\sin\alpha}$

Similarly, we can show that $R = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$

Hence, we have

$$\boxed{\frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} = R}$$

Now, $R = \frac{a}{2\sin\alpha}$

$= \frac{a}{2\left(\frac{2\blacktriangle}{bc}\right)}$ $\left(\because\ \blacktriangle = \frac{1}{2}bc\sin\alpha\right)$

Therefore, $\boxed{R = \frac{abc}{4\blacktriangle}}$ where, $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$.

Example: Find the value of R in triangle ABC where $a = 10\ cm$, $b = 8\ cm$ and $c = 5\ cm$

Solution:

Here, $s = \frac{1}{2}(a + b + c)$

$= \frac{1}{2}(10 + 8 + 5) = \frac{23}{2} = 11.5\text{cm}$

and $s - a = 11.5 - 10 = 1.5$, $s - b = 3.5$, $s - c = 6.5$

Now, $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{11.5(1.5)(3.5)(6.5)}$

$= \sqrt{392.4375}$

$\blacktriangle = 19.81$ square cm

We know that $R = \frac{abc}{4\blacktriangle}$

So, $R = \frac{(10)(8)(5)}{4(19.81)} = \frac{100}{19.81} = 5.04cm$

(ii) In-Radius of a triangle

Consider the figure (11.14). An in-circle is a circle inscribed in a triangle ABC we find the in-radius for the circle as follows:

Draw the bisectors of angles A, B and C so that they meet at I, the in-centre. Draw perpendiculars $\overline{IF}, \overline{IE}$ and $\overline{ID}$ on $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively.

If r is the in-radius, then we have

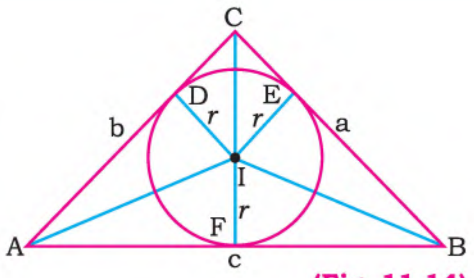
$$m\overline{IF} = m\overline{IE} = m\overline{ID} = r.$$

Since Area of ΔABC=▲ABI+▲BCI+▲CAI,

$$= \frac{1}{2}\ cr + \frac{1}{2}\ ar + \frac{1}{2}\ br$$

$$= \frac{1}{2}\ r(a + b + c)$$

$$▲\ = r.\ s$$



(Fig. 11.14)

Therefore, $$r = \frac{▲}{s}$$

Example: Find the value of in- radius of a circle inscribed in a triangle, where $a = \sqrt{2}\ cm,\ and\ \ b = c = 1\ cm$

Solution: We know that $r = \frac{\Delta}{s}$

$$▲= \sqrt{s(s-a)(s-b)(s-c)}$$

Hence,

$$s = \frac{1}{2}(a + b + c)$$

$$s = \frac{1}{2}\left(\sqrt{2} + 1 + 1\right) = \frac{1}{2} = (1.41 + 2) = \frac{3.41}{2} = 1.71$$

$$s - a = 1.71 - \sqrt{2} = 1.71 - 1.41 = 0.30$$

$$s - b = 1.71 - 1 = 0.71$$

$$s - c = 1.71 - 1 = 0.71$$

Now, $$▲\ = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{1.71(0.30)(0.71)(0.71)}\ \ = \sqrt{2586}$$

$$▲\ \ = 0.5085 \text{ square cm}$$

Now, $$r = \frac{▲}{s} = \frac{0.5085}{0.71} = 0.29cm$$

(iii) Radii of e-circles of a triangle

Consider a triangle ABC (Fig. 11.15). Let I_1, be the excentre of e-circle opposite to the vertex A of the triangle. Draw perpendiculars $\overline{I_1P_1}$, $\overline{I_1E_1}$ and $\overline{I_1F_1}$ on $\overline{CB}$, $\overrightarrow{AO}$ and $\overrightarrow{AR}$ respectively.

then $m\overline{F_1I_1} = m\overline{E_1I_1} = m\overline{P_1I_1} = r_1$.

where r_1 is the radius of the e-circle, we have

Area of triangle ABC =

Area of triangle I_1CA +Area of triangle I_1BA – Area of triangle I_1BC

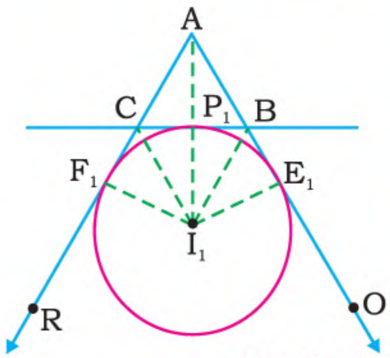
Since, $\blacktriangle = \frac{1}{2}br_1 + \frac{1}{2}cr_1 - \frac{1}{2}ar_1$

Therefore,

$\frac{1}{2}r_1(-a+b+c) = \frac{1}{2}r_1(2s-2a) = r_1(s-a)$

So, $$r_1 = \frac{\blacktriangle}{s-a}$$

Similarly, $r_2 = \frac{\blacktriangle}{s-b}$ and $r_3 = \frac{\blacktriangle}{s-c}$



(Fig. 11.15)

Example: Find the radii r_1, r_2, r_3 of the escribed circles of a triangle ABC in

(i) $a = b = 5.5\ cm$ and $c = 9\ cm$ (ii) $a = b = c$

Solution (i):

Here $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(5.5+5.5+9) = 10\text{cm}$

and $s-a = 10-5.5 = 4.5$, $s-b = 10-5.5 = 4.5$, $s-c = 10-9 = 1$

Now, $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{10(4.5)(4.5)(1)} = 4.5\sqrt{10}$

We know that $r_1 = \frac{\blacktriangle}{s-a} = \frac{4.5\sqrt{10}}{4.5} = 3.16$ cm, $r_2 = 3.16$ cm and $r_3 = \frac{\blacktriangle}{s-c} = 14.23$ cm

Solution (ii):

Here, $a = b = c = x$

So, $S = \frac{1}{2}(a+b+c) = \frac{3x}{2}$

and $s-a = \frac{3x}{2} - x = \frac{x}{2}$, $s-b = s-c = \frac{x}{2}$

Now $\blacktriangle = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{3x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}} = \frac{x^2\sqrt{3}}{4}$ sq. units

Hence, $r_1 = \frac{\blacktriangle}{s-a} = \frac{\frac{x^2\sqrt{3}}{4}}{\frac{x}{2}} = \frac{x^2\sqrt{3}}{4}.\frac{2}{x} = \frac{\sqrt{3}}{2}x$,

Similarly, $r_2 = r_3 = \frac{\sqrt{3}x}{2}$ units.

Exercise 11.4

1. Find the values of R and r of ΔABC, when

(i) $a = 5cm$, $b = 10cm$, $c = 12cm$

(ii) $a = 10.5cm$, $b = 11.5cm$, $c = 20.5cm$

(iii) $a = 40cm$, $b = 12cm$, $c = 38cm$

2. Find r_1, r_2 and r_3, if the measures of sides of triangle ABC are

(i) $a = 23cm,\quad b = 24cm,\quad c = 28cm$

(ii) $a = 24.4cm,\quad b = 34.8cm,\quad c = 42.5cm$

(iii) $a = 75cm,\quad b = 62cm,\quad c = 53cm$

3. If $a = b = c$, then prove that

$$r_1 : \mathrm{R} : r = 3:2:1$$

4. Show that:(i) $r_1 r_2 r_3 = rs^2$ (ii) $r = 4\mathrm{R}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ (iii)$r_1 = \frac{a.\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

5. Prove that $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\blacktriangle^2}$

6. Show that $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

Review Exercise 11

1. **Select correct answer.**

i. From the top of a cliff 80m high the angle of depression of a boat is α. If the distance between the boat and foot of cliff is $80\sqrt{3}$ m, then angle α is: (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

ii. To solve an oblique triangle, we use:
(a) law of sines (b) Laws of cosines (c) Laws of tangents (d) All of these

iii. A circle which touches all the sides of a triangle is called __________.
(a) circum-circle (b) in-circle (c) ex-circle (d) tri-circle

iv. In a triangle ABC, if $\beta = 90°$, then $b^2 = c^2 + a^2 - 2ac\cos\beta$ becomes:
(a) Law of sines (b) Law of tangents
(c) Pythagoras theorem (d) None of these

v. In any triangle ABC, $\sqrt{\frac{(s-a)(s-c)}{ac}}$ is equal to:
(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\sin\frac{\beta}{2}$ (d) $\sin\frac{\gamma}{2}$

vi. In any triangle ABC, $\cos\frac{\gamma}{2}$ is equal to:
(a) $\sqrt{\frac{s(s-a)}{ab}}$ (b) $\sqrt{\frac{s(s-b)}{ac}}$ (c) $\sqrt{\frac{s(s-a)}{bc}}$ (d) $\sqrt{\frac{s(s-c)}{ab}}$

vii. In any triangle ABC, with usual notations, s is equal to:
(a) $a + b + c$ (b) $\frac{a+b+c}{2}$ (c) $\frac{a+b+c}{3}$ (d) $\frac{abc}{2}$

viii. $\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \cdots$ (a) $\sin\frac{\beta}{2}$ (b) $\cos\frac{\beta}{2}$ (c) $\tan\frac{\beta}{2}$ (d) $\cot\frac{\beta}{2}$

ix. In any triangle ABC, $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ is equal to:

(a) $\sin\frac{\gamma}{2}$ (b) $\cos\frac{\gamma}{2}$ (c) $\tan\frac{\gamma}{2}$ (d) $\cot\frac{\gamma}{2}$

x. We can solve an oblique triangle, if:

(a) One side and two angles are known

(b) Three sides are known

(c) Two sides and their included angles are known

(d) All of these

xi. In ΔABC, inradius = ------------------

(a) $\frac{▲}{s-a}$ (b) $\frac{▲}{s-b}$ (c) $\frac{▲}{s-c}$ (d) $\frac{▲}{s}$

xii. In any triangle ABC, Area of triangle is:

(a) $bc\sin\alpha$ (b) $\frac{1}{2}ca\sin\alpha$ (c) $\frac{1}{2}ab\sin\gamma$ (d) $\frac{1}{2}ab\sin\beta$

xiii. Hero's formula is:

(a) $▲= s(s-a)(s-b)(s-c)$ (b) $▲=\sqrt{(s-a)(s-b)(s-c)}$

(c) $▲=\sqrt{s(s-a)(s-b)(s-c)}$ (d) $▲=\frac{a+b+c}{2}$

xiv. The circle passing through the three vertices of a triangle is called:

(a) Circum-circle (b) In-circle (c) Ex-centre (d) Escribed circle

xv. The point of intersection of the right bisectors of the sides of a triangle is called:

(a) Circum-centre (b)In-centre (c)Escribed centre (d) Ortho-centre

xvi. Radius of the circle which passes through all the vertices of a triangle is:

(a) Circum-radius (b) In-radius (c) e-Radius (d) Diameter

xvii. In any triangle ABC, with usual notations, $abc =$

(a) R (b) Rs (c) 4R▲ (d) $\frac{▲}{s}$

xviii. The point of intersection of the internal bisectors of angles of a triangle is:

(a) In-centre (b) e-centre (c) Circum-centre (d) Ex-centre

xix. In any equilateral triangle ABC, with usual notations, $r:R:r_1$

(a) 1: 2: 3 (b) 3: 2: 1 (c) 1: 3: 2 (d) 1: 1: 1

xx. Circum radius of ΔABC is:

(a) $\frac{▲}{S}$ (b) $\frac{▲}{S-b}$ (c) $\frac{▲}{S-a}$ (d) $\frac{abc}{4▲}$

2. Prove that:

(i) $r_2 = \frac{b\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$ (ii) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

3. Find the area of triangle ABC when
(i). $a = 5.2cm$, $\beta = 30°$, $\gamma = 40°$ (ii)$a = 125cm$, $b = 120$, $\gamma = 150°$

4. Find the R and r of ΔABC when:
(i) $a = 18.8cm, b = 24.5cm$, $c = 30.2c$ (ii) $a = 13.8cm, b = 13.8cm, c = 10.4cm$

5. Find r_1, r_2 and r_3, of the escribed circle if the measures of sides of triangle ABC are $a = 3$cm, $b = 4$cm and $c = 5$cm.

