Based on National Curriculum of Pakistan 2022-23

Textbook of

Mathematics

Science Group

12

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A Textbook of Mathematics for Grade 12
based on National Curriculum of Pakistan (NCP) 2022-23

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Preface

This Textbook for Mathematics Grade 12 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building the foundation of learning from the previous grades. A key emphasis of the present textbook is creating real life linkage of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they grow up in the learning curve and also to fully grasp the conceptual basis that will be built in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its textbooks. The present textbook features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement, the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this textbook.

May Allah guide and help us (Ameen).

Dr. Kamran Jahangir
Managing Director

Application of Mathematics

Functions and Graphs: Functions represent relationships between variables and their graphs provide a visual representation of these relationships. They are used to study trends like profit versus cost in businesses, population growth over time or changes in speed in physics. Functions help in predicting outcomes and analyzing real-world data effectively.

Limit, Continuity and Derivative: Limits describe how functions behave near specific points, continuity ensures smooth graphs without breaks, and derivatives measure rates of change like speed or growth. These concepts are essential in physics to calculate instantaneous velocity, in economics for cost optimization and in biology for population growth modeling.

Integration: Integration helps calculate areas under curves, volumes, or accumulated quantities. It is widely used to determine total distance from velocity, analyze energy consumption and compute areas in construction projects. Integration also has applications in physics, such as finding work done by a variable force.

Differential Equations: Differential equations describe changes in dynamic systems, modeling real-world processes like population growth, chemical reactions and motion. They are used in engineering to design systems, in physics to describe heat flow or wave motion and in biology to model disease spread.

Kinematics of Motion in a Straight Line: This studies the motion of objects along a straight path using concepts like displacement, velocity and acceleration. Applications include calculating stopping distances of vehicles, analyzing free-fall under gravity and predicting the motion of objects in linear transport systems.

Analytical Geometry: Analytical geometry combines algebra and geometry to study the properties of shapes on the coordinate plane. It is used in designing structures, solving distance and slope problems and planning urban layouts. It plays a significant role.in architecture and engineering.

Conic Section: Conic sections include circles, ellipses, parabolas, and hyperbolas, which have various applications. Ellipses describe satellite orbits, parabolas are used in designing headlights and bridges and hyperbolas are applied in communication systems and radar design.

Inverse Trigonometric Functions and Their Graphs: Inverse trigonometric functions calculate angles from trigonometric values, essential in navigation, surveying, and architecture. Their graphs help solve problems involving slopes, elevation angles and real-world measurements requiring precision.

Solution of Trigonometric Equations: Trigonometric equations model periodic phenomena such as sound and light waves. Solving these equations is crucial in designing musical instruments, analyzing alternating electrical currents and studying wave patterns in physics and engineering.

Numerical Methods: Numerical methods approximate solutions for complex problems using algorithms. They are widely used in weather forecasting, structural analysis of buildings and financial risk modeling. These methods make it possible to solve equations that are difficult to handle analytically.

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

شروع اللہ کے نام سے جو بڑا مہربان نہایت رحم والا ہے۔

CONTENT

UNIT
01

FUNCTIONS AND GRAPHS

After studying this unit, students will be able to:

- Recall definition of function, find its domain, codomain, range and its types.
- Find inverse of a function and demonstrate its domain and range with examples.
- Know linear, quadratic and square root functions.
- Sketch graphs of linear and non-linear functions.
- Plot graph of function of the type $y = x^n$ when (i) n is a + ve or – ve integer and $x \neq 0$, (ii) n is a rational number for $x > 0$.
- Plot graph of quadratic function of the form $y = ax^2 + bx + c$, where a, b, c are integers and $a \neq 0$.
- Draw graph using factors and predict functions from their graphs.
- Find the intersecting points graphically when intersection occurs between (i) a linear function and coordinate axes (ii) two linear functions (iii) a linear function and a quadratic function.
- Draw the graph of modulus functions.
- Solve graphically appropriate problems from daily life.
- Classify algebraic and transcendental functions and describe trigonometric, inverse trigonometric, logarithmic and exponential functions.
- Define logarithm, and derive and apply laws of logarithm.
- Graph and analyse exponential and logarithmic functions.
- Apply the concept of exponential functions to find compound interest.
- Solve problems involving exponential and logarithmic equations.
- Identify the domain and range of transcendental functions through graphs.
- Interpret the relation between a one-one function and its inverse through a graph.
- Demonstrate the transformation of a graph through horizontal shift, vertical shift and scaling.

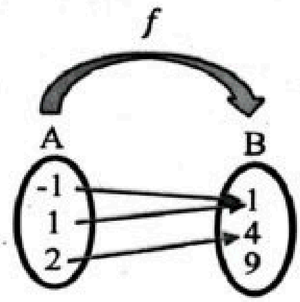
Functions have many applications in real life. One is the use of function in signal processing applications in engineering, including noise reduction, modulation, and filtering. For example, functions in audio processing are used to analyze and alter sound waves, which makes it possible to design devices like equalizers and noise-canceling headphones. Similar to this, functions are essential to the encoding, transmitting, and decoding of signals for wireless communication systems in the telecommunications industry.

1.1 Function

A function f from a set A to a set B assigns to each element of A exactly one element of B. The set A is called the domain of the function and the set B is called the co-domain of the function.

Mathematically, it is written as $f : A \rightarrow B$ and is read as f is a function from A to B.

For example, if we are given two sets $A = \{-1, 1, 2\}$ and $B = \{1, 4, 9\}$,



then $f = \{(-1, 1), (1, 1), (2, 4)\}$ is a function because each element of the set A is assigned to exactly one element of the set B. i.e. there is no repetition in the first element of ordered pairs in f. The first element of each ordered pair in f is called pre-image while its corresponding second element is called image of the first element. For example, in $(2, 4)$, 2 is the pre-image of 4 and 4 is the image of 2.

If x is independent variable and y is dependent variable, then in general, a function f from A to B is written as $f(x) = y$.

For example, in the above example,

$f(-1) = 1, f(1) = 1$ and $f(2) = 4$

Explanation: As discussed above, a function relates an input to an output.

For example, if a tree grows 15 cm every year and the height h of the tree is related to its age as follows:

$$h(\text{age}) = \text{age} \times 15$$

then the height of the tree after 10 year is $h(10) = 10 \times 15 = 150$ cm

$\therefore$ '$h(10) = 150$' is like saying 10 is related to 150 or $10 \rightarrow 150$

Here, 10 is the input and 150 is the output of the function.

1.1.1 Domain of a Function

The set of all possible values of independent variable which qualify as inputs to a function is known as the domain of the function. In the above example,

Domain of function $f = \text{Dom } f = \{-1, 1, 2\}$

Key Facts

A function in which real numbers are used is called a real valued function.

How to Find the Domain of a Function

To find the domain, we ensure that there is no zero in the denominator of a fraction and no negative sign inside a square root. In general, the set of all real numbers is considered as the domain of a function subject to some restrictions. For example:

(i) When the given function is of the form $f(x) = 3x + 8$ or $f(x) = x^3 + 2x - 5$, the domain will be "the set of all real numbers".

(ii) When the given function is of the form $f(x) = \frac{1}{x-2}$ the domain will be the set of all real numbers except 2.

(iii) In some cases, the interval be specified along with the function such as, $f(x) = x + 1$, $0 < x < 10$. Here, x can take the values between 0 and 10 in the domain.

(iv) Domain restrictions refer to the values for which the given function cannot be defined.

1.1.2 Range of a Function

The set of all the outputs of a function is known as the range of the function or after substituting the domain, the entire set of all possible values as outcomes of the dependent variable.

In a function $y = f(x)$, the spread of all the values y from minimum to maximum is the range of the function. In the above example,

Range of function f = Rang $f = \{1, 4\}$

How to Find the Range of a Function

- Substitute all the values of x in the function to check whether it is positive, negative or equal to other values. Eliminate the values of x for which the function is not defined.
- Find the minimum and maximum values for y.

1.1.3 Codomain of a Function

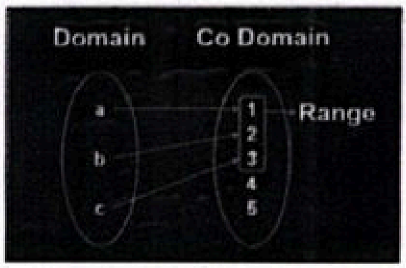
The codomain is the set of all possible outcomes of the given function.

In general, the range is the subset of the codomain. But sometimes the codomain is also equal to the range of the function. In above example,

Codomain of function $f = \{1, 4, 9\}$

In short:

- What can go into a function is called the domain.
- What may possibly come out of a function is called the codomain.
- What actually comes out of a function is called the range.



In the adjoining figure,

- The set $\{a, b, c\}$ is the domain.
- The set $\{1, 2, 3, 4, 5\}$ is the codomain.
- The set $\{1, 2, 3\}$ is the range.

Example 1: Find the domain and range of a function $f(x) = 2x^2 - 4$.

Solution: $f(x) = 2x^2 - 4$

The given function has no undefined values of x.

Thus, the domain is the set of all real numbers.

Domain $= (-\infty, \infty) = R$

If we put $x = 0$ in the given function, we get $f(x) = -4$.

For all real values of x, other than 0, we get an output greater than -4.

Hence, the range of $f(x)$ is $[-4, \infty)$.

Key Facts

A function is like a machine that takes an input and produces a corresponding output. For example, the distance a car has traveled (the output) is dependent on how long that car has been driving (the input).

Example 2:

Find the domain and range of function $g(x) = \sqrt{x-3}$.

Solution: $g(x) = \sqrt{x-3}$

The given function is defined for all real numbers x greater than or equal to 3.

Thus, the domain of $g(x)$ is $[3, \infty)$.

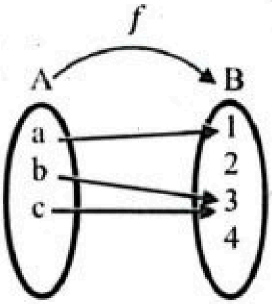
If we put $x = 3$ in the given function, we get $g(3) = 0$.

For all real values of x, greater than 3, we get an output greater than 0.

Hence, the range of $g(x)$ is $[0, \infty)$.

1.2 Types of Functions

1.2.1 Into Function



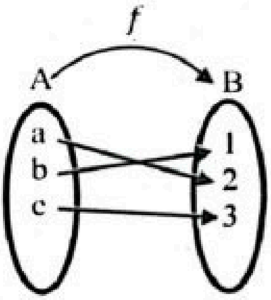
A function f: A → B is said to be into function if there exists at least one element or more than one element in B, which does not have any pre-images in A, which simply means that every element of the codomain is not mapped with elements of the domain. i.e., range$(f) \neq$ B.

In the adjoining diagram, f = {(a, 1), (b, 3), (c, 3)} is an into function.

Some examples of into functions are:

- $f(x) = \sin x$ where f: R → R is into function because it doesn't cover all values in the interval R.
- $g(x) = x^2$ where g: R → R is into function because it doesn't map to any negative real numbers.
- $h(x) = e^x$ where h: R → $[0, \infty)$ is into function because it doesn't map to zero.

1.2.2 Onto (Surjective) Function



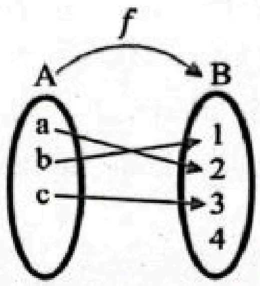
For any two non-empty sets A and B, a function f : A → B will be onto if every element of set B is an image of some element of set A. i.e., for every $y \in$ B there exists an element x in A such that $f(x) = y$ which implies rang(f) = B.

In the adjoining diagram, f = {(a, 2), (b, 1), (c, 3)} is an onto function.

Some examples of onto functions are:

- $f(x) = x$ (Identity function)
- $g(x) = e^x$ when g: R → R^+ (Exponential function)
- $h(x) = x^2$ (Square function)
- $m(x) = x^3$ (Cubic function)
- $p(x) = c$ (Constant function)

1.2.3 One to One (Injective) Function



For any two non-empty sets A and B, a function $f: A \to B$ will be one-to-one if distinct elements of set A have distinct images in set B. In the adjoining diagram, $f = \{(a, 2), (b, 1), (c, 3)\}$ is a one-to-one function.

A function $f: A \to B$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

i.e., an image of a distinct element of A under f mapping (function) is distinct.

Some examples of one-one functions are:

- $f(x) = x$ (Identity function)
- $g(x) = 2x + 3$ (Linear function)
- $h(x) = e^x$ (Exponential function)
- $m(x) = \sqrt{x}$ (Square root function, defined for $x \geq 0$)

1.2.4 Injective Function

A function which is both into and one-one is called an injective function.

1.2.5 Bijective Function

A function which is both onto and one-one is called a bijective function.

Bijective function shows one-one correspondence between the elements of two sets.

Key Facts

Various types of functions are mentioned in the below table:

Based on elements	One-one function, Many-one function, Onto function, Bijective Function, Into function, constant Function
Based on the equation	Identity function, Linear function, Quadratic function, Cubic function, Polynomial functions
Based on the range	Modulus function, Rational function, Even and odd functions, Periodic functions, Greatest and smallest integer function, Inverse function, Composite functions
Based on the domain	Algebraic functions, Trigonometric functions, Logarithmic functions, Exponential functions

Example 3: Check whether the function $f(x) = 2x + 3$, is one-to-one or not if domain = $\{0.5, 1, 2\}$ and codomain = $\{4, 5, 7\}$

Solution: Putting 1, 2 and 0.5 in $f(x) = 2x + 3$, we get $f(0.5) = 4$, $f(1) = 5$ and $f(2) = 7$

As, for every value of x, we get a unique $f(x)$ thus, the function $f(x)$ is one to one.

Example 4: Check whether the function is one-to-one or not: $f(x) = 2x^2 + 1$.

Solution: To check whether the function is one to one or not, let:

$$f(x_1) = f(x_2)$$
$$2(x_1)^2 + 1 = 2(x_2)^2 + 1$$
$$(x_1)^2 = (x_2)^2$$

Since $(x_1)^2 = (x_2)^2$ is not always true, therefore the function is not one to one function.

Example 5: Check the type of function $f(x) = x^2 - 1$ if Dom $f(x) = \{1, -1, 2, -2\}$ and Codom $f(x) = \{0, 3, -3\}$

Solution: Given $f(x) = x^2 - 1$ with Dom $f(x) = \{1, -1, 2, -2\}$ and Codom $f(x) = \{0, 3, -3\}$

Substituting the elements of the domain in the function, we get:

$$f(1) = 1^2 - 1 = 0$$
$$f(-1) = (-1)^2 - 1 = 0$$
$$f(2) = 2^2 - 1 = 3$$
$$f(-2) = (-2)^2 - 1 = 3$$

Therefore, Rang $f(x) = \{0, 3\}$. As, Rang $f(x) = \{0, 3\} \neq \{0, 3, -3\} =$ Codom $f(x)$.
So, the given function is an into function.

Example 6: Find the type of the function $f(x) = 3x + 2$ defined on $f: R \to R$.

Solution: Let, $f(x) = y \quad \Rightarrow \quad y = 3x + 2 \quad \Rightarrow \quad y - 2 = 3x \quad \Rightarrow \quad x = \frac{y-2}{3}$

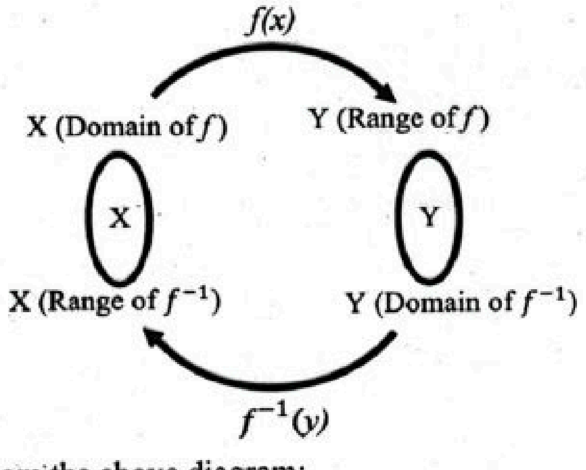
Substituting the value of x in the given function $f(x)$, we get:

$$f(x) = f\left(\frac{y-2}{3}\right) = 3\left(\frac{y-2}{3}\right) + 2 = y - 2 + 2 = y$$

Since, we get back y after putting the value of x in the function. Hence the given function is an onto function.

1.3 Inverse Function

The inverse of any function $f(x)$ is a function denoted by $f^{-1}(x)$ which reverses the effect of $f(x)$ and it undoes what $f(x)$ does. In mathematics, the inverse function is also denoted by f^{-1}. If $f: X \to Y$, then $f^{-1}: Y \to X$. i.e., If the application of a function f to x as input gives an output of y, then the application of inverse function f^{-1} to y should give back the value of x.
It can be illustrated in the following diagram as:



Key Facts

- If $y = f(x)$ is bijective function then $x = f^{-1}(y)$.
- If $fog(x) = gof(x) = x$, then $g = f^{-1}$ and $f = g^{-1}$
- $(f^{-1})^{-1} = f$

From the above diagram:

$$\text{dom } f = \text{rang } f^{-1} \quad \text{and} \quad \text{rang } f = \text{dom } f^{-1}$$

i.e., The domain of the given function becomes the range of the inverse function, and the range of the given function becomes the domain of the inverse function.

Note that f^{-1} is not the reciprocal of f and not every function has an inverse. If a function $f(x)$ has an inverse, then $f(x)$ never takes the same value twice. In simple words, the inverse function exists only when f is both one-one and onto function. Can we say that the inverse function is also a bijective function?

Moreover, the composition of the function f and the inverse function f^{-1} gives the domain value of x.

$$fof^{-1}(x) = f^{-1}of(x) = x$$

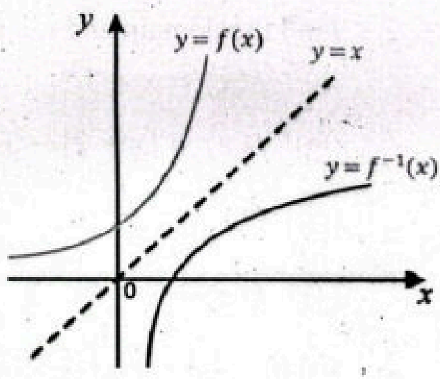
1.3.1 Steps to Find an Inverse Function

Consider a function $f(x) = ax + b$.

- Replace $f(x)$ with y, to obtain $y = ax + b$.
- Solve the expression for x to obtain $x = \frac{y-b}{a}$.
- Replace x with $f^{-1}(y)$ to get $f^{-1}(y) = \frac{y-b}{a}$.
- Interchange y with x in the function $f^{-1}(y) = \frac{y-b}{a}$ and get inverse function $f^{-1}(x) = \frac{x-b}{a}$.

1.3.2 Graph of an Inverse Function

If the graphs of both functions are symmetric with respect to the line $y = x$ then we say that the two functions are inverses of each other. This is because of the fact that if (x, y) lies on the function, then (y, x) lies on its inverse function.



Example 7:

Find the inverse function of $f(x) = \frac{x}{x-2}$ defined on $f: R \to R$.

(i) Find domain and range of function and its inverse.

(ii) Prove that $fof^{-1}(x) = f^{-1}of(x) = x$

Solution:

(i) Given function is $f(x) = \frac{x}{x-2}$

Dom $f(x) = R - \{2\}$

To find inverse function, let:

$$y = \frac{x}{x-2} \Rightarrow y(x-2) = x \Rightarrow xy - x = 2y$$

$$\Rightarrow x(y-1) = 2y \Rightarrow x = \frac{2y}{y-1}$$

$$\Rightarrow f^{-1}(y) = \frac{2y}{y-1} \quad \text{...... } (x = f^{-1}(y))$$

$$\Rightarrow f^{-1}(x) = \frac{2x}{x-1} \quad \text{...... Replacing } y \text{ with } x.$$

From the inverse function, we see that:

Dom $f^{-1}(x) = R - \{1\}$

Hence,

Dom $f = R - \{2\} =$ Rang f^{-1} and Dom $f^{-1} = R - \{1\} =$ Rang f

(ii) $fof^{-1}(x) = f(f^{-1}(x)) = f\left(\frac{2x}{x-1}\right) = \frac{\frac{2x}{x-1}}{\frac{2x}{x-1}-2} = \frac{2x}{2x-2x+2} = \frac{2x}{2} = x$ (i)

$f^{-1}of(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{x}{x-2}\right) = \frac{2\left(\frac{x}{x-2}\right)}{\frac{x}{x-2}-1} = \frac{2x}{x-x+2} = \frac{2x}{2} = x$ (ii)

From (i) and (ii), we get:

$$fof^{-1}(x) = f^{-1}of(x) = x$$

Exercise 1.1

1. Find the domain of following functions.
   (i) $f(x) = x^2 - 6$ (ii) $g(x) = \frac{x}{x+3}$ (iii) $h(x) = \frac{x+4}{x^2-9}$
   (iv) $i(x) = \frac{x}{5x+2}$ (v) $j(x) = \frac{x}{x^2+4}$ (vi) $k(x) = \sqrt{x+1}$
2. Find the domain and range of the functions.
   (i) $f(x) = x + 7$ (ii) $f(x) = 2x^2 + 1$ (iii) $f(x) = 2\sqrt{x-5}$
   (iv) $f(x) = |x-2| - 3$ (v) $f(x) = 1 + \sin x$ (vi) $f(x) = 3 + \sqrt{x-2}$
   (vii) $f(x) = \frac{3e^x}{7}$ (viii) $f(x) = \frac{x^2-16}{x+4}$ (ix) $f(x) = (x-1)^2 + 1$
   (x) $f(x) = \frac{1}{x-1}$ (xi) $f(x) = \frac{x-2}{x+3}$ (xii) $f(x) = \frac{x^2-x-6}{x-3}$
3. Given that A = {0, 1, 2, 3}, B = {p, q, r, s} and f = {(0, p), (1, q), (2, r), (3, s)}. Check whether the function is one to one, onto and/or into.
4. A = {2, 3, 4, 5}, B = {b, c, d, e}. The function is defined as f = {(2, b), (3, c), (4, e), (5, e)}. Check whether the function is one to one, into or onto.
5. Check whether the functions are one-to-one or not.
   (i) $f(x) = 4x - 7$ (ii) $f(x) = 6x^2 + 2$ (iii) $f(x) = \frac{x^3-1}{2}$
6. Check the type of function $g(x) = 2x^2 + 3x + 1$ if Dom $g(x) = \{0, 1, 2, 3\}$ and Rang $g(x) = \{1, 6, 15, 28, 35\}$
7. Find the type of the function $h(x) = 2x + 1$ defined on $h: R \to R$.
8. If $f: A \to B$ is defined by $f(x) = \frac{x-2}{x-3}$ for all $x \in A$ where $A = R - \{3\}$ and $B = R - \{1\}$.

   Then show that the function f is bijective.

9. Find the domain and range of inverse functions when:

(i) $f(x) = 4x - 3$ (ii) $f(x) = \frac{x}{x-5}$ (iii) $f(x) = \frac{x+2}{x-1}$

(iv) $f(x) = \sqrt{x+2}$ (v) $f(x) = x^2 + 6$ (vi) $f(x) = \frac{2x-1}{x+4}$

Also prove that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

1.4 Linear, Quadratic and Square Root Functions

1.4.1 Linear Function

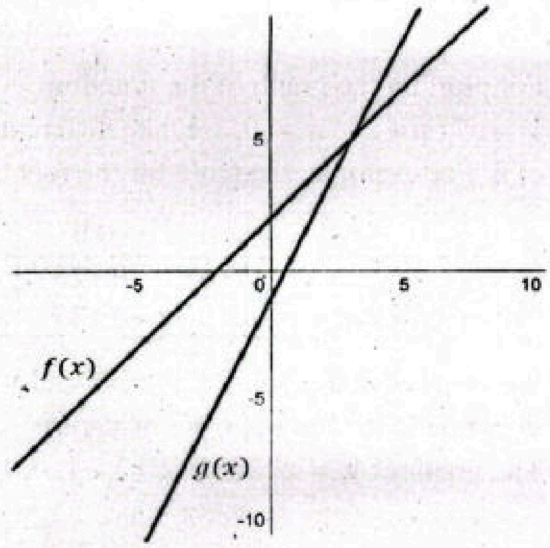
The function of the form,

$$y = ax + b;\ a, b \in R \wedge a \neq 0$$

is called linear. It is a polynomial function of degree one.

For example, $f(x) = x + 2,\ g(x) = 2x - 1$ are linear functions.

The graph of a linear function is a straight line and the slope of any two points on the line is the same. The domain and range of the linear function is R.



1.4.2 Non-linear Function

A function that is not linear is called a non-linear function. A nonlinear function is a function whose plotted graph form a curved line. For example, quadratic function, cubic function, square root function and exponential function etc. The slope of every two points on the graph of non-linear is not the same. Let us recall the shapes of the graphs of quadratic and square root functions here.

(a) Quadratic Function

The function is of the form,

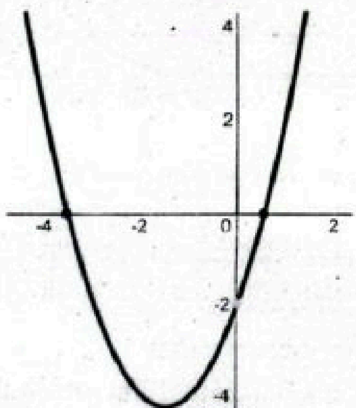
$$y = ax^2 + bx + c;\ a, b, c \in R \wedge a \neq 0$$

is called quadratic.

