Matrices and Determinants

Unit 2

2.1 Matrices

Matrix is a Latin word which means a place where something develops or originates.

J. J. Sylvester (1814-1897), was the first British mathematician, who formed squares containing rows and columns which he extracted from a rectangular arrangement of objects and called it a matrix (plural matrices).

Arthur Cayley (1821-1895) developed the theory of matrices and used it in algebra of matrices.

Matrices are used to solve the system of linear equations. These have wide applications in the fields of mathematics, statistics, engineering, physical and social sciences also in other various disciplines.

2.1.1 Recall the concept of

- **a matrix and its notation**
- **order of a matrix**
- **equality of two matrices**

(a) A matrix and its notation

A rectangular array of numbers, symbols, or expressions, arranged in rows and columns is called a matrix.

Matrices are generally denoted by capital letters of English alphabet. Small letters of English alphabet or numbers generally denote its elements or entries.

The following notations are used to enclose the elements of a matrix.

$$[\quad],\ (\quad)$$

Following are the examples of matrices:

$$A = \begin{bmatrix} a \\ b \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 6 \\ 8 & 7 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} i & 2i \\ 3i & -4i \end{bmatrix} \text{ and } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The general form of a matrix 'A' with 'm' rows and 'n' columns is represented as follows: j^{th} column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \\ \\ \\ i^{th}\text{ row} \\ \\ \\ \\ \end{array}$$

It is noted that the element a_{ij} is lying on the intersection of the i^{th} row and j^{th} column of matrix A. It is referred to as the $(i,j)th$ element. Hence, the above given matrix A can be represented by $A = [a_{ij}]_{(m,n)}$

where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$

2.1.1 (b) Order of a matrix

The order or dimension of a matrix having m rows and n columns is denoted by $m \times n$ (read as m by n).

A matrix of order $m \times n$ can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots \ \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}$$

Examples:

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2×2 matrix or matrix of order 2.

(ii) $\begin{bmatrix} i & 4i & 7i \\ 2i & 5i & 8i \\ 3i & 6i & 9i \end{bmatrix}$ is a 3×3 matrix or matrix of order 3.

(iii) $\begin{bmatrix} i \\ 2i \\ 3i \end{bmatrix}$ is a matrix of order 3×1.

(iv) $[1 \quad 2 \quad 3]$ is a matrix of order 1×3.

2.1.1 (c) Equality of two matrices

Two matrices A and B are said to be equal if they have the same order or dimension and the corresponding elements are equal.

For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2-1 & 1+1 \\ 4-1 & 3+1 \end{bmatrix}$ are equal matrices.

but, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ are not equal, because $a_{22} \neq c_{22}$ i.e. $4 \neq 5$

Similarly, $\begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \neq [4 \quad -3 \quad 5]$ because of different order.

2.1.2 Know row matrix, column matrix, square matrix, rectangular matrix, zero/null matrix, diagonal matrix, scalar matrix, identity matrix

(i) Row Matrix

A matrix having only one row is called a row matrix. i.e. matrix of order $\mathbf{1} \times \boldsymbol{n}$ is row matrix.

For example,

$A = [a \quad b]$ is a row matrix of order 1×2;

$B = [\alpha \quad \beta \quad \gamma]$ is a row matrix of order 1×3;

The matrix A has two columns and B has three columns but both have one row.

(ii) Column Matrix

A matrix having only one column is called a column matrix. i.e. matrix of order $m \times 1$ is a column matrix.

For example:

$A = \begin{bmatrix} a \\ b \end{bmatrix}$ is a column matrix of order 2×1;

$B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ is a column matrix of order 3×1;

The matrix A has two rows and B has three rows but both have one column.

(iii) Square Matrix

A matrix in which the number of rows and columns are equal is called a square matrix. i.e., matrix of order $m \times n$ is square matrix if $m = n$.

Example: $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ are square matrices.

The order of matrix B is 2×2 and the order of matrix C is 3×3.

(iv) Rectangular Matrix

A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix. i.e., matrix of order $m \times n$ is rectangular matrix if $m \neq n$.

Example: $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ -3 & 5 \\ 6 & 0 \end{bmatrix}$ are rectangular matrices.

(v) Zero/Null Matrix

A matrix in which every element is zero is called a zero or null matrix.

Symbolically, a null matrix of order $m \times n$ is denoted by $0_{m,n}$.

So, $\quad 0_{m,n} = [0]_{(m,n)}$ i.e., $a_{ij} = 0$

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix of order 2 and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null matrix of order 2×3.

(vi) Diagonal Matrix

A square matrix is said to be a diagonal matrix if all the non-diagonal elements of the matrix are zero and at least one diagonal element is non-zero, i.e., $\boldsymbol{a_{ij} = 0}$ **where** $\boldsymbol{i \neq j}$ **and at least one** $\boldsymbol{a_{ij} \neq 0}$ **where** $\boldsymbol{i = j}$.

Example: $\mathbf{D} = \begin{bmatrix} \boldsymbol{d_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{d_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{d_3} \end{bmatrix}$ is a diagonal matrix of order 3.

The entries d_1, d_2, d_3 are of principal or leading or main diagonal of the matrix D and these entries are called diagonal elements.

The matrix D can also be denoted as; $\mathrm{D} = diag\,(d_1, d_2, d_3)$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are few examples of diagonal matrix of order 3.

(vii) Scalar Matrix

A diagonal matrix, in which all the diagonal elements are equal, is called a scalar matrix. i.e., $a_{ij} = 0$ where $i \neq j$, and $a_{ij} = k$ where $i = j$ and k is a scalar. For example, $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ is a scalar matrix of order 2.

and $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ is a scalar matrix of order 3.

(viii) Unit or Identity Matrix

A diagonal matrix in which each diagonal element is 1, is called a unit or identity matrix. The unit matrix of order $n \times n$ is denoted by I_n.

For example, $\mathrm{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

$\mathrm{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: Every scalar matrix is also a diagonal matrix.
Every identity matrix is also a scalar matrix.

2.1.3 Define upper and lower triangular matrix, transpose of a matrix, symmetric matrix and skew-symmetric matrix, Idempotent, Nilpotent, Involutory, Periodic, Hermitian matrix and Skew Hermitian matrix of order up to 4

(i) Upper Triangular Matrix

A square matrix, whose all elements below the main diagonal are zero, is called upper triangular matrix,

i.e., $a_{ij} = 0, \ \forall \, i > j$

For example, $\begin{bmatrix} 5 & 8 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

is an upper triangular matrix of order 3.

(ii) Lower Triangular Matrix

A square matrix, whose all elements above the main diagonal are zero, is called lower triangular matrix,

i.e., $a_{ij} = 0, \ \forall \, i < j$.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

is a lower triangular matrix of order 3.

Note: If a matrix is upper triangular or lower triangular then it is said to be a triangular matrix.

(iii) Transpose of a Matrix

The matrix obtained from any given matrix A by interchanging its rows and columns is called transpose of A. It is denoted by $\mathbf{A}^t$; read as "**A** transpose".
i.e., $\mathbf{A}^t$ of order $\boldsymbol{n} \times \boldsymbol{m}$ is the transpose of matrix A of order $\boldsymbol{m} \times \boldsymbol{n}$.
Symbolically, if $\mathrm{A} = [a_{ij}]_{(m,n)}$ then $\mathrm{A}^t = [a_{ji}]_{(n,m)}$

For example, If $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \end{bmatrix}$, then $\mathrm{A}^t = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}$.

(iv) Symmetric Matrix and Skew-Symmetric Matrix

A square matrix **A** is called symmetric matrix if $\mathbf{A}^t = \mathbf{A}$.

For example, $\mathrm{A} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is a symmetric matrix because $\mathrm{A}^t = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \mathrm{A}$

$\mathrm{B} = \begin{bmatrix} a & d & c \\ d & b & f \\ c & f & e \end{bmatrix}$ is also a symmetric matrix because $\mathrm{B}^t = \begin{bmatrix} a & d & c \\ d & b & f \\ c & f & e \end{bmatrix} = \mathrm{B}$

A square matrix **A** is called skew-symmetric matrix if $A^t = -A$

For example, If $A = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

then $A^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$

$= -\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} = -A$

i.e., $A^t = -A$

So, A is skew-symmetric matrix.

Note: In skew-symmetric matrix, all the diagonal elements are always zero.

(v) Idempotent matrix

A square matrix **A** is called idempotent if $A^2 = A$.

Example: Show that the following matrix is an idempotent matrix.

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Since $A^2 = A$

Therefore, A is idempotent matrix.

Note: Matrix multiplication will be discussed in detail in section 2.2.1.

(vi) Nilpotent matrix

A nilpotent matrix is a square matrix A such that $A^p = 0$ for some positive integer p. The smallest p is called the index or degree of nilpotent matrix.

Example: Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} (1)(1)+(2)(1)+(3)(-1) & (1)(2)+(2)(2)+(3)(-2) & (1)(3)+(2)(3)+(3)(-3) \\ (1)(1)+(2)(1)+(3)(-1) & (1)(2)+(2)(2)+(3)(-2) & (1)(3)+(2)(3)+(3)(-3) \\ (-1)(1)+(-2)(1)+(-3)(-1) & (-1)(2)+(-2)(2)+(-3)(-2) & (-1)(3)+(-2)(3)+(-3)(-3) \end{bmatrix}$$

$$\Rightarrow \qquad A^2 = \begin{bmatrix} 1+2-3 & 2+4-6 & 3+6-9 \\ 1+2-3 & 2+4-6 & 3+6-9 \\ -1-2+3 & -2-4+6 & -3-6+9 \end{bmatrix}$$

$$\Rightarrow \qquad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

i.e., $A^2 = O$, so A is a nilpotent matrix of index 2.

(vii) Involutory matrix

A square matrix A is said to be involutory matrix if $A^2 = I$.

Example:

Let, $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24 & 40-40 & 0 \\ -15+15 & -24+25 & 0 \\ -5+6-1 & -8+10-2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

i.e., $A^2 = I$, so A is an involutory matrix.

(viii) Periodic matrix

A square matrix A is called a periodic matrix if $A^{k+1} = A$ for some positive integer $k \geq 1$ where k is called the period of A.

Example: Show that $\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is a periodic matrix of period 2.

Let $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+6-12 & -2-4-0 & -6-18+18 \\ -3-6+18 & 6+4+0 & 18+18-27 \\ 2-0-6 & -4+0+0 & -12+0+9 \end{bmatrix} = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}$$

again we multiply by A

So, $A^2 \times A = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

$$\text{or} \quad A^3 = \begin{bmatrix} -5+18-12 & 10-12-0 & 30-54+18 \\ 9-30+18 & -18+20+0 & -54+90-27 \\ -4+12-6 & 8-8-0 & 24-36+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = A$$

i.e., $A^3 = A$, so A is periodic matrix of period 2. Hence shown.