6. A hiker walks due east at 4 km per hour and a second hiker, starting at the same point, walks 55° north-east at the rate of 5 km per hour. How far apart will they be after 3 hours?

7. Three points A, B, C form a triangle such that the ratio of the measures of their angles is 1 : 2 : 3. Find the ratio of the lengths of the sides.

8. A piece of plastic strip 1 meter long is bent to form an isosceles triangle with 95° as measure of its largest angle. Find the length of the sides.

9. Two airplanes leave a field at the same time. One flies 30° East of North at 250km/h, the other 45° East of South at 300km/h. How far apart are they at the end of 2 hours?

10. Two men are on the opposite sides of a 100m high tower. If the measures of the angles of elevation of the top of the tower are 18° and 22° respectively. Find the distance between them.

11. A man standing 60m away from a tower notices that the angles of elevation of the top and the bottom of a flagstaff on the top of the tower are 64° and 62° respectively. Find the length of the flagstaff.

12. The angle of elevation of the top of a 60m high tower from a point A, on the same level as the foot of the tower, is 25°. Find the angle of elevation of the top of the tower from a point B, 20m nearer to A from the foot of the tower.

13. Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20°. The angle of elevation from the base of the building B to the top of the building A is 50°. Find the height of the building B.

Graphs of Trigonometric and Inverse Trigonometric Functions and Solution of Trigonometric Equations

Unit 12

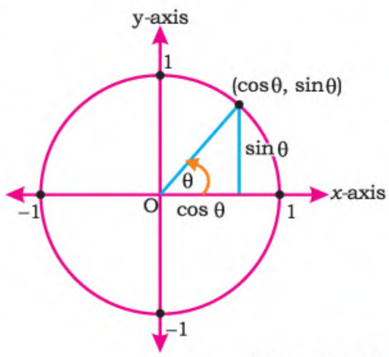
12.1 Period of Trigonometric Functions

12.1.1 Find the domain and range of the trigonometric functions

The method of finding domain and range of a function has already been discussed in section 8.1.1. Here, we discuss domain and range of trigonometric functions.

Domain and Range of sinθ, cosθ and tanθ

(i) Function $y = \sin\theta$ is defined as the ordinate (y-coordinate) of a point on a unit circle that corresponds to an angle of θ radians. Therefore, the domain of this function is the set of all real numbers as θ can be any real number and the range is set of all real numbers from −1 to +1 as the maximum and minimum values of y are 1 and −1 respectively. (Fig 12.1)



(Fig. 12.1)

Example: Find the domain and range of $2\sin 5x$

Solution: Here, $y = 2\sin 5x$

∵ Given function is defined for all real numbers.

∴ Domain of $2\sin 5x = \mathbb{R}$

As Range of $\sin 5x = \{y | y \in \mathbb{R} \wedge -1 \le y \le 1\}$

So, Range of $2\sin 5x = \{y | y \in \mathbb{R} \wedge -2 \le y \le 2\}$

(ii) Function $y = \cos\theta$ is defined as the abscissa (x-coordinate) of a point on a unit circle that corresponds to an angle of θ radians. Therefore, the domain of this function is the set of all real numbers as θ can be any real number and the range is set of all real numbers from −1 to +1 as the maximum and minimum values of y are 1 and −1 respectively. (Fig 12.1)

Example: Find the domain and range of $\cos 7x$

Solution: Here, $y = \cos 7x$

$\because$ Given function is defined for all real numbers

$\therefore$ Domain of $\cos 7x = \mathbb{R}$

As the range of $\cos 7x$ is same as the range of $\cos x$

So, range of $\cos 7x = \{y | y \in \mathbb{R} \wedge -1 \le y \le 1\}$

(iii) Function $y = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is defined when $\cos\theta \ne 0$. The domain of this function is the set of all real numbers except those where $\cos\theta = 0$ and the angles where $y = \tan\theta$ is undefined are $\dfrac{n\pi}{2}$ radians where n is an odd integer. The range is the set of all real numbers between $-\infty$ and $+\infty$ as the minimum and maximum values of $\dfrac{\sin\theta}{\cos\theta}$ are $-\infty$ and $+\infty$ respectively.

Example: Find the domain and range of $\tan 4x$.

Solution: Here $y = \tan 4x$

We know that $\tan 4x = \dfrac{\sin 4x}{\cos 4x}$ is undefined if $\cos 4x = 0$

Let $\cos 4x = 0$

$\Rightarrow 4x = n\dfrac{\pi}{2}$ (where n is an odd integer)

$\Rightarrow x = n\dfrac{\pi}{8}$

So, domain of $\tan 4x = \mathbb{R} - \left\{n\dfrac{\pi}{8} \middle| n \text{ is an odd integer}\right\}$

$\because$ The range of $\tan 4x$ is same as the range of $\tan x$

$\therefore$ Range of $\tan 4x = \mathbb{R}$

For the trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\text{cosec}\theta$, $\sec\theta$, and $\cot\theta$ the domain and range are given in the following table.

Function	Domain	Range
$f(\theta) = \sin\theta$	$\mathbb{R}$	$-1 \le \sin\theta \le 1$
$f(\theta) = \cos\theta$	$\mathbb{R}$	$-1 \le \cos\theta \le 1$
$f(\theta) = \tan\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an odd integer}\}$	$\mathbb{R}$
$f(\theta) = \text{cosec}\,\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an even integer}\}$	All real numbers ≥ 1 and ≤ -1
$f(\theta) = \sec\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an odd integer}\}$	All real numbers ≥ 1 and ≤ -1
$f(\theta) = \cot\theta$	$\mathbb{R} - \{n\frac{\pi}{2} \mid n \text{ is an even integer}\}$	$\mathbb{R}$

12.1.2 Define even and odd functions

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in the domain of f and a function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x in the domain of f.

Example: Decide whether each of the following function is even, odd or neither. In the following functions, check whether the function is even or odd.

(i) $f(x) = \sqrt{x^4 + 5}$ (ii) $h(x) = x^5$ (iii) $k(x) = x + |x|$

Solution:

(i) $f(x) = \sqrt{x^4 + 5}$

$f(-x) = \sqrt{(-x)^4 + 5} = \sqrt{x^4 + 5}$ (Replacing x by $-x$)

$\because$ $f(-x) = f(x)$

$\therefore$ $f(x)$ is an even function.

(ii) $h(x) = x^5$

$h(-x) = (-x)^5 = -x^5$ (Replacing x by $-x$)

$\because$ $h(-x) = -h(x)$

$\therefore$ $h(x)$ is an odd function.

(iii) $k(x) = x + |x|$

$k(-x) = -x + |-x| = -x + |x|$ (Replacing x by $-x$)

i.e., $k(-x) \neq k(x)$ and $k(-x) \neq -k(x)$

Hence, $k(x)$ is neither even nor odd function.

Even and Odd Trigonometric functions

Consider the basic trigonometric functions $\sin x$, $\cos x$, and $\tan x$

(i) $f(x) = \sin x$

Replacing x by $-x$

We get, $f(-x) = \sin(-x)$

$= -\sin(x)$ ($\because$ $\sin(-x) = -\sin x$)

$= -f(x)$

Hence, $\sin x$ is an odd function

(ii) $f(x) = \cos x$

We get, $f(-x) = \cos(-x)$

$= \cos(x)$ ($\because$ $\cos(-x) = \cos x$)

$= f(x)$

Hence, cos x is even function

(iii) $f(x) = \tan x$

Replacing x by $-x$

We get, $f(-x) = \tan(-x)$

$= -\tan(x)$ $(\because \tan(-x) = -\tan x)$

$= -f(x)$

Hence, tan x is odd function.

Note: $\cot x$ and $\operatorname{cosec} x$ are odd functions, where $\sec x$ is an even function.

Example: Determine whether the following trigonometric functions are even, odd or neither.

(a) $f(x) = \sec x \tan x$ (b) $g(x) = x^3 \sin x \cos 2x$ (c) $h(x) = \cos x + \sin x$

Solution:

a) $f(x) = \sec x \tan x$

Here, $f(-x) = \sec(-x)\tan(-x)$

$= [\sec x][-\tan x]$ $(\because \sec(-x) = \sec x$ and $\tan(-x) = -\tan x)$

$= -\sec x \tan x$

$= -f(x)$

Hence, $f(x)$ is odd

b) $g(x) = x^3 \sin x \cos 2x$

Here, $g(-x) = (-x)^3 \sin(-x)\cos 2(-x)$

$= (-x^3)(-\sin x)(\cos 2x)$ $(\because \cos(-x) = \cos x$ and $\sin(-x) = -\sin x)$

$= x^3 \sin x \cos 2x$

$= g(x)$

Hence, $g(x)$ is even

c) $h(x) = \cos x + \sin x$

Here $h(-x) = \cos(-x) + \sin(-x)$ $(\because \sin(-x) = -\sin x$ and $\cos(-x) = \cos x)$

$= \cos x + (-\sin x)$

$= \cos x - \sin x$

$\neq -h(x)$ or $h(x)$

Hence, $h(x)$ is neither even nor odd

12.1.3 Discuss the periodicity of trigonometric functions. Find the maximum and minimum value of a given function of the type:

- $a + b\sin\theta$,
- $a + b\cos\theta$,
- $a + b\sin(c\theta + d)$,
- $a + b\cos(c\theta + d)$,
- the reciprocals of above

where *a*, *b*, *c* and d are real numbers.

12.1.3(a) Periodicity of the Trigonometric Functions

Let X and Y be the subsets of set of real numbers. A function $f : X \to Y$ is called a periodic function of period p if $f(x+p) = f(x)$, for all $x \in X$, and p is the smallest positive real number.

Since for any integer n, $\sin(\theta + 2n\pi) = \sin\theta$; $\cos(\theta + 2n\pi) = \cos\theta$ and $\tan(\theta + n\pi) = \tan\theta$. Therefore, sine and cosine are periodic functions of period 2π and tangent is a periodic function of period π.

Similarly, cosecant, secant and cotangent are periodic functions of period 2π, 2π and π respectively.

Since all trigonometric functions are periodic. Therefore, they repeat their values after specific interval and this property is called periodicity of trigonometric functions.

Example 1. Verify that $\sin x, \cos x$ and $\tan x$ have periods 2π, 2π and π respectively.

Solution:

(i) $f(x) = \sin x$

Replacing x by $x + 2\pi$

$$f(x+2\pi) = \sin(x+2\pi) = \sin x \cos 2\pi + \cos x \sin 2\pi$$
$$= \sin x(1) + \cos x(0)$$
$$= \sin x$$

i.e., $f(x+2\pi) = f(x)$

Hence, the period of $\sin x$ is 2π.

(ii) $f(x) = \cos x$

Replacing x by $x + 2\pi$

$$f(x+2\pi) = \cos(x+2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi$$
$$= \cos x(1) - \sin x(0)$$
$$= \cos x$$

Thus, $f(x+2\pi) = \cos x = f(x)$

Hence, the period of $\cos x$ is 2π.

(iii) $f(x) = \tan x$

Replacing x by $x + \pi$

$$f(x+\pi) = \tan(x+\pi) = \frac{\tan x + \tan\pi}{1 - \tan x \tan\pi}$$
$$= \frac{\tan x + 0}{1 - \tan x\,(0)} \qquad (\because \tan\pi = 0)$$
$$= \tan x$$

Thus, $f(x+\pi) = \tan x = f(x)$

Hence, the period of $\tan x$ is π.

