UNIT

06

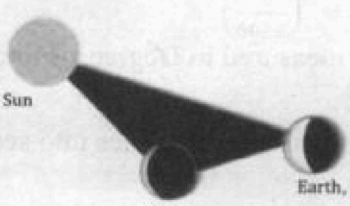
TRIGONOMETRY AND BEARING

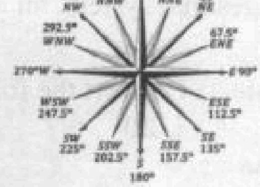
In this unit the students will be able to:

- Measure an angle in sexagesimal and circular systems.
- Convert an angle from D°M'S" to decimal form and vice versa.
- Convert an angle from sexagesimal to circular system and vice versa.
- Find length of circular arc when radius and central angle are given.
- Find area of the sector of a circle.
- Find the trigonometric ratios of 0°, 30°, 45°, 60°, 90°, 180°, 270° and 360°.
- Prove and apply trigonometric identities.
- Find angles of elevation and depression.
- Solve real life problems related to a right-angled triangle.
- Interpret and use three figure bearings.
- Find bearing of cardinal and intercardinal directions.
- Solve real life problems related to bearing and trigonometry.

Trigonometry is the branch of Mathematics that deals with the relationship of sides with angles in a triangle. With trigonometry, finding out the heights of big mountains or towers is possible. It is also used to find the distance between stars or planets in astronomy and is widely used in physics, architecture, engineering and GPS navigation systems. Trigonometry is based on the principle that **if two triangles have the same set of angles, their corresponding sides are in the same ratio.**

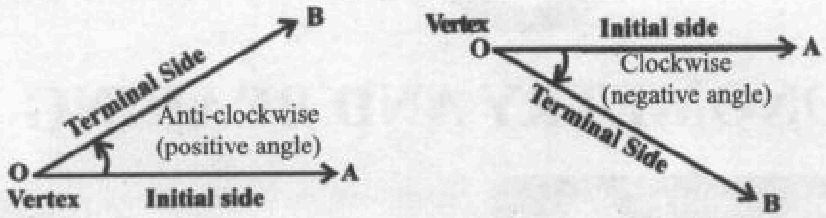
The measurement along with direction of angles plays an important role in satellite navigation. By using trigonometric and bearing principles, navigation has become easier for the military and hikers . Direction compass supports in getting clear picture of the angle between cardinal and intercardinal directions.





6.1 Measurement of an Angle

Union of two rays having common end point forms an angle. An angle is positive or negative if measured in anti-clockwise or clockwise direction respectively.



Angle in standard position

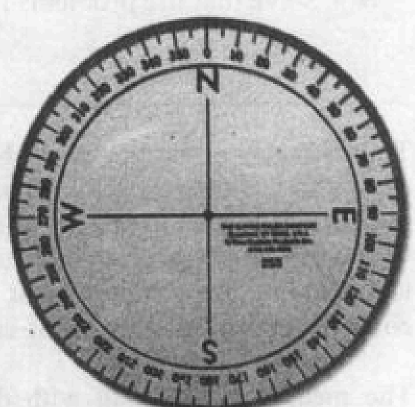
An angle when drawn in the cartesian plane(rectangular coordinate system) is in standard position if its vertex is at the origin and its initial arm is directed along positive x-axis. Its terminal arm may lie either on any of the axis or in any of the quadrants.

There are three systems of measurement of an angle.

i. Sexagesimal System or English System
ii. Radian measure or Circular System
iii. Centesimal or French System

Centesimal or French system is beyond the scope of this book.

6.1.1 Sexagesimal System



Draw a circle and divide its boundary into 360 equal arcs. The central angle of each arc is of measure 1 degree (1°). To draw an angle of 30°, we need 30 consecutive arcs of 1°. As 90° is one fourth of 360°, so 90 consecutive arcs of 1° form a quadrant of the circle. i.e., central angle of quadrant of a circle is 90°. There are 180 arcs of 1° in a half circle and all 360 arcs form a complete circle. i.e., central angle of half circle is 180° and that of full circle it is 360°. To measure the central angle of an arc whose length is smaller than the length of arc of 1°, we divide the arc of 1° into further 60 equal arcs. The central angle of each of these small arcs is of measure 1 minute denoted as $1'$. So, there are $60'$ in a degree. Similarly, there are 60 seconds($60''$) in a minute.

Thus, $1^\circ = 60'$, $1' = 60''$, $1^\circ = 3600''$.

$$1' = \left(\frac{1}{60}\right)^\circ, 1'' = \left(\frac{1}{60}\right)', 1'' = \left(\frac{1}{3600}\right)^\circ.$$

In sexagesimal system, an angle is measured in Degrees, Minutes and Seconds.

Example 1: Convert the following measures of angles into seconds.

i. $30'$ ii. 60° iii. $45^\circ 45' 45''$

Solution:

i. $30' = 30 \times 60'' = 1800''$

ii. $60^{\circ} = 60 \times 60' = 3600' = 3600 \times 60'' = 216000''$

iii. $45^{\circ}45'45'' = 45^{\circ} + 45' + 45''$
$= 45 \times 3600'' + 45 \times 60'' + 45''$
$= 162000'' + 2700'' + 45'' = 164745''$

Example 2: Convert the following measures of angles into minutes

i. 36° **ii.** $45''$ **iii.** $36^{\circ}45'30''$

Solution:

i. $36^{\circ} = 36 \times 60' = 2160'$

ii. $45'' = \left(45 \times \frac{1}{60}\right)' = \left(\frac{3}{4}\right)' = 0.75'$

iii. $36^{\circ}45'30'' = 36^{\circ} + 45' + 30''$
$= 36 \times 60' + 45' + \left(\frac{30}{60}\right)'$
$= 2160' + 45' + 0.5' = 2205.5'$

Example 3: Write the following measures of angles into $D^{\circ}M'S''$.

i. 60.5° **ii.** 45.125° **iii.** $3600''$ **iv.** $216000''$ **v.** $20.555'$

Solution:

i. $60.5^{\circ} = 60^{\circ} + 0.5^{\circ}$
$= 60^{\circ} + 0.5 \times 60'$
$= 60^{\circ} + 30' = 60^{\circ}30'$

ii. $45.125^{\circ} = 45^{\circ} + 0.125^{\circ}$
$= 45^{\circ} + 0.125 \times 60'$
$= 45^{\circ} + 7.5'$
$= 45^{\circ} + 7' + 0.5 \times 60'' = 45^{\circ}7'30''$

iii. $3600'' = \left(\frac{3600}{60}\right)' = 60' = 1^{\circ} = 1^{\circ}0'0''$

iv. $216000'' = \left(\frac{216000}{60}\right)' = 3600' = \left(\frac{3600}{60}\right)^{\circ} = 60^{\circ}0'0''$

v. $20.555' = 20' + 0.555'$
$= 20' + 0.555 \times 60''$
$= 20' + 33.3'' = 0^{\circ}20'33''$

Example 4: Write the following measures of angles in degrees correct to 4 decimal places.

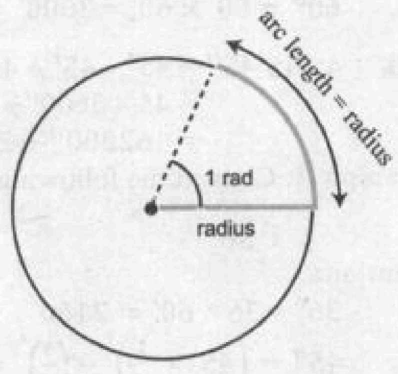
i. $60^{\circ}30'30''$ **ii.** $75^{\circ}25'35''$

Solution:

i. $60^{\circ}30'30'' = 60^{0} + \left(\frac{30}{60}\right)^{0} + \left(\frac{30}{3600}\right)^{0}$
$= 60^{0} + 0.5^{0} + 0.0083^{0} = 60.5083^{0}$

ii. $75^{\circ}25'35'' = 75^{0} + \left(\frac{25}{60}\right)^{0} + \left(\frac{35}{3600}\right)^{0}$
$= 75^{0} + 0.4167^{0} + 0.0097^{0} = 75.4264^{0}$

6.1.2 Circular System or Radian Measure



The ratio between length of circular arc and radius is called radian. If length of an arc is equal to radius of circle, then their ratio is 1 radian. i.e., one radian is measure of central angle of an arc whose length is equal to radius of the circle.

The ratio between circumference and radius of a circle is 2π, therefore measure of a complete angle in circular system is 2π radians.

In circular system, an angle is measured in radians.

In sexagesimal system measure of a complete angle is 360^0,

so $\quad 2\pi$ radians $= 360^0$

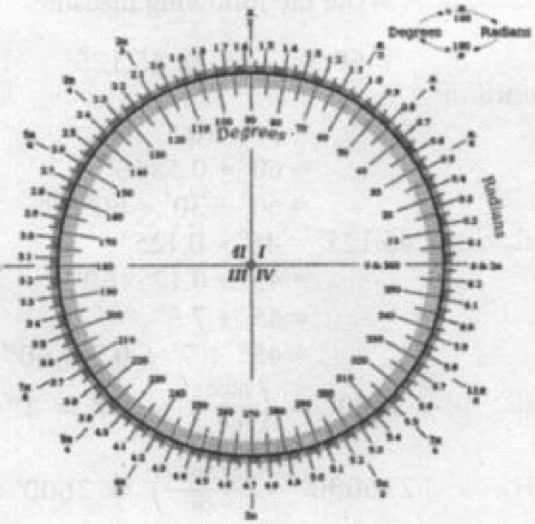
π radians $= 180^0$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^0$$

$$\approx 57^0 17' 45''$$

Similarly,

$$1^0 = \frac{\pi}{180} \text{ radians} \approx 0.0175 \text{ radians}$$



To convert measure of an angle given in radians to degrees, multiply its value by $\left(\frac{180}{\pi}\right)^0$ and to convert measure of an angle given in degrees to radians, multiply its value by $\frac{\pi}{180}$ radians.