(ix) Hermitian Matrix and Skew Hermitian Matrix

A square matrix over $\mathbb{C}$ is called Hermitian matrix if $(\bar{A})^t = A$

Whereas a square matrix over $\mathbb{C}$ is called Skew Hermitian matrix if $(\bar{A})^t = -A$

Note: $(\bar{A})^t = \overline{A^t}$

Example 1. Show that matrix A is Hermitian matrix.

where $$A = \begin{bmatrix} 1 & i & -2i & 9i \\ -i & 5 & 5i & 2i \\ 2i & -5i & 8 & -i \\ -9i & -2i & i & 4 \end{bmatrix}$$

Solution:

Here $$A = \begin{bmatrix} 1 & i & -2i & 9i \\ -i & 5 & 5i & 2i \\ 2i & -5i & 8 & -i \\ -9i & -2i & i & 4 \end{bmatrix}$$

Now $$\bar{A} = \begin{bmatrix} 1 & -i & 2i & -9i \\ i & 5 & -5i & -2i \\ -2i & 5i & 8 & i \\ 9i & 2i & -i & 4 \end{bmatrix}$$

and $$(\bar{A})^t = \begin{bmatrix} 1 & i & -2i & 9i \\ -i & 5 & 5i & 2i \\ 2i & -5i & 8 & -i \\ -9i & -2i & i & 4 \end{bmatrix} = A$$

$\because \ (\bar{A})^t = A$

$\therefore$ A is Hermitian matrix. Hence shown.

Note: In Hermitian matrix elements of main diagonal are real numbers, and symmetric elements are conjugate to each other.

Example 2. Show that A is skew Hermitian matrix.

Where $$A = \begin{bmatrix} 0 & i & 2i & 9i \\ i & 0 & 5i & 2i \\ 2i & 5i & 0 & i \\ 9i & 2i & i & 0 \end{bmatrix}$$

Solution:

Here, $$A = \begin{bmatrix} 0 & i & 2i & 9i \\ i & 0 & 5i & 2i \\ 2i & 5i & 0 & i \\ 9i & 2i & i & 0 \end{bmatrix}$$

Now $$\bar{A} = \begin{bmatrix} 0 & -i & -2i & -9i \\ -i & 0 & -5i & -2i \\ -2i & -5i & 0 & -i \\ -9i & -2i & -i & 0 \end{bmatrix}$$

and $$(\bar{A})^t = \begin{bmatrix} 0 & -i & -2i & -9i \\ -i & 0 & -5i & -2i \\ -2i & -5i & 0 & -i \\ -9i & -2i & -i & 0 \end{bmatrix}$$

$\Rightarrow$ $$(\bar{A})^t = -\begin{bmatrix} 0 & i & 2i & 9i \\ i & 0 & 5i & 2i \\ 2i & 5i & 0 & i \\ 9i & 2i & i & 0 \end{bmatrix}$$

i.e., $(\bar{A})^t = -A$

Hence, A is skew Hermitian matrix.

Note: In Skew Hermitian matrix elements of main diagonal are always zero, and symmetric elements are conjugate to each other.

Exercise 2.1

1. Specify the type of each of the following matrices.

(i) $\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}$ (ii) $\begin{bmatrix} \sqrt{3} \\ \frac{4}{7} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{i}{2} & 0 & \frac{i}{3} \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

(vii) $\begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{7} \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & i & 2i \\ -i & 0 & -4i \\ -2i & 4i & 0 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & -i & 5i \\ i & 3 & 7i \\ -5i & -7i & 4 \end{bmatrix}$

2. A newspaper agent of a town records the number of papers sold on each day of one week as follows:

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Daily Dawn	80	90	100	95	85	75	70
Daily Jang	100	110	90	95	105	85	80

Write this information in a matrix form and write its order.

3. Find the values of the unknowns in each of the following.

(i) $\begin{bmatrix} a & -4i \\ 8i & 6i \end{bmatrix} = \begin{bmatrix} 7i & b \\ c & d \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ a & 9 & 0 \\ b & c & -1 \end{bmatrix} = \begin{bmatrix} d & e & g \\ -2 & f & 0 \\ -4 & 7 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y & 0 & z \\ 9 & 2x+y & 6 \\ a & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3x \\ 9 & 4 & 6 \\ \frac{y}{2} & 1 & 3 \end{bmatrix}$

4. Find the transpose of each of the following matrices.

(i) $\begin{bmatrix} -4 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 2i & 5i & -3i \\ 0 & -6i & 2i \end{bmatrix}$ (iii) $\begin{bmatrix} 8 \\ 3 \\ -4 \end{bmatrix}$

(iv) $\begin{bmatrix} -2i & i \\ 5i & 7i \\ 2i & -5i \end{bmatrix}$ (v) $\begin{bmatrix} -8 & 7 \\ 3 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} -7 & -10 & 8 \\ 4 & 5 & 9 \\ -1 & 2 & 3 \end{bmatrix}$

5. Write down in tabular form:

(i) $A = [a_{ij}]_{(2,3)}$ (ii) $X = [x_{ij}]_{(3,4)}$ (iii) $B = [b_{ik}]_{(4,4)}$

6. Which of the following are symmetric or skew-symmetric matrices.

(i) $\begin{bmatrix} 0 & -5 & -6 \\ 5 & 0 & 7 \\ 6 & -7 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 5 & 8 \\ 5 & -1 & 6 \\ 8 & 6 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & -7 \\ 5 & -7 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$

7. Find the index of the following nilpotent matrices.

(i) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

8. Find the period of the following periodic matrix.

$$\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

9. Which of the following is idempotent or involutory matrix.

(i) $\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

10. Which of the following is Hermitian or Skew Hermitian matrix or neither.

(i) $\begin{bmatrix} 3 & 1-2i & 4+7i \\ 1+2i & -4 & -2i \\ 4-7i & 2i & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -i & 3i & i \\ 3i & 7i & -5i \\ i & -5i & -i \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$

11. Find real numbers x,y,z such that matrix A is Hermitian matrix.

$$A=\begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$$

2.2 Algebra of Matrices

2.2.1 Carryout scalar multiplication, addition/ subtraction of matrices, multiplication of matrices with real and complex entries (3 by 3)

(i) Scalar Multiplication of a Matrix

Let $A=[a_{ij}]$ is a matrix and k is a scalar then the scalar multiplication of the matrix A denoted by kA is defined as:

$$kA = k[a_{ij}] = [ka_{ij}]; \quad \forall_{i,j}$$

In other words, for a matrix A and a number k (also called a scalar), the matrix kA is obtained by multiplying each element of A by k.

If $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$.

(ii) Addition of Matrices

If A and B are two matrices of the same order (dimension) $m \times n$ then their sum $A+B$ is the matrix of the same order obtained by adding each element of A with the corresponding element of B.

Thus, if $A=[a_{ij}]_{(m,n)}$ and $B=[b_{ij}]_{(m,n)}$, then $A+B=[a_{ij}+b_{ij}]_{(m,n)}$.

Example: Let $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix}$. Find $A+B$.

Solution:

$$A+B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}+\begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix}=\begin{bmatrix} 1+4 & 2+7 & 3+8 \\ 4+3 & 5+2 & 6+1 \\ 7+9 & 8+5 & 9+6 \end{bmatrix}=\begin{bmatrix} 5 & 9 & 11 \\ 7 & 7 & 7 \\ 16 & 13 & 15 \end{bmatrix}$$

(iii) Subtraction of Matrices

If A and B are two matrices of the same order (dimension) $m \times n$ then their difference A – B is the matrix of the same order obtained by subtracting the elements of B from the corresponding elements of A.

Thus, if $A = [a_{ij}]_{(m,n)}$ and $B = [b_{ij}]_{(m,n)}$, then $A - B = [a_{ij} - b_{ij}]_{(m,n)}$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix}$; Find A – B

Solution:

$$A - B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 1 \\ 9 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1-4 & 2-7 & 3-8 \\ 4-3 & 5-2 & 6-1 \\ 7-9 & 8-5 & 9-6 \end{bmatrix} = \begin{bmatrix} -3 & -5 & -5 \\ 1 & 3 & 5 \\ -2 & 3 & 3 \end{bmatrix}$$

(iv) Multiplication of Matrices

Let $A = [a_{ij}]$ be a matrix of order $m \times p$ and $B = [b_{ij}]$ be a matrix of order $p \times n$. Then their product A·B or AB is the matrix $C = [c_{ij}]$ of the order $m \times n$ where $$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ip}b_{pj}$$

The following points may be followed in matrix multiplication.

(i) The product AB is defined only if the number of columns of matrix A is equal to the number of rows of matrix B.

(ii) The elements in the (i,j)th place of AB is the sum of the products of the corresponding elements of i^{th} row of A and j^{th} column of B.

Example 1.

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 2 \\ 8 & 6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 10 & 1 \\ 5 & 8 & 3 \\ 2 & 7 & 2 \end{bmatrix}$, compute AB and BA.

Solution:

$$AB = \begin{bmatrix} (2\times6)+(3\times5)+(1\times2) & (2\times10)+(3\times8)+(1\times7) & (2\times1)+(3\times3)+(1\times2) \\ (5\times6)+(7\times5)+(2\times2) & (5\times10)+(7\times8)+(2\times7) & (5\times1)+(7\times3)+(2\times2) \\ (8\times6)+(6\times5)+(4\times2) & (8\times10)+(6\times8)+(4\times7) & (8\times1)+(6\times3)+(4\times2) \end{bmatrix} = \begin{bmatrix} 29 & 51 & 13 \\ 69 & 120 & 30 \\ 86 & 156 & 34 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} (6\times2)+(10\times5)+(1\times8) & (6\times3)+(10\times7)+(1\times6) & (6\times1)+(10\times2)+(1\times4) \\ (5\times2)+(8\times5)+(3\times8) & (5\times3)+(8\times7)+(3\times6) & (5\times1)+(8\times2)+(3\times4) \\ (2\times2)+(7\times5)+(2\times8) & (2\times3)+(7\times7)+(2\times6) & (2\times1)+(7\times2)+(2\times4) \end{bmatrix} = \begin{bmatrix} 70 & 94 & 30 \\ 74 & 89 & 33 \\ 55 & 67 & 24 \end{bmatrix}$$

Example 2. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I_3 = O_3$.

Solution:

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_3$$

i.e., $A^2 - 4A - 5I_3 = O_3$. Hence, shown.