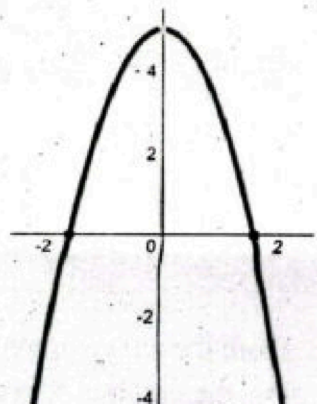
It is a polynomial function of degree two.

For example, $f(x) = x^2 + 3x - 2$ and $g(x) = 5 - 2x^2$ are quadratic functions.

The domain and range of the quadratic function is R. The graph of a quadratic equation is U-shaped and is parabolic in nature.



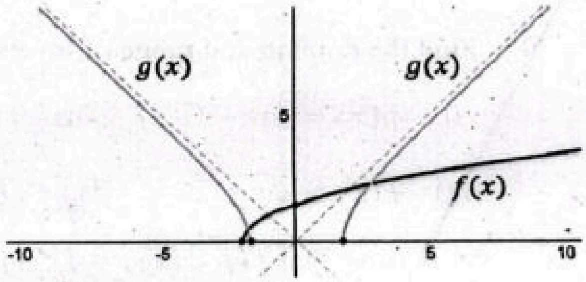
$y = f(x)$



$y = g(x)$

(b) Square Root Function

The function of the form $y = \sqrt{x}$, where $x \geq 0$ is called a square root function.
For example,
$f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{x^2-3}$
are square root functions.

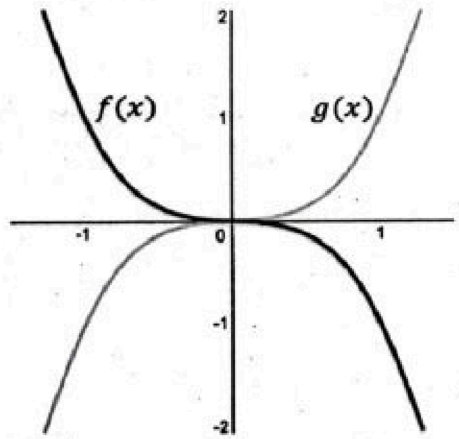


The *domain* of square root function depends upon its formation.

1.5 Plotting Graph of Function of the Type $y = x^n$

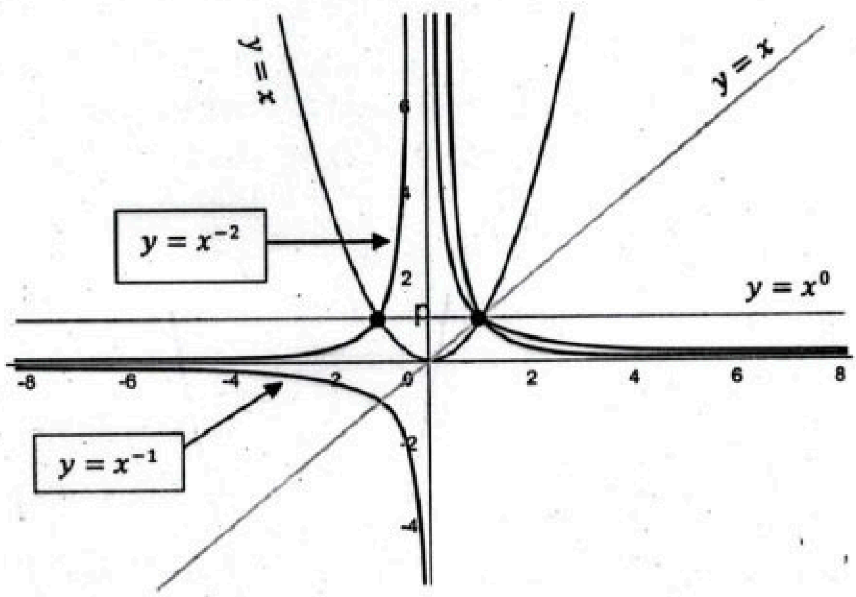
1.5.1 Graph of the Function $y = x^n$; $n \in Z \wedge x \neq 0$

For plotting the graph of the function $y = x^n$; $n \in Z \wedge x \neq 0$, we take different integral values of n. For example, the table for the function $y = \pm x^3$ is:



x	0	0.5	1	2
$f(x) = x^3$	0	0.125	1	8
$g(x) = -x^3$	0	-0.125	-1	-8

We observe that graphs of $y = x^3$ and $y = -x^3$ have the same shape but opposite behavior.
The graph of $y = x^n$ for $n = -2, -1, 0, 1, 2$ is shown below.



From the above graphs, we observe that:

- the graph of $y = x^0$ is a horizontal line passing through $y = 1$.
- the graph of $y = x^1$ is a straight line bisecting first and third quadrant.
- the graph of $y = x^{-1}$ is a hyperbola passing through first and third quadrant.

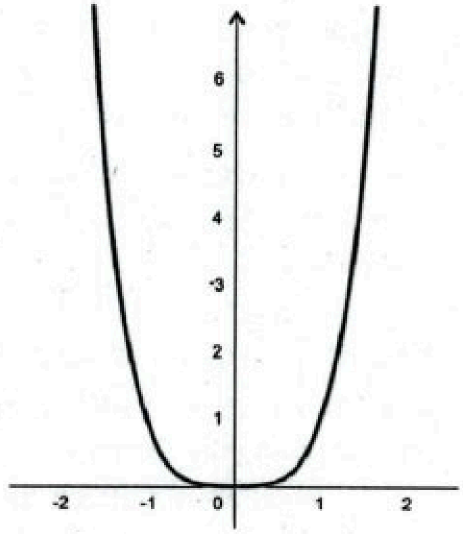
- the graph of $y = x^2$ is a parabola starting from origin and opening upwards.
- the graph of $y = x^{-2}$ has exponential behavior with two branches.
- The graphs of $y = x^n$ pass through $(1, 1)$.

Example 8: Draw the graph of $y = x^4$.

Solution:

Table for some values of x and y for the function is:

x	-1.5	-1	0	1	1.5
y	5.06	1	0	1	5.06

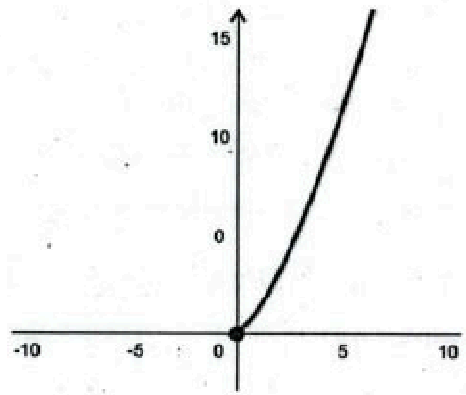


From the figure, we can see that the graph of $y = x^4$ is U-shaped which opens upward starting from origin. The value of y increases slowly for real numbers $-0.5 < x < 0.5$.
After that it increases abruptly.

1.5.2 Graph of the Function $y = x^n; n \in Q \wedge x > 0$

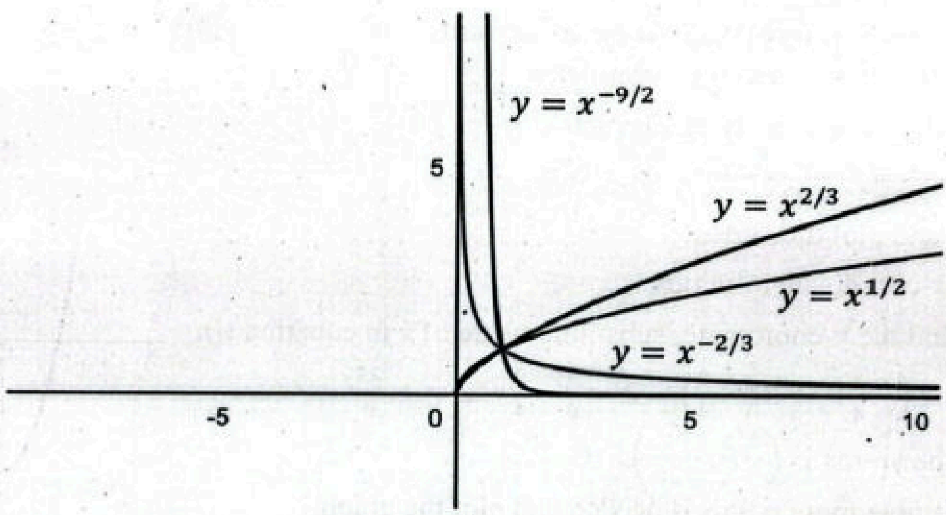
For plotting the graph of the function $y = x^n; n \in Q \wedge x > 0$, we take different integral values of n. For example, the table for the function $y = x^{3/2}$ is:

x	0.5	1	4	6
$f(x) = x^{3/2}$	0.35	1	8	14.7



We observe that graphs of $y = x^{3/2}$ has exponential behavior.

The graph of $y = x^n$ for $n = -\frac{9}{2}, -\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{2}$ is shown below.



From the above graphs, we observe that:

- the graph of $y = x^{-2/3}$ is closer to y-axis.
- the graph of $y = x^{-9/2}$ moves away from y-axis as compared with the graph of $y = x^{-2/3}$.
- the graph of $y = x^{1/2}$ is closer to x-axis.
- the graph of $y = x^{2/3}$ moves away from x-axis as compared with the graph of $y = x^{1/2}$.
- The graphs of $y = x^n$ pass through (1, 1).

1.5.3 Graph of Quadratic Function

We know that the polynomial function of degree two is called a quadratic function. This function is of the form:

$$f(x) = ax^2 + bx + c \,;\, a, b, c \in R \text{ and } a \neq 0$$

The graph of the quadratic equation is a parabola.

For example, $y = x^2 + 2x + 1$ and $y = 2 - 3x^2$ are quadratic functions.

Understanding the Graph

- a: The coefficient a, affects the direction and width of the parabola. If $a > 0$, the parabola opens upwards. If $a < 0$, the parabola opens downwards. The larger the absolute value of a, the narrower the parabola.
- b: This coefficient affects the position of the vertex horizontally (left or right) and slope of the parabola at the vertex.
- c: This is the y-intercept. So, the point $(0, c)$ is on the parabola.
- $x = -\frac{b}{2a}$ is equation of axis of symmetry and is also the x-coordinate of the vertex.

Example 9: Draw the graph of $y = 2x^2 + 3x - 2$.

Solution:

$y = 2x^2 + 3x - 2$ (i)

Comparing (i) with $y = ax^2 + bx + c$, we have $a = 2, b = 3$ and $c = -2$

Step 1: Determine whether parabola opens upwards or downwards.

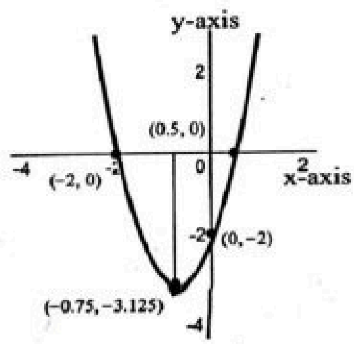
As $a > 0$, therefore parabola opens upwards.

Step 2: Find and draw the axis of symmetry.

Equation of axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{3}{2(2)} = -\frac{3}{4}$$

Step 3: Find and plot the vertex.



The x-coordinate of vertex is $x = -\frac{3}{4}$

To find the y-coordinate, substitute value of x in equation (i).

$$y = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 2 = \frac{9}{8} - \frac{9}{4} - 2 = -\frac{25}{8}$$

So, the vertex is $\left(-\frac{3}{4}, -\frac{25}{8}\right)$.

Step 4: Find some more points if needed and plot the graph.

Table for some values of x and y for the function is:

x	-2	-1	0	1
y	0	-3	-2	3

Form the figure, it is clear that the graph of function $y = 2x^2 + 3x - 2$ is parabola.

Example 10: Draw the graph of $y = 4 - 2x^2$

Solution: $y = -2x^2 + 0x + 4$ (i)

Comparing (i) with $y = ax^2 + bx + c$, we have $a = -2, b = 0$ and $c = 4$

Step 1: Determine whether parabola opens upwards or downwards.

As $a < 0$, therefore parabola opens downwards.

Step 2: Find and draw the axis of symmetry. Equation of axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{0}{2(-2)} = 0$$

Step 3: Find and plot the vertex.

The x-coordinate of vertex is $x = 0$. To find the y-coordinate, substitute value of x in equation (i). We get $y = -2(0)^2 + 0(0) + 4 = 4$

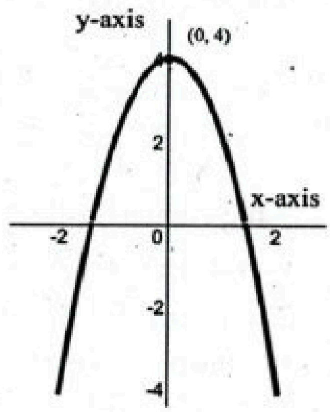
So, the vertex is $(0, 4)$.

Step 4: Find some more points if needed and plot the graph.

Table for some values of x and y for the function is:

x	-2	-1	0	1	2
y	-4	2	4	2	-4

The sketch of the graph is shown in the figure.



1.6 Drawing Graph Using Factors

Let us draw the graph of quadratic function using factors.

We know that $y = ax^2 + bx + c$ is a quadratic function where $a \neq 0$ and the graph of such a function is a parabola. These graphs can be tricky to sketch manually, but factoring the quadratic gives us all of the information we need to do so successfully.

Procedure:

The points where any parabola intersects the x-axis will be the solutions to the equation:

$$ax^2 + bx + c = 0 \quad \text{(i)}$$

Now if we can factor equation (i) in the format:

$$(x - x_1)(x - x_2) = 0$$

then by the zero product property, we get:

$$x = x_1 \quad \text{and} \quad x = x_2$$

This means that x_1 and x_2 are the x-intercepts. In other words, graph will intersect x-axis at $(x_1, 0)$ and $(x_2, 0)$. The constant c tells us what the y-intercept will be. More specifically, the constant term c places the y-intercept at $(0, c)$, giving us a third specific point on y-axis. Likewise, the value of a tells us whether our parabola will open up or down. If a is positive, the parabola opens upward. If it is negative, the parabola opens downward.

We can also figure out the vertex of the parabola by factoring the quadratic equation.

The x-coordinate of the vertex of a parabola is the arithmetic mean of two x-intercepts $= \frac{x_1+x_2}{2}$.

Once we have the x-coordinate, we can determine the y-coordinate by plugging the x value into the given function and solving for y. With these four specific points including both x-intercepts, y-intercept and the vertex of the parabola, we can create an accurate sketch of the graph quickly and easily.

Example 10: Draw the graph of $y = x^2 - 8x + 12$ using factors.

Solution:

Given function is: $y = x^2 - 8x + 12$

To get x-intercepts, put $x^2 - 8x + 12 = 0$

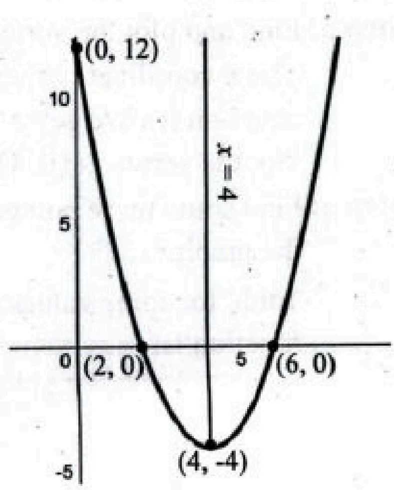
After factorization, we get:

$$x_1 = 2, \ x_2 = 6$$

$\Rightarrow$ The graph intersects x-axis at (2, 0) and (6, 0).

To find y-intercept, put $x = 0$ in the given function which gives $y = 12$.

Therefore y-intercept is (0, 12).



Vertex: x-coordinate of vertex $= \frac{x_1+x_2}{2} = \frac{2+6}{2} = 4$

Substituting, $x = 4$ in given function, we get:

y-coordinate of vertex $= 4^2 - 8(4) + 12 = -4$

$\therefore$ Vertex $= (4, -4)$

As $a = 1 > 0$, therefore parabola opens upward.

The graph is symmetric about $x = 4$. The sketch of the graph is shown in the figure.

Example 11: Draw the graph of $y = -x^2 + 4x - 4$ using factors.

Solution:

Given function is: $y = -x^2 + 4x - 4$

To get x-intercepts, put:

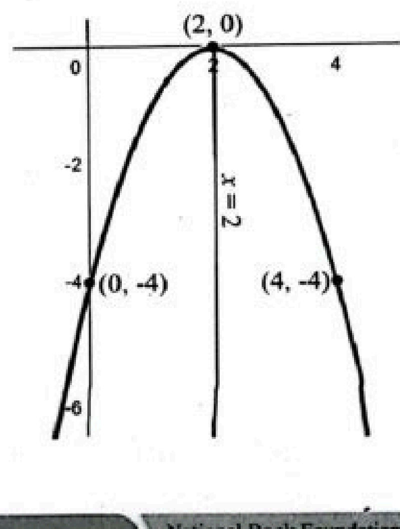
$-x^2 + 4x - 4 = 0 \Rightarrow -(x^2 - 4x + 4) = 0 \Rightarrow x^2 - 4x + 4 = 0$

After factorization, we get repeated roots:

$$x_1 = 2, \ x_2 = 2$$

This shows that the graph intersects x-axis at (2, 0), which is the vertex of the parabola.

To find y-intercept, put $x = 0$ in the given function which gives $y = -4$.



Therefore y-intercept is $(0, -4)$.
$a = -1 < 0$, shows thar the parabola opens downwards. As the graph is symmetric about $x = 4$, therefore parabola also passes through $(4, -4)$. The sketch of the graph is shown in the figure.

Check Point

Draw the graph of $y = 2x^2 + 6x + 4$ using factors.

1.7 Predicting Functions from Their Graphs

The method is explained with the help of following examples.

Example 12: Predict function from the graph.

Solution:

The graph shows a line passing through points $(2, 0)$ and $(0, -2)$.

Slope of the line is:

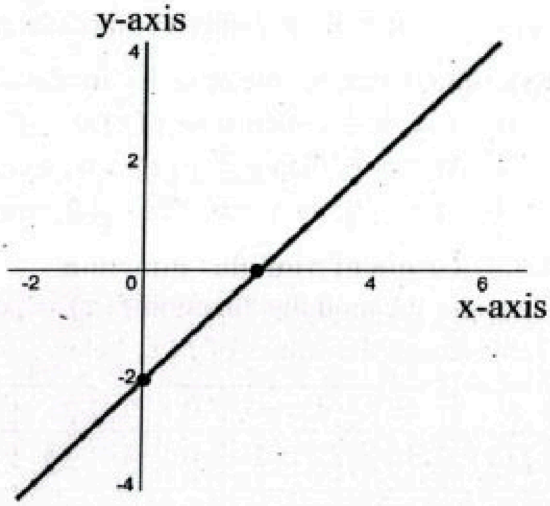
$m = \frac{0+2}{2-0} = 1$

Equation of the line is:

$y - 0 = 1(x - 2)$ (Point slope form of the line)

$y = x - 2$

Which is the required function.



Example 13: Predict function from the graph.

Solution:

The graph shows a parabola passing through points $(2, 0)$ and $(-1, 0)$ and $(0, -4)$. We know that the equation of the parabola is quadratic.

Now, the equation of the parabola passing through $(p, 0)$ and $(q, 0)$ is of the form:

$y = a(x - p)(x - q)$ where $p = 2, q = -1$ and $a > 0$ as the parabola opens upwards.

Substituting the values of p and q in the above equation, we get:

$y = a(x - 2)(x + 1)$ (i)

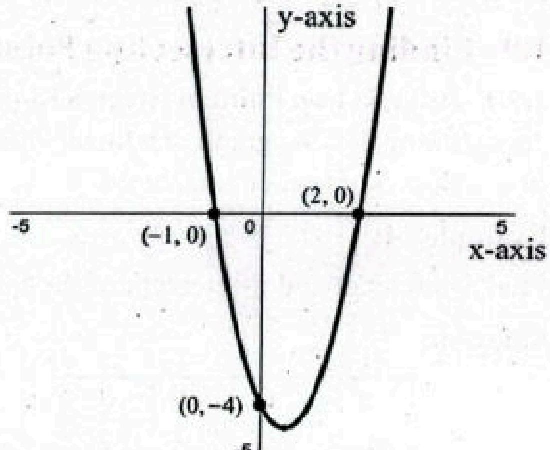
Since the parabola passes through $(0, -4)$, therefore from (i), we have:

$-4 = a(0 - 2)(0 + 1) \Rightarrow -4 = a(-2) \Rightarrow a = 2$

Therefore from (i):

$y = 2(x - 2)(x + 1) \Rightarrow y = 2(x^2 - x - 2)$

Which is the required function.



Key Facts

Equation of parabolic function passing through (p, r) and (q, r) is:
$y - r = a(x - p)(x - q)$ where p, q and r are positive.

1.8 Graph of Modulus Functions

A modulus function (absolute valued function) determines a number's magnitude regardless of its sign. If x is a real number, then the modulus function is denoted by:

$$y = |x| \quad \text{or} \quad f(x) = |x| \text{ where } x \in \mathbb{R}.$$

The modulus function takes the actual value of x if it is more than or equal to 0 and the function takes the minus of the actual value x if it is less than 0.

1.8.1 Domain and Range of Modulus Function

The domain of modulus function is R while its range is the set of non-negative real numbers, denoted as $[0, \infty)$. Any real number can be modulated using the modulus function.

Example: Consider the modulus function $f(x) = |x|$. Then:

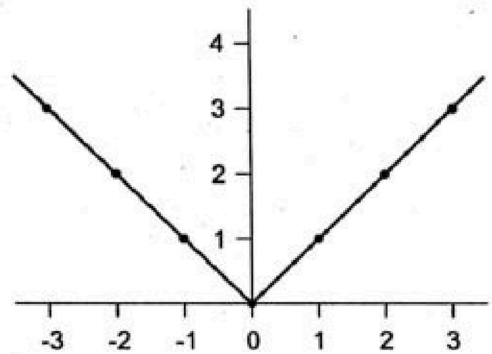
- If $x = -5$, then $y = f(x) = -(-5) = 5$, since x is less than zero.
- If $x = 6$, then $y = f(x) = 6$, since x is greater than zero.
- If $x = 0$, then $y = f(x) = 0$, since x is equal to zero.

1.8.2 Graph of Modulus Function

Consider the modulus function $f(x) = |x|$

Table shows the values of $f(x)$ below:

x	-3	-2	-1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3



It can be inferred that for all possible values of x, the function $f(x)$ remains positive.

1.9 Finding the Intersecting Points Graphically

1.9.1 Intersection Point between a Linear Function and Coordinate Axes

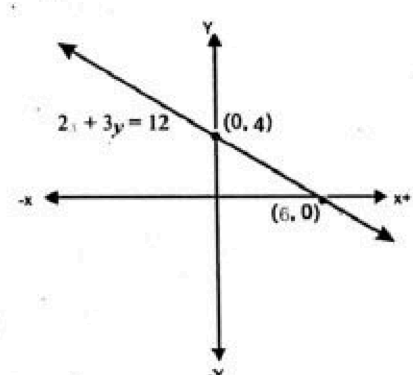
As we know that the graph of a linear equation is a straight line and the points of intersection of the line with axes are called intercepts.

Example 14:

Find x-intercept and y-intercept of the function: $f(x) = \frac{12-2x}{3}$

Solution:

x	6	0	3
$f(x)$	0	4	2



The graph of line $f(x) = \frac{12-2x}{3}$ is shown in the adjoining figure. From the graph it is clear that:

- The line crosses the x-axis at (6, 0). So, its x-intercept is 6.
- The line crosses the y-axis at (0, 4). So, its y-intercept is 4.

Check Point

x and y-intercepts of a line are -3 and -5 respectively. Find points of intersections of line with axes.

1.9.2 Intersection Point between two Linear Functions

While solving simultaneous linear functions graphically, keep in mind the following points.

1. Draw each linear function on the same set of axes.
2. Find the coordinates where the lines intersect.

Example 15: Find the graphical solution of $f(x) = \frac{6-x}{2}$ and $g(x) = x - 3$.

Solution:

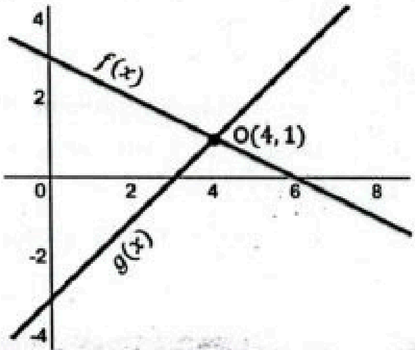


Table of values of function $f(x) = \frac{6-x}{2}$ is:

x	6	0	4
$f(x)$	0	3	1

Table of values of function $g(x) = x - 3$ is:

x	3	0	5
$g(x)$	0	−3	2

The graph of functions $f(x) = \frac{6-x}{2}$ and $g(x) = x - 3$ is shown in the adjoining figure.