Example 5:

Convert the following measures of angles in sexagesimal system ($D^oM'S''$).

i. 2.125 radians **ii.** 0.5 radians **iii.** $\frac{\pi}{3}$ radians **iv.** $\frac{3\pi}{4}$ radians

Solution:

i.
$$2.125 \text{ radians} = 2.125 \times \left(\frac{180}{\pi}\right)^0$$
$$\approx 2.125 \times 57.2958^0$$
$$\approx 121.7535^0$$
$$= 121^0 + 0.7535 \times 60'$$
$$= 121^0 + 45.21'$$
$$= 121^0 + 45' + 0.21 \times 60'' = 121^0 45' 13''$$

ii. $0.5 \text{ radians} = 0.5 \times \left(\frac{180}{\pi}\right)^0$

$\approx 0.5 \times 57.2958^0$

$\approx 28.6479^0$

$= 28^0 + 0.6479 \times 60'$

$= 28^0 + 38.874'$

$= 28^0 + 38' + 0.874 \times 60'' = 28^0 38' 52''$

iii. $\frac{\pi}{3} \text{ radians} = \frac{\pi}{3} \times \left(\frac{180}{\pi}\right)^0$

$= \left(\frac{180}{3}\right)^0 = 60^0$

iv. $\frac{3\pi}{4} \text{ radians} = \frac{3\pi}{4} \times \left(\frac{180}{\pi}\right)^0$

$= \left(\frac{540}{4}\right)^0 = 135^0$

Example 6:

Convert the following measures of angles in radians.

i. 45^0 **ii.** 150^0 **iii.** 300^0 **iv.** $30^0 30' 30''$

Solution:

i. $45^0 = 45 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{4} \text{ radians} \approx 0.7854 \text{ radians}$

ii. $150^0 = 150 \times \frac{\pi}{180} \text{ radians}$

$= \frac{5\pi}{6} \text{ radians} \approx 2.618 \text{ radians}$

iii. $300^0 = 300 \times \frac{\pi}{180} \text{ radians}$

$= \frac{5\pi}{3} \text{ radians} \approx 5.236 \text{ radians}$

iv. $30^0 30' 30'' = 30.5083^0$

$= 30.5083 \times \frac{\pi}{180} \text{ radians}$

$\approx 30.5083 \times 0.0175 \text{ radians}$

$\approx 0.5325 \text{ radians}$

EXERCISE 6.1

1. Convert the following measure of angles in seconds.
   i. 5^o ii. $30'$ iii. $10^o 30'$ iv. $20^o 20' 20''$
2. Convert the following measure of angles in minutes.
   i. 75^o ii. $120''$ iii. $50^o 40'$ iv. $30^o 30' 30''$
3. Convert the following measure of angles in degrees and write the answer correct to 4 decimals places.
   i. $135'$ ii. $150''$ iii. $60^o 60'$ iv. $45^o 45' 45''$

4. Write the following measures of angles in $D^oM'S''$.

i. 60.125^o ii. $135.375''$ iii. $60.85'$ iv. 255.45^o

5. Convert the following in radians. Write the answers in terms of π.

i. 45^o ii. 150^o iii. $60^o30'$ iv. 120^o

6. Convert the following in radians. Use $\pi = 3.1416$.

i. 270^o ii. 60^o iii. $180^o45'$ iv. $75^o30'45''$

7. Write the following radian measures of angles in $D^oM'S''$.

i. $\frac{\pi}{4}$ ii. $\frac{5\pi}{6}$ iii. $\frac{\pi}{12}$ iv. $\frac{7\pi}{40}$

v. 1 vi. 3.1416 vii. 12π viii. 5

6.2 Sector of Circle

6.2.1 Length of an Arc

Length of a circular arc and radian measure of its central angle are directly proportional. i.e., ratio between length of an arc and radian measure of its central angle remains constant.

Consider a circle of radius r with centre O. Choose an arc of length l on circumference. Let θ be the radian measure of central angle of that arc. Now choose another arc whose central angle is of 1 radian. The length (l_1) of such arc is equal to the radius. i.e. $l_1 = r$.

$$l : \theta :: l_1 : 1$$
$$l : \theta :: r : 1$$
$$l \times 1 = r \times \theta$$
$$l = r\theta$$

So, length of a circular arc is product of radius and radian measure of central angle.

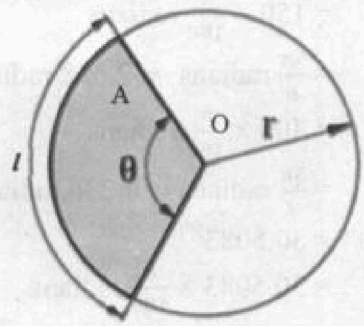


Fig. (i)

6.2.2 Area of Sector of a Circle

The union of a circle and its interior region is the circular region. The circular region bounded by an arc and its two corresponding radial segments is called sector of the circle. In figure (i) shaded region is a sector of the circle with radius r and θ the radian measure of central angle. Like arc length, area(A) of sector is directly proportional to the radian measure of central angle.

area of sector : θ : : area of circle : complete angle

$$A : \theta :: \pi r^2 : 2\pi$$
$$2\pi A = \pi r^2 \theta$$
$$A = \frac{\pi r^2\theta}{2\pi}$$
$$= \frac{r^2\theta}{2},$$

where θ is in radians.

Do You Know?

From elementary geometry:

$$\frac{Arc\ length(l)}{2\pi r} = \frac{Central\ angle\ of\ sector(\theta)}{2\pi} = \frac{Area\ of\ sector\ of\ circle\ (A)}{\pi r^2}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

$$\Rightarrow \frac{l}{2\pi r} = \frac{\theta}{2\pi} \quad \text{and} \quad \frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

$\Rightarrow$ Arc length(l) = $r\theta$ and Area of sector(A) = $\frac{1}{2}r^2\theta$

Example 7:

Find length of an arc of a circle whose central angle and radius are given below.

i. $\theta = \frac{\pi}{4}$ rad, $r = 2$ cm

ii. $\theta = 60^0$, $r = 30$ cm

Solution:

i. $l = r\,\theta$

$= 2 \times \frac{\pi}{4}$

$= \frac{\pi}{2}$ cm

$= \frac{3.14}{4}$

$= 1.57$ cm

ii. $\theta = 60^0 = \frac{\pi}{3}$ rad, $r = 30$ cm

Now, $l = r\,\theta$

$= 30 \times \frac{\pi}{3}$

$= 10\pi$ cm

$= 10 \times 3.14$

$= 31.4$ cm

Example 8:

Find area of the sector of a circle whose central angle is 120^0 and radius 5 cm.

Solution:

$r = 5$ cm,

$\theta = 120^0 = \frac{2\pi}{3}$ rad

$A = \frac{r^2\theta}{2} = \frac{1}{2}(5)^2\left(\frac{2\pi}{3}\right)$

$= \frac{25\pi}{3}\ \text{cm}^2 = 26.18\ \text{cm}^2$

Example 9:

The length of an arc and area of sector of a circle are 4 cm and 16 cm^2 respectively. Find radius and central angle of sector.

Solution:

$l = 4$ cm, $A = 16$ cm^2, $r = ?$, $\theta = ?$

As, $l = r\,\theta$

$\Rightarrow r\,\theta = 4$

$\therefore \theta = \frac{l}{r} = \frac{4}{r}$

Now $A = \frac{1}{2}r^2\theta$

$16 = \frac{1}{2}r^2 \times \frac{4}{r}$ (substituting the value of θ)

$16 = 2r$

$r = 8$ cm

$\therefore \theta = \frac{4}{r}$

$= \frac{4}{8}$

$= \frac{1}{2}$ rad.

EXERCISE 6.2

1. Find length of arc and area of sector for given radius and central angle.
   i. $r = 5$ cm, $\theta = \frac{\pi}{3}$ radians
   ii. $r = 12$ m, $\theta = 120^0$
   iii. $r = 6$ dm, $\theta = 60^0 45' 30''$

2. A central angle in a circle of radius 4 cm is 75^0. Find the length of the intercepted arc and area bounded by the sector.

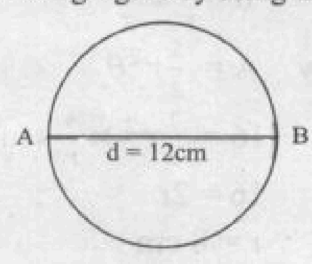
3. Find radian measure of the central angle of a circular sector for the following data.
   i. $l = 8$ cm, $r = 4$ cm
   ii. $l = 8.5$ m, $r = 2.25$ m
   iii. $A = 45\text{ cm}^2$, $r = 10.70$ cm
   iv. $A = 100\text{ cm}^2$, $l = 10$ cm

4. Find radius of circle for the following information.
   i. $l = 4$ cm, $\theta = \pi$ radians
   ii. $l = 6$ m, $\theta = 15^0$
   iii. $A = 200\text{ cm}^2$, $\theta = \frac{\pi}{4}$ radians
   iv. $A = 100\text{ dm}^2$, $l = 10$ dm

5. A 30 inch pendulum swings through an angle of 30^o. Find the length of the arc in inches through which the tip of the pendulum swings.

6. A motorcycle is traveling on a curve along a highway. The curve is an arc of a circle with radius of 10 km. If the motorcycle's speed is 42 km/h, what is the angle in degrees through which the motorcycle will turn in 21 minutes?

7. Find perimeter and area of a half circle in the following figure by using the formulae $l = r\theta,\ A = \frac{1}{2}r^2\theta$.

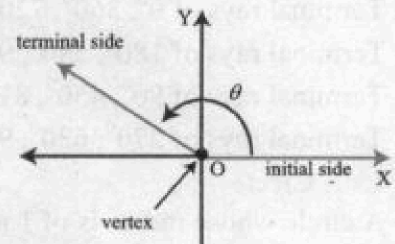


8. Find the circular measure of the angle between the hour hand and minute hand of a clock at:
   i. 9'o clock ii. 02:30 iii. 06:45

6.3 Trigonometric Ratios

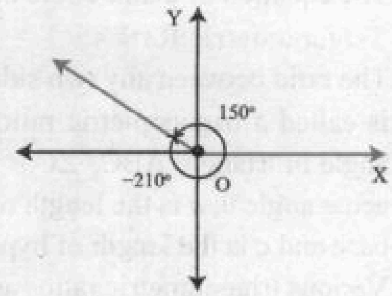
6.3.1 General Angle

In coordinate plane, the two axes divide the plane in four equivalent parts called quadrants. If we place an angle in the coordinate plane such that its vertex coincides with origin and one of its rays lies along $\overrightarrow{OX}$ (+ve x-axis), then this angle is said to be in standard position. The ray along $\overrightarrow{OX}$ is called initial ray and second ray in the plane is called terminal ray. Such an angle is called general angle.



Measure of an angle is positive in anticlockwise direction and is negative in clockwise direction.

In the adjoining figure two angles in standard position are drawn. Both have same terminal ray. One of them is 150^0 (in anticlock wise direction) and the other is -210^0(in clock wise direction).