2.2.2 Show that commutative property:

(i) holds under addition i.e., $A + B = B + A$

(ii) does not hold under multiplication, in general

(i) Commutative property holds under addition i.e., $A + B = B + A$

If the matrices A and B are conformable for addition then commutative property under addition holds i.e.,

$$A + B = B + A$$

Example: If $A = \begin{bmatrix} i & 5i \\ 9i & -7i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3i \\ 8i & i \end{bmatrix}$, verify the commutative property under addition.

Solution:

$$A + B = \begin{bmatrix} i & 5i \\ 9i & -7i \end{bmatrix} + \begin{bmatrix} 0 & 3i \\ 8i & i \end{bmatrix} = \begin{bmatrix} i+0 & 5i+3i \\ 9i+8i & -7i+i \end{bmatrix} = \begin{bmatrix} i & 8i \\ 17i & -6i \end{bmatrix} \quad \ldots\text{(i)}$$

and $$B + A = \begin{bmatrix} 0 & 3i \\ 8i & i \end{bmatrix} + \begin{bmatrix} i & 5i \\ 9i & -7i \end{bmatrix} = \begin{bmatrix} 0+i & 3i+5i \\ 8i+9i & i-7i \end{bmatrix} = \begin{bmatrix} i & 8i \\ 17i & -6i \end{bmatrix} \quad \ldots\text{(ii)}$$

From (i) and (ii), we get $A + B = B + A$, Hence verified.

(ii) Commutative property does not hold under multiplication, in general

If the matrices A and B are conformable for multiplication then commutative property under multiplication does not hold in general

i.e., $AB \neq BA$

Example: If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, show that $AB \neq BA$

Solution:

$$AB = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3+4+14 & 4+8+7 & 1+4+7 \\ 6+5+16 & 8+10+8 & 2+5+8 \\ 9+6+18 & 12+12+9 & 3+6+9 \end{bmatrix} = \begin{bmatrix} 21 & 19 & 12 \\ 27 & 26 & 15 \\ 33 & 33 & 18 \end{bmatrix} \quad \ldots(i)$$

$$\text{Now } BA = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3+8+3 & 12+20+6 & 21+32+9 \\ 1+4+3 & 4+10+6 & 7+16+9 \\ 2+2+3 & 8+5+6 & 14+8+9 \end{bmatrix} = \begin{bmatrix} 14 & 38 & 62 \\ 8 & 20 & 32 \\ 7 & 19 & 31 \end{bmatrix} \quad \ldots(ii)$$

From (i) and (ii),

we get $AB \neq BA$. Hence shown.

2.2.3 Verify that $(AB)^t = B^tA^t$ (3 by 3)

If the matrices are conformable for multiplication then, we can verify: $(AB)^t = B^tA^t$

$$\text{Let} \quad A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 7 & 9 \\ -2 & 3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 & 5 \\ -5 & -7 & -9 \\ 2 & 3 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 7 & 9 \\ -2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 3 & 4 & 5 \\ -5 & -7 & -9 \\ 2 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6+15+8 & 8+21+12 & 10+27+24 \\ 15-35+18 & 20-49+27 & 25-63+54 \\ -6-15-8 & -8-21-12 & -10-27-24 \end{bmatrix} = \begin{bmatrix} 29 & 41 & 61 \\ -2 & -2 & 16 \\ -29 & -41 & -61 \end{bmatrix}$$

$$\text{Thus, } (AB)^t = \begin{bmatrix} 29 & -2 & -29 \\ 41 & -2 & -41 \\ 61 & 16 & -61 \end{bmatrix} \quad \ldots(i)$$

$$\text{Now,} \quad A^t = \begin{bmatrix} 2 & 5 & -2 \\ -3 & 7 & 3 \\ 4 & 9 & -4 \end{bmatrix} \text{ and } B^t = \begin{bmatrix} 3 & -5 & 2 \\ 4 & -7 & 3 \\ 5 & -9 & 6 \end{bmatrix}$$

$$B^t.A^t = \begin{bmatrix} 3 & -5 & 2 \\ 4 & -7 & 3 \\ 5 & -9 & 6 \end{bmatrix}\begin{bmatrix} 2 & 5 & -2 \\ -3 & 7 & 3 \\ 4 & 9 & -4 \end{bmatrix} = \begin{bmatrix} 6+15+8 & 15-35+18 & -6-15-8 \\ 8+21+12 & 20-49+27 & -8-21-12 \\ 10+27+24 & 25-63+54 & -10-27-24 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -2 & -29 \\ 41 & -2 & -41 \\ 61 & 16 & -61 \end{bmatrix} \quad \ldots(ii)$$

From (i) and (ii), we get $(AB)^t = B^t.A^t$. Hence verified.

Properties of matrix operations

(i) Properties of Matrix Addition

Following properties are satisfied by the matrices, A, B and C of the same order w.r.t matrix addition.

(i) $A + B$ is also a matrix of the same order.

(ii) $A + B = B + A$

(iii) $(A + B) + C = A + (B + C)$

(iv) For any matrix A, there exist a matrix of the same order, that is null matrix 0, such that $A + 0 = 0 + A = A$

(v) For any matrix A, there exists a matrix B of the same order, such that $A + B = B + A = 0$

where O is the null matrix of same order. The matrix B is called the additive inverse of A and is denoted by $-A$.

(ii) Properties of Scalar Multiplication

Following properties are satisfied by the matrices A and B of the same order and two scalars with respect to scalar multiplication.

(i) k_1A is also a matrix of same order.

(ii) $(k_1k_2)A = k_1(k_2A)$

(iii) $(k_1 + k_2)A = k_1A + k_2A$ and $k_1(A + B) = k_1A + k_1B$

(iv) $1A = A$ and $-1A = -A$

(v) $oA = 0 = Ao$ and $k_1 0 = 0 k_1 = 0$

(iii) Properties of Matrix Multiplication

If the matrices A, B and C are conformable for addition and multiplication, then

(i) $(AB)C = A(BC)$.

(ii) $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$.

(iii) $AI = IA = A$ where A and I are of the same order.

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

(v) Let A be a square matrix of order n, there exist a matrix B of the same order n, such that $AB = BA = I_n$, then B is called an inverse of A and is written as $B = A^{-1}$.

(iv) Properties of Transposed Matrices

If two matrices A and B are conformable for addition and multiplication, then

(i) $(A \pm B)^t = A^t \pm B^t$

(ii) $(kA)^t = kA^t$ where k is scalar

(iii) $(A^t)^t = A$

(iv) $(AB)^t = B^t A^t$

Exercise 2.2

1. If $A = \begin{bmatrix} 1+i & 2 & 3i \\ 4i & 5-i & 2+3i \\ 0 & 5 & 1-i \end{bmatrix}$ and $B = \begin{bmatrix} 2+i & 3-i & 4 \\ i & 0 & 5-i \\ 6+i & 2 & 2+3i \end{bmatrix}$ then find:

(i) $A+B$ (ii) $A-B$ (iii) $2A-B$ (iv) $3A+B+\text{I}$

where I is unit matrix of order 3.

2. Let $A = \begin{bmatrix} 3 & -4 & 1 \\ 4 & 5 & 7 \\ -2 & -3 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & -1 \\ 2 & -3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & 4 \\ 6 & -3 \\ -2 & 7 \end{bmatrix}$

wherever possible, compute the following:

(i) AC (ii) BC (iii) AB (iv) BA (v) A^2

(vi) CB (vii) $(AB)C$ (viii) C^tB^t (ix) C^tA^t (x) $(CA)\,B$

3. Let $\text{X} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, \text{Y} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & -1 \\ 3 & -5 & -1 \end{bmatrix}$ and $\text{Z} = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

then show that XY = XZ.

4. If $\text{A} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 4 & 3 \\ 0 & 8 & 5 \end{bmatrix}$ then find: $\text{A}^2 - 5\text{A} + 4\text{I}$.

5. Prove the identity: $\left\{\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

6. If $\text{A} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -2 & 5 \\ -1 & 0 & -7 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 9 & 4 & 6 \\ 7 & -8 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ then verify:

(i) commutative property under addition

(ii) $(\text{AB})^t = \text{B}^t\text{A}^t$

2.3 Determinants

2.3.1 Describe determinant of a square matrix, minor and cofactor of an element of a matrix

(i) Determinant of a square matrix

Determinant is the number which is associated with any square matrix. Determinant of a square matrix A is denoted by det(A), det A, or $|\text{A}|$.

For 2×2 matrix $\text{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the number $a_{11}a_{22} - a_{12}a_{21}$ is its determinant. Similarly,

for 3×3 matrix $\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ (Expansion by first row)}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Example: Let $A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$, Find its determinant.

Solution: $|A| = \begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 5 & 7 \\ 1 & 6 \end{vmatrix} - 9\begin{vmatrix} 3 & 7 \\ 8 & 6 \end{vmatrix} + 2\begin{vmatrix} 3 & 5 \\ 8 & 1 \end{vmatrix}$

$$= 4\,(30 - 7) - 9(18 - 56) + 2(3 - 40) = 360.$$

(ii) Minors and Cofactors of an element of a Matrix

(a) Minor

The minor of an element a_{ij} of a matrix A is the determinant of a square sub-matrix, obtained by deleting ith row and jth column. It is denoted by M_{ij}.

(b) Cofactor

The co-factor of an element a_{ij} of a matrix A, denoted as A_{ij}, is defined as $A_{ij} = (-1)^{i+j}M_{ij}$

Example 1. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

Find M_{12}, A_{12}, M_{31} and A_{31}.

Solution:

M_{12}= Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

A_{12} = Co-factor of $a_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{31} = Minor of $a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

A_{31} = Co-factor of $a_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$.

Example 2. Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & -4 \\ 0 & 2 & 3 \end{bmatrix}$. Compute M_{23} and A_{32}.

Solution:

$M_{23} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = (2) \times (1) - (4) \times (0) = 2.$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = (-1)\{(-4) + (4)\} = 0$

2.3.2 Evaluate determinant of square matrix using cofactors

Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & -4 \\ 0 & 2 & 3 \end{bmatrix}$, Using cofactors for each element of 1st row.

$$A_{11} = (-1)^{1+1}\begin{vmatrix}3 & -4\\2 & 3\end{vmatrix} = [(3)(3) - (-4)(2)] = 17$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}2 & -4\\0 & 3\end{vmatrix} = -[(2)(3) - (-4)(0)] = -6$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}2 & 3\\0 & 2\end{vmatrix} = [(2)(2) - (3)(0)] = 4$$

A_{11}, A_{12}, and A_{13} are three cofactors.

Now, $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1(17) + (4)(-6) + (-2)(4) = -15$

Similarly, we can also calculate det A by using other rows,

$$\det A = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = -15 \text{ (By } R_2\text{)}$$

$$\det A = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = -15 \text{ (By } R_3\text{)}$$

we can also calculate det A by using columns.