Example 2. Find the periods of the following functions.

i. $5\sin x$ **ii.** $\cos 3x$ **iii.** $\frac{1}{6}\tan\frac{x}{6}$

Solution:

i. Here $f(x) = 5\sin x$

So, $f(x) = 5\sin(x+2\pi) = f(x+2\pi)$

$\because f(x) = f(x+2\pi)$

$\therefore$ period of $f(x)$ is 2π.

ii. Here, $f(x) = \cos 3x$

$$= \cos(3x + 2\pi)$$
$$= \cos 3\left(x + \frac{2\pi}{3}\right)$$
$$= f\left(x + \frac{2\pi}{3}\right)$$
$$\because f(x) = f\left(x + \frac{2\pi}{3}\right)$$

$\therefore f(x)$ has period of $\frac{2\pi}{3}$.

iii. Here, $f(x) = \frac{1}{6}\tan\frac{x}{6}$

$$= \frac{1}{6}\tan\left(\frac{x}{6} + \pi\right)$$
$$= \frac{1}{6}\tan\left(\frac{x+6\pi}{6}\right)$$
$$= f(x + 6\pi)$$
$$\because f(x) = f(x+6\pi)$$

$\therefore$ Period of $f(x)$ is 6π.

12.1.3(b) Find the maximum and minimum value of a given functions of the type

(i) $a + b\sin\theta$, **(ii)** $a + b\cos\theta$

(iii) $a + b\sin(c\theta + d)$, **(iv)** $a + b\cos(c\theta + d)$

(v) **the reciprocals of the above,**

where a, b, c and d are real numbers

Consider types (i) and (ii), the expressions attain their maximum values when both $\sin\theta$ and $\cos\theta$ are at the maximum. i.e., $\sin\theta = 1$ and $\cos\theta = 1$ provided that b is non-negative.

If $b < 0$ then we get maximum values at $\sin\theta = -1$ and $\cos\theta = -1$.

So, maximum value of $(a + b\sin\theta) = a + |b|$ (maximum value of $\sin\theta$)

$$= a + |b|\ (1) = a + |b|$$

Thus, $\boxed{\text{Maximum value of } (a + b\sin\theta) = a + |b|}$ ....(i)

Similarly, $\boxed{\text{Maximum value of } (a + b\cos\theta) = a + |b|}$ ....(ii)

Now, the functions attain their minimum values when both $\sin\theta$ and $\cos\theta$ are at the minimum. i.e., $\sin\theta = -1$ and $\cos\theta = -1$ provided that b is non-negative.

If $b < 0$ then we get minimum values at $\sin\theta = 1$ and $\cos\theta = 1$.

So, minimum value of $(a + b\sin\theta) = a + |b|$ (minimum value of $\sin\theta$)

$$= a + |b|(-1) = a - |b|$$

Thus, $\boxed{\text{Minimum value of } (a + b\sin\theta) = a - |b|}$ ... (iii)

Similarly, $\boxed{\text{Minimum value of } (a + b\cos\theta) = a - |b|}$ ...(iv)

Now, consider the types (iii) and (iv), that is $a + b\sin(c\,\theta + d)$ and $a + b\cos(\mathrm{c}\,\theta + \mathrm{d})$. In these types, the values of c and d do not affect the function. So, we get the same results as for types (i) and (ii) which are:

Maximum value of $\big(a + b\sin(c\theta + d)\big) = a + |b|$ ...(v)

Maximum value of $\big(a + b\cos(c\theta + d)\big) = a + |b|$ ...(vi)

Minimum value of $\big(a + b\sin(c\theta + d)\big) = a - |b|$ ...(vii)

Minimum value of $\big(a + b\cos(c\theta + d)\big) = a - |b|$ ...(viii)

Thus, we conclude that, if M and m respectively represent the maximum and minimum values of the expression of types (i), (ii), (iii) and (iv). Then we have the following formulae.

$M = a + |b|$ and $m = a - |b|$

Now, let M' and m' respectively represent the maximum and minimum values of the reciprocals of the expressions.

Then, for $m > 0, M > 0$ and $m < 0, M < 0$, we have $M' = \frac{1}{m}$ and $m' = \frac{1}{M}$

And for $m < 0,\ M > 0$, we have $M' = \frac{1}{M}$ and $m' = \frac{1}{m}$

Example: Find the maximum and minimum values of the functions.

(i) $y = 3 - 5\sin\theta$ (ii) $y = 2 + 3\cos(5\theta + 10)$ (iii) $y = \frac{1}{4-5\sin(7\theta-8)}$

Solution:

(i) $y = 3 - 5\sin\theta$, Here, $a = 3$ and $b = -5$

Maximum value of $y = a + |b| = 3 + |-5| = 3 + 5 = 8$

Minimum value of $y = a - |b| = 3 - |-5| = 3 - 5 = -2$

(ii) $y = 2 + 3\cos(5\theta + 10)$, Here, $a = 2$ and $b = 3$

Maximum value of $y = M = a + |b| = 2 + |3| = 2 + 3 = 5$

Minimum value of $y = m = a - |b| = 2 - |3| = 2 - 3 = -1$

(iii) $y = \frac{1}{4-5\sin(7\theta-8)}$

Here, $a = 4$ and $b = -5$ then, $M = a + |b| = 4 + |-5| = 4 + 5 = 9$

and $m = a - |b| = 4 - |-5| = 4 - 5 = -1$

Let M' and m' respectively, represent the maximum and minimum value of the reciprocals of the functions

$\because\ m < 0,\ M > 0$

$\therefore M' = \frac{1}{M}$ and $m' = \frac{1}{m}$

So, $M' = \frac{1}{9}$ and $m' = \frac{1}{-1} = -1$

Exercise 12.1

1. Find the domain and range of each of the following functions:

(i) $2\sin 3x$ (ii) $5\cos 4x$ (iii) $8\tan 2x$ (iv) $\operatorname{cosec} 6x$

(v) $5\cot 2x$ (vi) $\sin\frac{x}{3}$ (vii) $\operatorname{cosec}\frac{x}{4}$ (viii) $\tan\frac{x}{7}$

2. Determine whether the following trigonometric functions are even, odd or neither.

i. $f(x) = \sin x\cos x$ ii. $g(x) = \frac{Sin^2 x}{1+\tan x}$

iii. $h(x) = \frac{\tan x}{x+\sin x}$ iv. $k(x) = x^3(\sin x + \cos x)$

3. Find the period of the following functions.

(i) $\sin 3x$ (ii) $\cos 4x$ (iii) $\tan\frac{x}{3}$ (iv) $\sec\frac{x}{5}$

(v) $\operatorname{cosec} 8x$ (vi) $\cot\frac{x}{6}$ (vii) $\sqrt{5}\cos\frac{3x}{2}$ (viii) $\cot\sqrt{2}x$

(ix) $\sin\frac{x}{3}$ (x) $\cos\frac{x}{4}$ (xi) $\frac{7}{2}\cot\frac{2\pi x}{3}$ (xii) $-\frac{2}{5}\sec\frac{3x}{\pi}$

4. Find the maximum and minimum values of each of the following functions.

(i) $y = 4 + 3\sin\theta$ (ii) $y = \frac{2}{3} - 4\cos\theta$

(iii) $y = 6 - \frac{1}{3}\sin(3\theta + 2)$ (iv) $y = 8 + 5\cos(\theta - 25)$

(v) $y = \frac{1}{25-12\sin(3\theta-2)}$ (vi) $y = \frac{1}{1+6\cos(5\theta-4)}$

12.2 Graphs of Trigonometric Functions

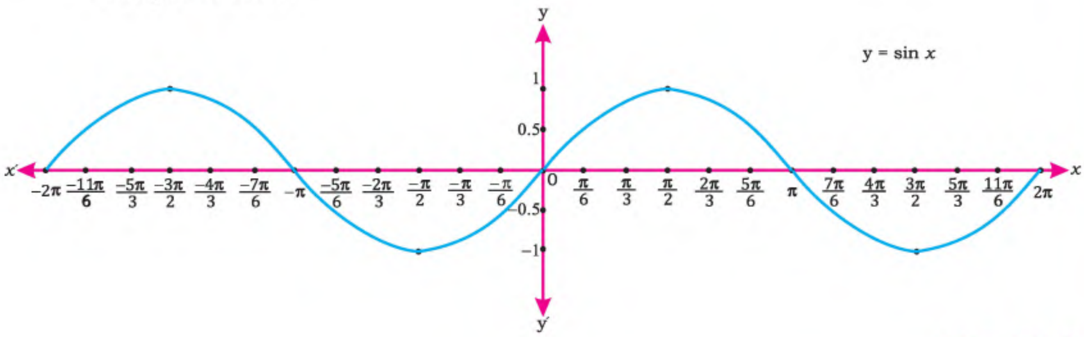
The procedure for plotting the graphs of trigonometric functions is similar as that of graphs of algebraic functions.

In order to draw the graph of trigonometric function we take the angles x on x-axis and the corresponding values of trigonometric functions are taken on y-axis.

12.2.1 Recognize the shapes of the graphs of sine, cosine and tangent for all angles

The graphs of trigonometric functions represent curves and the trigonometric functions are periodic, so their curves repeat after a specific interval. The shapes of the graphs of trigonometric functions sine, cosine and tangent are as follows.

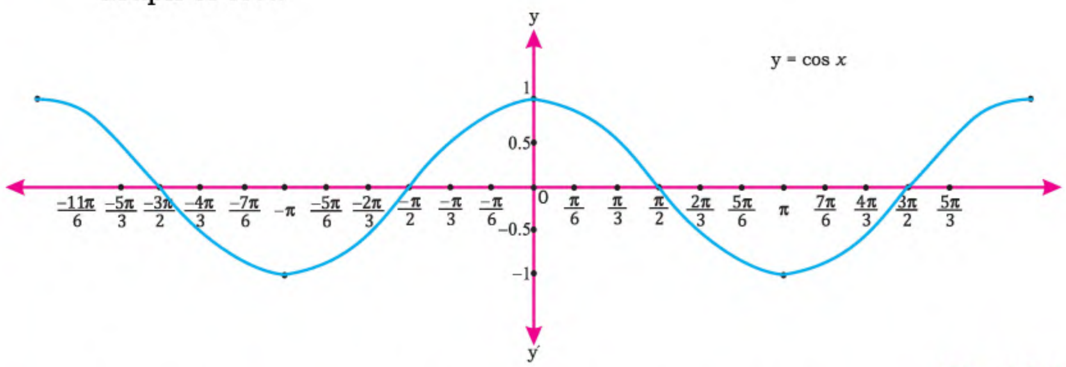
Graph of $\sin x$



(Fig. 12.2)

The shape of sine function is sinusoidal or up-down curve which repeats after every 2π radians. It passes through origin, heads up to 1 by $\frac{\pi}{2}$ radians and then heads down to -1 at $\frac{3\pi}{2}$ radians. Its cycle completes at 2π radians. The cycles repeat after the interval of 2π. The graph is symmetric about the origin.