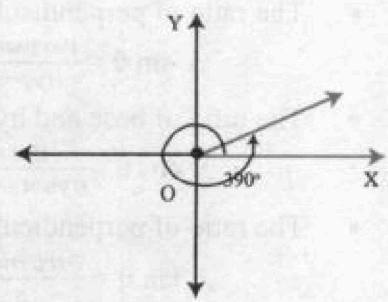


Two angles having same terminal ray in standard position are called co-terminal angles.

The measure of a general angle can be greater than 360^0.

To draw a general angle whose measure is greater than 360^0, we subtract any multiple of 360^0 from its value and try to find a value less than 360^0. Terminal ray of 360^0 or that of any of its multiple coincides with 0^0 , i.e., $\overrightarrow{OX}$.

In general, two angles are co-terminal if their measures differ by some multiple of 360^o. e.g., to draw an angle of 390^0, first find $390^0 - 360^0 = 30^0$, then draw 30^0 in anticlockwise direction.



390^0 and 30^0 have same terminal ray so they are co-terminal angles.

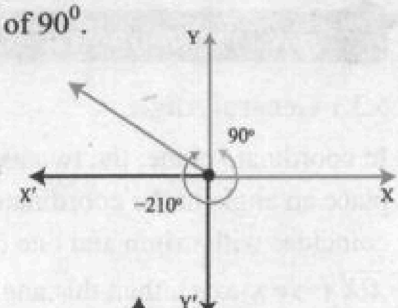
Similarly, 150^0 and -210^0 are co-terminal angles.

Thinking Corner!
How many co-terminal angles can be of an angle?

6.3.2 Quadrantal Angles

If terminal ray of an angle in standard position coincides with any of the axes, then it is called quadrantal angle. The measure of a quadrantal angle is a multiple of 90^0.

e.g,. $0^0, 90^0, 180^0, 270^0, 360^0, 450^0, \ldots$ are the quadrantal angles.



In radian measure, quadrantal angle is a multiple of $\frac{\pi}{2}$.

e.g., $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi \ldots$ are quadrantal angles.

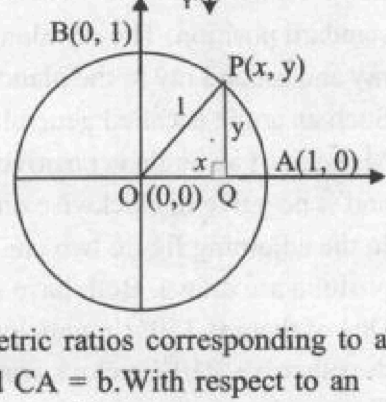
Terminal rays of $0^0, 360^0, 720^0, \ldots$ are along $\overrightarrow{OX}$.

Terminal rays of $180^0, 540^0, 900^0, \ldots$ are along $\overrightarrow{OX'}$.

Terminal rays of $90^0, 450^0, 810^0, \ldots$ are along $\overrightarrow{OY}$.

Terminal rays of $270^0, 630^0, 990^0, \ldots$ are along $\overrightarrow{OY'}$.

Unit Circle



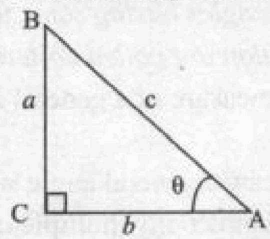
A circle whose radius is of 1 unit is called a unit circle.

In standard position centre of a unit circle is O (Origin).

The equation of a unit circle centered at O is $x^2 + y^2 = 1$.

Trigonometric Ratios

The ratio between any two sides of a right-angled triangle is called a trigonometric ratio. There are six possible trigonometric ratios corresponding to an angle.In triangle ABC, $\angle C = 90^0$, $\angle A = \theta$, AB = c, BC = a and CA = b.With respect to an acute angle θ, *a* is the length of perpendicular, *b* is the length of base and c is the length of hypotenuse.



Various trigonometric ratios are defined as:

$$\frac{Perpendicular}{Hypotenuse} = \frac{a}{c}, \frac{Base}{Hypotenuse} = \frac{b}{c}, \frac{Perpendicular}{Base} = \frac{a}{b},$$

$$\frac{Hypotenuse}{Perpendicular} = \frac{c}{a}, \frac{Hypotenuse}{Base} = \frac{c}{b}, \frac{Base}{Perpendicular} = \frac{b}{a}$$

- The ratio of perpendicular and hypotenuse is sine of θ denoted by sin θ.

  $$\therefore \sin\theta = \frac{Perpendicular}{Hypotenuse} = \frac{a}{c}$$

- The ratio of base and hypotenuse is cosine of θ denoted by cos θ.

  $$\therefore \cos\theta = \frac{Base}{Hypotenuse} = \frac{b}{c}$$

- The ratio of perpendicular and base is tangent of θ denoted by tan θ.

  $$\therefore \tan\theta = \frac{Perpendicular}{Base} = \frac{a}{b}$$

- The ratio of hypotenuse and perpendicular is cosecant θ denoted by cosec θ.

  $$\therefore \operatorname{cosec}\theta = \frac{Hypotenuse}{Perpendicular} = \frac{c}{a} = \frac{1}{\sin\theta}$$

- The ratio of hypotenuse and base is secant of θ denoted by sec θ.

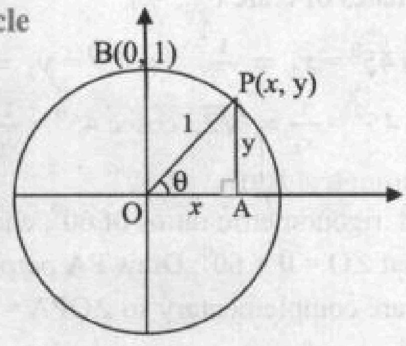
  $$\therefore \sec\theta = \frac{Hypotenuse}{Base} = \frac{c}{b} = \frac{1}{\cos\theta}$$

- The ratio of base and perpendicular is cotangent of θ denoted by cot θ.

  $$\therefore \cot\theta = \frac{Base}{Perpendicular} = \frac{b}{a} = \frac{1}{\tan\theta}$$

- cosec θ, sec θ and cot θ are respectively reciprocals of sin θ, cos θ and tan θ.

6.3.3 Trigonometric Ratios with the Help of Unit Circle



Consider a unit circle centered at O.
Mark a point P(x, y) on the circle. Join P with O.
Draw PA perpendicular to x-axis.
OA = x, PA = y and OP = 1(radius of circle).
If $\angle AOP = \theta$, then $\sin\theta = \frac{y}{1} = y$ and $\cos\theta = \frac{x}{1} = x$.
So, coordinates of P are $(\cos\theta, \sin\theta)$.
i.e., $x = \cos\theta$ and $y = \sin\theta$. Also, $\tan\theta = \frac{y}{x}$.

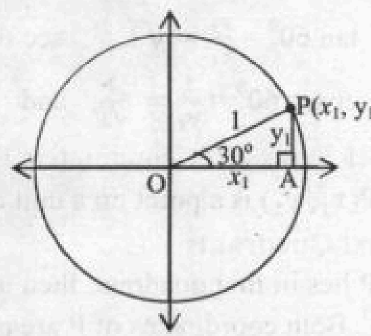
Reciprocal of y is cosec θ, reciprocal of x is sec θ and reciprocal of $\frac{y}{x}$ is cot θ.

If P is a point on a unit circle centered at O and $\angle AOP = \theta$, then coordinates of P are $(\cos\theta, \sin\theta)$.

6.3.4 Trigonometric Ratios of 30°, 45° and 60°

Trigonometric Ratios of 30°

To find trigonometric ratios of 30°, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle O = \theta = 30^o$. Draw PA perpendicular to x-axis. As in a right-angled triangle length of side opposite to 30° is half of hypotenuse, so in right angled triangle OAP, $y_1 = \frac{1}{2}$.



Equation of unit circle is $x^2 + y^2 = 1$.
Put $x = x_1$ and $y = y_1 = \frac{1}{2}$ in above equation to get

$$x_1^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$x_1^2 = 1 - \frac{1}{4}$$

$$x_1^2 = \frac{3}{4} \text{ or } x_1 = \frac{\sqrt{3}}{2} \ (\because \text{P lies in quadrant I})$$

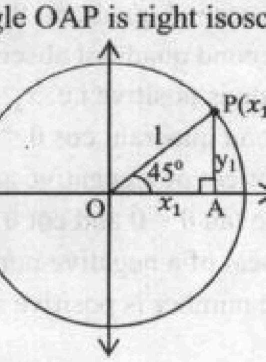
Coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

So, $\cos 30^0 = x_1 = \frac{\sqrt{3}}{2}$, $\sin 30^0 = y_1 = \frac{1}{2}$, $\tan 30^0 = \frac{y_1}{x_1} = \frac{1}{\sqrt{3}}$

$\sec 30^0 = \frac{1}{x_1} = \frac{2}{\sqrt{3}}$, $\text{cosec } 30^0 = \frac{1}{y_1} = 2$, and $\cot 30^0 = \frac{x_1}{y_1} = \sqrt{3}$.

Trigonometric Ratios of 45°

To find trigonometric ratios of 45°, choose a point $P(x_1, y_1)$ on a unit circle in standard position such that $\angle O = \theta = 45^0$. Draw PA perpendicular to x-axis. Triangle OAP is right isosceles.
Therefore, AP = OA or $x_1 = y_1$.
Equation of unit circle is $x^2 + y^2 = 1$.
Put $x = x_1$ and $y = y_1 = x_1$ in above equation to get



$$x_1^2 + x_1^2 = 1$$

$$2x_1^2 = 1$$

$$x_1^2 = \frac{1}{2} \Rightarrow x_1 = \frac{1}{\sqrt{2}}, \text{ also } y_1 = \frac{1}{\sqrt{2}},$$

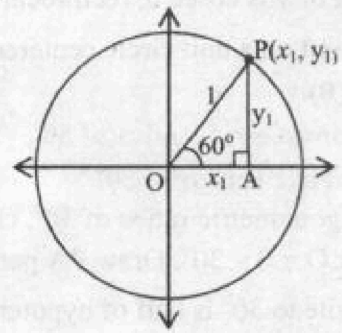
Coordinates of P are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

So, $\cos 45^0 = x_1 = \frac{1}{\sqrt{2}}$, $\sin 45^0 = y_1 = \frac{1}{\sqrt{2}}$, $\tan 45^0 = \frac{y_1}{x_1} = 1$,

$\sec 45^0 = \frac{1}{x_1} = \sqrt{2}$, $\operatorname{cosec} 45^0 = \frac{1}{y_1} = \sqrt{2}$, and $\cot 45^0 = \frac{x_1}{y_1} = 1$.