Using cofactors for each element of 1st column

$$A_{11} = (-1)^{1+1}\begin{vmatrix}3 & -4\\2 & 3\end{vmatrix} = [(3)(3) - (-4)(2)] = 17$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}4 & -2\\2 & 3\end{vmatrix} = -[(4)(3) - (-2)(2)] = -16$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}4 & -2\\3 & -4\end{vmatrix} = [(4)(-4) - (-2)(3)] = -10$$

A_{11}, A_{21}, and A_{31} are three cofactors.

Now, $\det A = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = 1(17) + (2)(-16) + (0)(-10) = -15$

Similarly, we can also calculate det A by using other columns,

$$\det A = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = -15 \text{ (By } C_2\text{)}$$

$$\det A = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = -15 \text{ (By } C_3\text{)}$$

In general, det A can be calculated by using any row or column. The evaluation of a determinant with the help of cofactors is known as Laplacian expansion.

2.3.3 Define singular and non-singular matrices

(a) Singular Matrix:

A square matrix A is said to be a singular or non-invertible matrix if its determinant is zero.

Example: If $A = \begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$, then $|A| = \begin{vmatrix}1 & 2\\3 & 6\end{vmatrix} = 0$.

$\because |A| = 0$

$\therefore$ A is a singular matrix.

(b) Non-Singular Matrix:

A square matrix A is said to be a non-singular or invertible matrix if its determinant is not equal to zero.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$.

$\because |A| \neq 0$

$\therefore$ A is a non-singular matrix.

2.3.4 Describe the Adjoint of a square matrix and a diagonal matrix

(a) Adjoint of square matrix

The adjoint of a square matrix A is the transpose of the matrix formed by all the cofactors of the corresponding elements of A and is denoted by *adj* A,

Symbolically, $adj\ A = [A_{ij}]^t = [A_{ji}]$

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The matrix of the co-factors of the above matrix is

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \text{ and adj } A = [A_{ij}]^t = [A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Example: Find adjoint of matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 9 & 8 & 6 \end{bmatrix}$

Solution: We know that adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$, so we find all the cofactors.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 7 \\ 8 & 6 \end{vmatrix} = (-1)^{1+1}(-32) = -32$; $\quad A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ 9 & 6 \end{vmatrix} = (-1)^{1+2}(-51) = 51$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 9 & 8 \end{vmatrix} = (-1)^{1+3}(-20) = -20$; $\quad A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 5 \\ 8 & 6 \end{vmatrix} = (-1)^{2+1}(-22) = 22$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ 9 & 6 \end{vmatrix} = (-1)^{2+2}(-39) = -39$; $\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 9 & 8 \end{vmatrix} = (-1)^{2+3}(-19) = 19$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} = (-1)^{3+1}(1) = 1$; $\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} = (-1)^{3+2}(-3) = 3$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (-1)^{3+3}(-2) = -2$

$$\text{Hence, adj } A = \begin{bmatrix} -32 & 51 & -20 \\ 22 & -39 & 19 \\ 1 & 3 & -2 \end{bmatrix}^t = \begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix}$$

(b) Adjoint of diagonal matrix

Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

The matrix of cofactors of matrix A is: $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$

where $A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & 0 \\ 0 & a_{33} \end{vmatrix} = a_{22}a_{33}$

$A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & 0 \\ 0 & a_{33} \end{vmatrix} = a_{11}a_{33}$

$A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}$

Hence, adj $A = \begin{bmatrix} a_{22}a_{33} & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}^t = \begin{bmatrix} a_{22}a_{33} & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$

Example: Find adjoint of matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The matrix of cofactors of matrix A is: $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$

where $A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 0 \\ 0 & 1 \end{vmatrix} = 5 \times 1 = 5$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \times 1 = 2$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 2 \times 5 = 10$

Hence, adj $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}^t = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

2.3.5 Use adjoint method to calculate inverse of a square matrix and verify

The inverse of a square matrix A, denoted by A^{-1}, is another matrix such that the product of A and A^{-1} is the identity matrix.

i.e., $AA^{-1} = A^{-1}A = I$ (where I is identity matrix of the same order)

A^{-1} exists only if A is non singular matrix.

Note: (i) If $B = A^{-1}$, then $B^{-1} = A$

(ii) $(A^{-1})^{-1} = A$, i.e., inverse of the inverse of a matrix A is A itself.

Adjoint Method for computing A^{-1}

The inverse of a matrix A by adjoint method is defined as

$$A^{-1} = \frac{\text{adj } A}{|A|}; \text{ where } |A| \neq 0$$

Example 1. Find the inverse of a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by adjoint method.

Solution: Here $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$.

Thus, A is non-singular, so its inverse exists.

We know that for 2×2 matrix A, $\text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^t = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

So, $\text{adj } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

One can verify that $A\,A^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Example 2. Find the inverse of $A = \begin{bmatrix} 9 & 8 & 6 \\ 2 & 4 & 7 \\ 1 & 3 & 5 \end{bmatrix}$ by adjoint method.

Solution: We know that, $A^{-1} = \frac{adj\ A}{|A|}$, where $|A| \neq 0$.

Here, $|A| = \begin{vmatrix} 9 & 8 & 6 \\ 2 & 4 & 7 \\ 1 & 3 & 5 \end{vmatrix} = 9(20 - 21) - 8(10 - 7) + 6(6 - 4) = -21 \neq 0$.

Cofactors of A:

$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 7 \\ 3 & 5 \end{vmatrix} = (-1)^{1+1}(-1) = -1$; $\quad A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ 1 & 5 \end{vmatrix} = (-1)^{1+2}(3) = -3$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = (-1)^{1+3}(2) = 2$; $\quad A_{21} = (-1)^{2+1}\begin{vmatrix} 8 & 6 \\ 3 & 5 \end{vmatrix} = (-1)^{2+1}(22) = -22$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 9 & 6 \\ 1 & 5 \end{vmatrix} = (-1)^{2+2}(39) = 39$; $\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 9 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^{2+3}(19) = -19$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 8 & 6 \\ 4 & 7 \end{vmatrix} = (-1)^{3+1}(32) = 32$; $\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 9 & 6 \\ 2 & 7 \end{vmatrix} = (-1)^{3+2}(51) = -51$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 9 & 8 \\ 2 & 4 \end{vmatrix} = (-1)^{3+3}(20) = 20$

So, $\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & -22 & 32 \\ -3 & 39 & -51 \\ 2 & -19 & 20 \end{bmatrix}$

Thus $\quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-21}\begin{bmatrix} -1 & -22 & 32 \\ -3 & 39 & -51 \\ 2 & -19 & 20 \end{bmatrix}$

2.3.6 Verify the result $(AB)^{-1} = B^{-1}A^{-1}$

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, then show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: L.H.S = $(AB)^{-1}$

First we find the product of A and B

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Now, $|AB| = \begin{vmatrix} 19 & 22 \\ 43 & 50 \end{vmatrix} = (19)(50) - (22)(43) = 4$

We know that $(AB)^{-1} = \frac{adj(AB)}{|AB|}$

$$\text{So,} \qquad (AB)^{-1} = \frac{1}{4}\begin{bmatrix} 50 & -22 \\ -43 & 19 \end{bmatrix} = \begin{bmatrix} \frac{50}{4} & \frac{-22}{4} \\ \frac{-43}{4} & \frac{19}{4} \end{bmatrix} = \begin{bmatrix} \frac{25}{2} & \frac{-11}{2} \\ \frac{-43}{4} & \frac{19}{4} \end{bmatrix}$$

R.H.S = $B^{-1}A^{-1}$

First we find A^{-1} and B^{-1} .

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3)$

$= 4 - 6 = -2 \neq 0$

$adj\, A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Now, $A^{-1} = \frac{adj\, A}{|A|} = \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

$|B| = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = (5)(8) - (6)(7)$

$= 40 - 42 = -2 \neq 0$

$adj\, B = \begin{bmatrix} 8 & -6 \\ -7 & 5 \end{bmatrix}$

Now, $B^{-1} = \frac{adj\, B}{|B|} = \frac{-1}{2}\begin{bmatrix} 8 & -6 \\ -7 & 5 \end{bmatrix}$

$= \begin{bmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{bmatrix}$

$$\text{So, } B^{-1}A^{-1} = \begin{bmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 8+\frac{9}{2} & -4-\frac{3}{2} \\ -7-\frac{15}{4} & \frac{7}{2}+\frac{5}{4} \end{bmatrix} = \begin{bmatrix} \frac{25}{2} & \frac{-11}{2} \\ \frac{-43}{4} & \frac{19}{4} \end{bmatrix}$$

$\because$ L.H.S = R.H.S

$\therefore$ $(AB)^{-1} = B^{-1}A^{-1}$.

Hence verified.

Exercise 2.3

1. Evaluate the following determinants:

(i) $\begin{vmatrix} 5 & 4 & 3 \\ 3 & -4 & 0 \\ 2 & 3 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} -2 & 4 & 3 \\ 5 & 4 & -2 \\ 2 & 7 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 2c & c & c \\ a & a & 2a \\ b & 2b & b \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & 0 & 1-i \\ 0 & 1 & i \\ 1+i & i & 1 \end{vmatrix}$ (v) $\begin{vmatrix} 7 & -2 & 1 \\ 2 & 2 & 4 \\ 4 & 3 & 7 \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}$

2. Identify the singular or non-singular matrix.

(i) $\begin{bmatrix} 4 & 0 & 1 \\ 7 & 5 & 5 \\ -12 & -6 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 2 & 0 \\ 3 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 13 & -5 & 4 \\ 8 & 1 & 3 \\ 7 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 6 & 3 \\ -2 & 1 & 0 \\ 6 & 4 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 20 & 10 & 30 \\ 2 & 1 & 3 \\ 0 & 20 & 1 \end{bmatrix}$

3. Find the value of x for which the following matrices are singular.

(i) $\begin{bmatrix} 10 & 5 \\ 6 & x \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 4 \\ x & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & 3 & 7 \\ 3 & -4 & 2 \\ 5 & x & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} x & -2 & 1 \\ 2 & -3 & 4 \\ x & -2 & -1 \end{bmatrix}$

4. Find the adjoint of the following matrices.

(i) $\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -0.3 & 0.5 \\ 1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 6 & 8 \\ 1 & 3 & 2 \\ 2 & 7 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 1-i \\ 1+i & i & 1 \\ 0 & 1 & -i \end{bmatrix}$ (v) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

5. If $A = \begin{bmatrix} 11 & 10 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ then verify:

(i) $(A^{-1})^{-1} = A$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$ (iii) $\text{adj}\,(AB) = (\text{adj}\,B)(\text{adj}\,A)$

6. Verify the following:

(i) $\left[\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}$

(ii) $\left[\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1}$

7. Use adjoint method to calculate the inverse of the following square matrices, if possible.

(i) $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 3 \\ 9 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 0 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ -5 & 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

2.4 Properties of Determinants

2.4.1 State and verify the properties of determinants

The properties given in this section are very useful in evaluating the determinants. The properties of determinants of order three are also valid for determinants of any order. All the properties which hold for rows are also valid for columns.