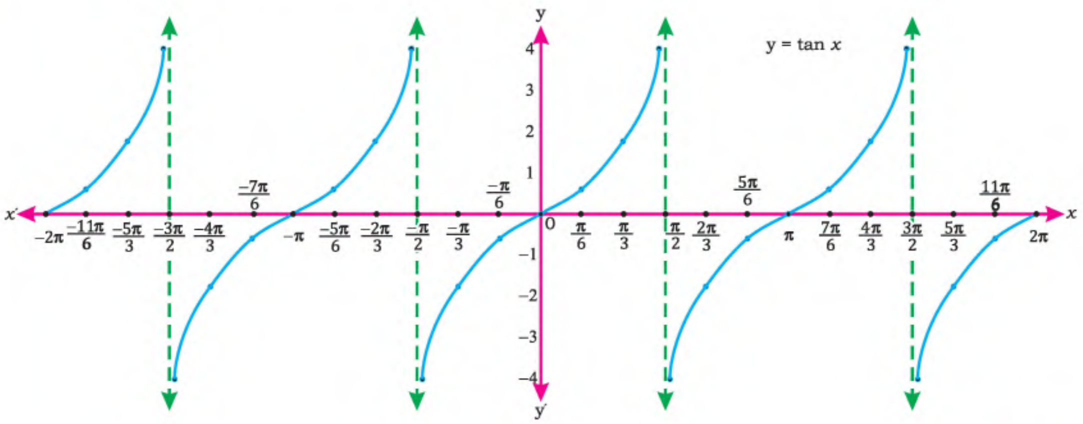
Graph of $\cos x$



(Fig. 12.3)

The shape of cosine function is also sinusoidal or up-down curve which repeats after every 2π radians. It does not pass-through origin. Its maximum value is 1 and minimum value is -1. Its cycle completes in interval of 2π radians. The cycles repeat after interval of a 2π. The graph is symmetric about the y-axis.

Graph of $\tan x$



(Fig. 12.4)

The tangent function has a completely different shape. It goes between $-\infty$ and $+\infty$. It passes through origin. At $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ radians ..., etc., the function is undefined. It is discontinuous curve and is symmetric about origin.

12.2.2 Draw the graphs of the six basic trigonometric functions within the domain from -2π to 2π

(a) Graph of $y = \sin x, -2\pi \le x \le 2\pi$

Table of values of $\sin x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	−360°	−330°	−300°	−270°	−240°	−210°	−180°	−150°	−120°	−90°	−60°	−30°	0°
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0

(Fig. 12.5)

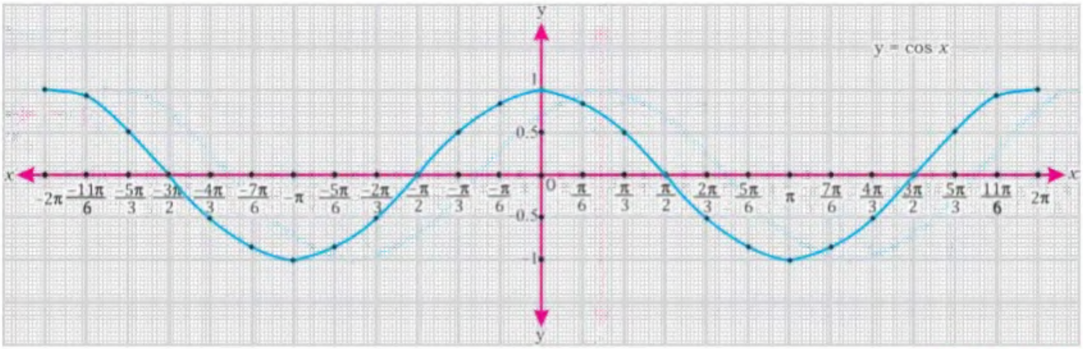
Graph of $y = \sin x, -2\pi \le x \le 2\pi$

The graph of $\sin x$, is sinusoidal curve and it is also called sine wave.

(b) Graph of $y = \cos x, -2\pi \le x \le 2\pi$

Table of values of $\cos x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	−360°	−330°	−300°	−270°	−240°	−210°	−180°	−150°	−120°	−90°	−60°	−30°	0°
$y = \cos x$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.87	0.5	1
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \cos x$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1



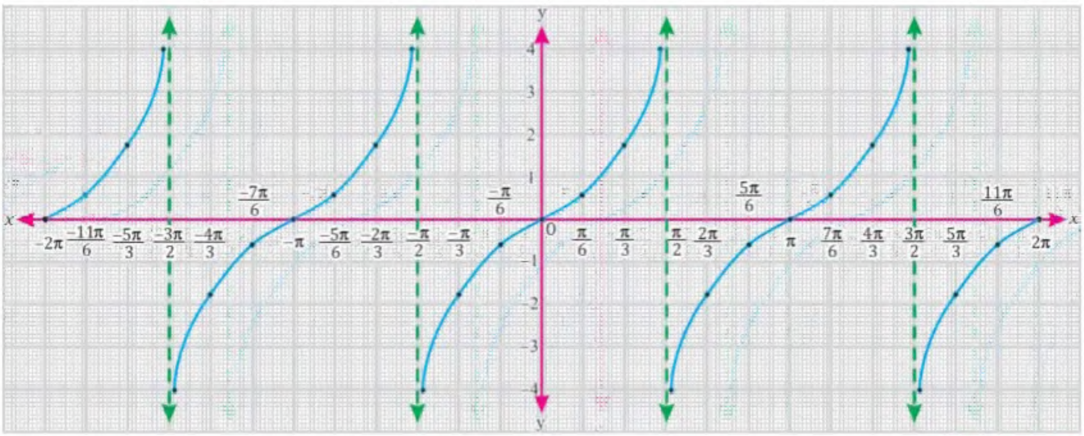
(Fig. 12.6)

Graph of $y = \cos x, -2\pi \le x \le 2\pi$

(c) Graph of $y = \tan x, -2\pi \le x \le 2\pi$

Table of values of $\tan x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	−360°	−330°	−300°	−270°	−270°	−240°	−210°	−180°	−150°	−120°	−90°	−90°	−60°	−30°	0°
$y = \tan x$	0	0.58	1.73	+∞	−∞	−1.73	−0.58	0	0.58	1.73	+∞	−∞	−1.73	0.58	0
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	0°	30°	60°	90°	90°	120°	150°	180°	210°	240°	270°	270°	300°	330°	360°
$y = \tan x$	0	0.58	1.73	+∞	−∞	−1.73	−0.58	0	0.58	1.73	+∞	−∞	−1.73	-0.58	0



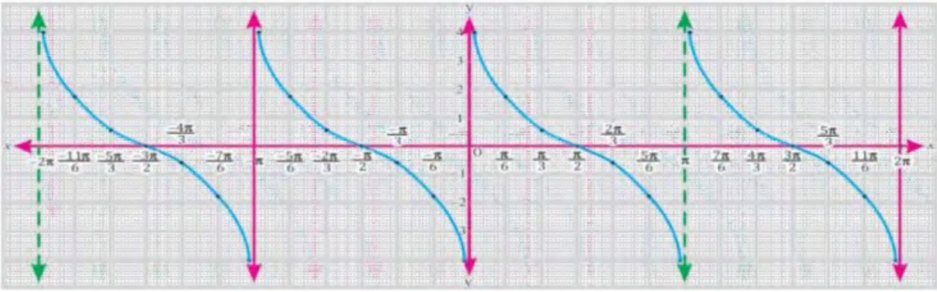
(Fig. 12.7)

Graph of $y = \tan x, -2\pi \le x \le 2\pi$

(d) Graph of $y = \cot x, -2\pi \le x \le 2\pi$

Table of values of $\cot x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	−360°	−330°	−300°	−270°	−240°	−210°	−180°	−180°	−150°	−120°	−90°	−60°	−30°	0°
$y = \cot x$	∞	1.73	0.58	0	−0.58	−1.73	+∞	−∞	1.73	0.58	0	−0.58	−1.73	−∞
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	0°	30°	60°	90°	120°	150°	180°	180°	210°	240°	270°	300°	330°	360°
$y = \cot x$	∞	1.73	0.58	0	−0.58	−1.73	+∞	−∞	1.73	0.58	0	−0.58	−1.73	−∞



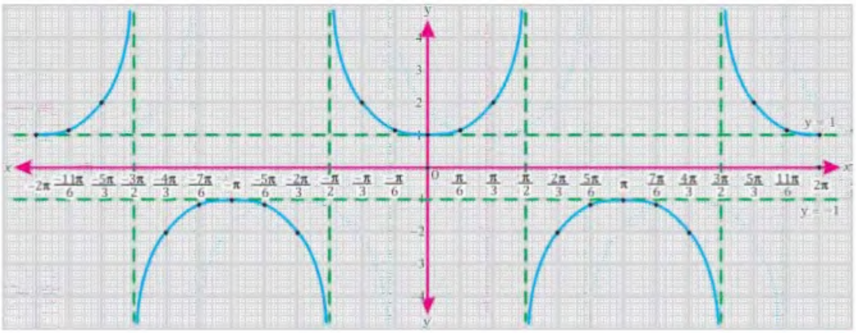
(Fig. 12.8)

Graph of $y = \cot x, -2\pi \le x \le 2\pi$

(e) Graph of $y = \sec x, -2\pi \le x \le 2\pi$

Table of values of $\sec x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	−360°	−330°	−300°	−270°	−270°	−240°	−210°	−180°	−150°	−120°	−90°	−90°	−60°	−30°	0°
$y = \sec x$	1	1.15	2	+∞	−∞	−2	−1.15	−1	−1.15	−2	+∞	−∞	2	1.15	1
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	0°	30°	60°	90°	90°	120°	150°	180°	210°	240°	270°	270°	300°	330°	360°
$y = \sec x$	1	1.15	2	+∞	−∞	−2	−1.15	−1	−1.15	−2	+∞	−∞	2	1.15	1



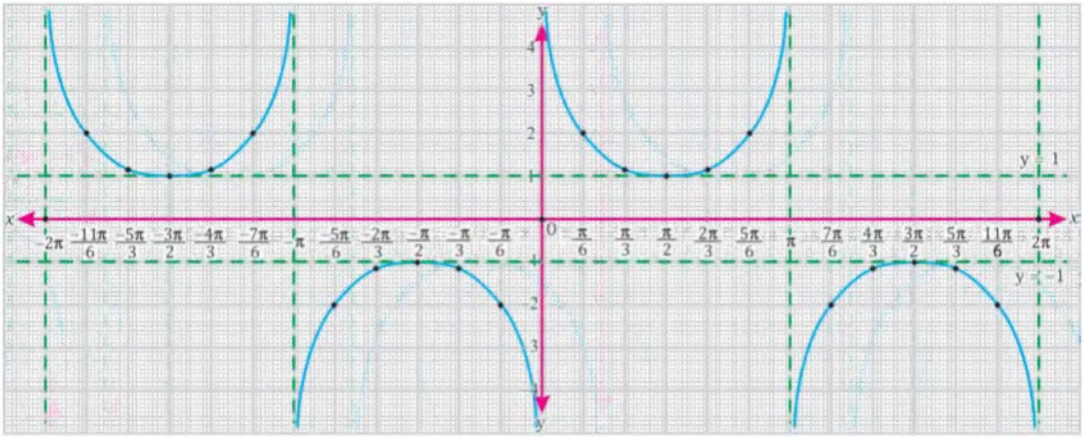
(Fig. 12.9)

Graph of $y = \sec x, -2\pi \le x \le 2\pi$

(f) Graph of $y = \operatorname{cosec} x, -2\pi \le x \le 2\pi$

Table of values of $\operatorname{cosec} x$ from -2π to 2π.