Trigonometric Ratios of 60^0

To find trigonometric ratios of 60^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle O = \theta = 60^0$. Draw PA perpendicular to x-axis. As in a right-angled triangle two acute angles are complementary so $\angle OPA = 30^0$. As in a right-angled triangle length of side opposite to 30^0 is half of hypotenuse, so $x_1 = \frac{1}{2}$.



Equation of unit circle is $x^2 + y^2 = 1$.

Put $x = x_1 = \frac{1}{2}$ and $y = y_1$ in above equation to get

$\left(\frac{1}{2}\right)^2 + y_1^2 = 1$

$y_1^2 = 1 - \frac{1}{4} \Rightarrow y_1^2 = \frac{3}{4} \Rightarrow y_1 = \frac{\sqrt{3}}{2}$, also $x_1 = \frac{1}{2}$,

Coordinates of P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

So, $\cos 60^0 = x_1 = \frac{1}{2}$, $\sin 60^0 = y_1 = \frac{\sqrt{3}}{2}$,

$\tan 60^0 = \frac{y_1}{x_1} = \sqrt{3}$, $\sec 60^0 = \frac{1}{x_1} = 2$,

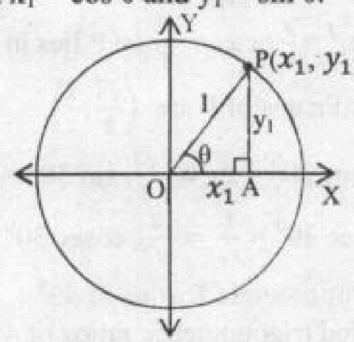
$\operatorname{cosec} 60^0 = \frac{1}{y_1} = \frac{2}{\sqrt{3}}$, and $\cot 60^0 = \frac{x_1}{y_1} = \frac{1}{\sqrt{3}}$.

6.3.5 Signs of Trigonometric Ratios in the Four Quadrants

If $P(x_1, y_1)$ is a point on a unit circle such that $\angle XOP = \theta$, then $x_1 = \cos\theta$ and $y_1 = \sin\theta$.

First Quadrant

If P lies in first quadrant, then θ lies between 0^0 and 90^0. Both coordinates of P are positive i.e., $x_1 > 0$ and $y_1 > 0$. So, in first quadrant $\cos\theta > 0$ and $\sin\theta > 0$. As the quotient and reciprocals of positive numbers are also positive so in first quadrant all trigonometric ratios are positive.



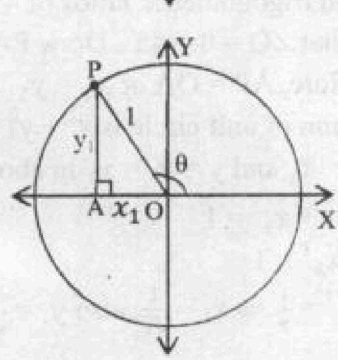
Second Quadrant

If P lies in second quadrant, then θ lies between 90^0 and 180^0. In second quadrant abscissa of a point is negative and ordinate is positive i.e. $x_1 < 0$ and $y_1 > 0$.

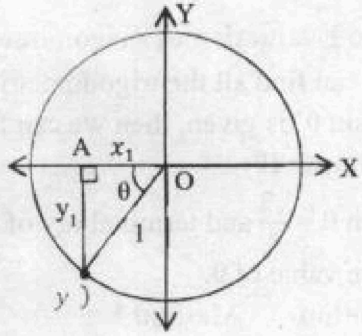
So, in second quadrant $\cos\theta < 0$ and $\sin\theta > 0$.

As the quotient of a negative and a positive number is negative so $\tan\theta < 0$ and $\cot\theta < 0$.

As reciprocal of a negative number is negative and that of positive number is positive so $\sec\theta < 0$ and $\operatorname{cosec}\theta > 0$.

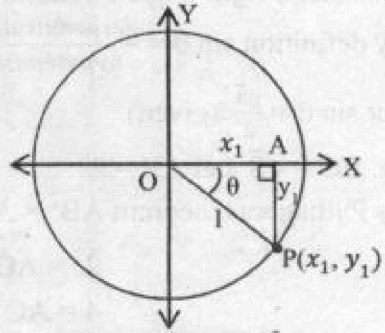


Third Quadrant



If P lies in third quadrant, then θ lies between 180^0 and 270^0. In third quadrant both abscissa and ordinate of a point are negative i.e., $x_1 < 0$ and $y_1 < 0$. So, in third quadrant $\cos\theta < 0$ and $\sin\theta < 0$. As the quotient of two negative numbers is positive so $\tan\theta > 0$ and $\cot\theta > 0$. As reciprocals of negative numbers are negative, so $\sec\theta < 0$ and $\text{cosec}\,\theta < 0$.

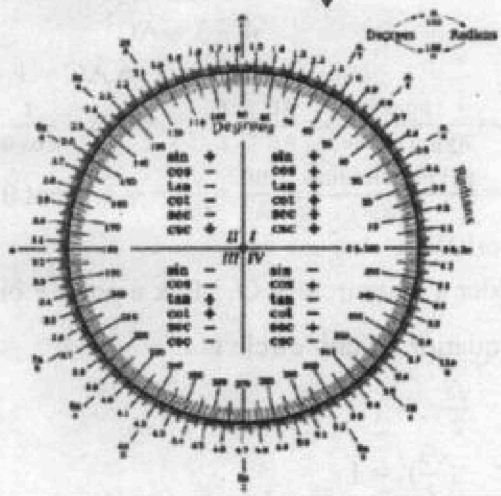
Fourth Quadrant



If P lies in fourth quadrant, then θ lies between 270^0 and 360^0. In fourth quadrant abscissa of a point is positive and ordinate is negative i.e. $x_1 > 0$ and $y_1 < 0$.

So in fourth quadrant $\cos\theta > 0$ and $\sin\theta < 0$. As the quotient of a negative and a positive number is negative so $\tan\theta < 0$ and $\cot\theta < 0$. As reciprocal of a negative number is negative and that of positive number is positive, so $\sec\theta > 0$ and $\text{cosec}\,\theta < 0$.

The figure at right would be helpful to understand the signs of trigonometric ratios in different quadrants.



To Be Observed!

When point P lies in any of the quadrants or on any of the axis, by joining it to origin, we get the angle θ formed by terminal arm OP in standard position, then:

$$\sin\theta = \frac{\textit{algebraic distance of P from x-axis}}{\textit{distance of P from origin}} = \frac{y}{r}$$

$$\cos\theta = \frac{\textit{algebraic distance of P from y-axis}}{\textit{distance of P from origin}} = \frac{x}{r}$$

$$\tan\theta = \frac{\textit{algebraic distance of P from x-axis}}{\textit{algebraic distance of P from y-axis}} = \frac{y}{x}$$

(due to the word 'algebraic', x and y will get +/− sign according to the quadrant of P)
We consider $r = 1$ in a unit circle.

6.3.6 Evaluation of Trigonometric Ratios of an Angle if One of Them is Given

We can find all the trigonometric ratios of a particular angle if one of them is given i.e. if value of '$\sin\theta$' is given, then we can find the remaining ratios, for the same value of θ.

Example 10:

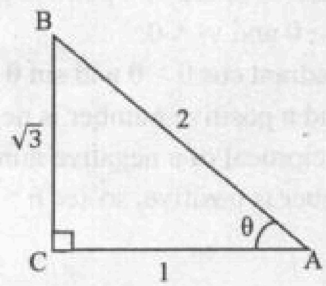
If $\sin\theta = \frac{\sqrt{3}}{2}$ and terminal ray of θ is in first quadrant, then find remaining trigonometric ratios for same value of θ.

Solution: **Method I**

Consider a right-angled triangle ABC, in which $m\angle C = 90^0$, $m\angle A = \theta$.

By definition $\sin\theta = \frac{perpendicular}{hypotenuse} = \frac{BC}{AB}$

But $\sin\theta = \frac{\sqrt{3}}{2}$ (given)



So, $BC = \sqrt{3}$ and $AB = 2$

By Pythagoras theorem $AB^2 = AC^2 + BC^2$

$$2^2 = AC^2 + (\sqrt{3})^2$$
$$4 = AC^2 + 3$$
$$4 - 3 = AC^2$$
$$AC^2 = 1 \Rightarrow AC = 1$$

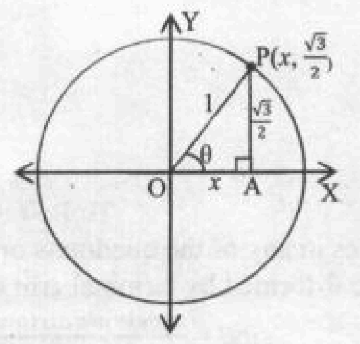
$\cos\theta = \frac{base}{hypotenuse} = \frac{AC}{AB} = \frac{1}{2}$, $\sec\theta = \frac{1}{\cos\theta} = 2$

$\tan\theta = \frac{perpendicular}{base} = \frac{BC}{AC} = \frac{\sqrt{3}}{1} = \sqrt{3}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$, $\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$

Method II

Consider a unit circle at O. Mark a point P on the circle in first quadrant whose y coordinate is $\frac{\sqrt{3}}{2}$.

The equation of unit circle is $x^2 + y^2 = 1$.



Put $y = \frac{\sqrt{3}}{2}$

$$x^2 + (\frac{\sqrt{3}}{2})^2 = 1$$
$$x^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

Since, P is in first quadrant so $x > 0$. i.e., $x = \frac{1}{2}$

So, $\cos\theta = x = \frac{1}{2}$

$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$, $\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$

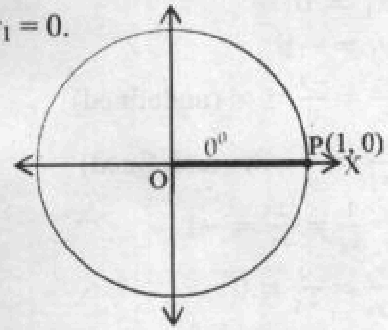
$\sec\theta = \frac{1}{\cos\theta} = 2$, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$

6.3.7 Trigonometric Ratios of Quadrantal Angles

Trigonometric Ratios of 0^0

To find trigonometric ratios of 0^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 0^0$.

As P lies on x-axis so its coordinates are $x_1 = 1$ and $y_1 = 0$.