Property 1. The values of the determinants of any matrix A and its transpose are always same.

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$. Verify that $|A| = |A^t|$.

Solution: $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$. Expanding $|A|$ by R_1, we get

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = (2-6) - 0 + 3(6-4) = 2.$$

Now, $A^t = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ then $|A^t| = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 3 & 2 & 2 \end{vmatrix}$. Expanding $|A|$ by R_1, we get

$$|A^t| = 1\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = (2-6) - 2(0-9) + 4(0-3) = 2.$$

So, $|A| = |A^t|$. Hence verified.

Property 2. The interchange of any two rows of a matrix A changes the sign of its determinant without altering its numerical value.

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$. Verify that $|B| = -|A|$, where B is a matrix obtained by interchanging any two rows of A.

Solution: Interchanging any two rows, say second and third, we get:

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix},$$

Now $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2$ ...(i)

and $|B| = \begin{vmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = -2$ ...(ii)

From (i) and (ii), we get $|B| = -|A|$.

Hence verified.

Property 3. If two rows of a matrix are identical, then its determinant is zero.

Example 1. Show that $|A| = 0$, where $A = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 3 & 2 \\ 1 & 3 & 0 \end{bmatrix}$.

Solution: $|A| = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 3 & 2 \\ 1 & 3 & 0 \end{bmatrix}$.

Expanding $|A|$ by R_1, we get

$$|A| = 1\begin{vmatrix}3 & 2\\3 & 0\end{vmatrix} - 3\begin{vmatrix}4 & 2\\1 & 0\end{vmatrix} + 0\begin{vmatrix}4 & 3\\1 & 3\end{vmatrix}$$
$$= 1(0-6) - 3(0-2) + 0 = 0$$

So, $|A| = 0$. Hence shown.

Alternatively, $\because$ Two rows are identical

$\therefore |A| = 0.$

Example 2. Show that $|A| = 0$, where $A = \begin{bmatrix}1 & 3 & 1\\2 & 2 & 2\\4 & 0 & 4\end{bmatrix}$.

Solution: Here $|A| = \begin{bmatrix}1 & 3 & 1\\2 & 2 & 2\\4 & 0 & 4\end{bmatrix}$.

Expanding $|A|$ by R_1 , we get

$$|A| = 1\begin{vmatrix}2 & 2\\0 & 4\end{vmatrix} - 3\begin{vmatrix}2 & 2\\4 & 4\end{vmatrix} + 1\begin{vmatrix}2 & 2\\4 & 0\end{vmatrix}$$
$$= 1(8-0) - 3(8-8) + 1(0-8) = 0$$

So, $|A| = 0$

Alternatively, $\because$ Two columns are identical

$\therefore |A| = 0$

Property 4. If all the elements of a row of a square matrix are zero, then its determinant is zero.

Example: Show that $|A| = 0$, where $A = \begin{bmatrix}2 & -3 & 5\\0 & 0 & 0\\9 & -4 & 2\end{bmatrix}$.

$$|A| = 2\begin{vmatrix}0 & 0\\-4 & 2\end{vmatrix} + 3\begin{vmatrix}0 & 0\\9 & 2\end{vmatrix} + 5\begin{vmatrix}0 & 0\\9 & -4\end{vmatrix}$$
$$= 2\,(0) + 3(0) + 5(0) = 0$$

Alternatively, $\because$ each element of R_2 is zero.

$\therefore |A| = 0.$

Property 5. If every element in a row of matrix A is multiplied by the same number k, then $|A|$ gets multiplied by k.

Example: Show that $5\begin{vmatrix}1 & 2 & 3\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix} = \begin{vmatrix}5 & 10 & 15\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix}$.

Solution: R.H.S $= \begin{vmatrix}5 & 10 & 15\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix}$

$$= 5\begin{vmatrix}3 & 7\\0 & 1\end{vmatrix} - 10\begin{vmatrix}2 & 7\\4 & 1\end{vmatrix} + 15\begin{vmatrix}2 & 3\\4 & 0\end{vmatrix} = 5\left(1\begin{vmatrix}3 & 7\\0 & 1\end{vmatrix} - 2\begin{vmatrix}2 & 7\\4 & 1\end{vmatrix} + 3\begin{vmatrix}2 & 3\\4 & 0\end{vmatrix}\right)$$

$$= 5\begin{vmatrix}1 & 2 & 3\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix} = \text{L.H.S}$$

$\because$ L.H.S = R.H.S

$$\therefore \quad 5\begin{vmatrix}1 & 2 & 3\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix} = \begin{vmatrix}5 & 10 & 15\\2 & 3 & 7\\4 & 0 & 1\end{vmatrix}.\text{ Hence shown.}$$

Property 6. If every element a row of a matrix A be expressed as the sum of two terms then |A| can be expressed as the sum of determinants of two matrices differing in the elements of that row but with remaining rows as the same as those of |A|.

Example: If

$$A = \begin{bmatrix}16 & 3 & 0\\20 & 5 & 1\\17 & 7 & 2\end{bmatrix}, B = \begin{bmatrix}16 & 3 & 0\\16 & 5 & 1\\16 & 7 & 2\end{bmatrix}, C = \begin{bmatrix}0 & 3 & 0\\4 & 5 & 1\\1 & 7 & 2\end{bmatrix},$$

then show that $|A| = |B| + |C|$

Solution: L.H.S

$$|A| = \begin{vmatrix}16 & 3 & 0\\20 & 5 & 1\\17 & 7 & 2\end{vmatrix} = \begin{vmatrix}16+0 & 3 & 0\\16+4 & 5 & 1\\16+1 & 7 & 2\end{vmatrix} = \begin{vmatrix}16 & 3 & 0\\16 & 5 & 1\\16 & 7 & 2\end{vmatrix} + \begin{vmatrix}0 & 3 & 0\\4 & 5 & 1\\1 & 7 & 2\end{vmatrix} = |B| + |C| = \text{R.H.S}$$

Property 7. If the elements of one row of a matrix A are k times the corresponding elements of its another row, then $|A| = 0$.

Let $A = \begin{bmatrix}2 & 5 & -3\\k(2) & k(5) & k(-3)\\7 & -2 & 11\end{bmatrix}$ where $R_2 = kR_1$

$$\text{Then } |A| = \begin{vmatrix}2 & 5 & -3\\k(2) & k(5) & k(-3)\\7 & -2 & 11\end{vmatrix} = k\begin{vmatrix}2 & 5 & -3\\2 & 5 & -3\\7 & -2 & 11\end{vmatrix} \quad \text{by property 5}$$

$$= k(0) \quad \text{by property 3 i.e., } R_1 = R_2$$

$$= 0$$

Corollary:

If $A = \begin{bmatrix}k_1a_{21} + k_2a_{31} & k_1a_{22} + k_2a_{32} & k_1a_{23} + k_2a_{33}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$, then $|A| = 0$.

Hint: $|A| = |B| + |C|$,

where $B = \begin{bmatrix}k_1a_{21} & k_1a_{22} & k_1a_{23}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$ and $C = \begin{bmatrix}k_2a_{31} & k_2a_{32} & k_2a_{33}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$

Property 8. If to each element of a row of a matrix A is added, a constant multiple of the corresponding element of another row, then the value of |A| is unaltered.

Let $A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$

and $\quad B = \begin{bmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

where (R_1 of B) = (R_1 of A) + k(R_2 of A).

Then $|B| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{21} & ka_{22} & ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, By using property 6

$= |A| + k\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, By property 5

$= |A| + k(0) = |A|$ By using the property 3 ($R_1 = R_2$)

Similarity, it can be shown that

$$\begin{vmatrix} a_{11}+k_1a_{21}+k_2a_{31} & a_{12}+k_1a_{22}+k_2a_{32} & a_{13}+k_1a_{23}+k_2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$$

2.4.2 Evaluate the determinant without expansion (i.e., using properties of determinants)

Example 1. Without expanding, show that $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$ vanishes.

Solution: Let $\Delta = \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$.

Adding C_2 to C_3, we get:

$$\Delta = \begin{vmatrix} 1 & x & x+y+z \\ 1 & y & x+y+z \\ 1 & z & x+y+z \end{vmatrix};$$

$$\Delta = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix} \quad \text{[Taking } (x+y+z) \text{ common from } C_3],$$

$= (x+y+z) \times 0 = 0;$ [By using property 3]

Example 2. Without expanding, show that $|A| = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$.

Solution:

$$|A| = \begin{vmatrix} 1 & \omega & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1+\omega+\omega^2 \\ \omega^2 & 1 & 1+\omega+\omega^2 \end{vmatrix}; \quad \text{[By adding } C_1 \text{ and } C_2 \text{ to } C_3]$$

$$= \begin{vmatrix} 1 & \omega & 0 \\ \omega & \omega^2 & 0 \\ \omega^2 & 1 & 0 \end{vmatrix}; \qquad [\because 1+\omega+\omega^2=0]$$

$= 0$ [By using property 4]

Example 3. Without expanding, show that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ vanishes.

Solution: Let $\Delta = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$\Delta = \Delta_1 - \Delta_2$ (say) ... (i) [By using property 6]

Now $\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}; \qquad [\text{Multiplying } R_1, R_2, R_3, \text{ by } a, b, c \text{ respectively}]$$

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}; \qquad [\text{Taking } abc \text{ common from } C_3]$$

$$= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} \qquad [\text{By using property 2}]$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \Delta_1 \qquad [\text{By using property 2}]$$

From (i), $\Delta = \Delta_1 - \Delta_2 = \Delta_1 - \Delta_1 = 0$ Hence shown. $[\because \Delta_2 = \Delta_1]$

Exercise 2.4

1. Let $A = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 4 & 6 \\ 1 & -5 & 1 \end{bmatrix}$, verify that $|A| = |A^t|$

2. Without expanding, prove, each of the following:

(i) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

(iii) $\begin{vmatrix} k & b & a & c+d \\ k & b & c & a+d \\ k & d & c & b+a \\ k & a & d & b+c \end{vmatrix} = 0$

3. Without expanding determinants, prove that

(i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x+1 & x+3 & x+5 \\ x+4 & x+6 & x+8 \\ x+7 & x+9 & x+11 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} x+1 & x+3 & x+5 \\ x+4 & x+6 & x+8 \\ x+7 & x+9 & x+11 \end{vmatrix} = 0$

(iv) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(v) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$

(vi) $\begin{vmatrix} \alpha & \beta\gamma & \alpha\beta\gamma \\ \beta & \gamma\alpha & \alpha\beta\gamma \\ \gamma & \alpha\beta & \alpha\beta\gamma \end{vmatrix} = \begin{vmatrix} \alpha & \alpha^2 & \alpha^3 \\ \beta & \beta^2 & \beta^3 \\ \gamma & \gamma^2 & \gamma^3 \end{vmatrix}$

4. Without expanding the determinants, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

5. If a, b, c are different and $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$,

then show that $1+abc=0$.