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$y = \operatorname{cosec} x$	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$+180°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$y = \operatorname{cosec} x$	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$



(Fig. 12.10)

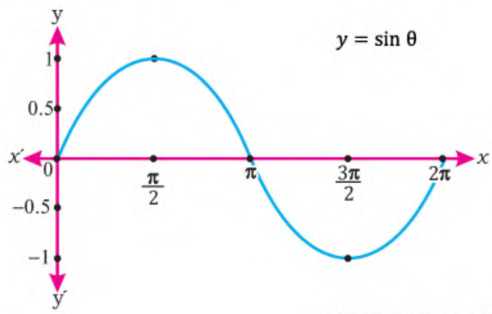
Graph of $y = \operatorname{cosec} x, -2\pi \le x \le 2\pi$

12.2.3 Guess the graphs of $\sin 2\theta, \cos 2\theta, \sin\frac{\theta}{2}, \cos\frac{\theta}{2}$ etc, without actually drawing them

We know that the graph of $y = \sin\theta$ and $y = \cos\theta$ both have period 2π. In $y = \sin n\theta$ and $y = \cos n\theta$, n is constant and indicates the number of cycles in the interval of 0 to 2π, i.e., $0 \le \theta \le 2\pi$

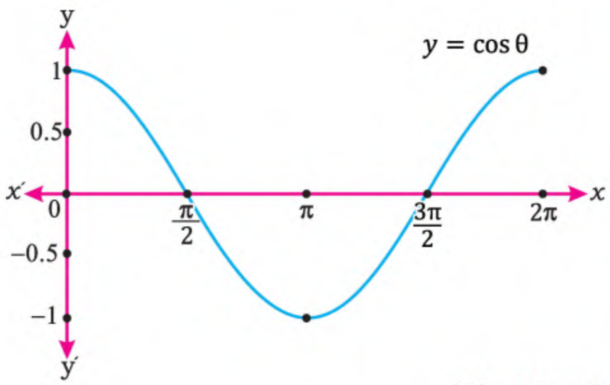
Now, we draw the graphs of $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$ without using table. If $n = 1$ in $\sin n\theta$ and $\cos n\theta$, it means that there is only one cycle in the interval: $0 \le \theta \le 2\pi$

Fig 12.11 shows the graph of $\sin\theta$, $0 \le \theta \le 2\pi$ and period of $\sin\theta$ is 2π.



(Fig. 12.11)

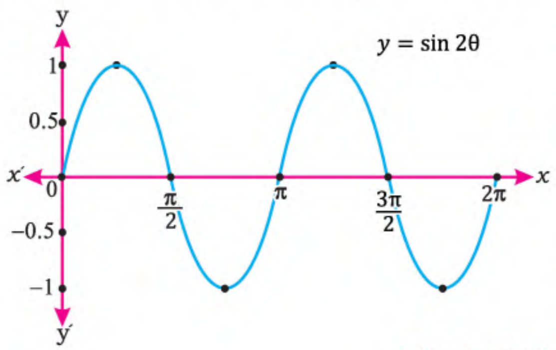
In Fig 12.12, there is one cycle in graph of $y = \cos\theta, 0 \le \theta \le 2\pi$ and period of $\cos\theta$ is 2π.



(Fig. 12.12)

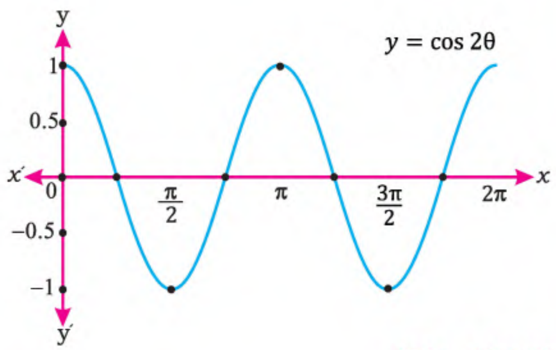
If $n = 2$ then we have, $y = \sin 2\theta$ and $y = \cos 2\theta$ which means that there are two cycles in the interval 0 to 2π. The graphs of $y = \sin 2\theta$ and $y = \cos 2\theta$ are the compressed forms of the graphs of $y = \sin\theta$ and $y = \cos\theta$ respectively. The graphs of $\sin 2\theta, \cos 2\theta$ are as follows:

In Fig 12.13, there are two cycles in the graph of $y = \sin 2\theta$, $0 \le \theta \le 2\pi$ and period of $\sin 2\theta$ is π.



(Fig. 12.13)

Fig 12.14 shows the graph of $y = \cos 2\theta, 0 \le \theta \le 2\pi$. It has two cycles and period of $\cos 2\theta$ is π.

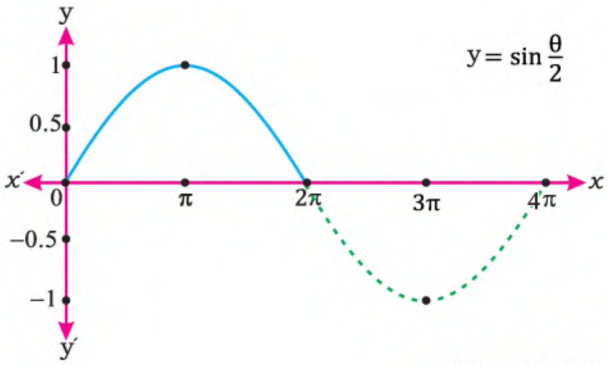


(Fig. 12.14)

Now, if $n = 3$ then we have $y = \sin 3\theta$ and $y = \cos 3\theta$, it means that there are three cycles in the interval 0 to 2π. The graphs of $y = \sin 3\theta$ and $y = \cos 3\theta$ are the compressed forms of the graphs of $y = \sin\theta$ and $y = \cos\theta$ respectively.

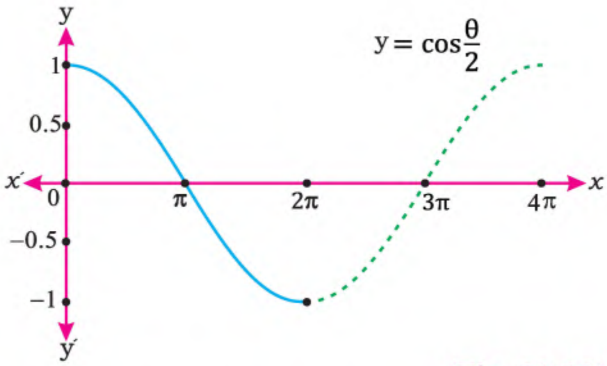
Now, if $n = \frac{1}{2}$ then we have $y = \sin\frac{1}{2}\theta$ and $y = \cos\frac{1}{2}\theta$, it means that there is only half a cycle in the interval 0 to 2π. The graphs of $y = \sin\frac{\theta}{2}$ and $y = \cos\frac{\theta}{2}$ are the expanded forms of the graph $y = \sin\theta$ or $y = \cos\theta$ respectively.

Fig. 12.15 shows the graph of $y = \sin\frac{\theta}{2}$, $0 \le \theta \le 2\pi$. It has half cycle in 0 to 2π and period of $\sin\frac{\theta}{2} = 4\pi$.



(Fig. 12.15)

Fig. 12.16 shows the graph of $y = \cos\frac{\theta}{2}, 0 \le \theta \le 2\pi$. If has half cycle in the interval 0 to 2π and period of $\cos\frac{\theta}{2} = 4\pi$



(Fig. 12.16)

Thus, we conclude that multiplying θ by a number greater than 1 compresses the graph of $\sin\theta$ or $\cos\theta$, while multiplying θ by a positive number less than 1, expands the graph. In this case the period is given by

$$Period = \frac{2\pi}{n}$$ where, n is the number of cycles.

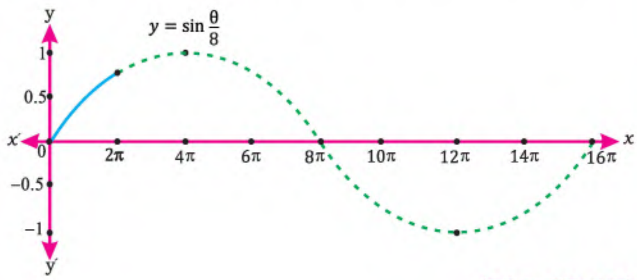
The period of sine and cosine functions can be found by the given formula.

Now, we also define the amplitude as the maximum height of the graph of sine or cosine functions from the horizontal axis. The functions $y = \sin\theta$ and $y = \cos\theta$ have amplitude 1.

Example 1. Guess the graph of $\sin\frac{\theta}{8}$. Also find its period, and amplitude.

Solution: Let $y = \sin\frac{1}{8}\theta$

Here, $n = \frac{1}{8} < 1$, so the graph of $\sin\frac{1}{8}\theta$ is an expanded type of the graph of $y = \sin\theta$, in the interval of 0 to 2π, there is one eighth of a cycle as shown in Fig. 12.17.



(Fig. 12.17)

Now, period of $\sin\frac{1}{8}\theta = \frac{2\pi}{n} = \frac{2\pi}{\frac{1}{8}} = 8 \times 2\pi = 16\pi$

The amplitude of the sine function is the coefficient of sine function, which is 1 in this case.

Hence, Amplitude of $\sin\frac{1}{8}\theta = 1$

Example 2. Guess the graph of $2\sin 2\theta$. Also find its period, and amplitude.

Solution: Let $y = 2\sin 2\theta$

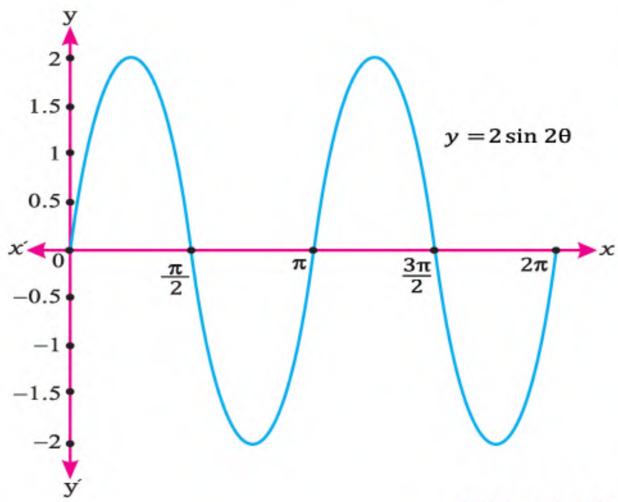
Here, $n = 2 > 1$, so the graph of $2\sin 2\theta$ is a compressed type of the graph of $y = \sin\theta$, in the interval of 0 to 2π, there are two cycles as shown in Fig. 12.18.

Now, Period of $2\sin 2\theta = \frac{2\pi}{2} = \pi$

The amplitude of the sine function is the coefficient of sine, which is 2.

Hence,

Amplitude of $2\sin 2\theta = 2$



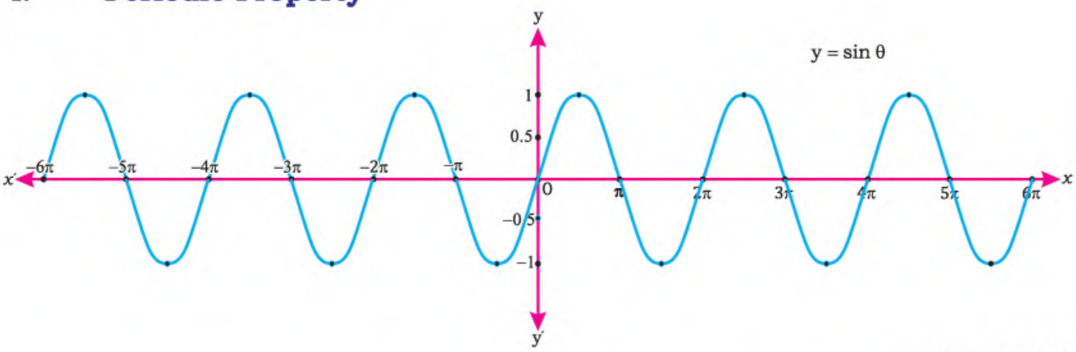
(Fig. 12.18)

12.2.4 Define periodic, even/odd and translation properties of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e., $\sin\theta$ has

- **Periodic property** $\sin(\theta \pm 2\pi) = \sin\theta$
- **Odd property** $\sin(-\theta) = -\sin\theta$
- **Translation property** $\begin{cases} \sin(\theta - \pi) = -\sin\theta \\ \sin(\pi - \theta) = \sin\theta \end{cases}$

a) Properties of the graph of $\sin\theta$

i. Periodic Property

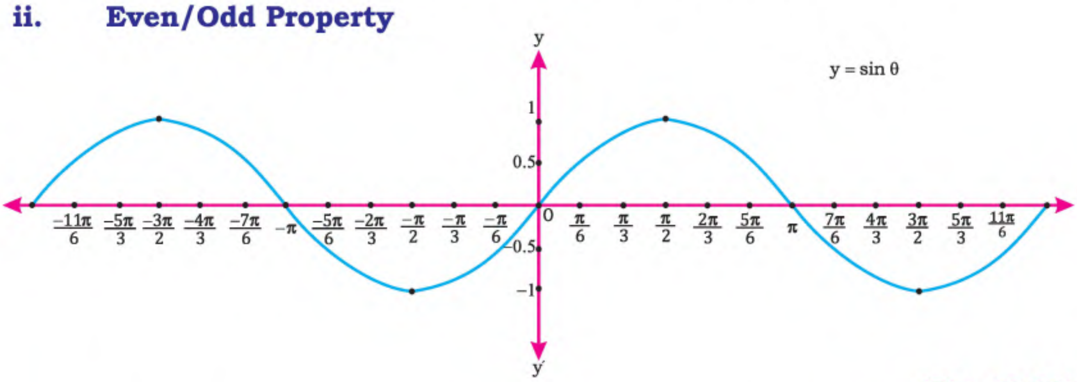


(Fig. 12.19)

We know that the graph of $\sin\theta$ repeats itself after a period of 2π as shown in Fig. 12.19. Therefore, $\sin(\theta \pm 2\pi) = \sin\theta$.