So, $\cos 0^0 = x_1 = 1$

$\sin 0^0 = y_1 = 0$

$\tan 0^0 = \frac{y_1}{x_1} = \frac{0}{1} = 0$

$\sec 0^0 = \frac{1}{x_1} = \frac{1}{1} = 1$

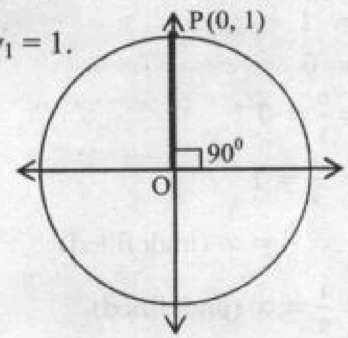
$\text{cosec } 0^0 = \frac{1}{y_1} = \frac{1}{0} = \infty$ (undefined)

$\cot 0^0 = \frac{x_1}{y_1} = \frac{1}{0} = \infty$ (undefined)

Trigonometric Ratios of 90^0

To find trigonometric ratios of 90^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 90^0$.

As P lies on y-axis so its coordinates are $x_1 = 0$ and $y_1 = 1$.



So, $\cos 90^0 = x_1 = 0$

$\sin 90^0 = y_1 = 1$

$\tan 90^0 = \frac{y_1}{x_1} = \frac{1}{0} = \infty$ (undefined)

$\sec 90^0 = \frac{1}{x_1} = \frac{1}{0} = \infty$ (undefined)

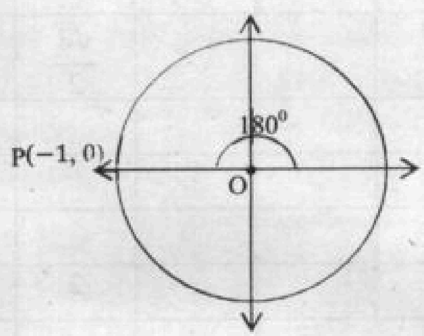
$\text{cosec } 90^0 = \frac{1}{y_1} = \frac{1}{1} = 1,$

$\cot 90^0 = \frac{x_1}{y_1} = \frac{0}{1} = 0$

Trigonometric Ratios of 180^0

To find trigonometric ratios of 180^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 180^0$.

As P lies on x-axis at left side of O so its coordinates are $x_1 = -1$ and $y_1 = 0$.



So, $\cos 180^0 = x_1 = -1$

$\sin 180^0 = y_1 = 0$

$\tan 180^0 = \frac{y_1}{x_1} = \frac{0}{-1} = 0$

$\sec 180^0 = \frac{1}{x_1} = \frac{1}{-1} = -1$

$\text{cosec } 180^0 = \frac{1}{y_1} = \frac{1}{0} = \infty$ (undefined)

$\cot 180^0 = \frac{x_1}{y_1} = \frac{-1}{0} = \infty$ (undefined)

Trigonometric Ratios of 270^0

To find trigonometric ratios of 270^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 270^0$.

As P lies on y-axis below origin, therefore its coordinates are $x_1 = 0$ and $y_1 = -1$.

So, $\cos 270^0 = x_1 = 0$

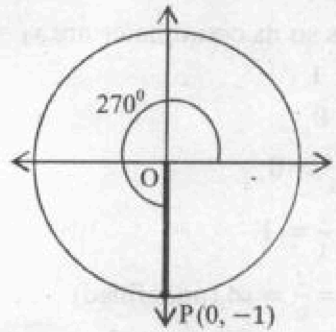
$\sin 270^0 = y_1 = -1$

$\tan 270^0 = \frac{y_1}{x_1} = \frac{-1}{0} = \infty$ (undefined)

$\sec 270^0 = \frac{1}{x_1} = \frac{1}{0} = \infty$ (undefined)

$\operatorname{cosec} 270^0 = \frac{1}{y_1} = \frac{1}{-1} = -1$

$\cot 270^0 = \frac{x_1}{y_1} = \frac{0}{-1} = 0$



Trigonometric Ratios of 360^0

To find trigonometric ratios of 360^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 360^0$. It coincides with 0^0.

As P lies on x-axis so its coordinates are $x_1 = 1$ and $y_1 = 0$.

So, $\cos 360^0 = x_1 = 1$

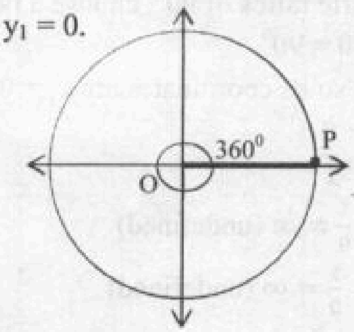
$\sin 360^0 = y_1 = 0$

$\tan 360^0 = \frac{y_1}{x_1} = \frac{0}{1} = 0$

$\sec 360^0 = \frac{1}{x_1} = \frac{1}{1} = 1$

$\operatorname{cosec} 360^0 = \frac{1}{y_1} = \frac{1}{0} = \infty$ (undefined)

$\cot 360^0 = \frac{x_1}{y_1} = \frac{1}{0} = \infty$ (undefined)



EXERCISE 6.3

1. Complete the following table.

θ	0^0	30^0	45^0	60^0	90^0	180^0	270^0	360^0
$\sin\theta$	0			$\frac{\sqrt{3}}{2}$			-1	
$\cos\theta$						-1		
$\tan\theta$			1					
$\operatorname{cosec}\theta$	∞					∞		
$\sec\theta$				2				1
$\cot\theta$		$\sqrt{3}$			0			

2. Evaluate the following.

i. $\sin 60^0 - \cos 30^0$ ii. $\sec 45^0 \operatorname{cosec} 45^0 - \sin 30^0 \cos 60^0$

iii. $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ iv. $\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$ v. $\sin\frac{\pi}{2}\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\sin\frac{\pi}{3}$

3. For $\theta = \frac{\pi}{6}$, verify the following statements.

i. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ ii. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

iii. $\cos 2\theta = 1 - 2\sin^2\theta$ iv. $\sec^2\theta - \tan^2\theta = 1$ v. $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

4. Write value of θ for the following ratios ($0 < \theta < 2\pi$).

i. $\sin\theta = \frac{\sqrt{2}}{2}$ ii. $\operatorname{cosec}\theta = -1$ iii. $\tan\theta = 1$

iv. $\sec\theta = \frac{\sqrt{12}}{3}$ v. $\cos\theta = 0$

5. If terminal ray of θ is in first quadrant and $\cos\theta = \frac{1}{2}$, then find remaining trigonometric ratios of θ.

6. If terminal ray of θ is in second quadrant and $\operatorname{cosec}\theta = 2$, then find the value of $\cos\theta\left(\dfrac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta}\right)$.

7. If $\tan\theta = -1$, and terminal ray of θ is not in second quadrant, then find remaining trigonometric ratios of θ.

6.4 Trigonometric Identities

Identity is an equation which remains true for all possible replacements of variable involved in it. e.g. $(x+1)(x-1) = x^2 - 1$ is an identity.

Trigonometric identity is an identity involving different trigonometric ratios in it. Trigonometric identities are of many types. Some of them are given below.

i. Reciprocal identities
ii. Quotient identities
iii. Pythagorean identities

6.4.1 Reciprocal Identities

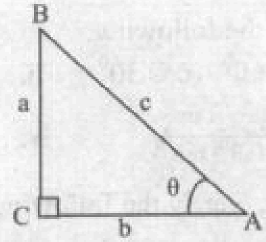
Cosec θ, sec θ and cot θ are reciprocals of sin θ, cos θ and tan θ respectively.

i.e. $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$

Proves of these identities are left for students.

6.4.2 Quotient Identities

i. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ ii. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$



Proof:

Consider a right-angled triangle ABC, in which $\angle C = 90^0$, $\angle A = \theta$, AB = c, BC = a and AC = b.

By definition, $\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$ and $\tan\theta = \frac{a}{b}$.

$\tan\theta = \frac{a}{b} = \dfrac{\frac{a}{c}}{\frac{b}{c}}$ (dividing numerator and denominator by c)

$= \dfrac{\sin\theta}{\cos\theta}$

Similarly, we can prove the second quotient identity.

6.4.3 Pythagorean Identities

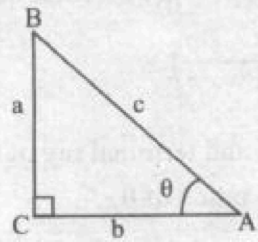
i. $\sin^2\theta + \cos^2\theta = 1$

ii. $\sec^2\theta - \tan^2\theta = 1$

iii. $cosec^2\theta - \cot^2\theta = 1$

Proof:

Consider a right-angled triangle ABC, in which $\angle C = 90^0$, $\angle A = \theta$, AB = c, BC = a and AC = b.



i. $\sin^2\theta + \cos^2\theta = 1$

By definition, $\sin\theta = \frac{a}{c}$ and $\cos\theta = \frac{b}{c}$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by c^2 we get

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$(\sin\theta)^2 + (\cos\theta)^2 = 1$ or $\sin^2\theta + \cos^2\theta = 1$ $(\because (sin\theta)^2 = sin^2\theta)$

ii. $\sec^2\theta - \tan^2\theta = 1$

By definition, $\sec\theta = \frac{c}{b}$ and $\tan\theta = \frac{a}{b}$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by b^2 we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$(\tan\theta)^2 + 1 = (\sec\theta)^2$$

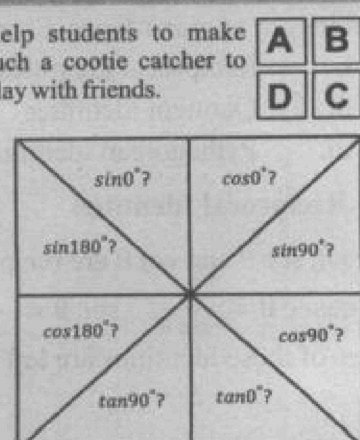
$$\tan^2\theta + 1 = \sec^2\theta \quad \text{or} \quad \sec^2\theta - \tan^2\theta = 1$$

iii. $cosec^2\theta - \cot^2\theta = 1$

By definition, $cosec\,\theta = \frac{c}{a}$ and $\cot\theta = \frac{b}{a}$

Funmatics

Help students to make such a cootie catcher to play with friends.