2.5 Row and Column Operations

Row and column operations are very useful in many applications in matrix theory, specially solving the homogenous and non-homogenous systems of linear equations.

2.5.1 Describe the elementary row and column operations on matrices

(a) Row operations on matrices:

If A is $m \times n$ matrix, then $m \times n$ matrix B obtained from A by performing elementary row operations on A is called row equivalent to A. Symbolically, we write B~A and read as "B is row equivalent to A."

Similarly, we can define column equivalent matrices, that is replacing the word "row" by "column" in the above definition. We also write B~A to denote B is column equivalent to A.

There are three elementary row operations:

(i) Interchange of any two rows. This is usually denoted by R_{ij} which means interchanging of R_i with R_j.

For example, $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix}$ by R_{12}

(ii) Multiplication of a row by a non-zero scalar. This is usually denoted by kR_i which means R_i multiplied by k.

For example, $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 8 & -6 \\ 0 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ by $2R_1$

(iii) Addition of any multiple of one row to another row of the matrix. This is usually denoted by $R_i + kR_j$ which means kR_j is added to R_i.

For example, $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 4 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ by $R_2 + 2R_1$

(b) Column operations on matrices

Three elementary column operations with notation are given as below:

(i) Interchanging any two columns, i.e., C_{ij} .

(ii) Multiplication of a column by any non-zero scalar k i.e., kC_i.

(iii) Addition of any multiple of one column to another column i.e., $C_i + kC_j$, where C_i, C_j are any two columns and k is any non-zero scalar.

2.5.2 Define echelon and reduced echelon form of a matrix

(a) Echelon form of a matrix

A matrix A of order $m \times n$ is called (row) echelon form, if it has the following structure.

(i) The first non-zero entry in any row is 1 that is leading entry.

(ii) All entries below the leading entry must be zeros.

(iii) Every non-zero row in a matrix precedes every zero row, if there is any

For example, $\begin{bmatrix} 0 & 1 & -3 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 5 & -1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form.

But $\begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -5 \\ 0 & 0 & -7 \\ 0 & 0 & 4 \end{bmatrix}$ are not in echelon form.

(b) Reduced Echelon form of a matrix

A matrix A of order $m \times n$ is called reduced echelon form if it is in echelon form, additionally all the elements of column which contain leading entry 1 are zero except that leading entry.

For example, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are in reduced echelon form.

But $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not in reduced echelon form.

2.5.3 Reduce a matrix to its echelon and reduced echelon form

Method of reducing a matrix in echelon form is explained with the help of the following example.

Example: Reduce the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -1 & -2 & -3 \\ 3 & -1 & 3 & 2 \end{bmatrix}$ to (row) echelon form.

Solution:

$$A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -1 & -2 & -3 \\ 3 & -1 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 2 & 3 & -1 & 4 \\ 3 & -1 & 3 & 2 \end{bmatrix} \quad \text{by } R_{12}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 5 & 3 & 10 \\ 0 & 2 & 9 & 11 \end{bmatrix} \quad \text{by } R_2 + (-2)R_1 \text{ and } R_3 + (-3)R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 2 & 9 & 11 \end{bmatrix} \quad \text{by } \frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 0 & \frac{39}{5} & 7 \end{bmatrix} \quad \text{by } R_3 + (-2)R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1 & \frac{-3}{5} & 2 \\ 0 & 0 & 1 & \frac{35}{39} \end{bmatrix} \quad \text{by } \frac{5}{39}R_3$$

It is an echelon form of the given matrix.

(ii) Reduced Echelon form of a matrix

Example: Find the reduced echelon form of the matrix $A = \begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}, \quad R_{13}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 6 & 3 & -4 \end{bmatrix}, \quad R_2 + 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix}, \quad R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -\frac{26}{9} \\ 0 & -9 & 26 \end{bmatrix}, \quad \frac{1}{9}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{26}{9} \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} R_1 - 2R_2 \\ R_3 + 9R_2 \end{matrix}$$

It is the reduced echelon form of A.

2.5.4 Recognize the rank of a matrix

The number of non-zero rows in echelon form/reduced echelon form of a matrix is called rank of that matrix.

For example, $\begin{bmatrix} 1 & 1 & -2 & 6i \\ 0 & 0 & 1 & 3i \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has rank 2, because there are 2 non-zero rows in echelon form.

Whereas, $\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has rank 3, because there are 3 non-zero rows in reduced echelon form.

Note: 1. If $|A| \neq 0$, then rank (A) = order of the matrix A.
2. Rank (A) ≥ 0.
3. Rank of a non-zero row or column matrix is 1.

2.5.5 Use row operations to find the inverse and the rank of a matrix

Let A be a non-singular matrix. If the application of elementary row operations in succession reduce A to I then same sequence of operations reduces I to A^{-1}. i.e. $[A:I] \sim [I:A^{-1}]$

Example 1. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{bmatrix}$ by using row operations.

Solution: $|A| = \begin{vmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{vmatrix}$

$= 2(-8-4) - 5(-6-2) - 1(6-4) = -24 + 40 - 2$

$= 40 - 26 = 14$

As $|A| \neq 0$, so A is non-singular and its inverse exists.

Appending I_3 on the right of the matrix A, we have $\begin{bmatrix} 2 & 5 & -1 & : & 1 & 0 & 0 \\ 3 & 4 & 2 & : & 0 & 1 & 0 \\ 1 & 2 & -2 & : & 0 & 0 & 1 \end{bmatrix}$

$$\begin{array}{c} A \qquad\qquad\qquad I_3 \\ \sim \begin{bmatrix} 1 & 2 & -2 & : & 0 & 0 & 1 \\ 3 & 4 & 2 & : & 0 & 1 & 0 \\ 2 & 5 & -1 & : & 1 & 0 & 0 \end{bmatrix} \text{ by } R_{13} \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & : & 0 & 0 & 1 \\ 0 & -2 & 8 & : & 0 & 1 & -3 \\ 0 & 1 & 3 & : & 1 & 0 & -2 \end{bmatrix} \text{ by } \begin{array}{l} R_2 + (-3)R_1 \\ R_3 + (-2)R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & : & 0 & 0 & 1 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & : & 1 & 0 & -2 \end{bmatrix} \text{ by } \left(-\frac{1}{2}\right)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & 0 & 1 & -2 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 7 & : & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \text{ by } \begin{array}{l} R_3 + (-1)R_2 \\ R_1 + (-2)R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & 0 & 1 & -2 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & : & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \text{ by } \frac{1}{7}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{6}{7} & \frac{4}{7} & 1 \\ 0 & 1 & 0 & : & \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ 0 & 0 & 1 & : & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \text{by } \begin{matrix} R_1 + (-6)R_3 \\ R_2 + 4R_3 \end{matrix}$$

Thus, $$A^{-1} = \begin{bmatrix} -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix}.$$

Example 2. Find the rank of $A = \begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$ by reducing it to echelon form.

Solution: $A = \begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & -5 & -3 \\ -3 & 5 & 6 \\ 5 & 9 & 3 \end{bmatrix} \text{by } R_{13}$$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ -3 & 5 & 6 \\ 5 & 9 & 3 \end{bmatrix} \text{by } (-1)\, R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 20 & 15 \\ 5 & 9 & 3 \end{bmatrix} \text{by } R_2 + 3R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 20 & 15 \\ 0 & -16 & -12 \end{bmatrix} \text{by } R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & -16 & -12 \end{bmatrix} \text{by } \frac{1}{20} R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \text{by } R_3 + 16R_2$$

This is an echelon form of the matrix A and number of its non-zero rows is 2. Hence, the rank of the matrix A is 2.

Example 3. Find the rank of the matrix $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$.

Solution: $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix}, \begin{matrix} \text{by } R_2 + (-2)R_1 \\ \text{by } R_3 + (-3)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 4 & 6 & -2 \end{bmatrix}, \text{ by } \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 + (-4)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_1 + R_2$$

This is in reduced echelon form and the number of non-zero rows are 2, so the rank of the given matrix A is 2.

Exercise 2.5

1. Reduce the following matrices into echelon form using elementary row operations.

(i) $\begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -4 & 0 & 9 \\ 2 & 4 & -1 & 0 \\ 10 & 0 & -2 & -4 \end{bmatrix}$

2. Reduce the following matrices into reduced echelon forms using elementary row operations.

(i) $\begin{bmatrix} 3 & 5 & 4 \\ 4 & 1 & 5 \\ 7 & 6 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -2 \\ 2 & -4 & 6 \\ 2 & 3 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 1 & -3 \\ -1 & -1 & 1 & -3 \end{bmatrix}$

3. Find the rank of the following matrices using elementary row operations.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -7 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

4. Find the inverse of the following matrices using elementary row operations.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

2.6 Solving System of Linear Equations

A system of linear equations is a collection of two or more linear equations for solving same set of variables.

For example, $\begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned}$ and $\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$

are the systems of linear equations in two and three variables respectively. An ordered triple (t_1, t_2, t_3) is called a solution of given system of three linear equations and three variables if all equations are satisfied by these values, the set of all solutions of linear system is called the solution set.

2.6.1 Distinguish between homogeneous and non-homogeneous systems of linear equations in 2 and 3 unknowns

Consider a system of three linear equations in three variables:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \dots (1)$$

The system (1) can be written as

$$\text{AX} = \text{B}, \quad \dots(2)$$

where $\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The matrix A is called the matrix of the coefficients of the system of equations, X is the column matrix of unknowns and B is the column matrix of constants.

In the above system (2), if $\text{B} = 0$, then the system is called homogeneous, otherwise non-homogeneous.

For example, the system of equations:

$$\begin{aligned} -3x_1 + 2x_2 &= 0, \\ 7x_1 - 5x_2 &= 13 \end{aligned}$$

can be written as $\text{AX} = \text{B}$

where $\text{A} = \begin{bmatrix} -3 & 2 \\ 7 & -5 \end{bmatrix}$, $\text{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \neq 0$ (i.e B = 0)

Thus, the system is a non-homogeneous system of two linear equations in two unknowns.