This property of graph of $\sin\theta$ is known as periodic property.

ii. Even/Odd Property



(Fig. 12.20)

We know that the graph of $y = \sin\theta$ is symmetrical about the origin as shown in Fig. 12.20. It means that if θ is replaced by $-\theta$ then the graph will be changed.

Therefore, $\sin(-\theta) = -\sin\theta$

Hence, $\sin\theta$ is an odd function and this property is called the odd property of graph of $\sin\theta$.

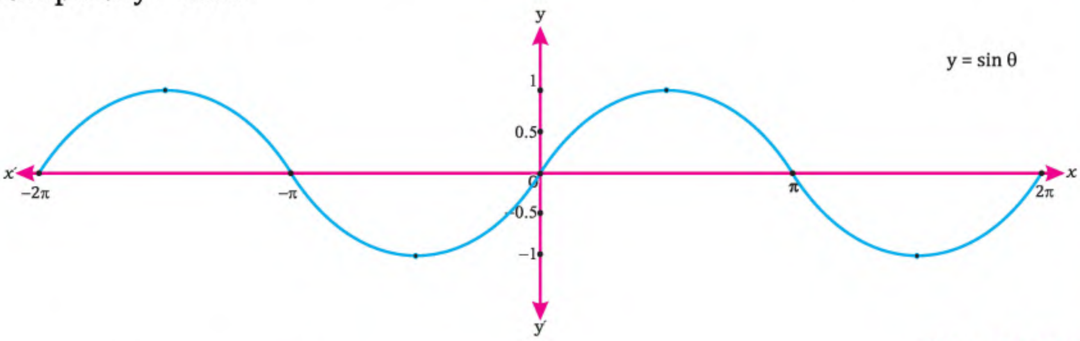
iii. Translation Property:

We know that

$$\left.\begin{aligned}\sin(\theta-\pi) &= -\sin\theta; \\ \sin(\pi-\theta) &= \sin\theta\end{aligned}\right\}$$

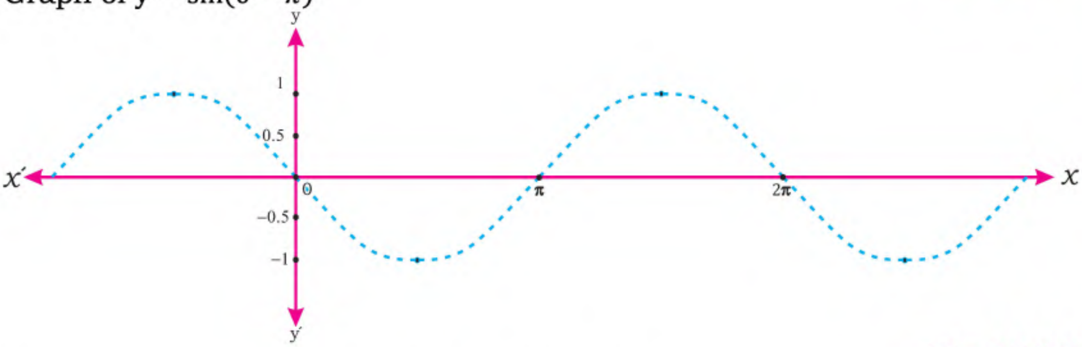
This property is called Translation property of graph of $\sin\theta$ because the graph of $y = \sin(\theta - \pi)$ is similar to the graph of original sine function but translated or shifted horizontally π units right to the graph of $y = \sin\theta$ as shown in Fig. 12.21 and Fig. 12.22.

Graph of $y = \sin\theta$



(Fig. 12.21)

Graph of $y = \sin(\theta - \pi)$



(Fig. 12.22)

We observe that graph of $y = \sin(\theta - \pi)$ is reflection of graph of $y = \sin\theta$ about x-axis.

So, the graph of $y = \sin(\theta - \pi)$ is same as the graph $y = -\sin\theta$

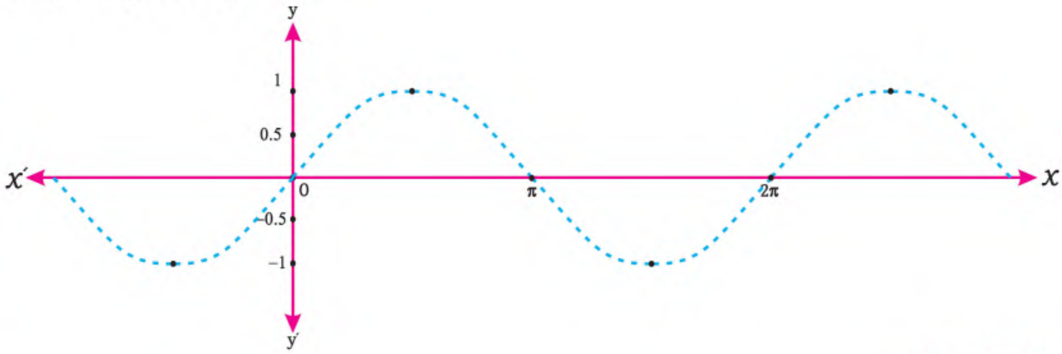
Hence, $\sin(\theta - \pi) = -\sin\theta$

Now, $\sin(\pi - \theta) = \sin[-(\theta - \pi)]$

$= -\sin(\theta - \pi)$ $\quad (\because \sin(-\theta) = -\sin\theta)$

So, graph of $\sin(\pi - \theta)$ is reflection of graph of $y = \sin(\theta - \pi)$ about x-axis as shown in Fig. 12.23.

Graph of $y = \sin(\pi - \theta)$



(Fig. 12.23)

We observe that the graph of $y = \sin(\pi - \theta)$ is same as the graph of $y = \sin\theta$

Hence, $\sin(\pi - \theta) = \sin\theta$

> **Note:** (i) The graph of $y = \sin(\theta - k)$ is similar to the graph of original sine function but translated or shifted horizontally k units right to the graph of $y = \sin\theta$.
>
> (ii) The graph of $y = \sin(\theta + k)$ is similar to the graph of original sine function but translated horizontally k units left to the graph of $y = \sin\theta$.

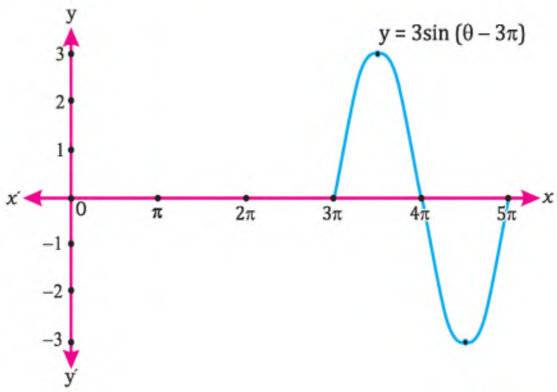
Example: Draw one cycle of the graph of $y = 3\sin(\theta - 3\pi)$

Solution: Here ,

Amplitude = 3

and period = 2π

According to the Translation property, the graph of $3\sin(\theta - 3\pi)$ is similar to graph of $\sin\theta$ but translated 3π units to the right and its amplitude is 3.



(Fig. 12.24)

Now, for initial point, we take $\theta - 3\pi = 0 \Rightarrow \theta = 3\pi$ and for terminal point, we take

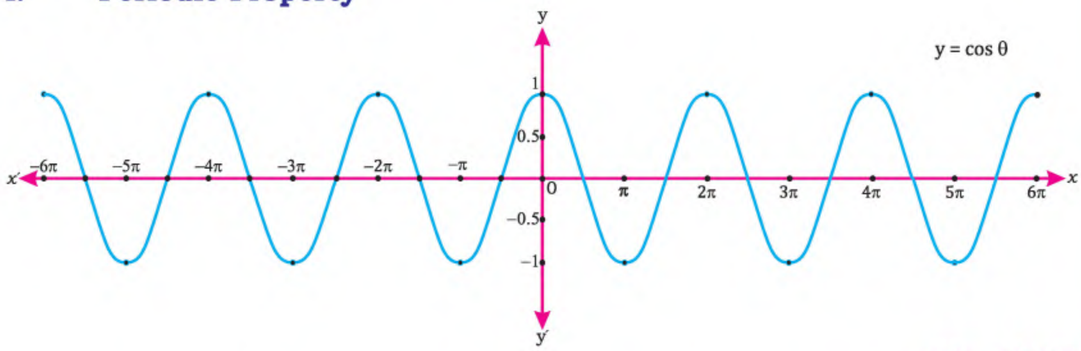
$$\theta - 3\pi = 2\pi \Rightarrow \theta = 5\pi$$

Therefore, the interval of the graph is $3\pi \le \theta \le 5\pi$

The graph of $y = 3\sin(\theta - 3\pi)$ is shown in Fig. 12.24.

b) Properties of the graph of cosθ

i. Periodic Property

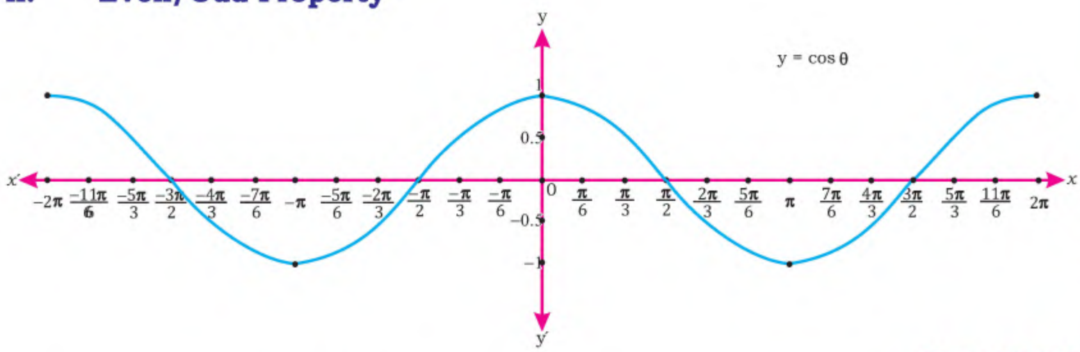


(Fig. 12.25)

We know that the graph of cos θ repeats itself after a period of 2π as shown in Fig. 12.25. Therefore, $\cos(\theta \pm 2\pi) = \cos\theta$.

This property of graph of cos θ is known as periodic property.

ii. Even/Odd Property



(Fig. 12.26)

We know that the graph of $y = \cos\theta$ is symmetrical about y-axis as shown I Fig. 12.26. It means that if θ is replaced by −θ then the value will be unchanged.