By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by a^2 we get

$$\frac{a^2}{a^2}+\frac{b^2}{a^2}=\frac{c^2}{a^2}$$

$$1+\left(\frac{b}{a}\right)^2=\left(\frac{c}{a}\right)^2$$

$$1+(\cot\theta)^2=(\operatorname{cosec}\theta)^2$$

$$1+\cot^2\theta=\operatorname{cosec}^2\theta$$

$$\text{or}\quad \operatorname{cosec}^2\theta-\cot^2\theta=1$$

Example 11:

Use basic trigonometric identities to prove the following relations.

i. $\cos\alpha+\sin\alpha\tan\alpha=\sec\alpha$

ii. $\dfrac{\tan\alpha}{\sec\alpha-1}=\dfrac{\sec\alpha+1}{\tan\alpha}$

iii. $\dfrac{\sin^2\alpha+2\cos\alpha-1}{\sin^2\alpha+3\cos\alpha-3}=\dfrac{1}{1-\sec\alpha}$

Solution:

i. $$\text{L.H.S.}=\cos\alpha+\sin\alpha\tan\alpha$$
$$=\cos\alpha+\sin\alpha\frac{\sin\alpha}{\cos\alpha}$$
$$=\cos\alpha+\frac{\sin^2\alpha}{\cos\alpha}$$
$$=\frac{\cos^2\alpha+\sin^2\alpha}{\cos\alpha}$$
$$=\frac{1}{\cos\alpha}=\sec\alpha=\text{R.H.S.}$$

Hence, $\cos\alpha+\sin\alpha\tan\alpha=\sec\alpha$

ii. $$\text{L.H.S.}=\frac{\tan\alpha}{\sec\alpha-1}$$
$$=\frac{\tan\alpha}{\sec\alpha-1}\times\frac{\sec\alpha+1}{\sec\alpha+1}$$
$$=\frac{\tan\alpha(\sec\alpha+1)}{\sec^2\alpha-1}$$
$$=\frac{\tan\alpha(\sec\alpha+1)}{\tan^2\alpha}$$
$$=\frac{\sec\alpha+1}{\tan\alpha}=\text{R.H.S.}$$

Hence, $\dfrac{\tan\alpha}{\sec\alpha-1}=\dfrac{\sec\alpha+1}{\tan\alpha}$

iii. $$\text{L.H.S.}=\frac{\sin^2\alpha+2\cos\alpha-1}{\sin^2\alpha+3\cos\alpha-3}$$
$$=\frac{1-\cos^2\alpha+2\cos\alpha-1}{1-\cos^2\alpha+3\cos\alpha-3}$$

$$=\frac{-\cos^2\alpha+2\cos\alpha}{-\cos^2\alpha+3\cos\alpha-2}$$

$$=\frac{-\cos\alpha(\cos\alpha-2)}{-(\cos^2\alpha-3\cos\alpha+2)}$$

$$=\frac{\cos\alpha(\cos\alpha-2)}{\cos^2\alpha-2\cos\alpha-\cos\alpha+2}$$

$$=\frac{\cos\alpha(\cos\alpha-2)}{\cos\alpha(\cos\alpha-2)-1(\cos\alpha-2)}$$

$$=\frac{\cos\alpha(\cos\alpha-2)}{(\cos\alpha-1)(\cos\alpha-2)}=\frac{\cos\alpha}{\cos\alpha-1}=\frac{\frac{\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha-1}{\cos\alpha}}$$

$$=\frac{1}{\frac{\cos\alpha}{\cos\alpha}-\frac{1}{\cos\alpha}}=\frac{1}{1-\sec\alpha}=\text{R.H.S.}$$

Hence $\dfrac{\sin^2\alpha+2\cos\alpha-1}{\sin^2\alpha+3\cos\alpha-3}=\dfrac{1}{1-\sec\alpha}$

Brain Buster

Check why

$\sin 30^\circ \cos 30^\circ \cos 90^\circ \sin 90 = 0$

Example 12:

Solve the following triangles. (Find unknown elements of the following triangles)

i. ΔABC, $\angle C = 90^0$, $\angle A = 30^0$, $a = 6$ cm

ii. ΔABC, $\angle C = 90^0$, $c = 12$ cm, $b = 6\sqrt{2}$ cm

i. Given Elements:

ΔABC, $\angle C = 90^0$, $\angle A = 30^0$, $a = 6$ cm

Required Elements:

$\angle B$, b, c.

Solution:

In ΔABC, $\angle A + \angle B + \angle C = 180^0$

$30^0 + \angle B + 90^0 = 180^0$

$\angle B = 180^0 - 90^0 - 30^0$

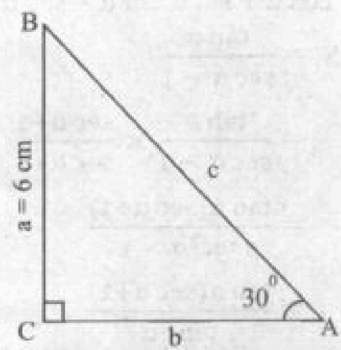
$\angle B = 60^0$

$\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$

$\sin 30^0 = \dfrac{a}{c}$

$\dfrac{1}{2} = \dfrac{6}{c}$

$c = 12$ cm



In ΔABC, by Pythagoras theorem,

$c^2 = a^2 + b^2$

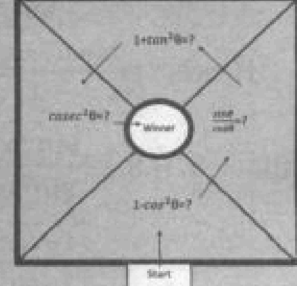
$12^2 = 6^2 + b^2$

$144 = 36 + b^2$

$b^2 = 144 - 36$

Math Play Ground

1. Make a hopscotch as given in the playground.
2. Ask a student to start hopping and finding answers.



$b^2 = 108$

$b = \sqrt{108} = 6\sqrt{3}$ cm

Therefore, required elements are $\angle B = 60^0$, $b = 6\sqrt{3}$ cm, $c = 12$ cm.

ii. Given Elements:

ΔABC, $\angle C = 90^0$, $c = 12$ cm, $b = 6\sqrt{2}$ cm

Required Elements:

$\angle A$, $\angle B$, a

Solution: In ΔABC, by Pythagoras theorem,

$c^2 = a^2 + b^2$

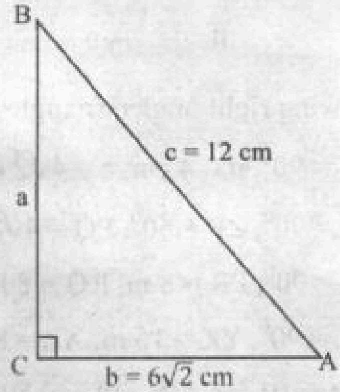
$12^2 = a^2 + (6\sqrt{2})^2$

$144 = a^2 + 72$

$a^2 = 144 - 72$

$a^2 = 72$

$a = \sqrt{72} = 6\sqrt{2}$ cm



$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$

$\cos\angle A = \frac{b}{c}$

$\cos\angle A = \frac{6\sqrt{2}}{12} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\angle A = 45^0$

In ΔABC, $\angle A + \angle B + \angle C = 180^0$

$45^0 + \angle B + 90^0 = 180^0$

$\angle B = 180^0 - 90^0 - 45^0 \Rightarrow \angle B = 45^0$

Therefore, required elements are $\angle A = 45^0$, $\angle B = 45^0$, $a = 6\sqrt{2}$ cm.

EXERCISE 6.4

1. Prove the following relations by using basic trigonometric identities.

   i. $(1 - \sin^2\theta)\sec^2\theta = 1$

   ii. $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

   iii. $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$

   iv. $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

   v. $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

   vi. $\frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

vii. $\sin^2\theta + \cos^4\theta = \sin^4\theta + \cos^2\theta$

viii. $\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$

ix. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$

x. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

2. If $x\cos\theta + y\sin\theta = m$ and $x\sin\theta - y\cos\theta = n$, prove that $x^2 + y^2 = m^2 + n^2$.

3. Find values of θ for the following equations in the interval $0 \le \theta \le \frac{\pi}{2}$.

   i $2\sin^2\theta = \frac{1}{2}$ ii. $\sin\theta = \cos\theta$ iii. $\sec^2\theta - 2\tan^2\theta = 0$

4. Solve the following right-angled triangles

   i. ΔABC, $\angle C = 90^0$, $a = 4$ cm, $c = 4\sqrt{2}$ cm.

   ii. ΔPQR, $\angle Q = 90^0$, $\angle P = 60^0$, $PQ = 4\sqrt{3}$ cm.

   iii. ΔPQR, $\angle R = 90^0$, $PR = 8$ m, $RQ = 8$ m.

   iv. ΔXYZ, $\angle X = 90^0$, $YZ = 16$ m, $XZ = 8$ m.

   v. ΔLMN, $LM = 10$ cm, $MN = 8$ cm, $NL = 6$ cm.

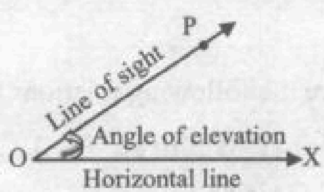
5. If $x = r\cos\alpha\sin\beta$, $y = r\cos\alpha\cos\beta$ and $z = r\sin\alpha$, then find value of $x^2 + y^2 + z^2$.

6.5 Angles of Elevation and Depression

Line of Sight

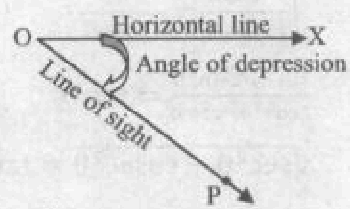
If an object P is observed from a point of observation O, then the line OP is called line of sight.

Angle of Elevation



If O is point of observation, OX is a horizontal line (ray)and an object P lies above the line OX, then the angle between the horizontal line and the line of sight to an object above the horizontal line is called angle of elevation . In the adjoining figure $\angle XOP$ is angle of elevation.

Angle of Depression



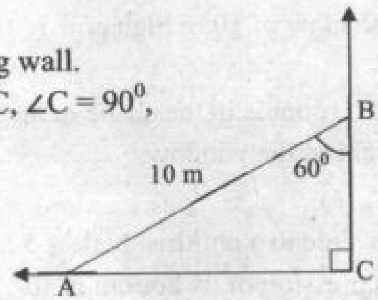
If O is point of observation, OX is a horizontal line (ray) and an object P lies below the line OX, then angle between the horizontal line and the line of sight to an object below the horizontal line is called angle of depression. In the adjoining figure $\angle XOP$ is angle of depression.

Applications

Example 13:

A ladder 10 m long leaning against a vertical wall makes an angle of 60^0 with wall. Find height of ladder along the wall.

Solution:



Let AB is a ladder. BC is height of ladder along wall.

AC is along horizontal ground. In triangle ABC, $\angle C = 90^0$,

$\angle B = 60^0$ and AB = 10 m.

$$\cos 60^0 = \frac{BC}{AB}$$

$$\frac{1}{2} = \frac{BC}{10}$$

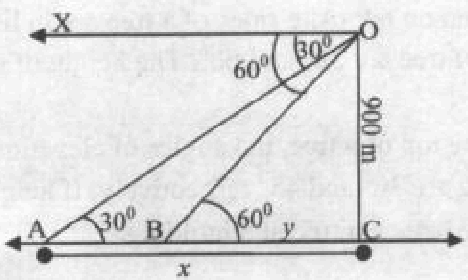
$$BC = 5 \text{ m}$$

So, height of ladder along wall = 10 m.