Similarly, the system of equations:

$$x_1 + 7x_2 - 3x_3 = 0,$$
$$11x_1 - 5x_2 + 2x_3 = 0,$$
$$-x_1 + 2x_2 + 3x_3 = 0 \quad \text{can be written as } AX = B,$$

where $A = \begin{bmatrix} 1 & 7 & -3 \\ 11 & -5 & 2 \\ -1 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$ (i.e $B = O$)

Hence, it is a homogeneous system of three linear equations with three unknowns.

2.6.2 Solve a system of three homogeneous linear equations in three unknowns

A system of homogeneous linear equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

is always satisfied by $x = 0, y = 0$ and $z = 0$. The solution (0,0,0) of the above system is called trivial solution or zero solution. Any other non-zero solution of the above system of equations is called a non-trivial solution.

We usually convert the matrix of the coefficients to echelon form by using elementary row operations to get simplified form of the system.
Finally, with help of free variable(s), we get non-zero solutions, if possible.

Note: If $AX = O$ is homogenous system of linear equations with "n" unknowns, then:
(i) it has only trivial solution if rank of $A = n$ or $|A| \neq 0$
(ii) it has trivial as well as infinitely many non-trivial solutions iff rank $(A) < n$ or $|A| = 0$

Example 1. Solve the following system of homogeneous linear equations:

$$x + y + z = 0$$
$$4x + 5y + 2z = 0$$
$$2x + 3y = 0$$

for non-trivial solutions if possible.

Solution: We change the above system of equations to the matrix form $AX = O$

i.e $$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \text{ by } \begin{matrix} R_2 - 4R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } \quad R_3 - R_2$$

Thus, we have $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (A)}$$

Here, Rank=2, n=3. Since Rank $(A) = 2 < 3$ so system has infinitely many solutions.

From system (A)

We get $$x + y + z = 0 \quad \text{...(i)}$$
$$y - 2z = 0 \quad \text{...(ii)}$$

From equation (ii), we get: $y = 2z$

Put $y = 2z$ in equation (i), we get: $x + 2z + z = 0$

or $x + 3z = 0$ or $x = -3z$

Here, we have only two equations with three variables, so we take one variable as a free-variable. The value of free-variable will be assumed as a non-zero real number.

Therefore we consider $z = k,\ k \neq 0$ as free variable.

By using above equations we get: $y = 2k$ and $x = -3k$

Now the solutions are $x = -3k, y = 2k$ and $z = k, \quad k \in \mathbb{R} - \{0\}$.

By putting different values of k, we will get the different solutions.

If $k = 1$, then $x = -3,\ y = 2$ and $z = 1$.

If $k = 2$, then $x = -6,\ y = 4$ and $z = 2$, and so on.

So non trivial solutions are (-3,2,1), (-6,4,2) at $k = 1$ and $k = 2$ respectively.

2.6.3 Define a consistent and inconsistent system of linear equations and demonstrate through examples

Consistent and Inconsistent Systems

1. Consistent System:

A system of equations is said to be consistent if it has one or more solutions, for example

$\left.\begin{matrix} x + 2y = 4 \\ 3x + 2y = 2 \end{matrix}\right\}$ is consistent because it has a unique solution $\left(-1, \frac{5}{2}\right)$

and $\left.\begin{matrix} x + 2y = 4 \\ 3x + 6y = 12 \end{matrix}\right\}$ is also consistent because it has infinite solutions $(0,2), (-2,3), (-4,4), \ldots$

2. Inconsistent System:

If a system of equations has no solution, it is said to be inconsistent, for example

$$\left.\begin{aligned} x + 2y &= 4 \\ 3x + 6y &= 5 \end{aligned}\right\} \text{ is inconsistent because it has no solution.}$$

Demonstration for consistency of a system of linear equations

Augmented Matrix:

Augmented matrix of the system of equations $AX = B$ is obtained by adding constant matrix as the last column of the coefficient matrix and it is denoted by A_B. Consider a system of a non-homogeneous linear equations in three variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix form, the above system of equations can be written as $AX = B$.

where $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Augmented matrix is obtained by adding the constant terms as the last column of the coefficient matrix.

We have $$\mathrm{A_B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Consistency Criteria

i. If rank of A = rank of $A_B = n$ then system has unique solutions

ii. If rank of A = rank of $A_B < n$ then system has infinite solutions

where n is the number of unknowns of the system of equations.

Inconsistency Criterion

If rank of $A \neq$ rank of A_B then the system has no solution.

Example: Check the following system of linear equations to be consistent or inconsistent.

$$x - y + 2z = 5,$$
$$3x + y + z = 8,$$
$$2x - 2y + 3z = 7$$

Solution: We write the above system of equations in matrix form:

$$AX = B$$

$$\text{i.e.,} \quad \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

Now we reduce the augmented matrix to echelon form to check consistency.

$$\text{Augmented Matrix} = A_B = \begin{bmatrix} 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 2 & -2 & 3 & 7 \end{bmatrix} \text{by } R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 0 & -1 & -3 \end{bmatrix} \text{by } R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{by } (-1)R_3$$

Here, we can see rank of A = rank of $A_B = 3$ =number of unknowns; so, the system has unique solutions. Hence the system is consistent.

2.6.4 Solve a system of 3 by 3 non-homogeneous linear equations using:

(i) matrix inversion method,

(ii) Cramer's rule

(iii) Gauss elimination method (echelon form)

(iv) Gauss-Jordan method (reduced echelon form)

(i) Matrix Inversion Method

We can solve the following system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

by using matrix inversion method which has the following steps:

- Write the system of linear equations in the matrix form $AX = B$.
- Find A^{-1} if exists.
- Find X by using $X = A^{-1}B$.

Note: The "Matrix Inversion Method" works when the given system is consistent and also has unique solution.

Example: Use matrix inversion method to solve the following system of linear equations, if possible.

$$3x_1 + 2x_2 - x_3 = 4$$
$$2x_1 - x_2 + 2x_3 = 10$$
$$x_1 - 3x_2 - 4x_3 = 5$$

Solution: We write the system of linear equations to the matrix form $AX = B$,

where, $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 10 \\ 5 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{vmatrix} = 3(4+6) - 2(-8-2) - 1(-6+1) = 55$$

For $adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{31} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$

We find cofactors of A:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = 10,$$
$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} = 10,$$
$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -5,$$
$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ -3 & -4 \end{vmatrix} = 11,$$
$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = -11,$$
$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = 11,$$
$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3,$$
$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} = -8,$$
$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7.$$

Now, $adj\ A = \begin{bmatrix} 10 & 10 & -5 \\ 11 & -11 & 11 \\ 3 & -8 & -7 \end{bmatrix}^t = \begin{bmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{bmatrix}$

Thus, $A^{-1} = \frac{1}{|A|}\ adj\ A = \frac{1}{55}\begin{bmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{bmatrix}$

Now, $X = A^{-1}B = A^{-1}\begin{bmatrix} 4 \\ 10 \\ 5 \end{bmatrix}$

i.e $$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{55}\begin{bmatrix}10 & 11 & 3\\10 & -11 & -8\\-5 & 11 & -7\end{bmatrix}\begin{bmatrix}4\\10\\5\end{bmatrix}=\frac{1}{55}\begin{bmatrix}40+110+15\\40-110-40\\-20+110-35\end{bmatrix}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

Thus, $x_1=3, x_2=-2$ and $x_3=1$ is the required solution.

(ii) Cramer's rule

Consider the system of linear equations,

$$\left.\begin{aligned}a_{11}x+a_{12}y+a_{13}z=b_1\\a_{21}x+a_{22}y+a_{23}z=b_2\\a_{31}x+a_{32}y+a_{33}z=b_3\end{aligned}\right\} \quad \dots \text{(i)}$$

We write the above system of linear equations in matrix form as

$$AX=B \quad \dots \text{(ii)}$$

where, $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

To solve the system of linear equations by Cramer's rule, following steps are used.

i. Calculate $|A|$, if $|A|\neq 0$ then go to step (ii) otherwise the method fails.

ii. Calculate $|A_1|$, $|A_2|$ and $|A_3|$ where,

$$A_1=\begin{bmatrix}b_1 & a_{12} & a_{13}\\b_2 & a_{22} & a_{23}\\b_3 & a_{32} & a_{33}\end{bmatrix},\ A_2=\begin{bmatrix}a_{11} & b_1 & a_{13}\\a_{21} & b_2 & a_{23}\\a_{31} & b_3 & a_{33}\end{bmatrix} \text{ and } A_3=\begin{bmatrix}a_{11} & a_{12} & b_1\\a_{21} & a_{22} & b_2\\a_{31} & a_{32} & b_3\end{bmatrix}$$

iii. Now find x, y and z by using $x=\frac{|A_1|}{|A|}, y=\frac{|A_2|}{|A|}$ and $z=\frac{|A_3|}{|A|}$

Note: This method works only when the given system has non-singular coefficient matrix .

Example: Use Cramer's rule to solve the following system of linear equations.

$$x+3y+2z=19 \quad \dots \text{(i)}$$
$$2x+y+z=13 \quad \dots \text{(ii)}$$
$$4x+2y+3z=31 \quad \dots \text{(iii)}$$

Solution:

Write the system of linear equations to the matrix form AX = B

i.e., $$\begin{bmatrix}1 & 3 & 2\\2 & 1 & 1\\4 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}19\\13\\31\end{bmatrix}$$

Now, we find the determinants of A, A_1, A_2 and A_3.

$$|A|=\begin{vmatrix}1 & 3 & 2\\2 & 1 & 1\\4 & 2 & 3\end{vmatrix}=1\begin{vmatrix}1 & 1\\2 & 3\end{vmatrix}-3\begin{vmatrix}2 & 1\\4 & 3\end{vmatrix}+2\begin{vmatrix}2 & 1\\4 & 2\end{vmatrix}=1-6+0=-5\neq 0.$$

$$|A_1| = \begin{vmatrix} 19 & 3 & 2 \\ 13 & 1 & 1 \\ 31 & 2 & 3 \end{vmatrix} = 19\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 13 & 1 \\ 31 & 3 \end{vmatrix} + 2\begin{vmatrix} 13 & 1 \\ 31 & 2 \end{vmatrix} = 19 - 24 - 10 = -15$$

$$|A_2| = \begin{vmatrix} 1 & 19 & 2 \\ 2 & 13 & 1 \\ 4 & 31 & 3 \end{vmatrix} = 1\begin{vmatrix} 13 & 1 \\ 31 & 3 \end{vmatrix} - 19\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 13 \\ 4 & 31 \end{vmatrix} = 8 - 38 + 20 = -10$$

$$|A_3| = \begin{vmatrix} 1 & 3 & 19 \\ 2 & 1 & 13 \\ 4 & 2 & 31 \end{vmatrix} = 1\begin{vmatrix} 1 & 13 \\ 2 & 31 \end{vmatrix} - 3\begin{vmatrix} 2 & 13 \\ 4 & 31 \end{vmatrix} + 19\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 5 - 30 + 0 = -25.$$

By Cramer's rule $x = \frac{|A_1|}{|A|} = \frac{-15}{-5} = 3$;$y = \frac{|A_2|}{|A|} = \frac{-10}{-5} = 2$ and $z = \frac{|A_3|}{|A|} = \frac{-25}{-5} = 5.$

Thus, $x = 3, y = 2$ and $z = 5$ is the required solution.