Therefore, $\cos(-\theta) = \cos\theta$

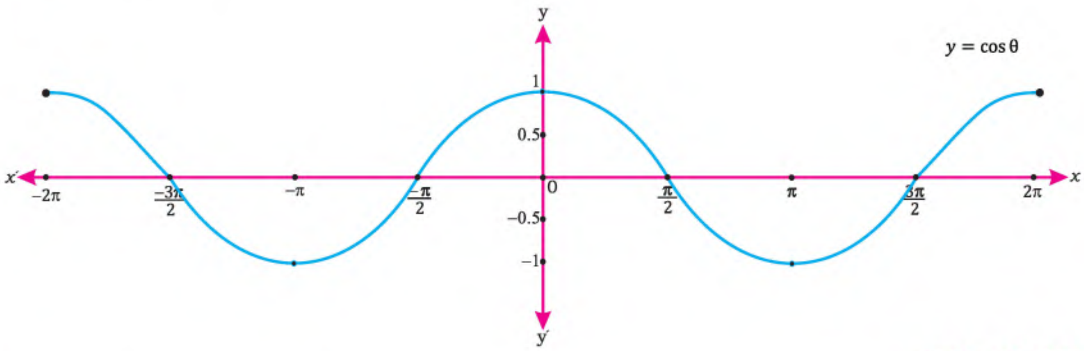
Hence, cos θ is an even function and this property is called even property of graph of cos θ.

iii. Translation Property: We know that

$$\left.\begin{aligned}\cos(\theta - \pi) &= -\cos\theta;\\ \cos(\pi - \theta) &= -\cos\theta\end{aligned}\right\}$$

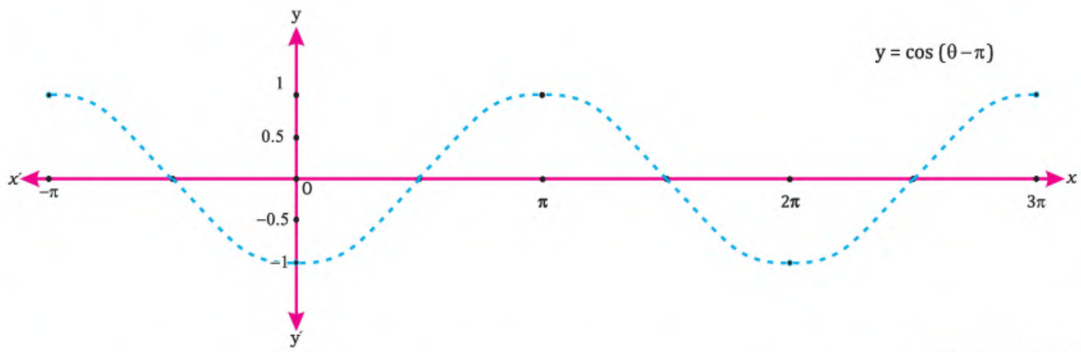
The property is called Translation property of graph of cos θ because the graph of $y = \cos(\theta - \pi)$ is similar to the graph of original cosine function but translated or shifted horizontally π units right to the graph of $y = \cos\theta$ as shown in Fig. 12.27 and Fig. 12.28.

Graph of $y = \cos\theta$



(Fig. 12.27)

Graph of $y = \cos(\theta - \pi)$



(Fig. 12.28)

We observe that the graph of $y = \cos(\theta - \pi)$ is reflection of graph of $y = \cos\theta$ about x-axis.

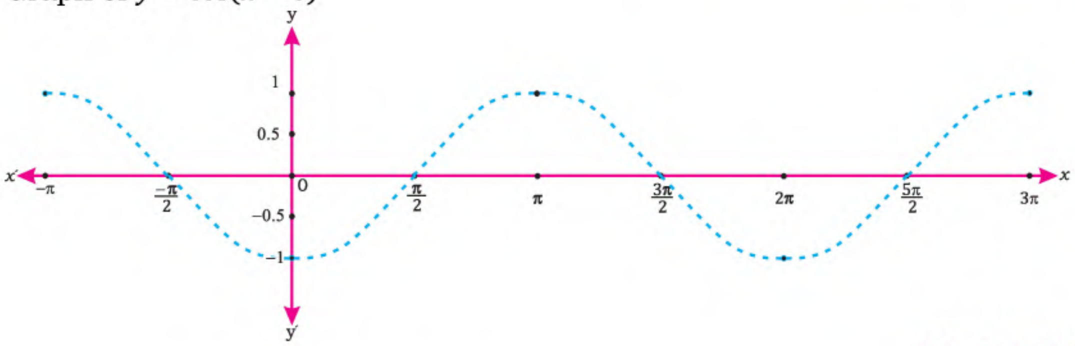
So, the graph of $y = \cos(\theta - \pi)$ is same as the graph of $y = -\cos\theta$.

Hence, $\cos(\theta - \pi) = -\cos\theta$

Since $\cos(\pi - \theta) = \cos(\theta - \pi)$

Therefore, graph of $\cos(\pi - \theta)$ is same as graph of $\cos(\theta - \pi)$ as shown in Fig. 12.29.

Graph of $y = \cos(\pi - \theta)$



(Fig. 12.29)

We observe that the graph of $y = \cos(\pi - \theta)$ is reflection of graph of $y = \cos\theta$ about x-axis, so, the graph of $y = \cos(\pi - \theta)$ is same as the graph $y = -\cos\theta$

Hence, $\cos(\pi - \theta) = -\cos\theta$

Note: (i) If we add constant $k > 0$ to θ as $y = \cos(\theta + k)$ then graph of cosine function will be translated k units to the left.

(ii) If we subtract constant $k > 0$ from θ as $y = \cos(\theta - k)$ then graph of cosine function will be translated k units to the right.

Example: Using graphs and its properties, show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Solution:

Let, $y = \sin\left(\frac{\pi}{2} - \theta\right) = -\sin\left(\theta - \frac{\pi}{2}\right)$

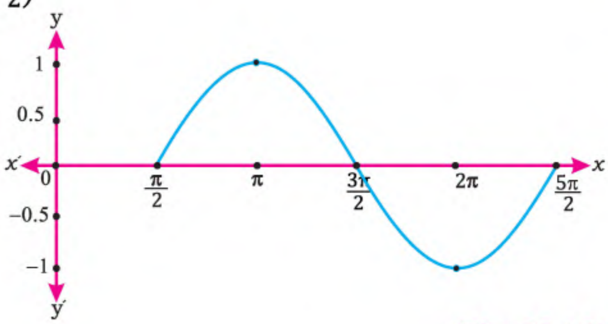
In order to draw graph of $\sin\left(\frac{\pi}{2} - \theta\right)$, we first draw graph of $\sin\left(\theta - \frac{\pi}{2}\right)$ and then reflect it about x-axis.

Here, amplitude $= |-1| = 1$

and period $= 2\pi$

For initial point of the graph:

$$\theta - \frac{\pi}{2} = 0 \implies \theta = \frac{\pi}{2}$$



(Fig. 12.30)

For terminal point of the graph: $\theta - \frac{\pi}{2} = 2\pi \implies \theta = \frac{5\pi}{2}$

Therefore, the interval of the graph is $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{2}$

By translation property, graph of $\sin\theta$ will be translated $\frac{\pi}{2}$ units to the right as shown in Fig. 12.30. Now, we reflect it about x-axis to get graph of

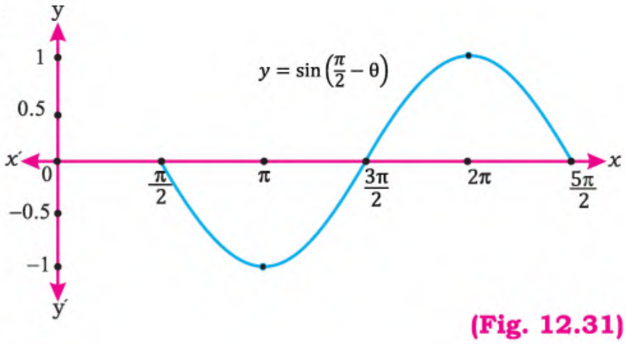
$y = \sin\left(\frac{\pi}{2} - \theta\right)$ as shown in Fig. 12.31.

We observe that the graph of $y = \sin\left(\frac{\pi}{2} - \theta\right)$ is as same as the graph of $y = \cos\theta$.

Hence,

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

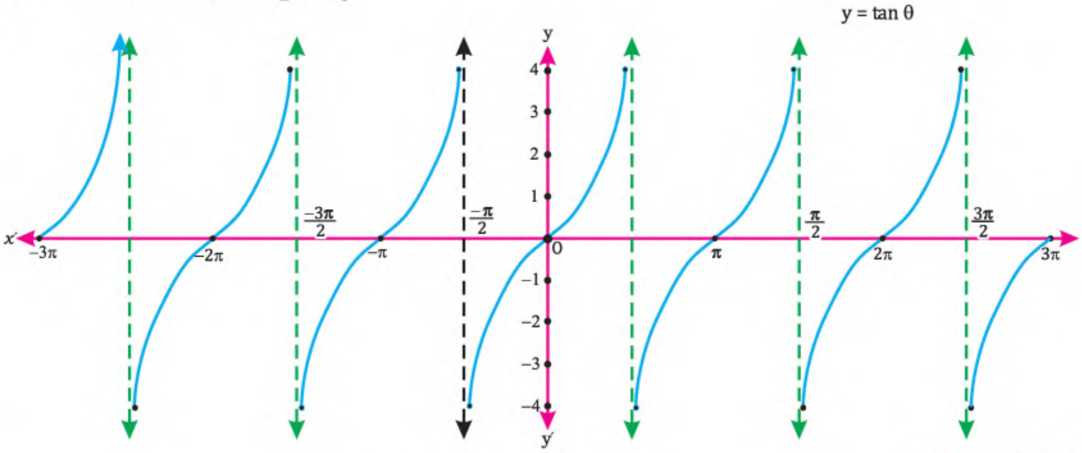
Hence proved.



(Fig. 12.31)

c) Properties of the graph of tanθ

i. Periodic Property



(Fig. 12.32)

We know that the graph of $\tan\theta$ repeats itself after a period of π as shown in Fig. 12.32. Therefore, $\tan(\theta \pm \pi) = \tan\theta$.

where, $\theta \neq \pm\frac{\pi}{2}, \pm3\frac{\pi}{2}, \pm5\frac{\pi}{2}, \pm7\frac{\pi}{2}, \ldots, (2k+1)\frac{\pi}{2}$; k is any integer

At these values of θ the tangent function is undefined.

This property of graph of $\tan\theta$ is known as periodic property.

ii. Even/Odd Property

The graph of $y = \tan\theta$ is symmetrical about origin as shown in Fig.12.32. It means that if θ is replaced by −θ then the graph will be changed.