Example 14:

An aeroplane at an altitude of 900 m finds that two ships are sailing towards it in the same direction. The angles of depression of ships as observed from plane are 30^0 and 60^0. Find distance between ships.

Solution:



In the figure O is position of plane. A and B are two ships. $\angle XOA$ and $\angle XOB$ are angles of depression. As alternate angles of parallel lines are equal.

So, $\angle A = 30^0$ and $\angle B = 60^0$

Let AC = x and BC = y.

In right angled triangle ACO,

$$\tan 30^0 = \frac{900}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{900}{x}$$

$$x = 900\sqrt{3} = 1558.85 \text{ m}$$

In right angled triangle BCO,

$$\tan 60^0 = \frac{900}{y}$$

$$\sqrt{3} = \frac{900}{y} \Rightarrow y = \frac{900}{\sqrt{3}} = 519.62 \text{ m}$$

Distance between ships $= x - y$

$= 1558.85 - 519.62$

$= 1039.23$ m

Check Point

One acute angle of a right triangle is a. Find both acute angles if $\sin a = \cos a$.

EXERCISE 6.5

1. From a point at a distance of 20 m from a tree, angle of elevation of top of a tree is 30^0. Find height of the tree.

2. The length of shadow of 10 m high pole is $10\sqrt{3}$ m. Find angle of elevation of the sun.

3. The window of a room is in the shape of an equilateral triangle. The length of each side is 1.6m. Find height of the window.

4. The height of a slide in a children park is 5 m. A girl standing at the top of the slide observes that angle of depression of its bottom is 30^0. Find length of the slide.

5. Two pillars of equal height stand on either side of a roadway which is 120 m wide. At a point on the road between pillars, the angles of elevation of the pillars are 60^0 and 30^0. Find height of each pillar and position of the point.

6. From the top of a tower of height 120 m, angles of depression of two boats on same side of tower at water level are 60° and 45°. Find distance between the boats.

7. Two men on opposite sides of a tree are in line with it. They observe that angles of elevation of top of tree are 30° and 60°. The height of tree is 15 m. Find distance between the men.

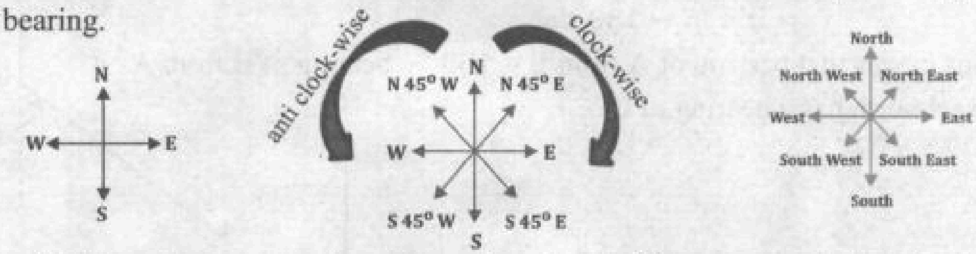
8. From the top of a tree, the angles of elevation and depression of the top and bottom of a building are 30^0 and 45^0 respectively. If height of tree is 12 m, find height of building and distance between tree and building.

9. From a point A on ground at a distance of 200m. angle of elevation of top of a tower is α. There is another point B 80m nearer to the tower. The angle of elevation of top of tower from B is β. If $\tan\alpha = \frac{2}{5}$, find height of tower and value of β.

10. A boat moving away from a light house 206.6 m high, takes 120 seconds to change the angle of elevation of top of light house from 60° to 45°. Find the speed of boat.

6.6 Bearing

When we want to move from one position to another, we need to know not only the distance to be traveled but also the direction of movement. Direction sense is the ability to know a person's current location and then find the correct way towards the destination. It helps us in getting clear picture of bearing.



The four cardinal directions are north, east, south and west. These are marked by the initials N, E, S and W. Starting from north with an angle of zero degree, east is at right angle to north and intercardinal direction north-east is at an angle of 45 degrees to north. Similarly south- east is at angle of 135 degrees to north and so on.

Bearings are the angles measured clockwise from due north and are expressed as three digit numbers. Bearings are mostly used by sailors and pilots to describe the direction they are travelling and to navigate from one point to another.

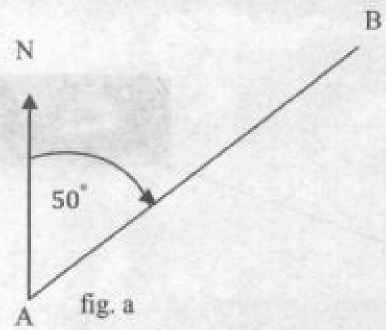


fig. a

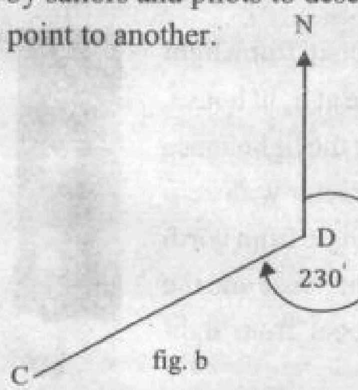


fig. b

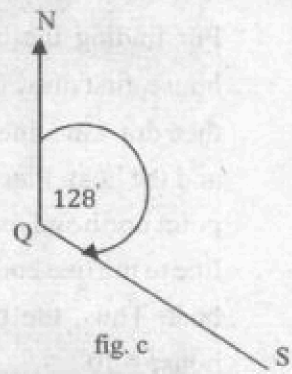


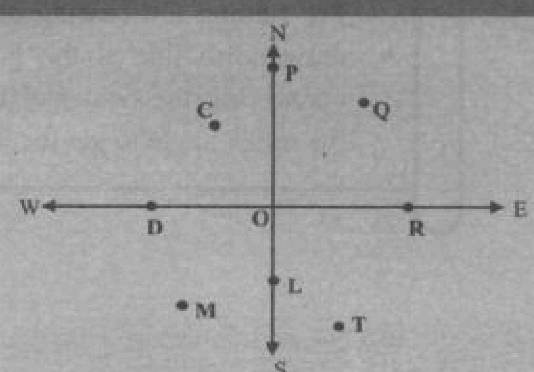
fig. c

In fig.a, the bearing of point B from point A is 050°.	In fig.b, the bearing of point C from point D is 230°.	In fig.c, the bearing of point S from point Q is 128°.

Above figures are not drawn upto scale.

Check Point

Join all the marked points to O, then find their bearing with the help of protractor. Also write the type of angle in each case. What is the bearing of north, south, east and west direction lines?

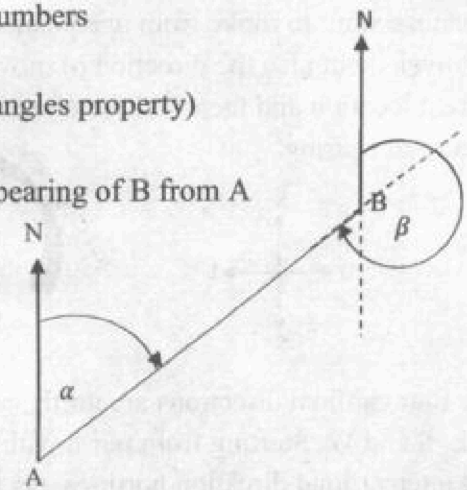


Points to ponder!

- Bearing is an angle expressed with three-digit numbers measured in clockwise direction from north.
- If α is acute, then $\beta = 180^{\circ} + \alpha$ (corresponding angles property)

  $$\Rightarrow \alpha = \beta - 180^{\circ}$$

  This means that bearing of A from B = 180° + bearing of B from A
- Exact north has a bearing of 0°.



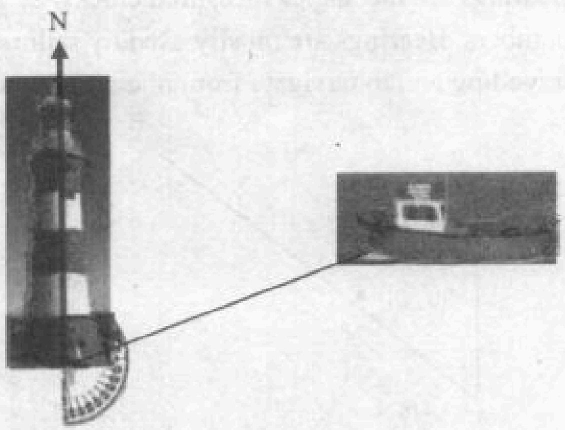
Example 15:

Find the bearing of boat from lighthouse in the given figure.

Solution:

For finding the bearing of boat from light house, first draw the north line at light house, then draw the line connecting the light house and the boat. Place your protractor with zero point north and measure the angle from north line to the line connecting light house and the boat. Thus, the bearing of boat from light house $= 70^{\circ}$.

(The figure is not drawn accurately)



Thinking Corner!

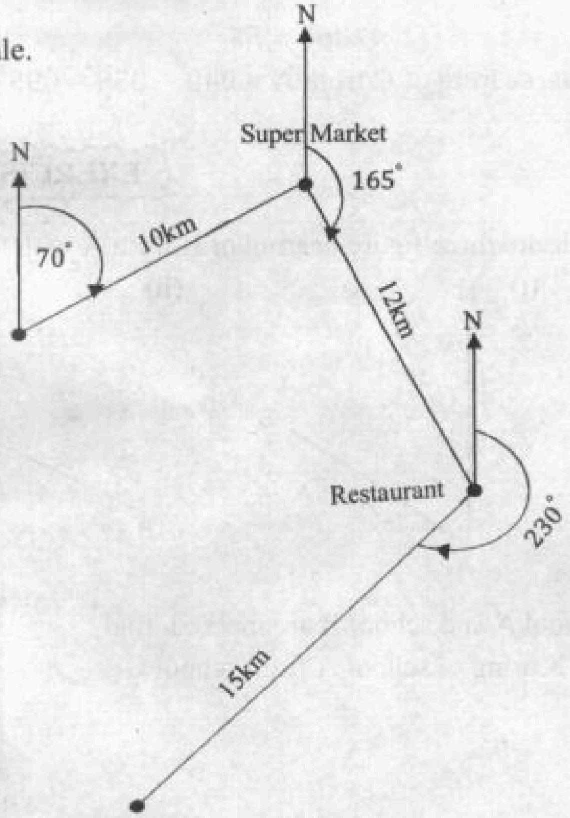
For finding the bearing of light house from boat, draw north line at the boat, then draw the line connecting the boat and the light house. Measure the reflex angle to get the required bearing.