(iii) Gauss elimination method (echelon form)

Gauss elimination method is an algorithm for solving system of linear equations. It is usually understood as a sequence of operations performed on the corresponding matrix of coefficients. The method is named after Carl Friedrich Gauss (1777-1855).

This method can be used to solve the non-homogeneous system of linear equations.

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

Following are the steps of Gauss elimination method:

- Write the system of linear equations to the matrix form AX = B.
- Form the augmented matrix by including the constant elements as an extra column in the coefficient matrix.
- Convert augmented matrix into echelon form by using elementary row operations.
- Find X by detaching the last column back to its original position i.e., on the right-hand side of the equivalent matrix equation to AX = B.

Example: Use Gauss elimination method to solve the following system of non-homogeneous linear equations:

$$\begin{aligned} x + 5y + 2z &= 9 \\ x + y + 7z &= 6 \\ -3y + 4z &= -2 \end{aligned}$$

Solution:

We write the system of linear equations in matrix form AX = B.

where, $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ -2 \end{bmatrix}$

Forming the augmented matrix by including the constant elements as an extra column in the coefficient matrix.

$$A_B = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix}$$

Converting augmented matrix into echelon form by using elementary row operations.

$$A_B \sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{3}{4} \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } \left(-\frac{1}{4}\right) R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \text{ by } R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } 4R_3$$

From the above matrix, we get equivalent matrix equation

$$x + 5y + 2z = 9 \quad \ldots \text{(i)}$$

$$y - \frac{5}{4}z = \frac{3}{4} \quad \ldots \text{(ii)}$$

$$z = 1 \quad \ldots \text{(iii)}$$

From (iii), we get $z = 1$. Putting $z = 1$ in (ii) we get $y = 2$ then from (i), we get:

$$x + 5(2) + 2 = 9 \implies x + 12 = 9 \implies x = -3$$

Thus, $x = -3$, $y = 2$ and $z = 1$ is the required solution.

(iv) Gauss - Jordan Method (reduced echelon form)

Gauss-Jordan method is the modified form of Gauss elimination method in which the augmented matrix is converted into the reduced echelon form.

Example: Use Gauss-Jordan method to solve the system of linear equations:

$$x + 5y + 2z = 9$$
$$x + y + 7z = 6$$
$$-3y + 4z = -2$$

Solution:

Changing the system of linear equations in the form AX = B.

where, $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ -2 \end{bmatrix}$.

Here

$$\text{Augmented matrix} = A_B = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -1 & 1 & -1 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 4 & -2 \end{bmatrix} \text{ by } (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_1 - 5R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ by } R_1 - 7R_3$$

Find X by detaching the last column back to its original position from the above matrix.

We get $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$x + 0y + 0z = -3 \qquad \ldots \text{(i)}$$

or $$0x + y + 0z = 2 \qquad \ldots \text{(ii)}$$

$$0x + 0y + z = 1 \qquad \ldots \text{(iii)}$$

From (i), (ii) and (iii) equations, we directly get $x = -3, y = 2$ and $z = 1$ as the required solution.

Exercise 2.6

1. Solve the following homogeneous system of linear equations for non-trivial solutions, if possible.

(i) $\begin{aligned} x + 2y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 4y + 8z &= 0 \end{aligned}$ (ii) $\begin{aligned} x + 4y + 2z &= 0 \\ 2x + y - 3z &= 0 \\ 3x + 2y - 4z &= 0 \end{aligned}$

2. Determine the consistency of non-homogeneous system of linear equations.

(i) $\begin{aligned} x - 2y - 2z &= -1 \\ 2x + 3y + z &= 1 \\ 5x - 4y - 3z &= 1 \end{aligned}$ (ii) $\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \\ 2x + 3y - z &= 9 \end{aligned}$

3. Solve the non-homogeneous system of linear equations using matrix inversion method.

(i) $\begin{aligned} x + 2y + z &= 8 \\ 2x - y + z &= 3 \\ x + y - z &= 0 \end{aligned}$ (ii) $\begin{aligned} 2x - y + 2z &= 4 \\ x + 10y - 3z &= 10 \\ -x + y + z &= -6 \end{aligned}$

4. Solve the non-homogeneous system of linear equations using Gauss elimination method.

(i) $\begin{aligned} -x + y + z &= 0 \\ x + 2y &= 5 \\ -3x + 2y - z &= -2 \end{aligned}$ (ii) $\begin{aligned} x + 2y + z &= 8 \\ 2x - y + z &= 3 \\ x + y - z &= 0 \end{aligned}$

5. Solve the non-homogeneous system of linear equations using Gauss-Jordan- method.

(i) $\begin{aligned} x - y + 4z &= 4 \\ 2x + 2y - z &= 2 \\ 3x - 2y + 3z &= -3 \end{aligned}$ (ii) $\begin{aligned} 2x + 2y - z &= 4 \\ x - 2y + z &= 2 \\ x + y &= 0 \end{aligned}$

6. Solve the non-homogeneous system of linear equations using Cramer's Rule.

(i) $\begin{aligned} x - 2y + z &= 2 \\ 2x + 2y - z &= 4 \\ x + y + 0.z &= 0 \end{aligned}$ (ii) $\begin{aligned} x - 2y + 0.z &= -4 \\ 3x + y + 0.z &= -5 \\ 2x + 0.y + z &= -1 \end{aligned}$

Review Exercise 2

1. Select correct option.

i. If a matrix **A** has m row and n column, then order of **A** is:

(a) $m \times n$ (b) $n \times m$ (c) mn (d) m^n

ii. Any matrix of order $m \times 1$ is called:

(a) Row matrix (b) Column matrix
(c) Square matrix (d) Zero matrix

iii. For the square matrix $A = [a_{ij}]$. If all $a_{ij} = 0, i \neq j$ and all $a_{ij} = k$ (non zero) for $i = j$, then A is called:

(a) Rectangular matrix (b) Scalar matrix
(c) Identity matrix (d) Null matrix

iv. The matrix [7] is:

(a) Square matrix (b) Row matrix
(c) Column matrix (d) all of these

v. $(k\,\mathrm{ABC})^t =$

(a) $k\mathrm{A}^t\mathrm{B}^t\mathrm{C}^t$ (b) $k\mathrm{C}^t\mathrm{B}^t\mathrm{A}^t$ (c) $k(\mathrm{BA})^t$ (d) $k^t(\mathrm{AB})$

vi. $\begin{vmatrix} 1 & 0 \\ 5 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$:______

(a) 4 (b) 8 (c) -2 (d) 10

vii. If AB = BA , then which one is true, where A and B are square matrices:

(a) A and B are multiplicative inverses of each other
(b) One of A or B is null matrix.
(c) One of A or B is identity matrix. (d) all of these

viii. If $A = [-7]$, then $|A|$ is equal to:

(a) 7 (b) -7 (c) 0 (d) Not possible

ix. Let $A = [a_{ij}]$ be a square matrix. Then cofactor of a_{ij} is equal to:

(a) M_{ij} (b) $(-1)^{i+j}M_{ij}$ (c) $(-1)^{ij}M_{ij}$ (d) $(-1)^{i+j}a_{ij}$

x. For any triangular matrix A, $|A|$ is equal to:

(a) Product of leading diagonal elements
(b) Sum of leading diagonal elements
(c) Sum of square of diagonal elements
(d) All of these

xi. A square matrix $A = [a_{ij}]$ for which all $a_{ij} = 0,\ i < j,$ then A is called:

(a) Upper triangular (b) Lower triangular
(b) Symmetric (d) Hermitian

xii. A triangular matrix is always a:

(a) Diagonal matrix (b) Scalar matrix
(c) Square matrix (d) all of these

xiii. A square matrix A is skew symmetric if:

(a) $A^t = A$ (b) $A^t = -A$ (c) $(\bar{A})^t = A$ (d) None

xiv. A square matrix A is Hermitian matrix if:

(a) $A^t = A$ (b) $A^t = -A$ (c) $(\bar{A})^t = A$ (d) $(\bar{A})^t = -A$

xv. Each diagonal element of main diagonal of a skew Hermitian matrix must be:

(a) 1 (b) 0 (c) Any non-zero number (d) Any complex number

xvi. If $\begin{vmatrix} a & b \\ 0 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & -9 \end{vmatrix}$ then

(a) $a = -3$ (b) $a = b$ (c) $a = \frac{1}{3}$ (d) $a = \frac{-1}{3}$

xvii. The number of non-zero rows in echelon form of a matrix is called:

(a) Order of a matrix (b) Rank of a matrix

(c) Leading Column (d) Leading row

xviii. If A is any square matrix and $\mathrm{A} = -\mathrm{A}^t$ then A is a:

(a) Symmetric matrix (b) Skew symmetric matrix

(c) Hermitian matrix (d) Skew Hermitian matrix

xix. If A is idempotent matrix then:

(a) $\mathrm{A}^2 = \mathrm{I}$ (b) $\mathrm{A}^2 = 0$

(c) $\mathrm{A}^2 = \mathrm{A}$ (d) $\mathrm{A}^2 = \mathrm{A}^t$

xx. The cofactor A_{22} of $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 5 \\ 0 & 1 & -1 \end{bmatrix}$ is:

(a) 0 (b) −1 (c) 1 (d) 2

2. Find the values of x and y if: $\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 8 & 14 \end{bmatrix}$.

3. Calculate AC, BC and (A + B)C. Also verify that (A + B)C = AC + BC for

$A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $\mathrm{A}^2 - 5A + 7I = 0$.

5. Find the degree or index of the nilpotent matrix $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$

6. Show that matrix $\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is periodic matrix of period 2.

7. Which of the following is idempotent or involutory matrix

(i) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

8. Which of the following matrices are Hermitian or Skew Hermitian

(i) $\begin{bmatrix} 1 & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 1-i & 7 \\ 1+i & 6 & -i \\ 7 & i & 5 \end{bmatrix}$

9. Let A be a square matrix. Show that

(a) $A + (\bar{A})^t$ is Hermitian,

(b) $A - (\bar{A})^t$ is skew Hermitian

10. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

11. Show that $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

12. Solve the following system of linear equations by Cramer's rule, Gauss elimination, Gauss-Jordan and matrix inversion methods.

$2x + 4y - z = 0$; $x - 2y - 2z = 2$ and $-5x - 8y + 3z = -2$.