Therefore, $\tan(-\theta) = -\tan\theta$

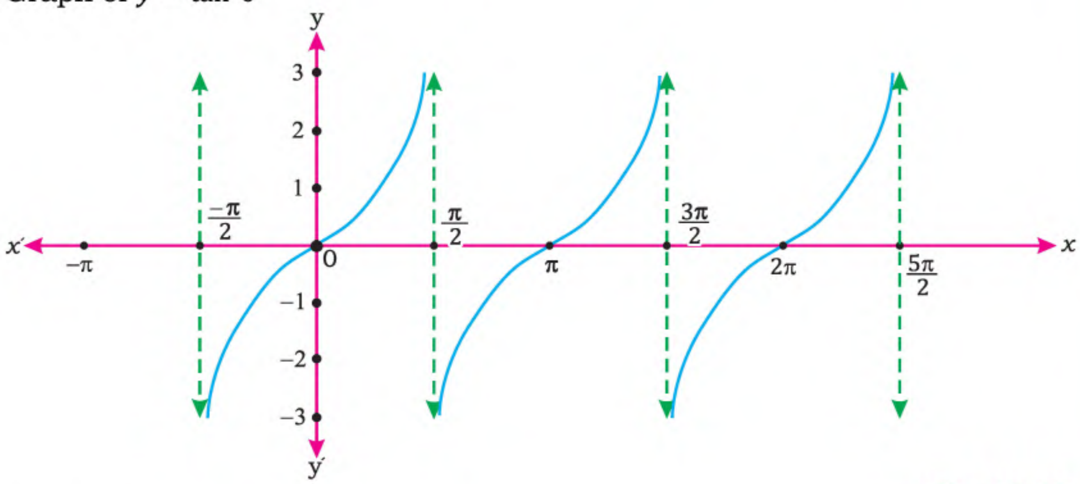
Hence, $y = \tan\theta$ is an odd function and this property of graph of $\tan\theta$ is called odd property.

iii. Translation Property: We know that

$$\left.\begin{aligned}\tan(\theta - \pi) &= \tan\theta\,;\\ \tan(\pi - \theta) &= -\tan\theta\end{aligned}\right\}$$

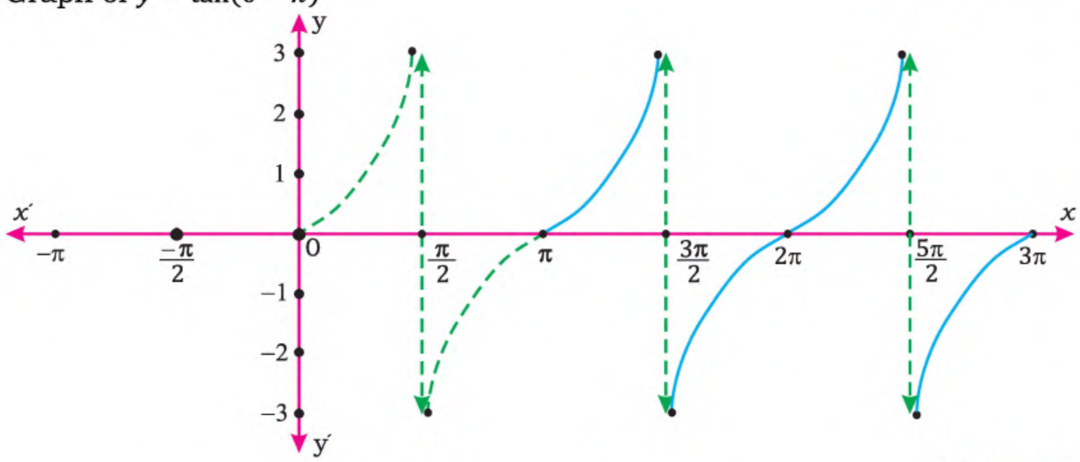
This property is called Translation property of $\tan\theta$ because the graph of $y = \tan(\theta - \pi)$ is similar to the graph of $\tan\theta$ but translated or shifted horizontally π units right to the graph of $y = \tan\theta$ as shown in Fig. 12.33 and Fig. 12.34.

Graph of $y = \tan\theta$



(Fig. 12.33)

Graph of $y = \tan(\theta - \pi)$



(Fig. 12.34)

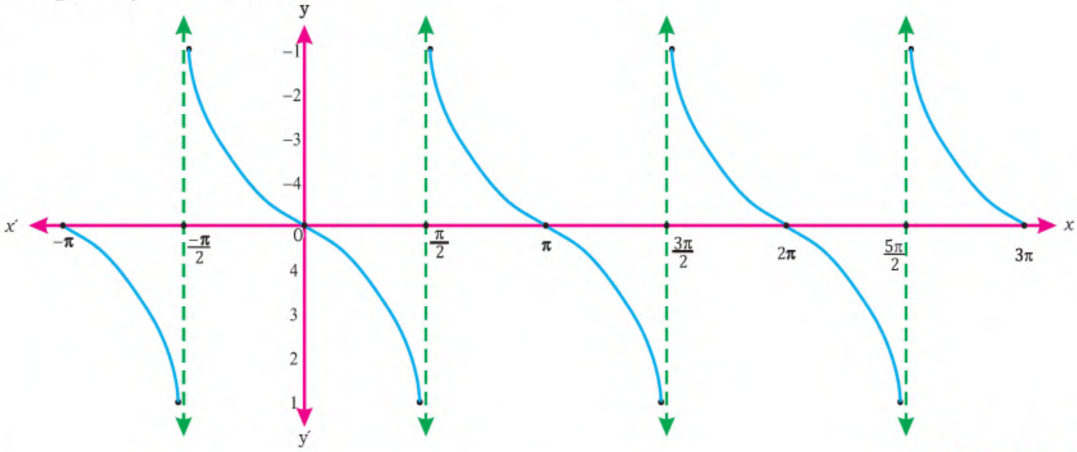
We observe that the graph of $y = \tan(\theta - \pi)$ is same as the graph of $y = \tan\theta$

Hence, $\tan(\theta - \pi) = \tan\theta$

Now, $\tan(\pi - \theta) = -\tan(\theta - \pi)$

So, graph of $y = \tan(\pi - \theta)$ is reflection of graph of $\tan(\theta - \pi)$ about x-axis as shown in Fig. 12.35.

Graph of $y = \tan(\pi - \theta)$



(Fig. 12.35)

We observe that the graph of $y = \tan(\pi - \theta)$ is same as the graph $y = -\tan\theta$

Thus, $\tan(\pi - \theta) = -\tan\theta$

Note: (i) If we add positive real number k to θ as $y = \tan(\theta + k)$, the graph of tangent function will be translated k units to the left.

(ii) If we subtract positive real number k from θ as $y = \tan(\theta - k)$ then the graph of tangent function will be translated k units to the right.

12.2.5 Deduce $\sin(\theta + 2k\pi) = \sin\theta$, where k is an integer

We know that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

So, $\sin(\theta + 2k\pi) = \sin\theta\cos 2k\pi + \cos\theta\sin 2k\pi$ where, $k \in Z$ ...(i)

$\because$ $\sin\theta$ and $\cos\theta$ are periodic functions of period 2π.

$\therefore$ for, any integer $k = 0, \pm 1, \pm 2, ...$

$$\cos 2k\pi = 1 \text{ and } \sin 2k\pi = 0$$

By using $\cos 2k\pi = 1$ and $\sin 2k\pi = 0$ in equation (i)

We get $\sin(\theta + 2k\pi) = \sin\theta(1) + \cos\theta(0)$

$\sin(\theta + 2k\pi) = \sin\theta + 0$

Hence, $\boxed{\sin(\theta + 2k\pi) = \sin\theta}$

Exercise 12.2

Draw the graph of the following trigonometric functions for the given interval.

1. $y = -\sin x$, $0 \le x \le 2\pi$ **2.** $y = \cos x$, $0 \le x \le 2\pi$

3. $y = \sin\frac{x}{2}$, $0 \le x \le 2\pi$ **4.** $y = \tan 2x$, $-\pi \le x \le 2\pi$

5. $y = 5\sec\frac{x}{3}$, $-2\pi \le x \le 2\pi$ **6.** $y = -4\cot x$, $-\pi \le x \le 2\pi$

7. Guess and draw the graph of the following functions without using table of values where $0 \le \theta \le 2\pi$

(i) $y = \cos 4\theta$ (ii) $y = 2\sin\frac{\theta}{2}$ (iii) $y = 3\cos\frac{\theta}{2}$

(iv) $y = 2\sin 3\theta$ (v) $y = \cos\frac{\theta}{3}$ (vi) $y = \sin 2\theta$

8. By using properties of graph of sine, cosine and tangent, show the following.

(i) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ (ii) $\sin\left(\frac{3\pi}{2} + \theta\right) = \cos\theta$

(iii) $\cos(\pi - \theta) = -\cos\theta$ (iv) $\tan(\pi + \theta) = \tan\theta$

(v) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$ (vi) $\cos(2\pi + \theta) = \cos\theta$

9. For any integer k, deduce that:

(i) $\cos(\theta + 2k\pi) = \cos\theta$ (ii) $\operatorname{cosec}(\theta + 2k\pi) = \operatorname{cosec}\theta$

(iii) $\tan(\theta + 2k\pi) = \tan\theta$ (iv) $\cot(\theta + 2k\pi) = \cot\theta$

12.3 Solving Trigonometric Equations Graphically

The equation, containing at least one trigonometric function is called trigonometric equation. For example, $\cos x = \frac{1}{2}$ and $\cos x + \sin x = 1$

Trigonometric equations have an infinite number of solutions due to the periodicity of the trigonometric functions. For example, if $\sin\theta = 0$ then it has infinite number of solutions i.e., $\theta = 0, \pm\pi, \pm 2\pi, \ldots$; which can be written as $\theta = n\pi,\ \forall\, n \in Z$.

12.3.1 Solve trigonometric equations of the type: $\sin\theta = k, \cos\theta = k$ and $\tan\theta = k$ using periodic, even/odd and translation properties

We know that sine and cosine functions are periodic and have period 2π. i.e., they repeat their values after every 2π units. Thus, if we want to find all solutions of the type of $\sin\theta = k$ and $\cos\theta = k$, we simply add and subtract integral multiple of 2π from the solution in the interval $0 \le \theta \le 2\pi$.

Similarly, tangent function is also periodic having period π. Thus, to find all the solution of the equation of the type $\tan\theta = k$, we add and subtract integral multiple of π from the solutions in the interval $0 \le \theta \le 2\pi$.

The method of solving trigonometric equations is explained through the following examples.

Example 1. Solve the equation: $\sin\theta = \frac{1}{2}$

Solution: We know that $\sin\frac{\pi}{6} = \frac{1}{2}$, so the reference angle is $\theta = \frac{\pi}{6}$.

Sine function is positive in 1st and 2nd quadrants, so equation has two solutions in the interval $0 \le \theta \le 2\pi$, one in 1st quadrant and the other in 2nd quadrant.

For 1st quadrant, $\theta = \frac{\pi}{6}$

For 2nd quadrant, $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ [Translation property]

Since, 2π is the period of sinθ.

Thus, $\theta = \frac{\pi}{6} + 2n\pi$ or $\theta = \frac{5\pi}{6} + 2n\pi,\ \forall\, n \in Z$ [Periodic property]

Hence, the solution set of $\sin\theta = \frac{1}{2}$ is $\left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\},\ \forall\, n \in Z$

Example 2. Solve the equation: $\cos\theta = \frac{-1}{\sqrt{2}}$

Solution: We know that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so the reference angle is $\theta = \frac{\pi}{4}$.

Cosine function is negative in 2nd and 3rd quadrant.

Thus, the equation has two solutions in the interval $0 \le \theta \le 2\pi$.

For 2nd quadrant, $\theta = \pi - \frac{\pi}{4}$ [Translational property]

$= \frac{3\pi}{4}$

For 3rd quadrant, $\theta = \pi + \frac{\pi}{4}$ [Translational property]

$= \frac{5\pi}{4}$

Since, 2π is the period of cosθ.

Thus, $\theta = \frac{3\pi}{4} + 2n\pi$, or $\theta = \frac{5\pi}{4} + 2n\pi,\ \forall n \in Z$ [Periodic property]

Hence, the solution set of $\cos\theta = \frac{-1}{\sqrt{2}}$ is $\left\{\frac{3\pi}{4} + 2n\pi\right\} \cup \left\{\frac{5\pi}{4} + 2n\pi\right\}$, $\forall\, n \in Z$

Example 3. Solve the equation: $\tan\theta = \sqrt{3}$

Solution: We know that $\tan\frac{\pi}{3} = \sqrt{3}$, so the reference angle is $\theta = \frac{\pi}{3}$.

Tangent function is positive in 1st quadrant and 3rd quadrant, so the equation