Example 16:

Arsalan travelled at a bearing of 70° for 10km to reach super market. Then, he travelled at a bearing of 165° for 12km to reach a restaurant. Finally, he travelled at a bearing of 230° for 15km to reach his friend's house. Make a labelled illustration to show his journey.

Solution:

In the figure, angles are not marked upto scale.



We can write bearing of 070° from starting point to supermarket as N 70°E.

Example 17:

A ship sails from port A to port B at a bearing of 040° for 35 km. Then, it sails at a bearing of 130° for 50 km from port B to port C. Find:

(a) the distance between port A and port C

(b) the bearing of C from A

Solution:

(a) By drawing the path according to given information, we observe that $\theta = 130^{\circ} - 40^{\circ} = 90^{\circ}$ (since two norths at A and B are parallel).

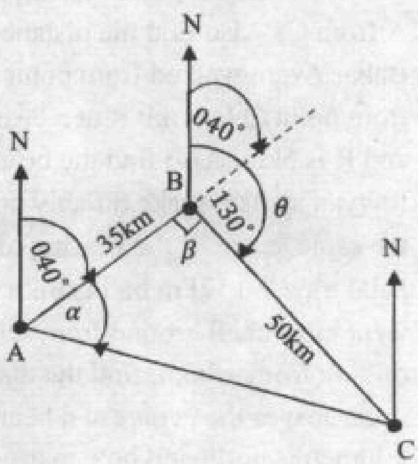
Therefore, $\beta = 180^{\circ} - 90^{\circ} = 90^{\circ}$.

Thus, triangle ABC is right with right angle at B.

In right ΔABC by using Pythagoras theorem,

$$m\overline{AC} = \sqrt{(m\overline{AB})^2 + (m\overline{BC})^2}$$

$$= \sqrt{(35)^2 + (50)^2} = \sqrt{3725} \approx 61.03 \text{ km}$$



(b) Bearing of C from A = $040^\circ + \alpha$

First we find α in right ΔABC,

$\tan \alpha = \frac{m\overline{BC}}{m\overline{AB}}$

$= \frac{\mathbf{50}}{\mathbf{35}} \approx 1.4286$

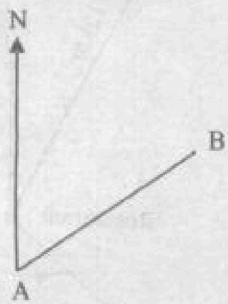
$\Rightarrow \alpha = tan^{-1}(1.4286) = 55^\circ$

Thus, bearing of C from A = $040^\circ + 055^\circ = 095^\circ$

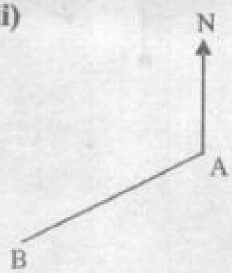
EXERCISE 6.6

1. Measure three figure bearing of B from A with the help of protractor in each of the following.

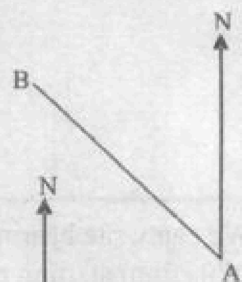
(i)



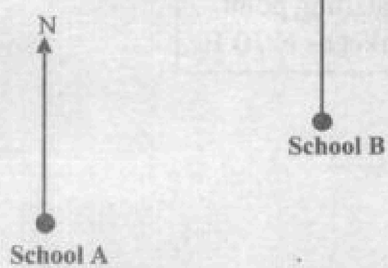
(ii)



(iii)



2. School A and school B are marked, find the bearing of school A from school B.



3. Two boats A and B are 5km apart and the bearing of boat B from boat A is 250°. Draw the diagram showing relative positions of both boats, then find the bearing of boat A from boat B.
4. The positions of three ships A, B and C are such that the bearing of B from A is 045° and ship C is to the east of A. If bearing of C from B is 180° and $m\overline{AB} = 10$km, what is the bearing of A from C? Also find the distance between ship A and ship C.
5. Babar Azam walked from point P to point Q at a bearing of 050° for 3km. Then, he walked from point Q to point R at a bearing of 140°. Find the distance QR, if the distance between P and R is 5km. Also find the bearing of R from P.
6. Rayyan spots a snake directly north from his location. Sarim is 25m east of Rayyan and spots the same snake. If the bearing of Sarim from snake is 122°,how far is Rayyan from the snake?
7. Bilal traveled 12km on his bike to reach school at a bearing of 060° from his home.Then, he went to football ground from school at a bearing of 150°.If the football ground is at a distance of 5km from school, find the distance he has to cover to reach home from football ground?
8. A car leaves the garage at a beaing of 040° and travels in a straight line for 13km. How many kilometres north and how many kilometres east has the car travelled from the garage?

KEY POINTS

- There are three systems of measurement of an angle.
  i. Sexagesimal System or English System
  ii. Radian measure or Circular System
  iii. Centesimal or French System
- If a circle is divided into 360 equal arcs. The central angle of each arc is of measure 1 degree (1°).
- There are 60 minutes in a degree. Similarly, there are 60 seconds in a minute.
- To convert the measure of an angle from degrees to minutes or from minutes to seconds we multiply its value by 60.
- To convert measure of an angle from seconds to minutes or from minutes to degrees we divide its value by 60.
- Ratio between length of circular arc and radius is called radian.
- One radian is measure of central angle of an arc whose length is equal to radius of the circle.
- Measure of a complete angle in circular system is 2π radians.
- 1 radian $= \left(\frac{180}{\pi}\right)^0 \approx 57^0 17' 45''$ and $1^0 = \frac{\pi}{180}$ radians ≈ 0.0175 radian.
- Length of a circular arc is product of radius and radian measure of central angle. i.e. $l = r\,\theta$
- Two angles having same terminal ray in standard position are called coterminal angles.
- If terminal ray of an angle in standard position coincides with any axis, then it is called quadrantal angle. Measure of a quadrantal angle is multiple of 90^0.
- A circle whose radius is 1unit is called a unit circle.
- Pythagorean Identities:
  (i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta - \tan\theta = 1$
  (iii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$
- Bearings are the angles measured clockwise from north line and are written as three digit numbers.
- North is at the bearing of 000^0, east is at 090^0, south is at 180^0, west is 270^0.

MISCELLANEOUS EXERCISE 6

1. Encircle the correct option in each of the following.

(i) If $\sin x = \dfrac{1}{4}$, what is the value of $\cos x$?

(a) $\dfrac{\sqrt{15}}{4}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{\sqrt{17}}{4}$ (d) $\dfrac{15}{4}$

(ii) $\cos\dfrac{2\pi}{3} = ?$

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\sqrt{\frac{3}{2}}$ (d) $\sqrt{\frac{3}{2}}$

(iii) $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=?$

(a) $\operatorname{cosec}^2\theta$ (b) $2\operatorname{cosec}^2\theta$ (c) $\sec^2\theta$ (d) $2\sec^2\theta$

(iv) $-3-3\tan^2\theta=?$

(a) $3\operatorname{cosec}^2\theta$ (b) $-3\operatorname{cosec}^2\theta$ (c) $-3\sec^2\theta$ (d) $3\sec^2\theta$

(v) If $\cos x = \sin x$, then the value of x is

(a) 30° (b) 45° (c) 60° (d) 90°

(vi) $4\operatorname{cosec}^2\theta-4\cot^2\theta-4\cos 0^\circ=?$

(a) 0 (b) 1 (c) -1 (d) 4

(vii) If $\tan x = \dfrac{a}{b}$, then $\sin x = ?$

(a) $\dfrac{a}{\sqrt{a^2-b^2}}$ (b) $\dfrac{b}{\sqrt{a^2-b^2}}$ (c) $\dfrac{a}{\sqrt{a^2+b^2}}$ (d) $\dfrac{b}{\sqrt{a^2+b^2}}$

(viii) $50^\circ 30' = ?$

(a) 50.2° (b) 50.3° (c) 50.4° (d) 50.5°

(ix) $(1-\cos^2\theta)\sec^2\theta=?$

(a) $\operatorname{cosec}^2\theta$ (b) $\cot^2\theta$ (c) $\sec^2\theta$ (d) $\tan^2\theta$

(x) In which quadrant do the angles between 90° and 180° lie?

(a) 1st (b) 2nd (c) 3rd (d) 4th

(xi) The system in which the angles are measured in radians is called

(a) CGS (b) sexagesimal (c) circular (d) centesimal

(xii) $270^\circ = ?$

(a) $\dfrac{3\pi}{2}$ (b) $\dfrac{\pi}{2}$ (c) $-\dfrac{\pi}{2}$ (d) $-\dfrac{3\pi}{2}$

(xiii) What is the bearing of south-west direction line?

(a) 45° (b) 135° (c) 225 (d) 270°

(xiv) If an automobile travels at the bearing of is 070° from city A to city B , then the bearing from city B to city A is

(a) 070° (b) 110° (c) 250° (d) 290°

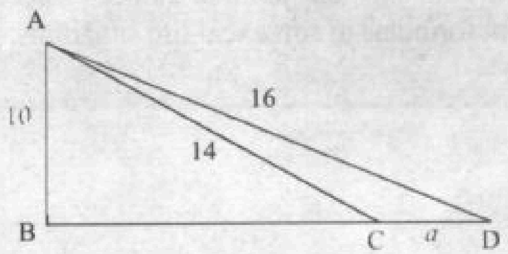
(xv) A boat sails 4km north from point L to M, then it sails 3km west from M to P. What is the bearing of P from L? Provide your answer to the nearest degree.

(a) 037° (b) 053° (c) 217° (d) 323°

2. Find $\sin\alpha$, $\tan\alpha$ and $\sec\alpha$, if $\cos\alpha = \frac{-5}{13}$, where $\frac{\pi}{2} < \alpha < \pi$.

3. The angle of elevation of a hot air balloon 500 m away, climbing up vertically, changes from 30° to 60° in 100 seconds. What is the upward speed in meters per second?

4. Find the value of a in the following figure.



5. The area of a right triangle is 50cm^2. One of its angles is 45°. Find the lengths of the sides and hypotenuse of the triangle.

6. If the shadow of a building increases by 12 meters when the angle of elevation of the sun rays decreases from 60° to 45°, what is the height of that building?

7. From the top of a 100 meters high building, the angle of depression to the bottom of a second building is 30°. From the same point, the angle of elevation to the top of the second building is 20°. Calculate the height of the second building.

8. A plane is 119km west and 100km south of an airport. At what bearing, the pilot wants to fly directly back to the airport?