From the graph it is clear that the both linear functions intersect each other at point O(4, 1).
Therefore, point O(4, 1) is the graphical solution of the given linear functions.

1.9.3 Intersection Point between a Linear Function and a Quadratic Function

As we know that the graph of a quadratic equation is a curve. The point of intersection of a linear function and a quadratic function is a point where both the graphs intersect each other.

Example 16: Solve $f(x) = 3x + 4$ and $g(x) = 5 + 3x - 2x^2$ graphically.

Solution:

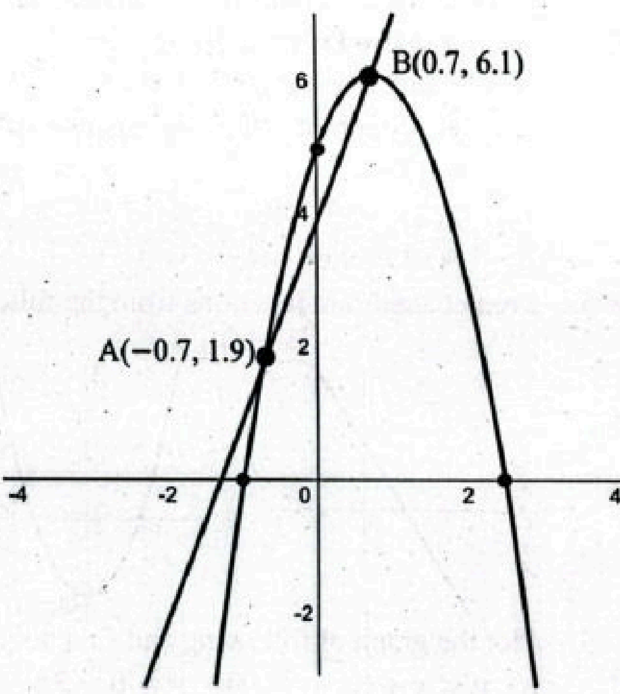
Table of values for $f(x) = 3x + 4$ is:

x	0	−1	1
$f(x)$	4	1	7

Comparing the graph of
$g(x) = 5 + 3x - 2x^2$, with
$y = ax^2 + bx + c$, we have:
$a = -2, b = 3$ and $c = 5$
Here, $a = -2 < 0$, so the curve will open downward.

Table of values for $g(x) = 5 + 3x - 2x^2$

x	−2	−1	0	1	2	3
$g(x)$	−9	0	5	6	3	−4

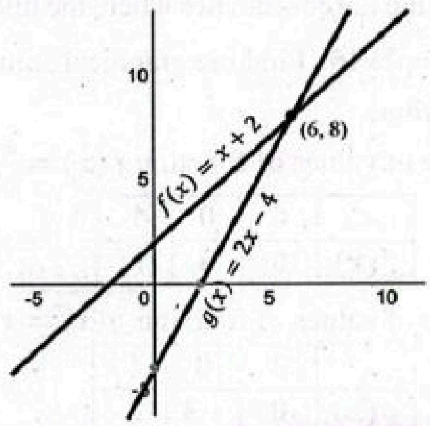


Both the graphs intersect each other at:
A(−0.7, 1.9) and B(0.7, 6.1).
Hence, solution set is: {(−0.7, 1.9), (0.7, 6.1)}

1.9.4 Solving Problems Graphically from Daily Life

See the following example to understand the method.

Example 17: Two airplanes are moving along the paths representing $f(x) = x + 2$ and $g(x) = 2x - 4$ respectively. Draw the graph of paths of both planes and find the point, where both planes pass.



Solution:

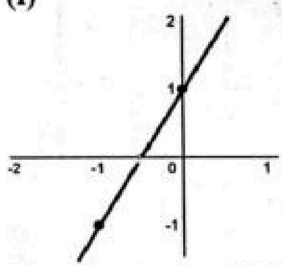
In the figure, red line represents the path of first plane while blue line represents the path of second plane.

The graph shows that both planes pass through the point (6, 8).

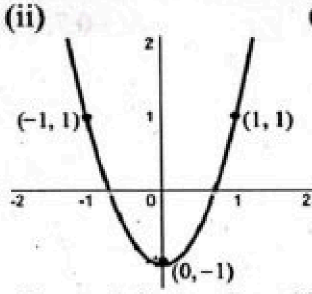
Exercise 1.2

1. Plot the graph of the functions:
   (i) $f(x) = 3x - 2$ (ii) $f(x) = 3x$ (iii) $f(x) = 1 - 2x$
   (iv) $g(x) = x^2 + 4$ (v) $g(x) = x^2 - x - 6$ (vi) $g(x) = \sqrt{2x+1}$
2. Plot the graph of following functions.
   (i) $f(x) = -x^2 + 1$ (ii) $f(x) = 2x^3$ (iii) $f(x) = 1 + x^{-2}$
   (iv) $f(x) = 3x^{\frac{1}{2}}$ (v) $f(x) = 2 - x^{-\frac{1}{2}}$ (vi) $f(x) = x^{\frac{5}{2}}$
3. Find possible x-intercept, y-intercept and vertex of the following functions and then plot.
   (i) $f(x) = x^2 + 2x + 1$ (ii) $f(x) = x^2 - 4x + 4$
   (iii) $f(x) = x^2 + 2x$ (iv) $f(x) = 9 - x^2$
4. Draw the graph of following function using factors.
   (i) $f(x) = x^2 - 2x + 1$ (ii) $f(x) = x^2 - 7x + 12$
   (iii) $f(x) = x^2 - 2x$ (iv) $f(x) = -2x^2 + x + 3$
   (v) $f(x) = 4x^2 - 4x$ (iv) $f(x) = 6 - x^2 - x$
5. Predict algebraic functions from the following graphs.

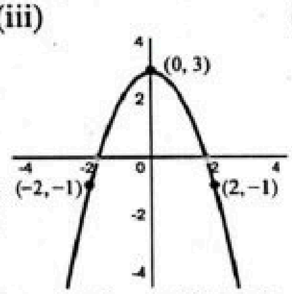
(i)



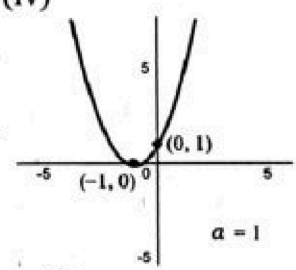
(ii)



(iii)



(iv)



6. Plot the graph of following and find point of intersection of function with axes.
   (i) $y = x + 3$ (ii) $y = 6 - 3x$ (iii) $y = x^2 - 5x$

7. Find graphical solution of:
   (i) $f(x) = 4 - 3x,\ g(x) = -x + 1$ (ii) $f(x) = 2(2 + x),\ g(x) = x^2 + 1$
   (iii) $f(x) = 5 + 3x,\ g(x) = -x^2 + 5$ (iv) $f(x) = 1,\ g(x) = -2x^2 + 2x + 5$
   (v) $f(x) = 2 + 3x + x^2,\ g(x) = 5 + 3x - 2x^2$
8. Draw the graph of following modulus functions.
   (i) $f(x) = -1.5|x|$ (ii) $f(x) = 1+2|x|$ (iii) $f(x) = 3|x| + x$
9. The equations for supply and demand are given by two linear equations:
   Supply equation: $S(x) = 2x + 10$; where x is quantity and $S(x)$ is the price.
   Demand equation: $D(x) = -3x + 40$; where x is quantity and $D(x)$ is the price.
   Find the equilibrium point where the price of supply equals the price of demand by drawing the graphs of both equations.
10. Suppose a ball is thrown into the air and its height h(t) after t seconds is given by the parabolic trajectory: $h(t) = -6t^2 + 10t + 5$. If this ball hits a wall 10 m high representing the equation: $h(t) = 9t$. By drawing the graphs, find out when and where the ball reaches the wall.
11. Two asteroids are following the parabolic paths represented by $f(x) = x^2 - 7x + 12$ and $g(x) = x(x - 3)$. By drawing the graphs of both trajectories, find out the place from where, both asteroids will pass.

1.10 Algebraic and Transcendental Functions

1.10.1 Algebraic Function

An algebraic function is a function that involves only algebraic operations. These operations include addition, subtraction, multiplication, division, and exponentiation.

Types of Algebraic Functions

Main types of algebraic functions are:

(i) Polynomial Functions

A function of the from $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_n$ are constants and n is integer, is called a polynomial function. Some examples are:

- $f(x) = 3x + 7$ (linear function)
- $f(x) = x^2 - 2x + 5$ (quadratic function)
- $f(x) = x^3 - 7x + 7$ (cubic function)
- $f(x) = x^4 - 5x^2 + 2x - 8$ (biquadratic function)
- $f(x) = x^5 - 7x + 3$ (quintic function)

(ii) Rational Functions

A function that is composed of two functions and expressed in the form of a fraction is a rational function. If $f(x)$ is a rational function, then $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. Some examples are:

$$f(x) = \frac{x - 4}{2x + 3}, \quad g(x) = \frac{7}{x^2 + 5x + 1}$$

(iii) Power Functions

The power functions are of the form $f(x) = k\,x^a$ where 'k' and 'a' are any real numbers. Since 'a' is a real number, the exponent can be either an integer or a rational number.

Some examples are:

$f(x) = x^2$, $f(x) = x^{-1}$ (reciprocal function) and $f(x) = \sqrt{x-2} = (x-2)^{\frac{1}{2}}$

Properties

- Algebraic functions are closed under addition, subtraction, multiplication, division and composition.
- Algebraic functions are easy to solve, differentiate and integrate.

Application

- Physics and Engineering: Simple mechanical system, the motion of objects under constant acceleration.
- Geometry: Many curves such as circles and ellipses.

1.10.2 Transcendental Functions

The functions which are not algebraic are called transcendental. These functions can only be expressed in terms of infinite series. Some examples are:

- Exponential functions: $f(x) = e^x,\ g(x) = a^{3x}$
- Logarithmic functions: $f(x) = log_a x,\ g(x) = lnx$; where base a is a positive constant.
- Trigonometric functions: $f(x) = \sin x,\ g(x) = \cos x,\ h(x) = \tan x$
- Inverse trigonometric functions: $f(x) = \sin^{-1}x,\ g(x) = \cos^{-1}x$
- Hyperbolic functions: $f(x) = \sinh x,\ g(x) = \cosh x,\ h(x) = \tanh x$
- Inverse hyperbolic functions: $f(x) = \sinh^{-1}x,\ g(x) = \cosh^{-1}x$
- Special functions: Bessel functions, Gamma functions, error functions etc.

Properties

- These functions are not expressible in terms of a finite combination of algebraic operation of addition, subtraction, division, multiplication, raising to a power and extracting a root.
- These functions often exhibit more complex behavior like periodicity (in the case of trigonometric functions) and rapid growth (in the case of exponential function).

Application

- Science and Engineering: Exponential and logarithmic functions are critical in modelling growth, decay and oscillation in natural systems.
- Signal processing: Trigonometric functions are fundamental in analysing waves, sounds and signals.
- Mathematical analysis: Many problems in calculus, differential equations and complex analysis involve transcendental functions.

1.10.3 Logarithmic Functions

Logarithmic functions form a fundamental class of transcendental functions. These functions are inverse of exponential functions. They play a crucial role in mathematics, science, engineering and many applied fields.

Definition: If you have an exponential function of the form $y = a^x$ where $a > 0$ and $a \neq 1$, then the logarithmic function is defined as:

$$x = \log_a(y)$$

Replacing y with x, we have:

$$y = \log_a(x)$$

Here, $\log_a(x)$ is read as logarithmic of x to the base a.

Base of the Logarithms

The base of logarithm determines its specific type. Some types are:

- Natural logarithm: It is written as $\log_e(x) = \ln x$ where $e = 2.71828...$ is called Euler's number.
- Common logarithm: It is written as $\log_{10}(x)$ where $a = 10$.
- Binary logarithm: It is written as $\log_2(x)$ where $a = 2$.

Properties

- The logarithm is the inverse of exponential. If $y = a^x$, then $x = \log_a(y)$. This means $\log_a(a^x) = x$ and $a^{\log_a(y)} = y$.
- The domain of $\log_a(x)$ is $x > 0$ because we cannot take the logarithm of zero or a negative real number.

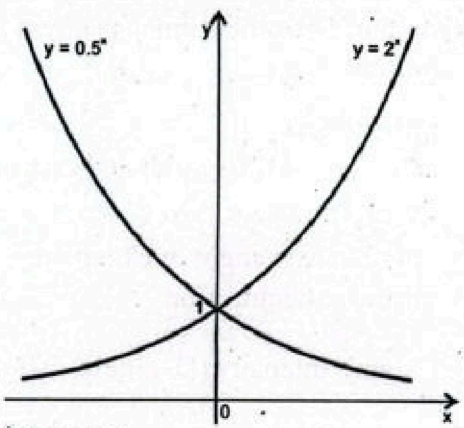
Laws of Logarithms

- Product Rule: $\log_a(xy) = \log_a(x) + \log_a(y)$
- Quotient Rule: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- Power Rule: $\log_a(x^n) = n\log_a(x)$
- Change of Base Rule: For any positive bases $a \neq 1$ and $b \neq 1$:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

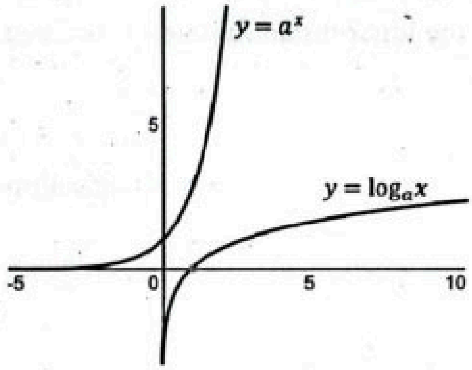
Note: As $a^x = 1$, therefore $\log_a(1) = 0$.

Graph of Exponential Function



- If the base, a is greater than 1, then the function increases exponentially at a growth rate of a. This is known as exponential growth.
- If the base, a is less than 1 (but greater than 0) the function decreases exponentially at a rate of a. This is known as exponential decay.
- If the base, a is equal to 1, then the function trivially becomes $y = 1$. This means exponential function always passes through (0, 1).
- The points (0, 1) and $(1, a)$ are always on the graph of the function $y = a^x$.
- Exponential function takes only positive values and its graph never touches x-axis.
- The domain of the exponential function is the set of all real numbers, whereas the range of this function is the set of positive real numbers.

Graph of Logarithmic Function



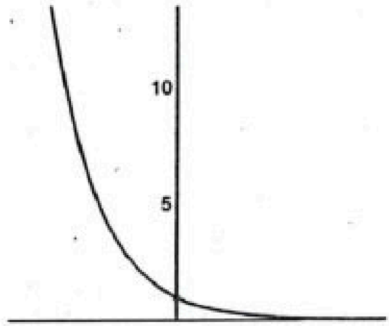
- When graphed, the logarithmic function is similar in shape to the square root function.
- The logarithmic function always passes through the point $(1, 0)$ because $\log_a(1) = 0$.
- The curve approaches to y-axis but never touches it.
- The domain of the logarithmic function is the set of all positive real numbers, whereas the range of this function is the set of all real numbers.
- For $a > 1$, the value of function increases as x increases.
- For $0 < a < 1$, the value of function decreases as x increases.

Example 18: Draw the graph of $f(x) = e^{-0.5x}$.

Solution:

Table of values for $f(x) = e^{-0.5x}$

x	-5	-2	-1	0	1	2	3
$g(x)$	12	2.7	1.6	1	0.6	0.4	0.2



Graph is shown in the adjoining figure.

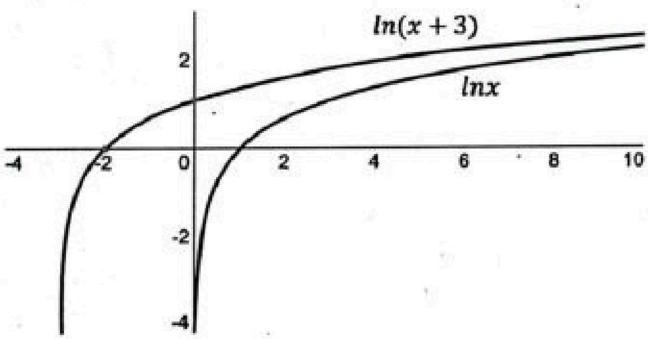
Example 19: Draw the graph of:

(i) $f(x) = ln x$ (ii) $g(x) = ln(x + 3)$

Solution:

Table of values for $f(x)$ and $g(x)$ is:

x	0	0.1	0.5	1	4	10
$f(x)$	-	-2.3	-0.7	0	1.4	2.3
$g(x)$	1.09	1.13	1.3	1.4	1.9	2.6



Graph of both functions is shown in the adjoining figure.

Applications

- Growth and Decay Model: Logarithm functions are used to model phenomenon that grow rapidly at first and then slow down such as population growth or the spread of deceases.
- pH Measurement in Chemistry: The pH of a solution is the logarithmic measure of the hydrogen in cm concentration.

$$\text{pH} = -\log_{10}[H^+]$$

- Sound Intensity (Decibels): The decibel scale which measures intensity, is a logarithmic scale:

$$\text{Decibels} = 10 \times \log_{10}\left(\frac{I}{I_0}\right)$$

here I is the intensity of the sound and I_o is the reference intensity.

- Information Theory: Logarithms are used in information theory to measure information content and entropy.

- Financial Models: Logarithmic functions are used in finance particularly in modelling the time value of money and compound interest.
- Computer Science: Logarithmic functions appear in algorithms and data structures.

Conclusions

Logarithmic functions are powerful tools for dealing with exponential growth and decay as well as for measuring and comparing quantities on vastly different scales. Their unique properties and applications make them essential in both theoretical and applied fields. Most of the applications, we find, are in the fields of engineering and computer technology.

Example 20: Suppose that Rs. 30,000 is invested at 8% interest compounded annually. In t years, it will grow to the amount $A(t)$ given by the function: $A(t) = 30{,}000\ (1.08)^t$

(i) How long will it take until then is Rs. 150,000 in the account?

(ii) Let T be the amount of time it takes for the Rs.30,000 to double itself. Find T.

Solution:

(i) We set $A(t) = 150{,}000$ and solve for t.

$$150{,}000 = 30{,}000\ (1.08)^t \quad \Rightarrow \quad (1.08)^t = \frac{150{,}000}{30{,}000} = 5$$

Taking natural log on both sides, we get:

$$ln(1.08)^t = ln5 \quad \Rightarrow \quad t\, ln(1.08) = \ln 5$$

$$\Rightarrow \quad t = \frac{\ln 5}{\ln(1.08)} = \frac{1.6094}{0.07696} \approx 20.9$$

Therefore, it will take almost 20.9 years for Rs. 30,000 to grow to Rs. 150,000.

(ii) To find the doubling time T, we set A(t) = Rs. 60,000, t = T and solve for T.

$$60{,}000 = 30{,}000\ (1.08)^T \quad \Rightarrow \quad (1.08)^T = \frac{60{,}000}{30{,}000} = 2$$

Taking natural log on both sides, we get:

$$ln(1.08)^T = ln2 \quad \Rightarrow T\ ln(1.08) = ln2$$

$$\Rightarrow \ T = \frac{\ln 2}{\ln(1.08)} = \frac{0.6931}{0.07696} \approx 9$$

Therefore, doubling time is about 9 years.

Example 21: In 2020, the population of the country was 249 million and the exponential growth rate was 0.9% per year. If $P(t) = P_o e^{rt}$ is exponential growth function, then:

(i) Find the exponential growth function for the given data.

(ii) What would you expect the population to be in the year 2028?

Solution:

(i) Here $P_o = 249,\ r = 9\% = 0.009$

The population growth function, gives:

$P(t) = 249 \times e^{0.009t}$ (a)

(ii) In 2028, we have $t = 8$.

To find the population in 2028, we substitute 8 for t in (a).

$$P(8) = 249 \times e^{0.009 \times 8} = 249 \times e^{0.072}$$
$$\approx 249 \times 1.0747 = 267.6$$

Therefore, population of the city in 2028, will be about 267.6 million.

Key Facts

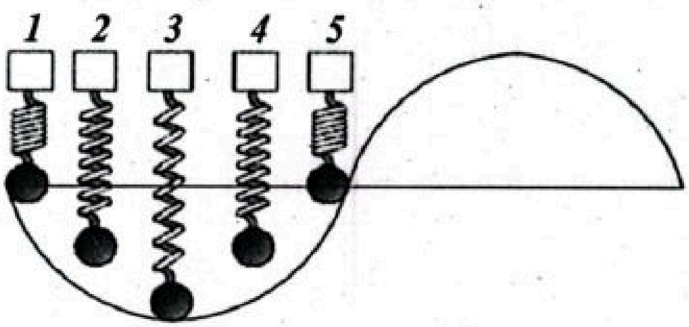
The function $P(t) = P_o e^{rt}$ models the growth in the quantity while the function $P(t) = P_o e^{-rt}$ models the decay or decline in the quantity where $r > 0$.

Exercise 1.3

1. Draw the graphs of functions.
   (i) $f(x) = e^{2x}$ (ii) $g(x) = e^{0.5x}$ (iii) $h(x) = 2 - e^{x}$
   (iv) $h(x) = 1 + e^{-2x}$ (v) $f(x) = \ln(2x)$ (vi) $g(x) = \log(x+1)$
   (vii) $h(x) = 3 + \log(x)$ (viii) $f(x) = e^{0.6x}$ and $g(x) = \ln(0.6x)$
2. The number of compact discs N (in million) purchased each year increasing exponentially is given by:
   $$N(t) = 7.5(6)^{0.5t}$$
   Where $t = 0$ corresponds to 2024, $t = 1$ corresponds to 2025 and so on, t being the number of years after 2024.
   a. After what amount of time will one billion compact discs be sold in a year?
   b. What is the doubling time on the sale of compact discs?
3. Suppose that Rs. 50,000 is invested at 6% interest compounded annually. After t years, it grows to the amount A given by the function:
   $$A(t) = 50{,}000(1.06)^{t}$$
   a. After what amount of time will Rs. 50,000 grows to Rs. 450,000?
   b. Find the doubling time.
4. The exponential growth rate of the population of the city is 1% per year. After how many years, the population will be doubled?
5. The population of the world was 5.2 billion in 1990. The exponential growth rate was 1.6% per year at that time.
   a. Find the exponential growth function. b. Find the population of the world in 2000.
   c. In which year the world population was 8 billion?
6. Students in a mathematics class took a final exam in monthly intervals thereafter. The average score $S(t)$, after t months was given by:
   $$S(t) = 68 - 20\log(t+1);\ \ t \geq 0$$
   a. What was the average score when they initially took the test ($t = 0$)?
   b. What was the average score (i) after 4 months (ii) after 24 months?
   c. Graph the function.
   d. After what time was the average score 50?
7. If $P(t) = P_0e^{kt}$ denotes the growth function of oil and the exponential growth rate of the demand for oil is 10% per year, when will the demand be doubled?
8. Approximately two third of all Aluminum cans distributed are recycled each year. A beverage company distributes 250,000 cans. The number still in use after t years is given by the function:
   $$N(t) = 250{,}000\left(\frac{2}{3}\right)^{t}$$
   a. After how many years will 60,000 cans be in use?
   b. After what amount of time will only 1,000 cans be in use?

1.11 Domain and Range of Transcendental Functions through Graphs

If a weight is attached to a spring and the weight is pushed up or pulled down and released, it tends to rise and fall alternately. The weight is said to be oscillating in harmonic motion. If the position of the weight y is graphed over time the result is the graph of a sine or cosine curve.

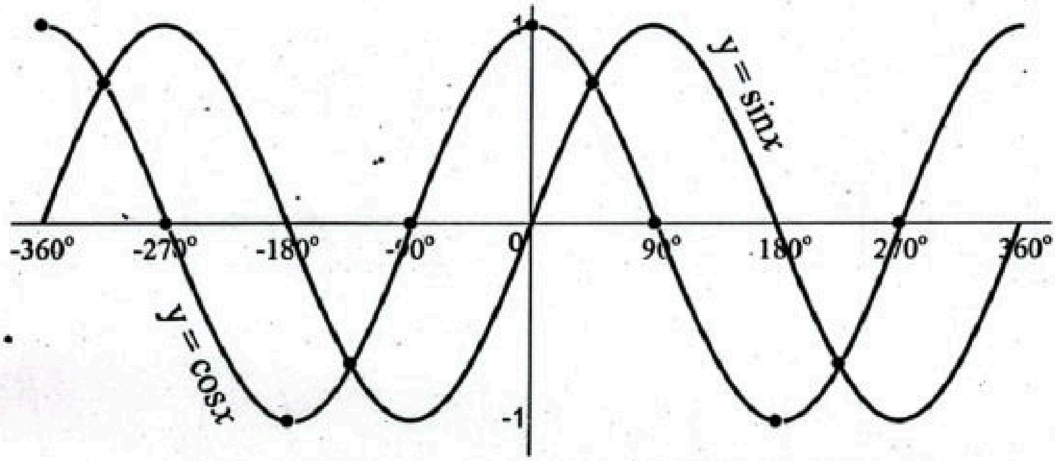


1.11.1 Graph of $y = \sin\theta$ and $y = \cos\theta$

To graph the sine or cosine function, we use the horizontal axis for the values of θ expressed in either degrees or radians and vertical axis for the values of $\sin\theta$ or $\cos\theta$. Ordered pairs for these points are of the form $(\theta, \sin\theta)$ or $(\theta, \cos\theta)$.

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$\sin\theta$	0	0.3	0.5	0.7	0.87	0.97	1	0.97	0.87	0.7	0.5	0.3	0
$\cos\theta$	1	0.97	0.87	0.7	0.5	0.3	0	-0.3	-0.5	-0.7	-0.87	-0.97	-1

θ	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
$\sin\theta$	- 0.3	-0.5	-0.7	-0.87	-0.97	-1	-0.97	-0.87	-0.7	-0.5	-0.3	0
$\cos\theta$	-0.97	-0.87	-0.7	-0.5	-0.3	0	0.3	0.5	0.7	0.87	0.97	1

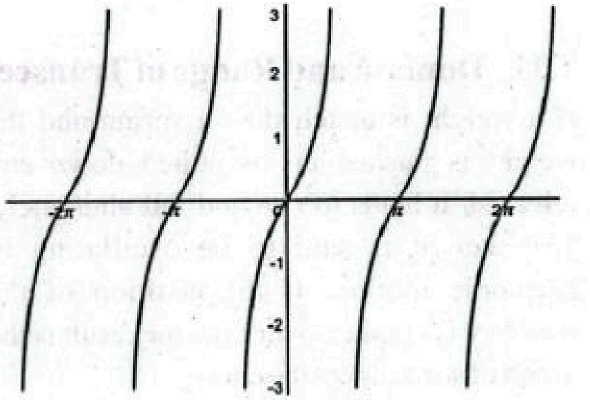


From the behavior of graphs of sine and cosine functions, we can easily predict the domain and range of both functions which are:

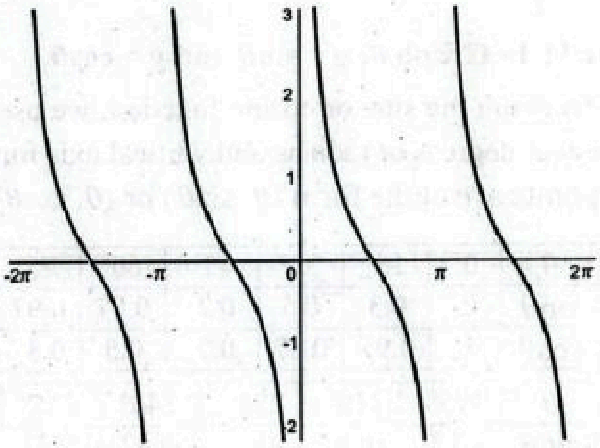
Function	Domain	Range
$y = \sin\theta$	$R = \theta \in (-\infty, \infty) = -\infty < \theta < \infty$	$y \in [-1, 1] = -1 \le y \le 1$
$y = \cos\theta$	$R = \theta \in (-\infty, \infty) = -\infty < \theta < \infty$	$y \in [-1, 1] = -1 \le y \le 1$

1.11.2 Graph of y = tanθ and y = cotθ

Similarly, by drawing the graph of y = tanθ and y = cotθ , we can easily predict the domain and range of both functions as follows.



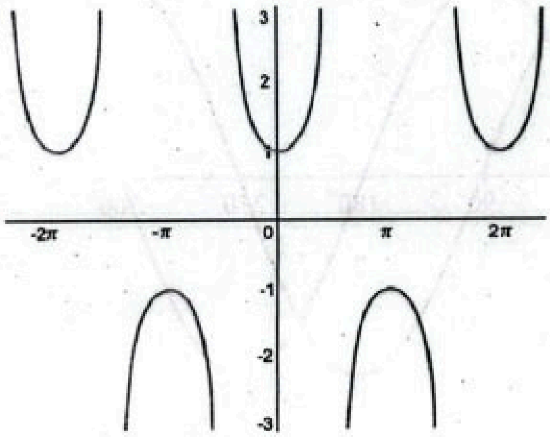
Function	$y = \tan\theta$
Domain	$\theta \neq (2n+1)\dfrac{\pi}{2}; n \in Z$
Range	R

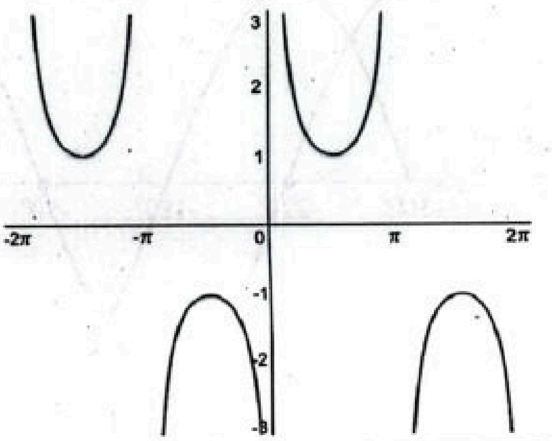


Function	$y = \cot\theta$
Domain	$\theta \neq n\pi; n \in Z$
Range	R

1.11.3 Graph of y = secθ and y = cosecθ

Domain and range of y = secθ and y = cosecθ is obvious form the graphs of both functions shown below.





Function	$y = \sec\theta$
Domain	$\theta \neq (2n+1)\dfrac{\pi}{2}; n \in Z$
Range	$y \leq -1, y \geq 1$ or $y \in (-\infty, -1] \cup [1, \infty)$

Function	$y = \csc\theta$
Domain	$\theta \neq n\pi; n \in Z$
Range	$y \leq -1, y \geq 1$ or $y \in (-\infty, -1] \cup [1, \infty)$

1.11.4 Inverse Trigonometric Functions

Inverse trigonometric functions, also known as arc functions or anti-trigonometric functions, are the inverse functions of the standard trigonometric functions (sine, cosine, tangent, cotangent, secant, and cosecant). They are used to find the angle when the trigonometric ratio is known. For example, arcsin(x) finds the angle whose sine is x.

Table below shows notation, domain and range of inverse trigonometric functions.

Name	Usual notation	Definition	Domain of x for real result	Range of usual principal value (radians)	Range of usual principal value (degrees)
arcsine	$y = \sin^{-1}(x)$	$x = \sin(y)$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$	$-90° \le y \le 90°$
arccosine	$y = \cos^{-1}(x)$	$x = \cos(y)$	$-1 \le x \le 1$	$0 \le y \le \pi$	$0° \le y \le 180°$
arctangent	$y = \tan^{-1}(x)$	$x = \tan(y)$	R	$-\pi/2 < y < \pi/2$	$-90° < y < 90°$
arccotangent	$y = \cot^{-1}(x)$	$x = \cot(y)$	R	$0 < y < \pi$	$0° < y < 180°$
arcsecant	$y = \sec^{-1}(x)$	$x = \sec(y)$	$\lvert x\rvert \ge 1$	$0 \le y < \pi/2$ or $\pi/2 < y \le \pi$	$0° \le y < 90°$ or $90° < y \le 180°$
arccosecant	$y = \csc^{-1}(x)$	$x = \csc(y)$	$\lvert x\rvert \ge 1$	$-\pi/2 \le y < 0$ or $0 < y \le \pi/2$	$-90° \le y < 0$ or $0° < y \le 90°$

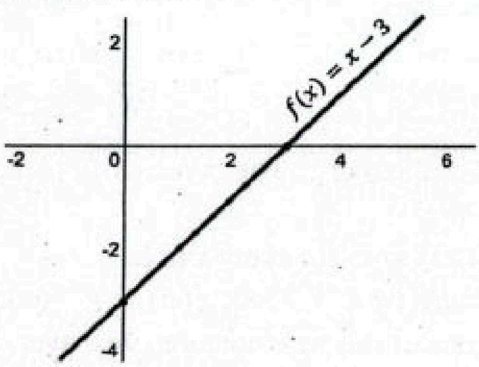
1.12 Relation Between a 1-1 Function and its Inverse through Graphs

1.12.1 One-One Function and its Graph

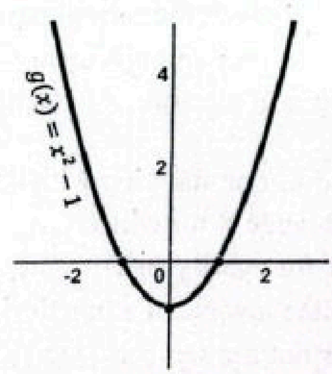
One to one function is a special function that maps every element of the range to exactly one element of its domain i.e., the outputs never repeat.

Examples: (i) The function $f(x) = x - 3$ is a one-to-one function since it produces a different answer for every input.

(ii) The function $g(x) = x^2 - 1$ is not a one-to-one function since it produces one output 0 for the two inputs 1 and -1.



One-One Function



Not a One-One Function

1.12.2 Horizontal Line Test

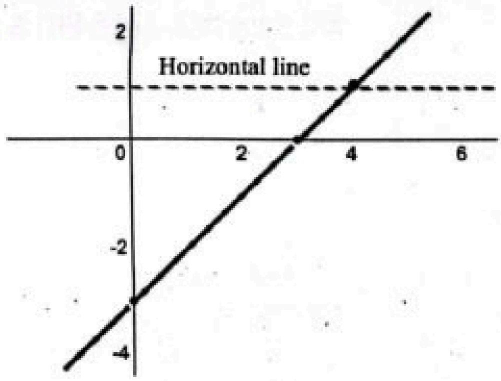
The horizontal line test is used to determine whether a function is one-one when its graph is given. To test whether the function is one-one from its graph just take a horizontal line (consider a horizontal stick) and make it pass through the graph.

- If the horizontal line does not pass through more than one point of the graph, then the function is one-one.
- If the horizontal line passes through more than one point of the graph, then the function is not one-one.

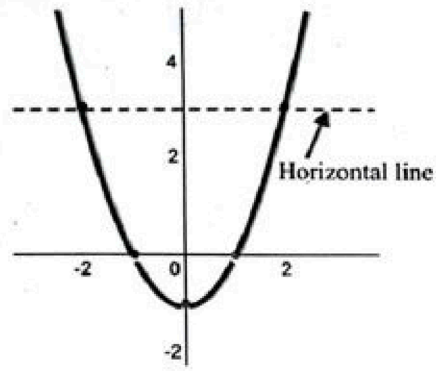
Examples: If we draw horizontal lines on the above graphs, we observe that:

(i) The graph of $f(x) = x - 3$ passes horizontal line test, so it is ono-one function.

(ii) The graph of $g(x) = x^2 - 1$ fails horizontal line test, so it is not ono-one function.



$f(x)$ is one-one function.



$g(x)$ is not a one-one function.

Check Point

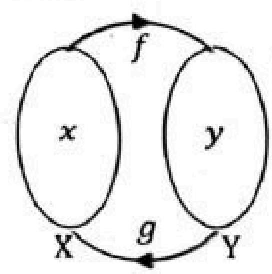
By using horizontal line test, check whether the function $y = x^3$ is 1-1 function or not.

1.12.3 Inverse of One-One Function

Suppose $f: X \to Y$ is a one-one function. Since every element y of Y corresponds with precisely one element x of X, the function f must determines a "**reverse function**" $g: Y \to X$ whose domain is Y and range is X. Then f and g imply that:

$f(x) = y$ and $g(y) = x$

$f(g(y)) = y$ and $g(f(x)) = x$



The function g is given the formal name as "**inverse of f**".

From the above discussion it is clear that:

Dom f = Rang g and Rang f = Dom g

Definition:

Let f be a one-to-one function with domain X and range Y. The inverse of f is a function g with domain Y and range X for which:

$f(g(y)) = y$ for every y in Y and $g(f(x)) = x$ for every x in X.

Symbolically the inverse of a function f is denoted by f^{-1}. Thus, $g(x) = f^{-1}(x)$. It is to be noted that $f^{-1}(x)$ is not the same as $[f(x)]^{-1}$. In terms of this new notation, we have:

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

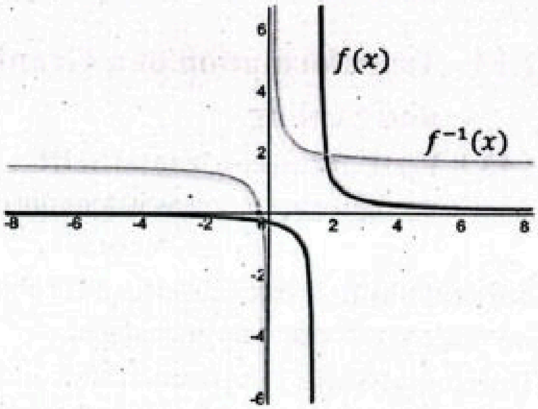
1.12.4 Properties of the Inverse of One to One Function

Here are the properties of the inverse of one to one function:

- The function f has an inverse function if and only if f is a one to one function.
- If the functions f and g are inverses of each other then, both these functions are one to one.
- f and g are inverses of each other if and only if $f(g(x)) = x, x$ in the domain of g and $g(f(x)) = x$, x in the domain of f.
- If f and g are inverses of each other then the domain of f is equal to the range of g and the range of g is equal to the domain of f.
- If f and g are inverses of each other then their graphs will make reflections of each other on the line $y = x$.
- If the point (a, b) is on the graph of f then point (b, a) is on the graph of f^{-1}.

Example 22: Find the inverse of $f(x) = \frac{1}{2x-3}; x \neq \frac{3}{2}$, then represent f and f^{-1} graphically.

Solution: Given that $f(x) = \frac{1}{2x-3}; x \neq \frac{3}{2}$



Since f is a one to one function, therefore:

$f(f^{-1}(x)) = \frac{1}{2f^{-1}(x)-3}$ [Replacing x with $f^{-1}(x)$]

Solving for $f^{-1}(x)$, we get:

$\Rightarrow \quad x = \frac{1}{2f^{-1}(x)-3} \qquad \Rightarrow \quad 2f^{-1}(x) - 3 = \frac{1}{x}$

$\Rightarrow \quad 2f^{-1}(x) = \frac{1}{x} + 3 \quad \Rightarrow \quad f^{-1}(x) = \frac{1+3x}{2x}$

Graph of function $f(x)$ and $f^{-1}(x)$ are shown in the adjoining figure. From the graph it is clear that if any point (a, b) is on the graph of $f(x)$ then point (b, a) is on the graph of $f^{-1}(x)$.

Challenge: Can you find inverse of $f(x)$ given in example 23, by any other method?

Example 23: Given that $f(x) = 3 - 4x$ is one to one. Find its inverse and represent f and f^{-1} graphically.

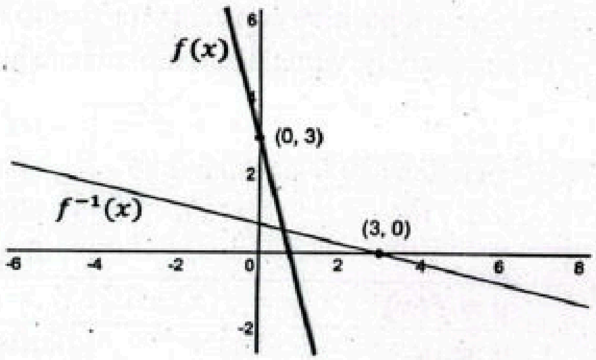
Solution: Given that $f(x) = 3 - 4x$ or $y = 3 - 4x$

Solving for x, we get:

$\Rightarrow \quad 4x = 3 - y \quad \Rightarrow \quad x = \frac{3-y}{4}$

$\Rightarrow \quad f^{-1}(y) = \frac{3-y}{4}$

$\Rightarrow \quad f^{-1}(x) = \frac{3-x}{4}$ [Replacing y with x.]



Graph of function $f(x)$ and $f^{-1}(x)$ are shown in the adjoining figure. From the graph it is clear that the point $(3, 0)$ is on the graph of $f(x)$ and the point $(0, 3)$ is on the graph of $f^{-1}(x)$.
Therefore, both the graphs are reflections of each other.

Exercise 1.4

1. Find the domain and range of the functions graphically.

   (i) $f(x) = \sin\left(\frac{x}{2}\right)$ (ii) $g(x) = 3\cos\left(\frac{x}{3}\right)$ (iii) $h(x) = 2\tan x$

   (iv) $y = \cot\left(\frac{x}{4}\right)$ (v) $y = 2\sec(2x)$ (vi) $y = \sin(2x)$

2. Determine whether the given function is one to one by examining its graph. If the function is one to one, find its inverse. Also draw the graphs of inverse function.

   (i) $f(x) = \frac{1}{3}x + 3$ (ii) $g(x) = x(x-5)$ (iii) $h(x) = (x+1)^2$

   (iv) $f(x) = x^3 - 8$ (v) $g(x) = \frac{4}{x}$ (vi) $h(x) = \frac{1}{3x+5}$

   (vii) $f(x) = x^4 + 2$ (viii) $g(x) = 5$ (ix) $h(x) = |x|$

1.13 Transformation of a Graph through Vertical Shift, Horizontal Shift and Scaling

1.13.1 Vertical and Horizontal Shift

A shift is a rigid translation as it does not change the shape or size of the graph of the function. A shift only changes the location of the graph.

Vertical Shift: A vertical shift adds/subtracts a positive constant to/from every y-coordinate while leaving the x-coordinate unchanged.

Horizontal Shift: A horizontal shift adds/subtracts a positive constant to/from every x-coordinate while leaving the y-coordinate unchanged.

Key Facts

Vertical and horizontal shifts can be combined into one expression.
Shifts are added/subtracted to the x or $f(x)$ components. If the positive constant is grouped with the x, then it is a horizontal shift, otherwise it is a vertical shift.

In this section, we will discuss the geometric effects on the graph of $y = f(x)$ by adding or subtracting a positive constant c to f or to its independent variable x.
The summary of vertical and horizontal shift is elaborated in the table 1.1 below.

Original function $y = f(x)$	Add a positive constant c to $f(x)$.	Subtract a positive constant c from $f(x)$.	Add a positive constant c to x.	Subtract a positive constant c from x.
$y = f(x)$	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
Geometric effects	Shifts the graph c units up.	Shifts the graph c units down.	Shifts the graph c units left.	Shifts the graph c units right.

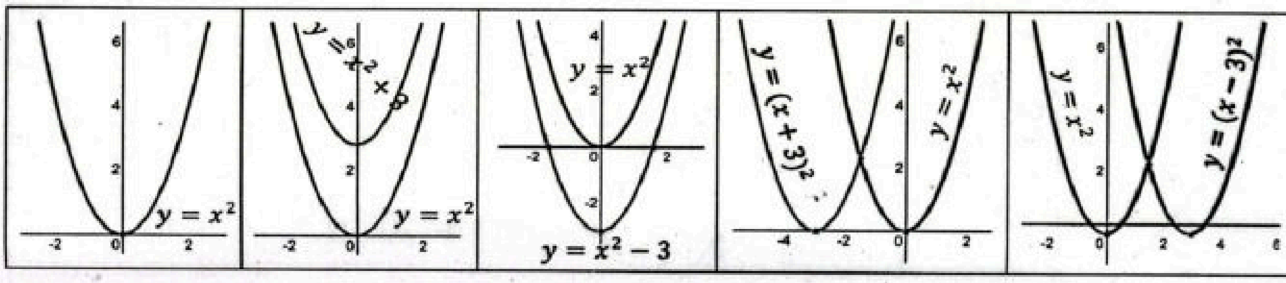
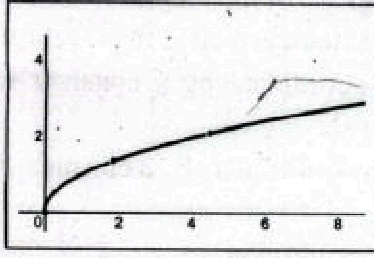
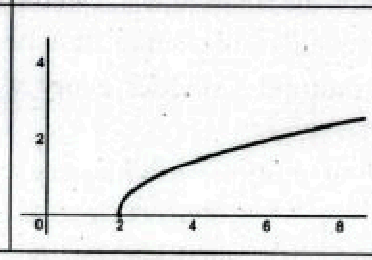
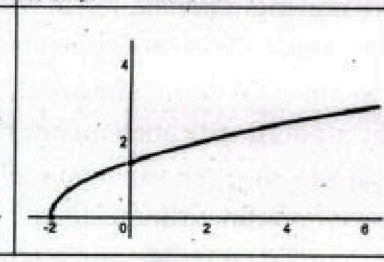


Table 1.1

Example 24: Sketch the graph of (a) $y=\sqrt{x}$ (b) $y=\sqrt{x-2}$ (c) $y=\sqrt{x+2}$
Which kind of shift did you observe after sketching the graphs.

Solution:

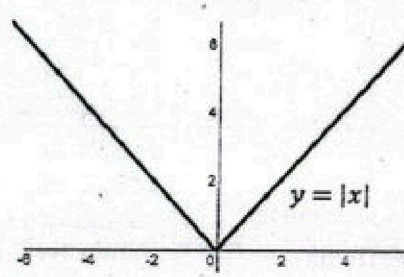
(a) $y=\sqrt{x}$	(b) $y=\sqrt{x-2}$	(c) $y=\sqrt{x+2}$
		

Above graphs show a horizontal shift. The graph of the function $y=\sqrt{x-2}$ can be obtained by transforming the graph of given function 2 units right to the origin while the graph of $y=\sqrt{x+2}$ can be obtained by transforming the graph of given function 2 units left to the origin.

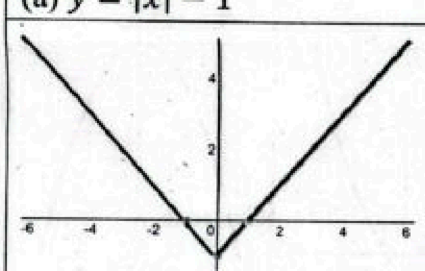
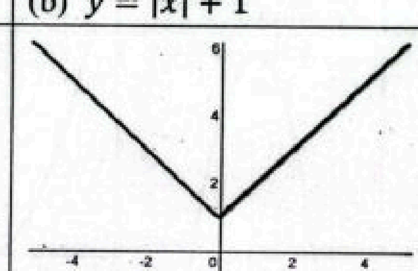
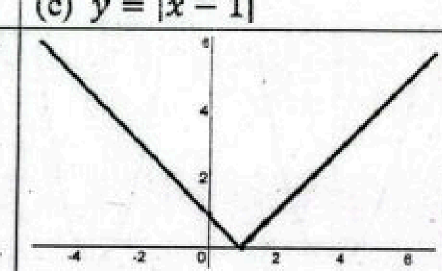
Example 25: Draw the graph of $y=|x|$ and then sketch the graphs of:
(a) $y=|x|-1$ (b) $y=|x|+1$ (c) $y=|x-1|$ (d) $y=|x+1|$
(e) $y=|x-1|-1$ (f) $y=|x+1|-1$
Which kind of shift did you observe after sketching the graphs.

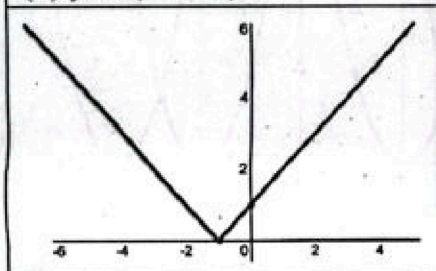
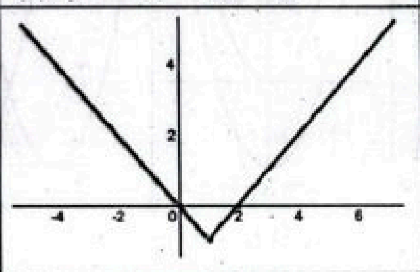
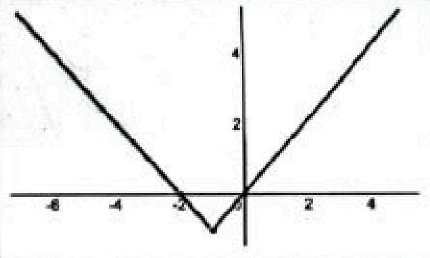
Solution: Table of the values of the function $y=|x|$ is given as:

x	0	±1	±2	±3	±4	±5	±6
$f(x)$	0	1	2	3	4	5	6



The graph is shown in the adjoining figure.
Sketch of other graphs is shown in the table below.

(a) $y=|x|-1$	(b) $y=|x|+1$	(c) $y=|x-1|$
		
Vertical shift 1 unit down	Vertical shift 1 unit up	Horizontal shift 1 unit right

(d) $y = \|x+1\|$	(e) $y = \|x-1\| - 1$	(f) $y = \|x+1\| - 1$
		
Horizontal shift 1 unit left	Horizontal shift 1 unit right Vertical shift 1 unit down	Horizontal shift 1 unit left Vertical shift 1 unit down

1.13.2 Scaling (Stretching/Compressing)

Scaling is a non-rigid translation in which the shape and size of the graph of the function is altered. A scale will multiply/divide coordinates and this will change the appearance as well as the location.

Vertical Scaling: A vertical scaling multiplies/divides every y-coordinate by a constant while leaving the x-coordinate unchanged.

Horizontal Scaling: A horizontal scaling multiplies/divides every x-coordinate by a constant while leaving the y-coordinate unchanged.

Note: The vertical and horizontal scaling can be combined into one expression.

In this section, we will discuss the geometric effects on the graph of $y = f(x)$ by multiplying or dividing with a positive constant c to f or to its independent variable x.

The summary of vertical and horizontal scaling is elaborated in the tables 1.2 and 1.3 below.

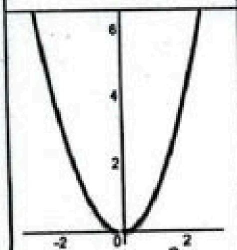
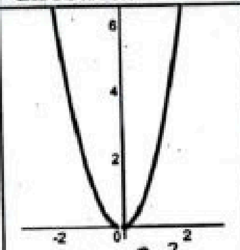
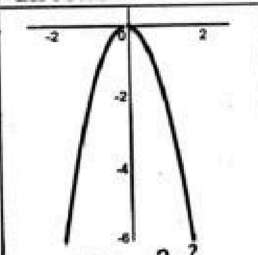
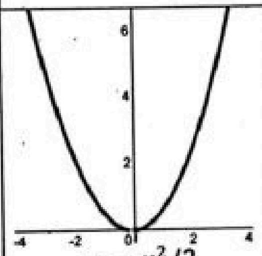
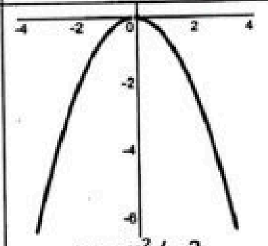
Original function $y = f(x)$	Multiply $f(x)$ by a positive constant c.	Multiply $f(x)$ by a negative constant c.	Divide $f(x)$ by a positive constant c.	Divide $f(x)$ by a negative constant c.
$y = f(x)$	$y = cf(x);$ $c > 0$	$y = cf(x);$ $c < 0$	$y = \frac{f(x)}{c}; c > 0$	$y = \frac{f(x)}{c}; c < 0$
Geometric effects	Figure is compressed by changing y-values by 2 in the same direction.	Figure is compressed by changing y-values by 2 in the opposite direction.	Figure is stretched by changing y-values by 2 in the same direction.	Figure is stretched by changing y-values by 2 in the opposite direction.
 $y = x^2$	 $y = 2x^2$	 $y = -2x^2$	 $y = x^2/2$	 $y = x^2/-2$

Table 1.2

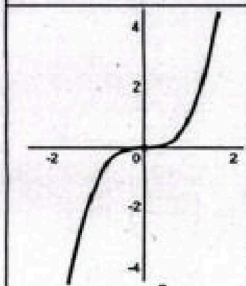
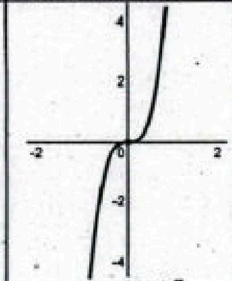
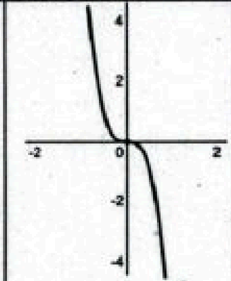
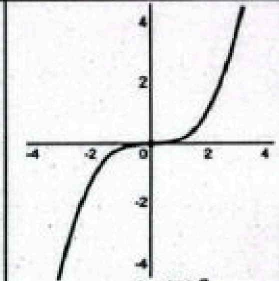
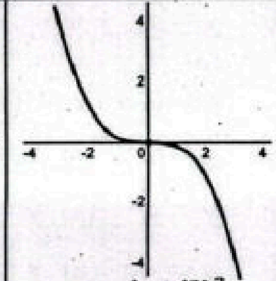
Original function $y = f(x)$	Multiply x by a positive constant c.	Multiply x by a negative constant c.	Divide x by a positive constant c.	Divide x by a negative constant c.
$y = f(x)$	$y = f(cx)$; $c > 0$	$y = f(cx)$; $c < 0$	$y = f(x/c)$; $c > 0$	$y = f(x/c)$; $c < 0$
Geometric effects	Figure is compressed by changing x-values by 2 in the same direction.	Figure is compressed by changing x-values by 2 in the opposite direction.	Figure is stretched by changing x-values by 2 in the same direction.	Figure is stretched by changing x-values by 2 in the opposite direction.
 $y = x^3$	 $y = (2x)^3$	 $y = (-2x)^3$	 $y = (x/2)^3$	 $y = (-x/2)^3$

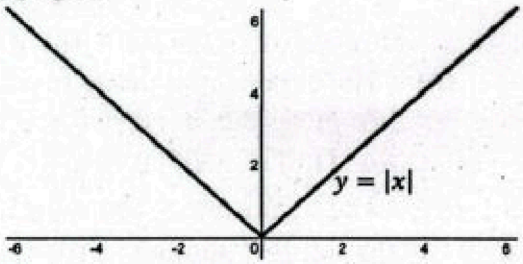
Table 1.3

Example 26: Draw the graph of $y = |x|$ and then sketch the graphs of:
(a) $y = |1.5x|$ and $y = |-1.5x|$ (b) $y = \frac{|x|}{1.5}$ (c) $y = \frac{|x|}{-1.5}$
Which kind of scaling did you observe after sketching the graphs.

Solution:

Table of the values of the function $y = |x|$ is given as:

x	0	± 1	± 2	± 3	± 4	± 5	± 6
$f(x)$	0	1	2	3	4	5	6



The graph is shown in the adjoining figure.
Sketch of other graphs is shown in the table below.

(a) $y = \|1.5x\| = \|-1.5x\|$	(b) $y = \|x\| \div 1.5$ and (c) $y = \|x\| \div (-1.5)$

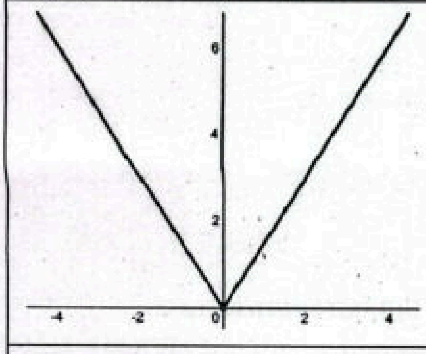
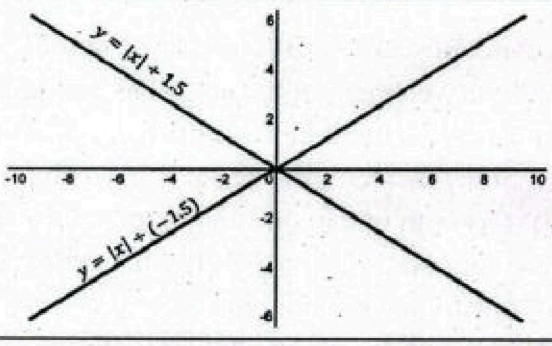

	

Figure is compressed by changing x-values by 1.5 in both cases.	Figure is stretched by changing y-values by 1.5 in both cases but with opposite behavior.

Exercise 1.5

Draw the graphs of the given functions and then sketch the graphs of other functions using translation. Verify the results using graphical calculator.

1. $y = |x|$ (a) $y = |x+2|$ (b) $y = |x-2|$ (c) $y = |x| + 2$ (d) $y = |x| - 2$
2. $y = x^2$ (a) $y = x^2 + 4$ (b) $y = x^2 - 4$ (c) $y = (x-4)^2$ (d) $y = (x+4)^2$
3. $y = \sqrt{x}$ (a) $y = \sqrt{x+3}$ (b) $y = \sqrt{x-3}$ (c) $y = \sqrt{x} + 3$ (d) $y = \sqrt{x} - 3$
4. $y = x$ (a) $y = x + 5$ (b) $y = x - 5$ (c) $y = 5x$ (d) $y = -5x$
5. $y = x^3$ (a) $y = x^3 + 1$ (b) $y = x^3 - 1$ (c) $y = (x-1)^3$ (d) $y = (x+1)^3$
6. $y = x^2 + 4$
   (a) $y = (x^2 + 4) - 3$ (b) $y = (x^2 + 4) + 3$
   (c) $y = (x-3)^2 + 4$ (d) $y = (x+3)^2 + 4$
7. $y = x^2$ (a) $y = 3x^2$ (b) $y = -3x^2$ (c) $y = \frac{x^2}{3}$ (d) $y = -\frac{x^2}{3}$
8. $y = x^2$ (a) $y = (3x)^2$ (b) $y = (-3x)^2$ (c) $y = \left(\frac{x}{3}\right)^2$ (d) $y = \left(-\frac{x}{3}\right)^2$
9. $y = \sqrt{x}$ (a) $y = \sqrt{2x}$ (b) $y = 2\sqrt{x}$ (c) $y = 2\sqrt{x} + 3$ (d) $y = \sqrt{2x+5}$

Review Exercise

1. Tick the correct option in each of the following.

   (i) Which of the following is an example of exponential growth function?
   (a) $f(x) = 3x + 4$ (b) $f(x) = 3^x \times 5$ (c) $f(x) = x^3$ (d) $f(x) = x^2$

   (ii) The exponential decay function is expressed by:
   (a) $f(x) = a.b^x; 0 < b < 1$ (b) $f(x) = a.b^x; b > 1$
   (c) $f(x) = a.b^x; 0 < a < 1$ (d) $f(x) = a.b^x; a > 1$

   (iii) The logarithmic function $f(x) = \log_b x$ is defined for:
   (a) all real numbers (b) $x < 0$ (c) $x > 0$ (d) $x \geq 0$

   (iv) What is the value of $\log_5 125$?
   (a) 25 (b) 5 (c) 4 (d) 3

   (v) A function $f: A \longrightarrow B$ is said to be onto if:
   (a) Every element of the set A has a unique image in the set B.
   (b) Every element in the set B has a preimage in the set A.
   (c) Some elements of the set B have no preimage in the set A.
   (d) f is both one to one and onto.

   (vi) The function $f(x) = x + 1$, where $f: \{1, 2, 3\} \longrightarrow \{2, 3, 4\}$, is:
   (a) one to one but not onto (b) onto but not one to one
   (c) both one to one and onto (d) neither one to one nor onto

(vii) The function $f: R \longrightarrow [0, \infty)$ defined by $f(x) = x^2 + 1$, is:

(a) onto but not on to one
(b) one to one but not onto
(c) neither one to one nor onto
(d) both one to one and onto

(viii) A function $f: \text{A} \longrightarrow \text{B}$ has an inverse if and only if:

(a) f is one to one
(b) f is onto
(c) f is both one to one and onto
(d) f is neither one to one and onto

(ix) The inverse function of $f(x) = x^3$, is:

(a) $f^{-1}(x) = x^{-3}$
(b) $f^{-1}(x) = \sqrt{x^{-3}}$
(c) $f^{-1}(x) = \sqrt[3]{x^3}$
(d) $f^{-1}(x) = \sqrt[3]{x}$

(x) The function $f(x) = \sin x$, where $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$, is:

(a) one to one but not onto
(b) onto but not one to one
(c) both one to one and onto
(d) neither one to one nor onto

(xi) The inverse function of $f(x) = \frac{1}{x}; x \neq 0$, is:

(a) $f^{-1}(x) = 1$
(b) $f^{-1}(x) = -x$
(c) $f^{-1}(x) = x$
(d) $f^{-1}(x) = \frac{1}{x}$

(xii) Scaling refers to:

(a) increasing the size of an object.
(b) decreasing the size of an object.
(c) maintaining the properties while resizing an object.
(d) changing the shape of an object.

(xiii) Which of the following statements is true for the uniform scaling?

(a) both width and height change proportionally.
(b) only the width changes.
(c) only the height changes.
(d) width and height remain unchanged.

(xiv) What is the effect on the graph of $f(x)$ when it is replaced by $f(x + 2)$?

(a) It shifts 2 units to the right.
(b) It shifts 2 units to the left.
(c) It shifts 2 units up.
(d) It shifts 2 unit down.

(xv) The domain of $y = \sin^{-1}(x)$, is:

(a) $[0, \infty)$
(b) $(-\infty, \infty)$
(c) $[-1, 1]$
(d) $[0, 1]$

2. Find the domain of the given functions.

(a) $f(x) = 4 + \sqrt{x + 2}$
(b) $f(x) = x\sqrt{2x - 3}$
(c) $f(x) = \frac{x}{x-2}$
(d) $f(x) = \sqrt{x^2 - 5x + 4}$

3. Find the domain and range of the given functions.

(a) $f(x) = 1 + x^2$
(b) $f(x) = (2x + 1)^2$
(c) $f(x) = 9 - \sqrt{x}$
(d) $f(x) = 3 + \sqrt{4 - x^2}$

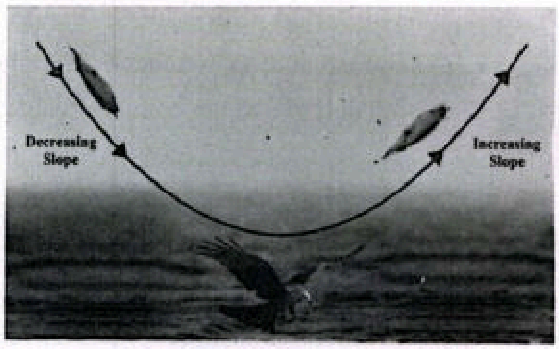
4. Draw the graph of $f(x) = \sqrt{x}$, then sketch the graphs of the following functions.
   (a) $f(x) = \sqrt{x-2}$ (b) $f(x) = \sqrt{x} + 4$ (c) $f(x) = -\sqrt{x}$
   (d) $f(x) = 1 + \sqrt{x-2}$ (e) $f(x) = 4\sqrt{x}$ (f) $f(x) = -\frac{1}{3}\sqrt{x}$
5. Graph the given functions.
   (a) $y = 2 + 2\sin x$ (b) $y = -\frac{1}{2}\tan x$
   (c) $y = 3 - \operatorname{cosec} x$ (d) $y = \cos(x + \pi)$
6. Find the domain and range of the inverse function of $f(x) = \log(x^2 + 1)$.
7. Show that $f(g(x)) = g(f(x)) = x$, when:
   $f(x) = e^x$ and $g(x) = \ln x$.
8. The population of a town grows exponentially according to the formula $P(t) = 1000\, e^{0.05t}$ where t is the time in years. After how many years, will the population reach 5000?
9. A company has the following cost and revenue functions:
   $C(x) = 5x + 10$; where x is the number of units produced.
   $D(x) = 15x$; where x is the number of units sold.
   Find the equilibrium point where cost equals revenue.

LIMIT, CONTINUITY AND DERIVATIVE

After studying this unit, students will be able to:

- Demonstrate and find the limit of a function.
- State and apply theorems on limit of sum, difference, product and quotient of functions to algebraic, exponential and trigonometric functions.
- Demonstrate and test continuity, discontinuity of a function at a point and in an interval.
- Apply concepts of transcendental functions, limit of a function and its continuity to real world problems.
- Calculate inflation over a period. Calculate depreciation with the help of straight-line method.
- Recognize the meaning of the tangent to a curve at a point.
- Calculate the gradient of a curve at a point. Identify the derivative as the limit of a difference quotient. Calculate the derivative of function. Estimate the derivative as rate of change of velocity, temperature and profit. Recognize the derivative function.
- State the connection between derivative and continuity.
- Find the derivative: function, square root, quadratic and logarithmic functions.
- Apply the differentiation rules to polynomials, rational and trigonometric functions.
- Apply the differentiation to state the increasing and decreasing function.
- Apply differentiation to real world problems.
- Find higher order derivatives of algebraic, implicit, parametric, trigonometric, inverse trigonometric functions. Describe the ability to approximate functions.
- Explain differentials to approximate the change in quantity. Calculate errors.
- Find extreme values by applying second derivative test. Explain and find critical point.
- Apply derivative and higher order derivative to real world problems.

The word calculus is a diminutive form of the Latin word calx, which means stone. In ancient civilization, small stones or pebbles were often used as a means for reckoning consequently, the word calculus can refer to any systematic method of computation. However, over the last. several hundred years, a definition of calculus means that the branch of mathematics concerned with the calculation and application of entities known as derivatives and integrals.



2.1 Limits of Functions

Two of the most fundamental concepts in the study of calculus are the notions of function and the limit of the function. In this first section, we shall be especially interested in determining whether the values $f(x)$ of a function f approach a fixed number L as x approaches a number 'a' using the symbol '$\rightarrow$' for the word **'approach'** we ask $f(x) \rightarrow L$ as $x \rightarrow a$.

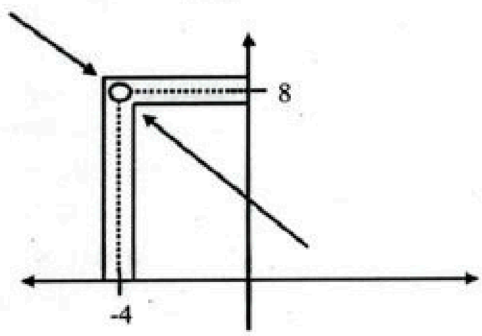
2.1.1 Limit of a Function as x Approaches to a Number

Consider a function: $$f(x)=\frac{16-x^2}{4+x}$$

Whose domain is set of all real numbers except -4. Although $f(-4)$ is not defined, nonetheless, $f(x)$ can be calculated for any value of x near -4. The table shows that, as x approaches to -4 from either the left or right, the functional values $f(x)$ approaches to 8. That is, when x is near -4, $f(x)$ is near 8. We say 8 is the limit of $f(x)$ as x approaches to -4. We can write as:

$$f(x)\rightarrow 8 \text{ as } x\rightarrow -4 \text{ or } \lim_{x\rightarrow -4}\frac{16-x^2}{4+x}=8$$

x	$f(x)$
-4.1	8.1
-4.01	8.01
-4.001	8.001
-3.9	7.9
-3.99	7.99
-3.999	7.999



For $x\neq -4$, f can be simplified by cancellation $f(x)=\frac{16-x^2}{4+x}=\frac{(4+x)(4-x)}{4+x}=4-x$.

The graph of f is essentially the graph of $y=4-x$ with the exception that the graph of f has a hole at the point that corresponds to $x=-4$. As x get closer and closer to -4, represented by the two arrowheads on the x-axis. The two arrowheads on the y-axis simultaneously get closer and closer to the number 8.

Intuitive Definition: If $f(x)$ can be made arbitrarily closer to a finite number by taking x sufficiently close to but different from a number a, from both the left and right side of a, then $\lim_{x\to a} f(x) = L$

$x\rightarrow a^-$ denote that x approaches a from the left and $x\rightarrow a^+$ denote that x approaches a from the right. Thus, if both sides have the common value L,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

We say that: $\lim_{x\to a} f(x)$ exist and write $\lim_{x\to a} f(x) = L$

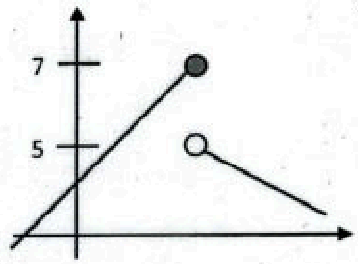
Note: The existence of a limit of a function f at a does not depend on whether f is actually defined for a but only on whether f is defined for near a.

Example 1: Using the graph, check whether the limit of the function exists or not.

$$f(x)=\begin{cases} x+2 & x\le 5 \\ -x+10 & x>5 \end{cases}$$

$\lim_{x\to 5^-} f(x) = x+2 = 7$

$x \to 5^-$	$f(x)$
4.9	6.9
4.99	6.99
4.999	6.999



$\lim_{x\to 5^+} f(x) = -x+10 = -5+10 = 5$

$x \to 5^+$	$f(x)$
5.1	4.9
5.01	4.99
5.001	4.999

Since $\lim_{x\to 5^-} f(x) \neq \lim_{x\to 5^+} f(x)$, we concluded that $\lim_{x\to 5} f(x)$ does not exist.

Example 2: Evaluate. a. $\lim_{x\to 0}\frac{\sin x}{x}$ b. $\lim_{x\to 0}\frac{1-\cos x}{x}$

Solution:

a.

$x \to 0^-$	$\frac{\sin x}{x}$
−0.1	0.998341
−0.01	0.9999833
−0.001	0.999998

a. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$x \to 0^+$	$\frac{\sin x}{x}$
0.1	0.998341
0.01	0.9999833
0.001	0.999998

b.

$x \to 0^-$	$\frac{1-\cos x}{x}$
−0.1	−0.0499583
−0.01	−0.0049999
−0.001	−0.0005001
−0.0001	−0.000510

$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

$x \to 0^+$	$\frac{1-\cos x}{x}$
0.1	0.0499583
0.01	0.0049999
0.001	0.0005001
0.0001	0.000510

Example 3: Evaluate: a. $\lim_{x\to 3} 15$ b. $\lim_{x\to 5} 10x$ c. $\lim_{x\to 5}(x^2-5x+6)$

d. $\lim_{x\to -1}\frac{3x-1}{6x+2}$ e. $\lim_{x\to 1}\frac{x-1}{x^2+x-2}$ f. $\lim_{x\to 2}(3x-2)^6$

Solution:

a. $\lim_{x\to 3} 15 = 15$ b. $\lim_{x\to 5} 10x = 10(5) = 50$

c. $\lim_{x\to 5}(x^2-5x+6)=25-25+6=6$

d. $\lim_{x\to -1}\frac{3x-1}{6x+2}=\frac{3(-1)-1}{6(-1)+2}=\frac{-3-1}{-6+2}=1$

e. $\lim_{x\to 1}\frac{x-1}{x^2+x-2}$, $\lim_{x\to 1}x^2+x-2=0$

By simplifying first we can apply theorem v,

$$=\lim_{x\to 1}\frac{x-1}{x^2+x-2}=\lim_{x\to 1}\frac{(x-1)}{(x-1)(x+2)}$$

$$=\lim_{x\to 1}\frac{1}{(x+2)}=\frac{1}{3}$$

f. $\lim_{x\to 2}(3x-2)^6=(3(2)-2)^6=(4)^6=4096$

Theorems on Limits:

i. If c is constant, then $\lim_{x\to a}c=c$.

ii. If c is constant, then

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

iii. $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$

$=L_1+L_2$

iv. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{L_1}{L_2}, L_2\neq 0$

v. $\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n=L^n$

Example 4: Evaluate: a. $\lim_{x\to 5}\frac{4x+5}{x^2-25}$ b. $\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$

Solution:

a.

$\lim_{x\to 5}\frac{4x+5}{x^2-25}$, $\lim_{x\to 5}4x+5=25,$

$\lim_{x\to 5}x^2-25=0$

$$\lim_{x\to 5}\frac{4x+5}{x^2-25}=\frac{25}{0}$$

We can't simplify to remove zero from the denominator, so limit $x\to 5$ doesn't exist.

b.

$\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$, $\lim_{x\to -8}2x+10=-6\neq 0$

$=\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$ (apply theorem iv)

$$=\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}=\frac{\lim_{x\to -8}x-\sqrt[3]{x}}{\lim_{x\to -8}2x+10}=\frac{-8-(-8)^{1/3}}{2(-8)+10}$$

$$=\frac{-8-((-2)^3)^{1/3}}{-6}=\frac{-8+2}{-6}=1$$

Exercise 2.1

1. Use a graph to find the given limit, if it exists.

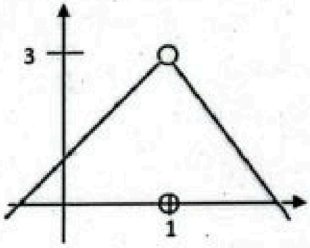
a. $\lim_{x\to 5}\sqrt{x-1}$ b. $\lim_{x\to 1}\frac{x^2-1}{x-1}$ c. $\lim_{x\to 0}\frac{x^2-3x}{x}$ d. $\lim_{x\to 0}\frac{|x|}{x}$

e. $\lim_{x\to 2}f(x)$, where $f(x)=\begin{cases}x & x<2\\ x+1 & x\geq 2\end{cases}$ f. $\lim_{x\to 0}f(x)$, where $f(x)=\begin{cases}x^2 & x<0\\ 2 & x=0\\ \sqrt{x}-1 & x>0\end{cases}$

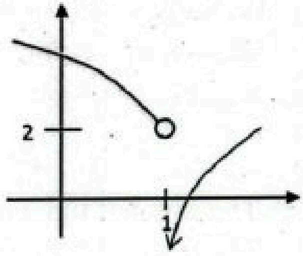
g. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

2. Use the given graph to find each limit ($x \to 1$), if it exists.

a.



b.



Evaluate the following.

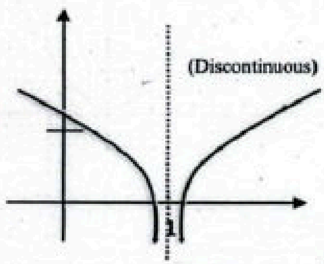
3. $\lim\limits_{x \to -2} \frac{x^3+8}{x+2}$
4. $\lim\limits_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
5. $\lim\limits_{x \to 7} \frac{x^2-21}{x+2}$
6. $\lim\limits_{x \to 0} \frac{x^2-6x}{x^2-7x+6}$
7. $\lim\limits_{y \to 1} \frac{y^3-1}{y-1}$
8. $\lim\limits_{x \to 3^+} \frac{(x+3)^2}{\sqrt{x-3}}$
9. $\lim\limits_{x \to 2} (x-4)^4(x^2-3)^{10}$
10. $\lim\limits_{x \to 0} \left(x - \frac{1}{x-2}\right)$
11. $\lim\limits_{x \to -3} \frac{2x+6}{4x^2-36}$
12. $\lim\limits_{x \to 0} \frac{\tan x}{x}$
13. $\lim\limits_{x \to 0} \frac{x}{\sin 3x}$

2.2 Continuity

In the case of limit, we have used the phrase "connect the points with smooth curve". The phrase provides the concept of graph that is a nice continuous curve that is, a curve with no gaps or breaks. Indeed, a continuous function is often described as one whose graph can be drawn without lifting pencil from paper.

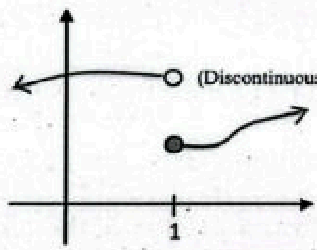
Before moving towards the precise definition of continuity, we demonstrate in figures some intuitive examples of functions that are not continuous or continuous at a number.

Fig (i)



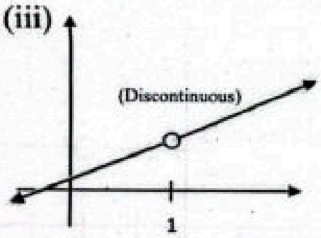
$\lim\limits_{x \to 1} f(x)$ does not exist and $f(1)$ is not defined.

Fig (ii)



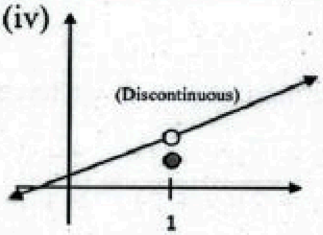
$\lim\limits_{x \to 1} f(x)$ does not exist and $f(1)$ is defined.

Fig (iii)



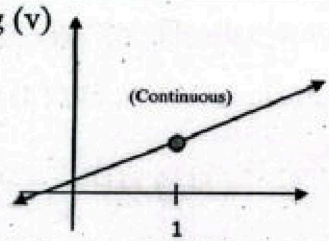
$\lim\limits_{x \to 1} f(x)$ exist but $f(1)$ is not defined.

Fig (iv)



$\lim\limits_{x \to 1} f(x)$ exist and $f(1)$ is defined but $\lim\limits_{x \to 1} f(x) \neq f(1)$

Fig (v)



$\lim\limits_{x \to 1} f(x)$ exist and $f(1)$ is defined and $\lim\limits_{x \to 1} f(x) = f(1)$

2.2.1 Continuity at a Number

Figures (i) - (v), at page 47, suggest the threefold conditions of continuity of a function at a number a (instead of 1 we consider a).

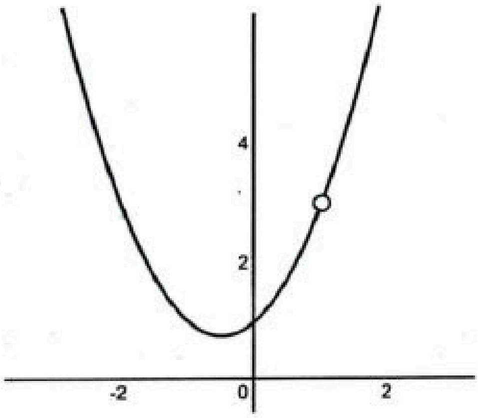
Definition: Continuity

A function is said to be continuous at a number a if

i. $f(a)$ is defined
ii. $\lim\limits_{x\to a} f(x)$ exits, and
iii. $\lim\limits_{x\to a} f(x) = f(a)$

Example 5: The rational function

$$f(x) = \frac{x^3 - 1}{x - 1}$$
$$= \frac{(x-1)(x^2 + x + 1)}{x - 1}$$
$$= x^2 + x + 1, x \neq 1$$

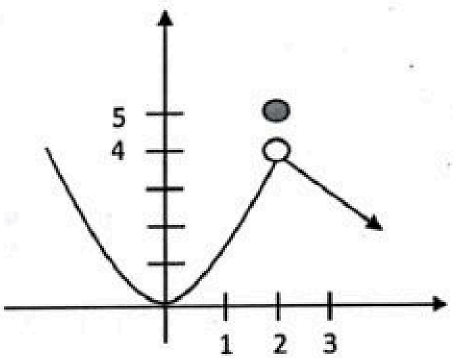


is discontinuous at 1 since $f(1)$ is not define.

From graph, we observe that $\lim\limits_{x\to 1} f(x) = 3$. We can also state that f is continuous at any other number $x \neq 1$.

Example 6: Given figure shows the graph of the piecewise function defined

$$f(x) = \begin{cases} x^2 & x < 2 \\ 5 & x = 2 \\ -x + 6 & x > 2 \end{cases}$$



Now $f(2)$ is defined and is equal to 5. Next, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x + 6 = 4$$

This implies limit exists: $\lim\limits_{x\to 2} f(x) = 4$.

Since $\lim\limits_{x\to 2} f(x) \neq f(2) = 5$, therefore f is discontinuous at 2.

Example 7: Let $f(x) = \frac{x^2+x-6}{x^2-4}$, for $x \neq 2$. Show how to define $f(2)$ in order to make f continuous function at 2.

Solution: Although $f(2)$ is not defined, if $x \neq 2$, we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

The function $f(x) = \frac{x+3}{x+2}$ is equal to $f(x)$ for $x \neq 2$, but is also

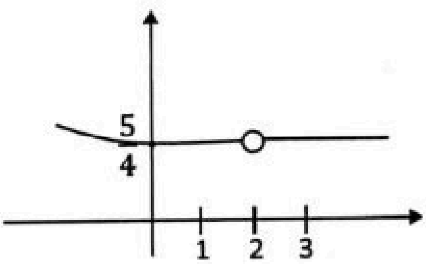


Fig (i)

continuous at $x = 2$ having the value of $\frac{5}{4}$. Thus f is the continuous extension of f to $x = 2$ and

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \lim_{x\to 2}\frac{x+3}{x+2} = \frac{5}{4}$$

The graph of f is shown in figure (i).

The graph of its continuous extension is shown in figure (ii).

$$f(x) = \frac{x+3}{x+2} = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$$

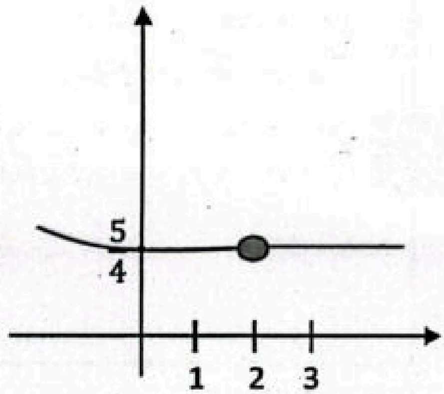


Fig (ii)

We can also observe that $x = 2$ is removable discontinuity for the $f(x) = \frac{x^2+x-6}{x^2-4}$.

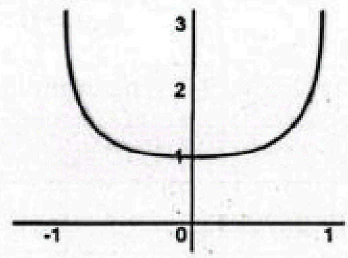
2.2.2 Continuity on an Interval

A function is said to be continuous on an open interval (a, b) if it is continuous at every number in the interval. A function f is continuous on a closed interval $[a, b]$ if it is continuous on $(a, b]$ and in addition, it is continuous on $[a, b)$

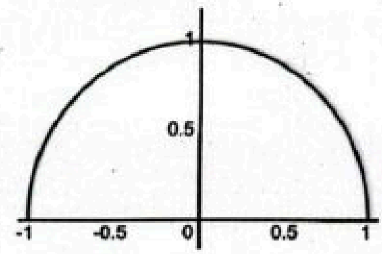
$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to b^-} f(x) = f(b)$$

Example 8:

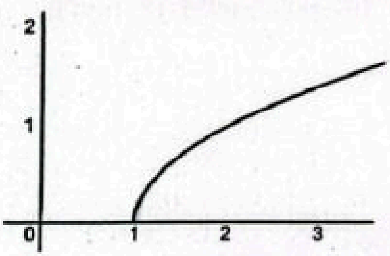
a. $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous on the open interval $(-1, 1)$ but is not continuous on the closed interval $[-1, 1]$, since neither $f(-1)$ nor $f(1)$ is defined.



b. $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$ we can observe from figure that $\lim_{x\to -1^+} f(x) = f(-1) = 0$ and $\lim_{x\to 1^-} f(x) = f(1) = 0$



c. $f(x) = \sqrt{x-1}$ is continuous on $[1, \infty)$ since $\lim_{x\to 1^+} f(x) = f(1) = 0$



Key Facts

Continuity of a Sum, Product and Quotient: If f and g are functions continuous at a number a, then cf(c a constant), $f+g, fg$ and $\frac{f}{g}, (g(a)\neq 0)$ are also continues at a.

Removable Discontinuity: If $\lim_{x\to a} f(x)$ exists but f is either not defined at a or $f(a)\neq \lim_{x\to a} f(x)$, then f is said to have a removable discontinuity at a. For example the function $\frac{x^2-1}{x-1}$ is not defined at 1 but $\lim_{x\to 1} f(x)=2$. By definition $f(1)=2$, the new function

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 2, & x=1\end{cases}$$

is continuous at every number.

Exercise 2.2

In problems 1-8, determine the numbers (if any), at which the given function is discontinuous.

1. $f(x)=x^2-5x+6$
2. $f(x)=\frac{2x}{x^2+5}$
3. $f(x)=\frac{1}{x^2-9x+8}$
4. $f(x)=\frac{x^2-1}{x^4-1}$
5. $f(x)=\frac{x-1}{\sin 2x}$
6. $f(x)=\begin{cases}x, & x<0\\ x^2, & 0\leq x\leq 2\\ x, & x\geq 2\end{cases}$
7. $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ \frac{1}{2}, & x=0\end{cases}$
8. $f(x)=\begin{cases}\frac{x^2-36}{x-6}, & x\neq 6\\ 12, & x=6\end{cases}$

In problems 9-14, determine whether the given function is continuous in the indicated intervals.

9. $f(x)=x^2+1$	a. $[-1, 3]$	b. $[3, \infty)$
10. $f(x)=\frac{1}{x}$	a. $(-3, 3)$	b. $(0, 10]$
11. $f(x)=\frac{1}{\sqrt{x}}$	a. $[1, 4)$	b. $[1, 9]$
12. $f(x)=\sqrt{x^2-9}$	a. $[-3, 3]$	b. $[3, \infty)$
13. $f(x)=\frac{x}{x^3+8}$	a. $[-4, -3]$	b. $[-10, 10]$
14. $f(x)=\sin\frac{1}{x}$	a. $[\frac{1}{\pi}, 5)$	b. $[\frac{\pi}{2}, \frac{3\pi}{2}]$

In problems 15-18, find the values of m and n so that the given function is continuous.

15. $f(x)=\begin{cases}mx, & x<4\\ x^2, & x\geq 4\end{cases}$
16. $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ m, & x=2\end{cases}$
17. $f(x)=\begin{cases}mx, & x<3\\ n, & x=3\\ -2x+9, & x>3\end{cases}$
18. $f(x)=\begin{cases}mx-n, & x<1\\ 5, & x=1\\ 2mx+n, & x>1\end{cases}$

19. Prove that the equation $\frac{x^2+1}{x+3}+\frac{x^4+1}{x-4}=0$ has a solution in the interval $(-3, 4)$.

20. Prove that $f(x)=\begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

is discontinuous at every real number. What does the graph of f look like?

2.3 Rate of Change of Functions

2.3.1 Tangent of a Graph

Suppose $y=f(x)$ is a continuous function. In the figure(i), the graph of f possesses a tangent line L at a point P, and then we would like to find its equation.

To do so we need: (i) the coordinates of P and

(ii) the slope m_{tan} of L.

The coordinates of P pose no difficulty since a point on a graph is obtained by specifying a value of x, say $x=a$ in domain of f. The coordinates of point of tangency are $(a, f(a))$.

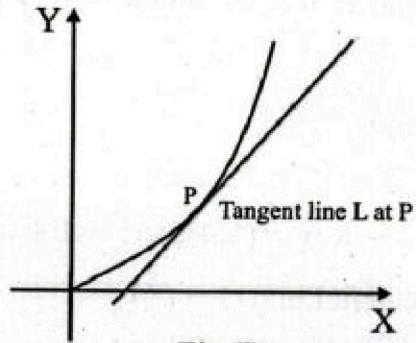


Fig (i)

As a means of approximating the slope m_{tan}, we find the slope of secant lines that pass through the fixed-point P and any other point Q on the graph.

If P has coordinates $(a, f(a))$ and if we let Q have coordinates $(a+\Delta x, f(a+\Delta x))$, then from fig (ii) the slope of the secant line through P and Q is

$$m_{sec}=\frac{\text{change in y-coordinate}}{\text{change in x-coordinate}}$$

$$=\frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}=\frac{\Delta y}{\Delta x}$$

Then, $m_{sec}=\frac{\Delta y}{\Delta x}$

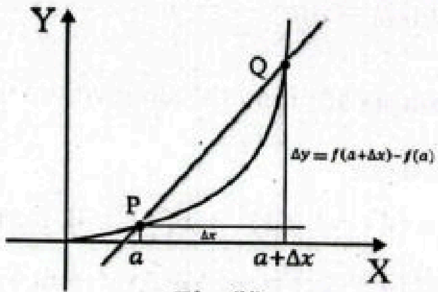


Fig (ii)

When the value of Δx is close to zero either positive or negative, we get points Q and Q' on the graph on each side of P, but close to the point P, we expect that the slopes m_{PQ} and $m_{PQ'}$ are very close to the slope of the tangent line L. See fig (iii)

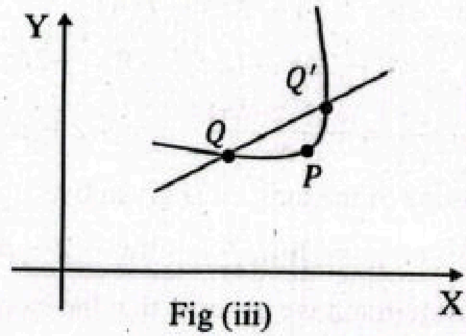


Fig (iii)

Definition: Tangent line

Let $y = f(x)$ be a continuous function. At a point (a, f(a)) the tangent line to the graph is the line that passes through the point with slope.

$$\text{Slope} = m_{tan} = \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

whenever the limit exists.

The slope of the tangent line at $(a, f(a))$ is also called the slope of the curve at the point. The tangent at $(a, f(a))$ is unique since a point and a slope determine a single line.

Example 9: Use definition to find the slope of the tangent line to the graph of $f(x) = x^2$ at $(1, f(1))$.

Solution:

i. $f(1) = 1^2 = 1$ for any $\Delta x \neq 0$

$f(1+\Delta x) = (1+\Delta x)^2 = 1 + 2\Delta x + (\Delta x)^2$

ii. $\Delta y = f(1+\Delta x) - f(1)$

$= 1 + 2\Delta x + (\Delta x)^2 - 1 = 2\Delta x + (\Delta x)^2 = \Delta x(2 + \Delta x)$

iii. $\frac{\Delta y}{\Delta x} = \frac{\Delta x(2+\Delta x)}{\Delta x} = 2 + \Delta x$

Slope of the tangent is given by:

iv. $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2 + \Delta x = 2$

We summarize the definition into 4 steps:

- Evaluate f at a and $a + \Delta x$: $f(a)$ and $f(a + \Delta x)$
- Find Δy
- Divide Δy by Δx, $\Delta x \neq 0$

$$\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

- Compute $\lim_{\Delta x \to 0}$

$$m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Example 10: Find the slope of the tangent line to the graph $f(x) = -x^2 + 6x$ at $(4, f(4))$.

Solution:

i. $f(4) = -(4)^2 + 6(4) = 8$, for any $\Delta x \neq 0$

$f(4+\Delta x) = -(4+\Delta x)^2 + 6(4+\Delta x) = 8 - 2\Delta x - (\Delta x)^2$

ii. $\Delta y = f(4+\Delta x) - f(4)$

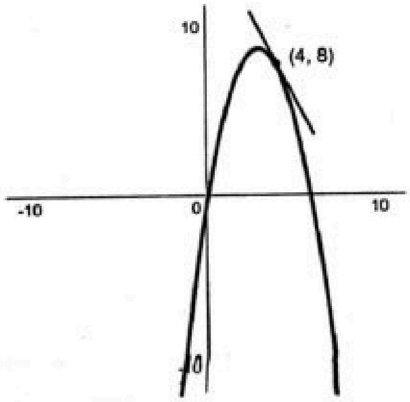
$= 8 - 2\Delta x - (\Delta x)^2 - 8 = -2\Delta x - (\Delta x)^2 = \Delta x(-2 - \Delta x)$

iii. $\frac{\Delta y}{\Delta x} = \frac{\Delta x(-2-\Delta x)}{\Delta x} = -2 - \Delta x$

Slope of the tangent is given by:

iv. $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} -2 - \Delta x = -2$

From graph we observe that the slope of line is -2 at $(4, 8)$.



Key Facts

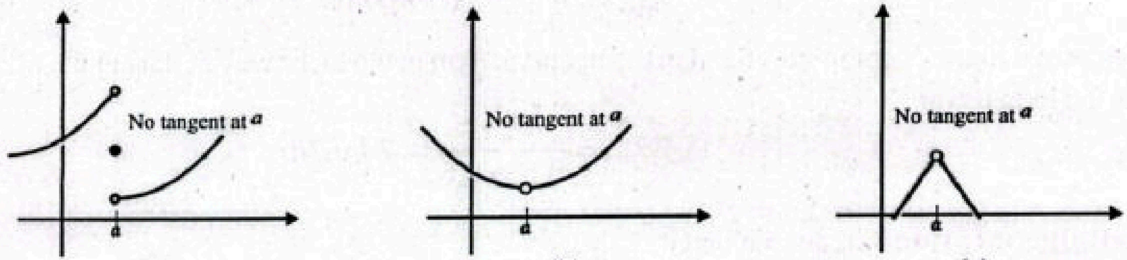
A Tangent May Not Exist: The graph of a function f will not have a tangent line at a point whenever.

i. f is discontinuous at $x = a$, or

ii. The graph of f has corner at $(a, f(a))$.

Moreover, the graph f may not have a tangent line at a point where

iii. The graph has a sharp peak.

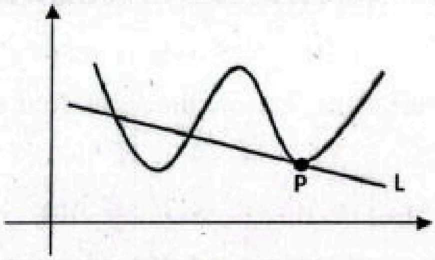


(a) (b) (c)

2.5.2 Rate of Change

The slope $\frac{\Delta y}{\Delta x}$ of a secant through $(a, f(a))$ is also called the average rate of change of f at a. The slope $m_{tan} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is said to be the instantaneous rate of change of the functions at a, if $m_{tan} = \frac{1}{10}$ at a point $(a, f(a))$, we would not expect the values of f to change drastically for x values near a.

Remark: The line L is tangent at P but intersects the graph of f at three points. but is not tangent to the graph.



2.4 Instantaneous Velocity

Almost everyone has an intuitive notion of speed or velocity as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 miles in one hour, the average velocity of the bus must have been 60 mil/hr. Of course, it is difficult to maintain the rate of 60 mil/hr for the entire trip because the bus slow down for towns and speeds up when it passes cars. In other words, the velocity changes with time. If a bus company's schedule demands that the bus travel the 60 miles from one town to another in one hour, the driver knows instinctively that he must compensate for velocities or speeds below 60 mil/hr by travelling at speeds greater than this at other points in journey. Knowing that the average velocity is 60 mil/hr doesn't, however, answer the questions, what is the velocity of the bus at a particular instant?

$$\text{Average velocity:} \qquad V_{ave} = \frac{\text{distance travelled}}{\text{time of travel}}$$

Consider a runner who finishes a 10 km race in an elapsed time of 1 hour and 15 min (1.25 hr). The runner's average velocity or average speed for the race was

$$V_{ave} = \frac{10}{1.25} = 8km/hr$$

But suppose we now wish to determine velocity at the instant the runner is one half hour into the race. If the distance run in the time interval from 0 hr to 0.5 hr is measured to be 5 km, then

$$V_{ave} = \frac{5}{0.5} = 10\ km/hr$$

Suppose if a runner's completes 5 km in 0.5 hr and 5.7 km in 0.6 hr, however, during the time interval from 0.5 hr to 0.6 hr

$$V_{ave} = \frac{5.7-5}{0.6-0.5} = 7\ km/hr$$

Definition: Instantaneous Velocity

Let $s = f(t)$ be a function that gives the position of an object moving in a straight line. The instantaneous velocity at time t_1 is

$$V(t_1) = \lim_{\Delta t \to 0} \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

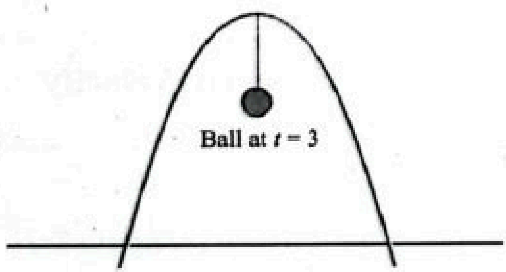
whenever the limit exists.

Example 11: The height s above ground of a ball dropped from the top of the tower is given by $s = -4.9t^2 + 192$ where s is measured in meters and t in seconds. Find the instantaneous velocity of the falling ball at $t_1 = 3\ sec.$

Solution: We use the same four step procedure:

Step 1: $f(3) = -4.9(3)^2 + 192 = 147.9$ for any $\Delta t \neq 0$

$$f(3 + \Delta t) = -4.9(3 + \Delta t)^2 + 192$$
$$= -4.9(\Delta t)^2 - 29.4\Delta t + 147.9$$



Step 2: $\Delta s = f(3 + \Delta t) - f(3)$

$$= [-4.9(\Delta t)^2 - 29.4\Delta t + 147.9] - 147.9$$
$$= \Delta t[-4.9\Delta t - 29.4]$$

Step 3: $\frac{\Delta s}{\Delta t} = \frac{\Delta t(-4.9\Delta t - 29.4)}{\Delta t}$

$$= -4.9\Delta t - 29.4$$

Step 4: $v(3) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0}(-4.9\Delta t - 29.4)$

$$= -29.4\ m/sec$$

The minus sign is significant because the ball is moving opposite to the positive or upward direction. The number $f(3) = 147.9\ m$ is the height of the ball above the ground at 3 seconds.

Exercise 2.3

In problem 1-6, find the slope of the tangent line to the graph of the given function at the indicated point.

1. $f(x) = 2x - 1$; $(x, \ f(x)) = (4, \ 7)$
2. $f(x) = -\frac{1}{2}x + 3$; $(a, \ f(a))$
3. $f(x) = x^2 + 4$; $(-1, \ 5)$
4. $f(x) = x^2 - 5x + 4$; $(2, \ -2)$
5. $f(x) = x^3$; $(1, \ f(1))$
6. $f(x) = \frac{1}{x}$; $\left(\frac{1}{3}, \ f\left(\frac{1}{3}\right)\right)$

In problem 7-8, find the average rate of change of the given function in the indicated interval.

7. $f(x) = x^3 + 2x^2 - 4x$; $[-1, \ 2]$
8. $f(x) = cosx$; $[-\pi, \ \pi]$

In problem 9-10, find the instantaneous velocity of the particle at the indicated time.

9. $f(t) = -4t^2 + 10t + 6$; $t = 3$
10. $f(t) = t^2 + \frac{1}{5t+1}$; $t = 0$
11. The height above ground of a ball dropped from an initial altitude of 122.5 m is given by $s(t) = 122.5 - 4.9t^2$, where s is measured in meters and t in seconds.
    i. What is the instantaneous velocity at $= \frac{1}{2}$?
    ii. At what time does the ball hit the ground?
    iii. What is the impact velocity?
12. The height of a projectile shot from ground level is given by $s(t) = -16t^2 + 256t$, where s is measured in feet and t in seconds.
    i. Determine the height of the projectile at $t = 2, \ t = 6, \ t = 9$ and $t = 10$.
    ii. What is the average velocity of the projectile between $t = 2$ and $t = 5$.
    iii. Show that the average velocity between $t = 7$ and $t = 9$ is zero, also interpret.
    iv. At what time does the projectile hit the ground?
    v. Determine the instantaneous velocity at time $t = 8$.
    vi. What is the maximum height that the projectile attains?

2.5 The Derivative Functions

In this section we will discuss the concept of a "derivative" which is the primary mathematics tool that is used to calculate and study rates of change.

We have studied a slope of tangent line: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

For any x, if the limit exists, then it can be interpreted either on the slope of a tangent line to the curve $y = f(x)$ as $x = x_0$ or as the instantaneous rate of change of y with respect to $x = x_0$. This limit is so important that it has special notations.

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x)}{\Delta x}$$

You can think of f'(read "f prime").

Definition: The Derivative Functions

The function f' defined by the formula: $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x)}{\Delta x}$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit exists.

Example 12: Find the derivative of $f(x) = x^2$, by definition.

Solution: We have: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - (x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Example 13: Find the derivative of

$$y = f(x) = -x^2 + 4x + 1$$

Solution: $\Delta y = f(x + \Delta x) - f(x)$

$$= \Delta x[-2x - \Delta x + 4]$$

Therefore $f'(x) = y' = \frac{\Delta y}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[-2x - \Delta x + 4]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [-2x - \Delta x + 4] = -2x + 4$$

Key Point: Notation

Many ways to denote the derivative of a function

$y = f(x)$

- y' "y prime"
- $\frac{dy}{dx} = \frac{df}{dx} = \frac{df(x)}{dx} = D_x f = y'$

We also read $\frac{dy}{dx}$ as "the derivative of y with respect of x" and $\frac{df}{dx}$ and $\left(\frac{d}{dx}\right) f(x)$ as "the derivative of f with respect of x".

- y' and f' (used by Newton).
- $\frac{d}{dx}$ (used by Leibniz).

Input

- Function $y = f(x)$. operator $\frac{d}{dx}$

Output

- Derivative $y' = \frac{df}{dx}$
- Process is also called differentiation.

Example 14:

a. Find the derivative of

$y = f(x) = \sqrt{x}$, by definition.

b. Find the slope of the tangent at $x = 9$.

Solution:

a. $f(x) = \sqrt{x}$, $f(x + \Delta x) = \sqrt{x + \Delta x}$

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} - \sqrt{x}} \text{(rationalise)} = \frac{1}{2\sqrt{x}}$$

b. The slope of the tangent at $x = 9$ is

$$\frac{dy}{dx}\Big|_{x=9} = \frac{1}{2\sqrt{x}}\Big|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

2.6 Rules of Differentiation

2.6.1 Power and Sum Rules

The definition of derivative has the obvious drawback of being rather clumsy and tiresome to apply. For example, to find the derivative of function like $f(x) = 5x^{100} + x^{\frac{7}{5}}$ is a time taking job. Here, we will develop some important theorems that will enable us to calculate derivatives more efficiently.

Theorem 2.1: Power Rule

If n is a positive integer, then: $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

Proof:

Let $f(x) = x^n, n$ a positive integer. By binomial theorem we can write:

$$f(x + \Delta x) = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-1}(\Delta x)^2 + \cdots + (\Delta x)^n$$

Thus: $\frac{d}{dx}[x^n] = f'(x) = \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-1}(\Delta x)^2 \cdots + (\Delta x)^n] - x^n}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(\Delta x) + \cdots + (\Delta x)^{n-1}]}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(\Delta x) + \cdots + (\Delta x)^{n-1}\right)$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

A power rule simply states that differentiate x^n: $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

Example 15: Find: $\frac{d}{dx}[x^4] = 4x^3$, $\frac{d}{dx}[x^7] = 7x^6$, $\frac{d}{dx}[x^{50}] = 50x^{49}$, $\frac{d}{dx}[x^{200}] = 200x^{199}$,

$$\frac{d}{dx}[x^{31}] = 31x^{30}$$

We can apply this formula for all real numbers like:

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[x^{\frac{4}{5}}\right] = \frac{4}{5}x^{\frac{4}{5}-1} = \frac{4}{5x^{\frac{1}{5}}}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left[\frac{1}{x^{50}}\right] = \frac{d}{dx}x^{-50} = -\frac{50}{x^{51}}$$

$$\frac{d}{dx}\left[10x^{\frac{1}{3}}\right] = 10\frac{d}{dx}x^{\frac{1}{3}} = \frac{10x^{\frac{-2}{3}}}{3} = \frac{10}{3x^{\frac{2}{3}}}$$

Derivative of constant function:

$$\frac{d}{dx}[c] = \frac{d}{dx}[cx^0] = c\frac{d}{dx}[x^0] = c.0x^{0-1} = 0$$

$\frac{d}{dx}[c] = 0$ like $\frac{d}{dx}[10] = 0$

Theorem:
If n is any real number

$$\frac{d}{dx}x^n = nx^{n-1}$$

Theorem:
If c is any real number

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Sum and Difference Rule:
If f and g are differentiable function, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

In words, the derivative of a sum equals to the sum of the derivatives and the derivative of difference is equal to the difference of the derivatives.

Example 16:

i. $\frac{d}{dx}[2x^6 + x^{-9}]$

$$= \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}]$$

$$= 2.\frac{d}{dx}[x^6] + (-9)x^{-9-1}$$

$$= 2(6)x^5 - 9x^{-10}$$

$$= 12x^5 - \frac{9}{x^{10}}$$

ii. $\frac{d}{dx}\left[4x^5 - \frac{1}{2}x^4 + 9x^3 + 7\right]$

$$= 4\frac{d}{dx}[x^5] - \frac{1}{2}\frac{d}{dx}[x^4] + 9\frac{d}{dx}[x^3] + \frac{d}{dx}[7]$$

$$= 4(5)x^4 - \frac{1}{2}(4)x^3 + 9(3)x^2 + 0$$

$$= 20x^4 - 2x^3 + 27x^2$$

iii. $\frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right]$

$$= 5\frac{d}{dx}[x^{-1}] - \frac{1}{5}\frac{d}{dx}[x]$$

$$= -5x^{-2} - \frac{1}{5}x^{1-1} = \frac{-5}{x^2} - \frac{1}{5}$$

Example 17: Find the derivative of the following w.r.t. x.

a. $y = (x+1)^2$ b. $y = (x+1)(x-2)$ c. $y = \frac{x^3+x^2}{3x}$

Solution:

a. $y = (x+1)^2 = x^2 + 2x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}[x^2 + 2x + 1] = \frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + \frac{d}{dx}(1) = 2x + 2 + 0 = 2x + 2$$

b. $y = (x+1)(x-2) = x^2 - x - 2$

$$\frac{dy}{dx} = \frac{d}{dx}x^2 - \frac{d}{dx}x - \frac{d}{dx}2$$

$$= 2x - 1 - 0 = 2x - 1$$

c. $y = \frac{x^3+x^2}{3x} = \frac{x^3}{3x} + \frac{x^2}{3x}$

$$y = \frac{x^2}{3} + \frac{x}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}\frac{d}{dx}x^2 + \frac{1}{3}\frac{d}{dx}x = \frac{1}{3}(2x) + \frac{1}{3} = \frac{2}{3}x + \frac{1}{3}$$

Note: In the different contents of science, engineering and business functions are often expressed in variable other than x and y. Correspondingly, we must adapt the derivative notation to new symbols, for example:

Function	Derivative	Function	Derivative
$V(t) = 4t$	$V'(t) = \frac{dV}{dt} = 4$	$H(z) = \frac{1}{4}z^6$	$H'(z) = \frac{dH}{dz} = \frac{3}{2}z^5$
$A(r) = \pi r^2$	$A'(r) = \frac{dA}{dr} = 2\pi r$	$r(\theta) = 4\theta^3 - 3\theta$	$r'(\theta) = \frac{dr}{d\theta} = 12\theta^2 - 3$

Exercise 2.4

1. Find the derivative of the functions w. r. t. x.

   a. $y = x^9$ b. $f(x) = 4x^{\frac{1}{3}}$ c. $f(x) = 9$ d. $f(x) = 6x^3 + 3x^2 - 10$

2. Determine $f'(x)$.

   a. $f(x) = \sqrt{5}$ b. $f(x) = \sqrt{5}x$ c. $f(x) = 5\sqrt{x}$ d. $f(x) = \sqrt{5x}$

3. Determine $f'(x)$.

   a. $f(x) = x^2(x^3 + 5)$ b. $f(x) = (x+9)(x-9)$ c. $f(x) = (x^2 + x^3)^3$

   d. $f(x) = -3x^{-8} + 2\sqrt{x}$ e. $f(x) = ax^3 + bx^2 + cx + d$, ($a$, b, c and d are constants)

   f. $f(x) = x^{24} + 2x^{\frac{1}{2}} + 3x^8 + 9x^4$

4. Find $\frac{dy}{dx}$.

   a. $y = \frac{x+2x^{\frac{3}{2}}}{\sqrt{x}}$ b. $y = (x^3 - 5)(2x + 3)$ c. $y = (4x^2 - 3)(7x^2 + x)$

5. Find slope of tangent at $x = 1$.

   a. $y = x^2 + 3x$ b. $y = x^4 - x^2$

2.7 The Product and Quotient Rules

We will develop techniques for differentiating products and quotients. If functions whose derivative are known.

2.7.1 Derivative of a Product.

You might be considered conjecture that the derivative of a product of two functions is the product of their derivatives. However, simple examples will show this not possible.

Consider:

$f(x) = x^2$ and $g(x) = x^3$

The product of their derivative is:

$f'(x)g'(x) = (2x)(3x^2) = 6x^3$

But their product is:

$y = f(x)g(x) = x^5$ and $\frac{dy}{dx} = y' = 5x^4 \neq 6x^3$

Thus, the derivative of the product is not equal to the product of their derivative.

Theorem: Product Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

The first function times the derivative of the second function plus the second function times the derivative of first function.

Example 18: Find $\frac{dy}{dx}$ if $y = (4x^2 - 1)(7x^3 + x)$.

Solution: We can use two methods to find $\frac{dy}{dx}$. We can either use the product rule or we can multiply out the factors in y and then differentiate. We provide both methods.

Method I: The Product Rule

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)]$$

$$\frac{dy}{dx} = \overbrace{(4x^2-1)}^{\text{First}}\overbrace{\frac{d}{dx}(7x^3+x)}^{\text{Derivative of second}} + \overbrace{(7x^3+x)}^{\text{Second}}\overbrace{\frac{d}{dx}(4x^2-1)}^{\text{Derivative of first}}$$

$$\frac{dy}{dx} = (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x)$$

$$\frac{dy}{dx} = 140x^4 - 9x^2 - 1$$

Method II: Multiplying First

$$y = (4x^2 - 1)(7x^3 + x) = 28x^5 - 3x^3 - x$$

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

Both derivatives are same.

Example 19: Find $\frac{dy}{dx}$ if $y = [(1+x^3)\sqrt{x}]$

Solution: Apply the product rule $\frac{dy}{dx} = \frac{d}{dx}[(1+x^3)\sqrt{x}]$

$$= (1+x^3)\frac{d}{dx}\sqrt{x} + \sqrt{x}\frac{d}{dx}(1+x^3) = (1+x^3)\frac{1}{2}x^{\frac{1}{2}-1} + \sqrt{x}(3x^2)$$

$$= \frac{(1+x^3)}{2\sqrt{x}} + 3x^{\frac{5}{2}} = \frac{1+x^3+6x^3}{2\sqrt{x}} = \frac{7x^3+1}{2\sqrt{x}}$$

2.7.2 Derivative of a Quotient

Just as the derivative of a product is not generally the product of derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship/method is given by the following.

Theorem: Quotient Rule

If f and g are differentiable functions and $g(x) \neq 0$, then,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The denominator times the derivative of numerator minus the numerator times the derivative of denominator all divided by the denominator square.

Example 20: Differentiate $y = \frac{3x^2-1}{2x^3+5x^2+7}$ w.r.t. x.

Solution: Apply the quotient rule:

$$\frac{dy}{dx} = \frac{\overbrace{(2x^3+5x^2+7)}^{\text{Denominator}}\frac{d}{dx}\overbrace{[3x^2-1]}^{\text{Derivative of numerator}} - \overbrace{(3x^2-1)}^{\text{Numerator}}\frac{d}{dx}\overbrace{[2x^3+5x^2+7]}^{\text{Derivative of Denominator}}}{(2x^3+5x^2+7)^2}$$

$$= \frac{(2x^3+5x^2+7)(6x) - (3x^2-1)(6x^2+10x)}{(2x^3+5x^2+7)^2}$$

$$= \frac{-6x^4+6x^2+52x}{(2x^3+5x^2+7)^2}$$

2.8 The Connection Between Derivatives and Continuity

- If a function is differentiable at a point, it is automatically continuous at that point.
- But the reverse is not always true. A function can be continuous at a point and still not be differentiable (like a sharp corner or cusp, for example | x | is continuous but not differentiable).

Exercise 2.5

Find $\frac{dy}{dx}$ if

1. $y = \frac{1}{x}$
2. $y = (x^2 - 7)(x^2 + 4x + 2)$
3. $y = (7x + 1)(x^4 - x^3 - 9x)$
4. $y = \frac{3x+4}{x^2+1}$ 5. $y = \frac{x-2}{x^4+x+1}$

Summary of Differentiation Rules:

- $\frac{d}{dx}[c] = 0$, $\frac{d}{dx}[cf] = cf'$, $\frac{d}{dx}[f \pm g] = f' \pm g'$
- $\frac{d}{dx}[f.g] = fg' + gf'$
- $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

6. $y = \frac{3x^2+5}{3x-1}$ 7. $y = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27)$ 8. $y = \frac{2-3x}{7-x}$ 9. $y = \frac{x^2-10x+2}{x^3-x}$

10. $y = \frac{x^4+2x^3-1}{x^2}$ 11. $y = \frac{10}{(x^3-10)^9}$ 12. $y = \frac{(x^2+1)^2}{3x-2}$ 13. $y = \frac{(x+1)^2}{(x-1)^2}$

Find the slope of the tangent line to the curve at the point whose abscissa is given.

14. $y = \frac{4x-1}{x}$, $x = -1$ 15. $y = \frac{54}{x^2+1}$, $x = 2$

16. $y = \frac{2x+5}{x+2}$, $x = 1$ 17. $y = (2\sqrt{x} + 1)(x^3 - 6)$, $x = 0$

2.9 Derivations of Trigonometric Functions

The main objective of this section is to obtain formulas for the derivatives of six basic trigonometric functions. We will assume in this section that the variable x in the trigonometric functions $\sin x, \cos x, \tan x, \cot x, \sec x$ and $\operatorname{cosec} x$ is measured in radians. We also need the limits in results and restated as follows:

$$\lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x} = 1 \text{ and } \lim_{\Delta x \to 0} \frac{1 - \cos\Delta x}{\Delta x} = 0$$

We start the problem of differentiating $f(x) = \sin x$. Using the definitions of derivative

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\sin x\left[\frac{\cos\Delta x - 1}{\Delta x}\right] + \cos x\left[\frac{\sin\Delta x}{\Delta x}\right]\right]$$

$$= \sin x \lim_{\Delta x \to 0}\frac{\cos\Delta x - 1}{\Delta x} + \cos x \lim_{\Delta x \to 0}\frac{\sin\Delta x}{\Delta x}$$

$$= \sin x(0) + \cos x(1)$$

$\sin x$ and $\cos x$ independent of Δx

Thus, we have $\boxed{\frac{d}{dx}\sin x = \cos x}$

In a similar manner it can be shown that $\boxed{\frac{d}{dx}\cos x = -\sin x}$

Example 21: Find $\frac{dy}{dx}$ if $y = x\sin x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}[x\sin x]$

$= x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x$ use product rule

$= x\cos x + \sin x(1) = x\cos x + \sin x$

Example 22: Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\sin x}{1+\cos x}\right]$

$$= \frac{1+\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}[1+\cos x]}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)\cos x - \sin x(0-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

The other Trigonometric Functions:

Let $y = \tan x$

$$\frac{dy}{dx} = \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{(\cos x)\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$

Similarly, $\boxed{\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x}$

For $y = \sec x$

$$\frac{dy}{dx} = \frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$

$$= \frac{\cos x\frac{d}{dx}(1) - (1)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x(0) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \sec x\tan x$$

$\boxed{\frac{d}{dx}\sec x = \sec x\tan x}$

Similarly, $\boxed{\frac{d}{dx}\mathrm{cosec}\, x = \mathrm{cosec}\, x\cot x}$

Example 23: Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{x-\cot x}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\cos x}{x-\cot x}\right]$

$$= \frac{(x-\cot x)\frac{d}{dx}\cos x - \cos x\frac{d}{dx}(x-\cot x)}{(x-\cot x)^2}$$

$$= \frac{(x-\cot x)(-\sin x) - \cos x(1-(-\mathrm{cosec}^2 x))}{(x-\cot x)^2}$$

$$= \frac{-x\sin x + \cot x\sin x - \cos x - \cos x\,\mathrm{cosec}^2 x}{(x-\cot x)^2}$$

$$= \frac{-x\sin x + \cos x - \cos x - \cos x\,\mathrm{cosec}^2 x}{(x-\cot x)^2}$$

$$= \frac{-x\sin x - \cos x\,\mathrm{cosec}^2 x}{(x-\cot x)^2}$$

Example 24: Find $\frac{dy}{dx}$ if $y = \sin x(2 + \sec x)$

Solution: $\frac{dy}{dx} = \frac{d}{dx}[\sin x(2 + \sec x)]$

$$= \sin x\frac{d}{dx}(2 + \sec x) + (2 + \sec x)\frac{d}{dx}(\sin x) = \sin x(0 + \sec x\tan x)) + (2 + \sec x)(\cos x)$$

$$= \sin x\sec x\tan x + 2\cos x + \sec x\cos x = \sin x\frac{1}{\cos x}\tan x + 2\cos x + \sec x\cos x$$

$$= \tan^2 x + 2\cos x + 1 = \tan^2 x + 1 + 2\cos x$$

$$= \sec^2 x + 2\cos x \qquad (1 + \tan^2 x = \sec^2 x)$$

2.10 Derivatives of Inverse Trigonometric Functions

The derivative of an inverse trigonometric function can be obtained. Research reveals that the inverse tangent and inverse cotangent are differentiable for all x. However the remaining four inverse trigonometric functions are not differentiable at either $x = -1$ or $x = 1$

Inverse sine function:

For $-1 < x < 1$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \sin^{-1}x$ if and only if $x = \sin y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\sin y$$

$$1 = \cos y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

Similarly, $\boxed{\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1}$

Inverse tangent function:

For $-\alpha < x < \alpha$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \tan^{-1}x$ if and only if $x = \tan y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\tan y$$

$$1 = \sec^2 y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \text{ for } x \in R}$

Similarly, $\boxed{\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}, \text{ for } x \in R}$

Inverse secant function:

For $|x| > 1$ and $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$

$y = \sec^{-1}x$ if and only if $x = \sec y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\sec y$$

$$1 = \sec y\tan y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y\tan y}$$

$\therefore 1 + \tan^2 y = \sec^2 y$

$\tan^2 y = \sec^2 y - 1$

$\tan y = \sqrt{\sec^2 y - 1}$

$$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}\ , \text{ for } |x| > 1$$

Similarly, $\frac{d}{dx}\text{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$, for $|x| > 1$

Example 25:

Differentiate $y = \sin^{-1}5x$ w.r.t. x.

Solution: $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}5x$

$$= \frac{1}{\sqrt{1-(5x)^2}} \cdot \frac{d}{dx}5x$$

$$\frac{d}{dx}\sin^{-1}5x = \frac{5}{\sqrt{1-25x^2}}$$

Example 26: Differentiate $y = \tan^{-1}12x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}12x$

$$= \frac{1}{1+(12x)^2} \cdot \frac{d}{dx}12x$$

$$\frac{d}{dx}\tan^{-1}12x = \frac{12}{1+144x^2}$$

Example 27: Differentiate $y = \sec^{-1}x^2$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\sec^{-1}x^2$

$$= \frac{1}{x^2\sqrt{(x^2)^2-1}} \cdot \frac{d}{dx}x^2$$

$$= \frac{1}{x^2\sqrt{(x^2)^2-1}} \cdot (2x)$$

$$\frac{d}{dx}\sec t x^2 = \frac{2x}{x^2\sqrt{x^4-1}}$$

Exercise 2.6

Find the derivative of the given functions w.r.t. x.

1. $y = x^2 - \cos x$
2. $y = 4x^3 + x + \sin x$
3. $y = 3\cos x - 5\cot x$
4. $y = \sin x\cos x$
5. $y = (x^2 + \sin x)\sec x$
6. $y = \frac{5-\cos x}{5+\sin x}$
7. $y = \frac{\sec x}{1+\tan x}$
8. $y = \frac{\sin x}{x^2+\sin x}$
9. $y = \frac{\cot x}{x+1}$
10. $y = (1 + \cos x)(x - \sin x)$

Find the derivative of the given functions w.r.t. x.

11. $y = \sin^{-1}(5x - 1)$
12. $y = 4\cot^{-1}\frac{x}{2}$
13. $y = \frac{\sin^{-1}x}{\sin x}$
14. $y = \frac{\sec^{-1}x}{x}$
15. $y = x\sin^{-1}x + x\cos^{-1}x$
16. $y = \frac{1}{\tan^{-1}x^2}$

2.11 Product Rule

In this section, we will derive a formula that expresses the derivative of a composition fog in terms of the derivative of f and g. This formula will enable us to differentiate complicated functions.

Suppose we wish to differentiate:

$$y = (x^5+1)^2 \quad ...... \quad \text{(i)}$$

We can write $y = (x^5+1)(x^5+1)$

$$\frac{dy}{dx} = (x^5+1)\frac{d}{dx}(x^5+1) + (x^5+1)\frac{d}{dx}(x^5+1)$$

$$= (x^5+1)(5x^4) + (x^5+1)(5x^4)$$

$$= 2(x^5+1)(5x^4) ...... \quad \text{(ii)}$$

2.11.1 Power Rule for Functions

From (i), $y = (x^5+1)^2$

$$\frac{dy}{dx} = 2(x^5+1)^{2-1}\frac{d}{dx}(x^5+1)$$

$$= 2(x^5+1)(5x^4) ...... \quad \text{(iii)}$$

Theorem: Power Rule for Functions

If n is an integer and g is a differentiable function then,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

From (ii) and (iii), both expressions are same.

Example 28: Differentiate w.r.t. x.

a. $y = (2x^3+4x+1)^4$

b. $y = \frac{1}{(7x^5-x^4+2)^{10}}$

Solution:

a. $\frac{dy}{dx} = \frac{d}{dx}(2x^3+4x+1)^4$

$$= 4(2x^3+4x+1)^{4-1}\frac{d}{dx}(2x^3+4x+1)$$

$$= 4(2x^3+4x+1)^3(6x^2+4)$$

b. $y = (7x^5-x^4+2)^{-10}$

$$\frac{dy}{dx} = \frac{d}{dx}(7x^5-x^4+2)^{-10}$$

$$= -10(7x^5-x^4+2)^{-10-1}\frac{d}{dx}(7x^5-x^4+2)$$

$$= -10(7x^5-x^4+2)^{-11}(35x^4-4x^3)$$

Example 29: Differentiate $y = \frac{(x^2-1)^3}{(5x+1)^8}$ w.r.t. x.

Solution: $\frac{dy}{dx} = \frac{d}{dx}\frac{(x^2-1)^3}{(5x+1)^8}$

$$= \frac{(5x+1)^8\frac{d}{dx}(x^2-1)^3 - (x^2-1)^3\frac{d}{dx}(5x+1)^8}{[(5x+1)^8]^2}$$

$$= \frac{(5x+1)^8 3(x^2-1)^2(2x) - (x^2-1)^3 8(5x+1)^7(5)}{(5x+1)^{16}}$$

$$= \frac{6x(5x+1)^8(x^2-1)^2 - 40(x^2-1)^3(5x+1)^7}{(5x+1)^{16}}$$

$$= \frac{(x^2-1)^2(5x+1)^7[6x(5x+1) - 40(x^2-1)]}{(5x+1)^{16}}$$

$$= \frac{(x^2-1)^2[-10x^2+6x+40]}{(5x+1)^9}$$

2.11.2 Chain Rule: A power of a function can be written as a composite function. If $f(x) = x^n$ and $u = g(x)$, then $f(x) = f(g(x)) = [g(x)]^n$ is a special case of the chain rule for differentiating composite function.

> **Theorem: Chain Rule**
> If $y = f(x)$ is a differentiable formula of u and $u = g(x)$ is a differentiable function, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f'(g(x)).g'(x)$

Example 30: Differentiate w.r.t. x.

a. $y = \tan^2 x$ b. $y = (9x^3 + 1)^2 \sin 5x$

Solution:

a. $y = \tan^2 x$

$$\frac{dy}{dx} = 2\tan^{2-1} x \frac{d}{dx}\tan x$$

$$= 2\tan x \sec^2 x$$

b. $y = (9x^3 + 1)^2 \sin 5x$

$$\frac{dy}{dx} = (9x^3 + 1)^2 \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}(9x^3 + 1)^2$$

$$= (9x^3 + 1)^2 \cos 5x(5) + \sin 5x.\,2(9x^3 + 1)27x^2$$

$$= (9x^3 + 1)[45x^3 \cos 5x + 5\cos 5x + 54x^2 \sin 5x]$$

2.12 Implicit Differentiation

2.12.1 Explicit and Implicit Functions

A function in which the dependent variable is expressed solely in terms of the independent variable x, namely $y=f(x)$ is said to be an explicit function, for example, $y = \frac{1}{4}x^3 - 1$ is an explicit function, whereas an equivalent equation $3y - x^3 - 4 = 0$ is said to define the function implicitly or y is an implicit of x.

2.12.2 Explicit Differentiation

To illustrate this, let us consider the simple equation:

$$xy = 1 \quad \ldots\ldots \quad \text{(i)}$$

One way to find $\frac{dy}{dx}$ is to rewrite this equation as:

$$y = \frac{1}{x}$$

From which it follows that: $\frac{dy}{dx} = -\frac{1}{x^2}$ ... (ii)

Another way to obtain this derivative is to differentiate both sides of (i) before solving for y in terms of x.

From (i) $\frac{d}{dx}(xy) = \frac{d}{dx}1$

$$x\frac{d(y)}{dx}+y\frac{d(x)}{dx}=0$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

If we take, $y=\frac{1}{x}$, we get

$$\frac{dy}{dx}=-\frac{1}{x^2}$$

This method of obtaining derivatives is called implicit differentiation.

Example 31: Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2+\sin y=x^2$

Solution: $\frac{d}{dx}[5y^2+\sin y]=\frac{d}{dx}[x^2]$

$$5\frac{d}{dx}y^2+\frac{d}{dx}\sin y=2x$$

$$5\left(2y\frac{dy}{dx}\right)+\cos y\frac{dy}{dx}=2x$$

$$(10y+\cos y)\frac{dy}{dx}=2x$$

Solving for $\frac{dy}{dx}$ we obtain: $\frac{dy}{dx}=\frac{2x}{10y+\cos y}$

2.13 Derivative of Exponential Functions

The derivative of exponential is: $\boxed{\frac{d}{dx}e^x=e^x}$ like $\frac{d}{dx}e^{3x}=e^{3x}.3$

Example 32: Differentiate $y=x^2e^{5x}$ w.r.t. x.

Solution: $\frac{dy}{dx}=\frac{d}{dx}[x^2e^{5x}]$

$$=x^2\frac{d}{dx}e^{5x}+e^{5x}\frac{d}{dx}x^2 \quad =x^2e^{5x}.5+e^{5x}.2x \quad =5x^2e^{5x}+2xe^{5x} \quad =xe^{5x}(5x+2)$$

2.14 Derivative of Logarithmic Functions

We find the derivative of common logarithmic which is continuous functions.

$\boxed{\frac{d}{dx}\ln x=\frac{1}{x}}$ like $\frac{d}{dx}\ln(x^3+1)=\frac{1}{x^3+1}\frac{d}{dx}(x^3+1)=\frac{3x^2}{x^3+1}$.

Example 33: Differentiate $\ln(4x^3+2x^2+9)$ w.r.t. x.

Solution: $y=\ln(4x^3+2x^2+9)$

$$\frac{dy}{dx}=\frac{1}{4x^3+2x^2+9}\frac{d}{dx}(4x^3+2x^2+9)=\frac{1}{4x^3+2x^2+9}(12x^2+4x)=\frac{4x(3x+1)}{4x^3+2x^2+9}$$

Derivative of $y=a^x$: $\frac{d}{dx}a^x=a^x.\frac{1}{\ln a}$

We will apply the chain rule to find the derivative of parametric equations.

Example 34: Differentiate $y=4^{3x^2+5}$ w.r.t. x.

Solution: Taking $\ln$ both sides

$$\ln y=\ln 4^{3x^2+5}$$

$$\ln y=(3x^2+5).\ln 4$$

$$\frac{1}{y}\frac{dy}{dx}=\ln 4.\frac{d}{dx}(3x^2+5),\ \frac{dy}{dx}=y\ln 4\,(6x)=\ln 4\left(4^{3x^2+5}\right)6x=6\ln 4\left(4^{3x^2+5}\right)x$$

Example 35: Find $\frac{dy}{dx}$ if $x = \tan t,\ y = 4t^3 + 1$

Solution: $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{dy}{dt} = \frac{d}{dt}(4t^3 + 1) = 12t^2$$

$$\frac{dx}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t$$

$$\frac{dy}{dx} = \frac{12t^2}{\sec^2 t}$$

Example 36: Find $\frac{dy}{dx}$ if $x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$

Solution: $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$= \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$= \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}$$

$$= \frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}$$

$$= \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{(t^2-1)}{2t}$$

2.15 Differentials

We have already discussed the derivative of finding slope of a tangent line to the graph of a function $y = f(x)$.

$$m_{\text{sec}} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

For small values of Δx ,

$$m_{sec} \cong m_{tan} \text{ or } \frac{\Delta y}{\Delta x} = m_{tan} = f'(x)$$

We have: $\frac{\Delta y}{\Delta x} = f'(x)$

$$\Delta y = f'(x)\Delta x$$

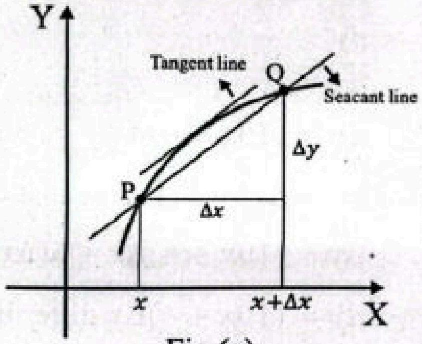


Fig (a)

Fig (a): Geometric representation of the derivative at a point on the curve.

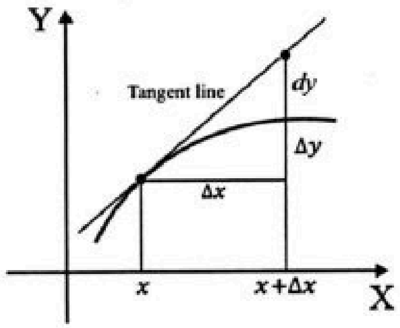
Definition: The increment Δx is called the differential of the independent variable x and is denoted by dx. i.e.

The function $f'(x)\Delta x$ is called differential of the dependent variable y and is denoted by dy.

i.e. $dy = f'(x)\Delta x = f'(x)dx$

Since the slope of a tangent to graph is

$$m_{\tan} = \frac{rise}{run} = f'(x) = \frac{f'(x)\Delta x}{\Delta x}, \Delta x \neq 0$$



It follows that the rise of the tangent line can be interrupted in dy

$\Delta y \cong dy$

Example 37: **a)** Find Δy and dy for $y = 5x^2 + 4x + 1$

b) Compare the values of Δy and dy for $x = 6, \Delta x = dx = 0.02$

Solution:

a) $\Delta y = f(x + \Delta x) - f(x)$

$= [5(x+\Delta x)^2 + 4(x+\Delta x) + 1] - [5x^2 + 4x + 1]$

$= 10x\Delta x + 4\Delta x + 5(\Delta x)^2$

$$\frac{\Delta y}{\Delta x} = \frac{10x\Delta x + 4\Delta x + 5(\Delta x)^2}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{(10x + 4 + 5\Delta x)\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = 10x + 4$$

$$\frac{dy}{dx} = 10x + 4$$

$$dy = (10x + 4)dx$$

Since $dx = \Delta x$. We observe that

$\Delta y = (10x + 4)\Delta x + 5(\Delta x)^2$ and

$dy = (10x + 4)\Delta x$ differ by the amount $5(\Delta x)^2$.

b) When $x = 6, \Delta x = 0.02$

$\Delta y = 10(6)(0.02) + 4(0.02) + 5(0.02)^2$

$= 1.282$

Whereas $dy = (10(6) + 4)(0.02) = 1.28$

$\Delta y \cong dy$

$1.282 \cong 1.28$

The difference in answers is, of course

$5(0.02)^2 = 0.002$

2.16 Approximations

When $\Delta x = 0$, differentials give a means of "predicting" the value of $f(x + \Delta x)$ by knowing the value of the function and its derivative at x. From fig if x is changes by an amount Δx, then the corresponding change in the function is

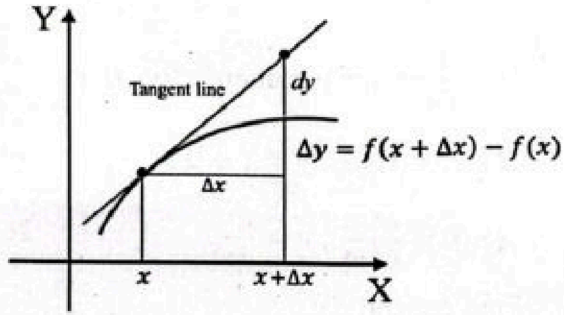
$\Delta y = f(x+\Delta x) - f(x)$ and so

$f(x+\Delta x) = f(x) + \Delta y$

For small change in x, take $\Delta y = dy$

$f(x+\Delta x) = f(x) + dy$

$= f(x) + f'(x)dx$



Example 38: Find the approximated value of $\sqrt{25.4}$.

Solution: First, identify the function $f(x) = \sqrt{x}$

We wish to calculate the approximated value of $f(x+\Delta x) = \sqrt{x+\Delta x}$ when $x = 25$ and $\Delta x = 0.4$; $(25.4 = 25 + 0.4)$

Now, $dy = \frac{1}{2}x^{\frac{-1}{2}}dx = \frac{1}{2\sqrt{x}}\Delta x$

We have; $f(x+\Delta x) = f(x) + dy$

$$= f(x) + \frac{1}{2\sqrt{x}}\Delta x = \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x = \sqrt{25} + \frac{1}{2\sqrt{25}}(0.4) = 5.04$$

Exercise 2.7

Find the derivative of functions w.r.t. variable involved.

1. $y = \left(x - \frac{1}{x^2}\right)^5$
2. $f(x) = \left(\frac{x^2-1}{x^2+1}\right)^2$
3. $y = (3x-1)^4(-2x+9)^5$
4. $f(\theta) = (2\theta+1)^3\tan^2\theta$
5. $y = \sin 2x \cos 3x$
6. $f(x) = (\sec 4x + \tan 2x)^5$
7. $h(t) = \frac{t + \sin 4t}{10 + \cos 3t}$
8. $f(x) = \tan\left(\cos\frac{x}{2}\right)$

Use implicit differentiation to find $\frac{dy}{dx}$.

9. $4x^2 + y^2 = 8$
10. $x + xy - y^2 - 20 = 0$
11. $y^4 - y^2 = 10x - 3$
12. $x^3y^2 = 2x^2 + y^2$
13. $xy = \sin x + y$
14. $x + y = \cos xy$
15. $x\sin y - y\cos x = 1$
16. $\sin y = y\cos 2x$

Find $\frac{dy}{dx}$.

17. $y = x^3e^{5x}$
18. $y = e^{4x}(1 + \ln x)$
19. $y = \frac{e^{2x}}{e^{-2x}+1}$
20. $y = \ln(e^x + e^{-x})$
21. $y = \ln\left(x + \sqrt{x^2+1}\right)$
22. $y = e^{-3x}\cos x$

Find $\frac{dy}{dx}$ of the parametric functions.

23. $x = t + \frac{1}{t}, y = t + 1$
24. $x = t^2 + \frac{1}{t^2}, y = t - \frac{1}{t}$
25. $x = \frac{\theta^2 - 1}{\theta^2 + 1}, y = \frac{\theta - 1}{\theta + 1}$
26. $x = \sin 2\theta, y = \cos 4\theta$

Find Δy and dy.

27. $y = x^2 + 1$
28. $y = \sin x$

Use the concept of the differential to find the approximated value of the given expressions.

29. $(1.8)^5$
30. $\sqrt{37}$
31. $\sin 31°$
32. $\tan\left(\frac{\pi}{4} + 0.1\right)$

2.17 Higher Order Derivatives

2.17.1 The Second Derivative

The derivative $f'(x)$ is a function derived from a function $y = f(x)$. By differentiating the first derivative $f'(x)$, we obtain another function called the second derivative, which is denoted by $f''(x)$. In terms of the operation symbol $\frac{d}{dx}$ we define the second derivative with respect to x as the function obtained by differentiating $y = f(x)$ twice is successive.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The second derivative is commonly denoted by

$$f''(x), \quad y'', \quad \frac{d^2y}{dx^2}, \quad D^2y$$

Normally, we shall use one of the first three symbols.

Example 39: Find the second derivative of $y = x^3 - 2x^2$ w.r.t. x.

Solution: The first derivative is: $\frac{dy}{dx} = 3x^2 - 4x$

The second derivative follows from differentiating the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x) = 6x - 4$$

Example 40: Find the second derivative:

a. $\sin 3x$ b. $(x^3+1)^4$ c. e^{2x}

Solution:

a. The first derivative is: $y' = \frac{dy}{dx} = \frac{d}{dx}(\sin 3x) = 3\cos 3x$

The second derivative is: $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(3\cos 3x) = -9\sin 3x$

b. The first derivative is:

$$y' = \frac{dy}{dx} = \frac{d}{dx}(x^3+1)^4 = 4(x^3+1)^3\frac{d}{dx}x^3 = 12x^2(x^3+1)^3$$

To find the second derivative, we will use product and power rule

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}[12x^2(x^3+1)^3] = 12\left[x^2\frac{d}{dx}(x^3+1)^3 + (x^3+1)^3\frac{d}{dx}x^2\right]$$

$$= 12[x^2 3(x^3+1)^2 3(x^2) + (x^3+1)^3(2x)] = 12x(x^3+1)^2[11x^3+2]$$

c. The first derivative is: $y' = \frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = 2e^{2x}$

The second derivative is: $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(2e^{2x}) = 4e^{2x}$

2.18 Higher Derivatives

Assuming all derivatives exist, we can differentiate a function $y = f(x)$ as many times as we want. The third derivative is the derivative of the second derivative. The fourth derivative is the derivative of the third derivative and so on. We denote the third and fourth derivative, by $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$, respectively and define them by:

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

In general, if n is a positive integer, then the nth derivative is denoted by:

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

Other notations for the first n derivatives are:

$$f'(x), f''(x), f'''(x), f^{(4)}(x), \dots\dots\dots f^{(n)}(x)$$

$$y', \; y'', \; y''', \; y^{(4)}, \dots\dots\dots \; y^{(n)}$$

$$D_x y, \; D^2{}_x y, \; D^3{}_x y, \; D^4{}_x y, \dots\dots\dots D^n{}_x y$$

Example 41: Find the first five derivatives of $f(x) = 2x^4 - 6x^3 + 7x^2 + 5x - 10$ w.r.t. x.

Solution: We have

$f'(x) = 8x^3 - 18x^2 + 14x + 5$

$f''(x) = 24x^2 - 36x + 14$

$f'''(x) = 48x - 36$

$f^{(4)}(x) = 48$

$f^{(5)}(x) = 0$

Example 42: Find the third derivatives of $y = \frac{1}{x^3}$

Solution: We have $y = \frac{1}{x^3} = x^{-3}$

$$\frac{dy}{dx} = -3x^{-4}$$

$$\frac{d^2y}{dx^2} = (-3)(-4)x^{-5} = 12x^{-5}$$

$$\frac{d^3y}{dx^3} = (12)(-5)x^{-6} = -60x^{-6} = \frac{-60}{x^6}$$

Exercise 2.8

Find the second derivative of the functions w.r.t. the variable involved.

1. $y = -x^3 + 6x + 9$
2. $f(x) = 30x^2 - x^3$
3. $f(x) = (-5x + 9)^2$
4. $y = 2x^6 + 5x^3 - 6x^2$
5. $y = 20x-3$
6. $y = \frac{2}{x^4}$
7. $f(x) = x^2(3x - 4)^3$
8. $f(x) = (x^2 + 5x - 1)^4$
9. $f(x) = \cos 10x$
10. $f(x) = \tan\frac{x}{2}$
11. $f(\theta) = \sin^2 5\theta$
12. $f(\theta) = \frac{1}{3+2\cos\theta}$
13. $f(x) = e^{2x}(x^2 + 1)$
14. $f(x) = (x^2 + 1)\ln(x^2 + 1)$

Find the indicated derivative.

15. $y = 4x^7 + x^6 - x^4;\ \frac{d^4y}{dx^4}$
16. $y = \frac{2}{x};\ \frac{d^5y}{dx^5}$
17. $f(x) = \cos\pi x;\ f'''(x)$
18. $f(x) = \frac{1}{\sec(2x+1)};\ f^{(4)}(x)$
19. Let $f(x) = x^3 + 2x$
    a. Find $f'(x)$ and $f''(x)$
    b. In general; $f''(x) = \lim\limits_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$

    provided limit exists. Use $f''(x)$ obtained in part (a) and use definition to find $f''(x)$.
20. Show that $\frac{d^2}{dx^2}(fg) = f''g + 2f'g' + fg''$

    $\frac{d^3}{dx^3}(fg) = f'''g + 3f''g' + 3f'g'' + fg'''$

2.19 Extrema of Functions

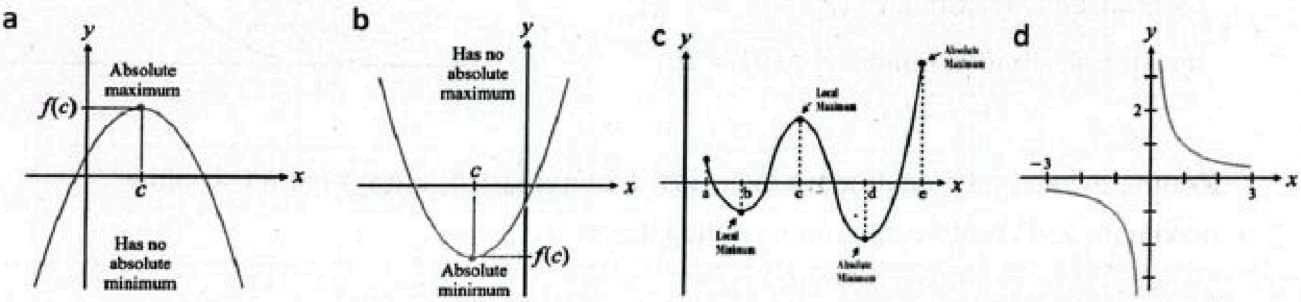
Suppose a function f is defined on an interval I. The maximum and minimum values of f on I (if exist) are said to be extrema of the functions. We have two kinds of extrema.

Definition: Absolute Extrema

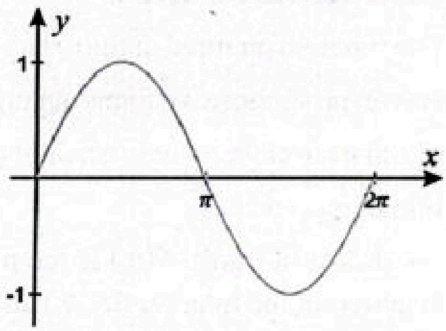
i. A number $f(c)$ is an **Absolute Maximum** of a function f if $f(x) \le f(c)$ for every x in the domain of f.

ii. A number $f(c)$ is an **Absolute Minimum** of a function f if $f(x) \ge f(c)$ for every x in the domain of f.

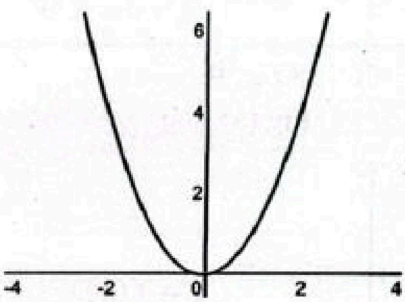
Absolute extrema are called global extrema. Figure shows several possibilities:



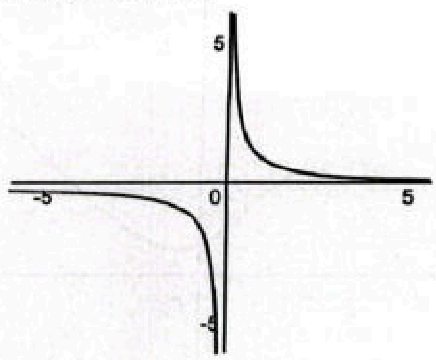
Example 43:



i. For $f(x) = \sin x$, $f\left(\frac{\pi}{2}\right) = 1$ is its absolute maximum and $f\left(\frac{3\pi}{2}\right) = -1$ is its absolute minimum.



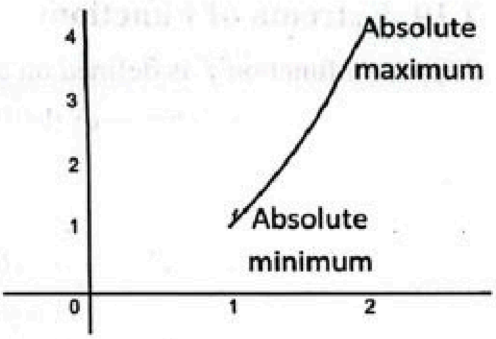
ii. The function $f(x) = x^2$ has the absolute minimum $f(0) = 0$ but has no absolute maximum.



iii. $f(x) = \frac{1}{x}$ has neither an absolute maximum nor an absolute minimum.

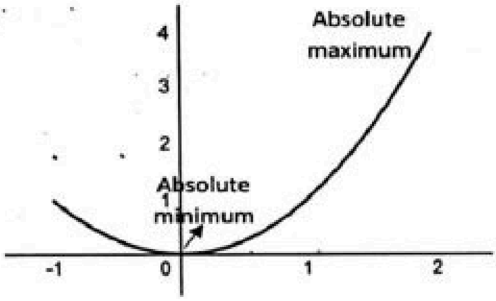
Example 44:

i. $f(x) = x^2$ defined only on the closed interval at $[1, 2]$ has the absolute maximum $f(2) = 4$ and the absolute minimum $f(1) = 1$



ii. On the other hand, if $f(x) = x^2$ is defined on the interval $(1, 2)$. f has no absolute extrema.

iii. $f(x) = x^2$ is defined on the interval $[-1, 2]$. f has absolute maximum $f(2) = 2$ and now the absolute minimum is $f(0) = 0$.



Result: A function f continuous on a closed interval $[a, b]$ always has an absolute maximum and absolute minimum on the interval.

2.19.1 Relative Extrema

The function pictured in fig(a) has no absolute extrema. However, suppose we focus our attention on values of x that are close to, or in a neighborhood of the numbers c_1 and c_2.

As shown in fig(b), $f(c_1)$ is the maximum value of the function in the interval (a_1, b_1) and $f(c_2)$ is a minimum value in the interval (a_2, b_2). These local or relative, extrema are defined as follows:

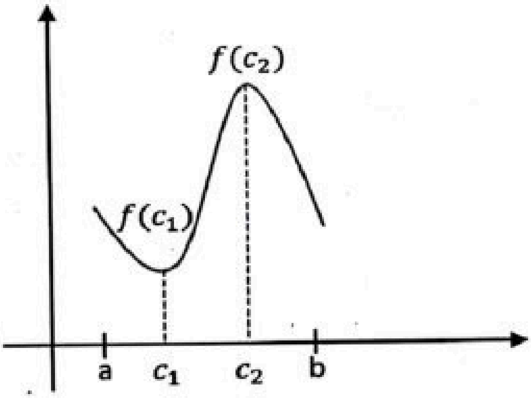
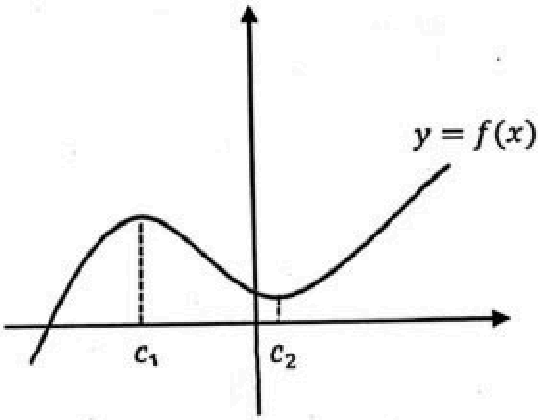


Fig (a)



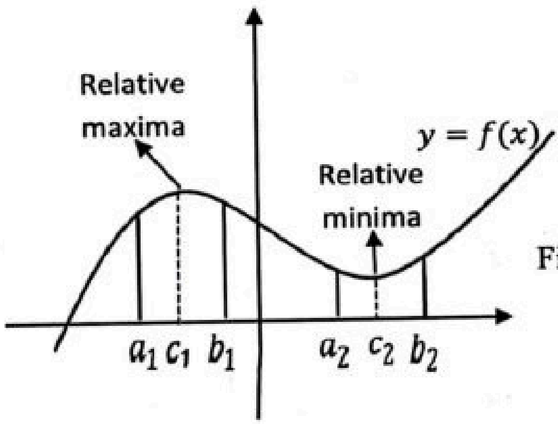


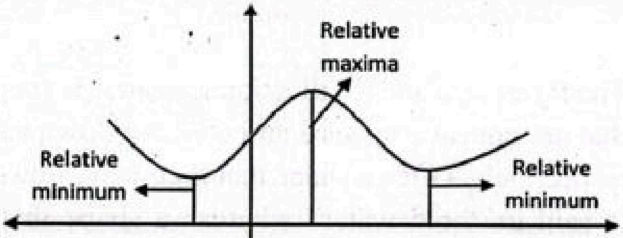
Fig (b)

Definition: Relative Extrema

i. A number $f(c_1)$ is a **Relative Maximum** of a function f if $f(x) \leq f(c_1)$ but every x in some open interval that contains c_1.

ii. A number $f(c_1)$ is a **Relative Minimum** of a function f if $f(x) \geq f(c_1)$ for every x in some open interval that contains c_1.

Result: From fig , we suggest if c is a value at which a function f has a relative extremum, then either $f'(c) = 0$ or $f'(c)$ does not exist.



Critical values: A critical value of a function f is a number in c in its domain for which $f'(c) = 0$ or $f'(c)$ does not exist.

Example 45: Find the critical values of

a. $f(x) = x^3 - 15x + 6$ c. $f(x) = \frac{x^2}{x-1}$ b. $f(x) = (x+4)^{\frac{2}{3}}$

Solution:

a. $f(x) = x^3 - 15x + 6$

$f'(x) = 3x^2 - 15$

$f'(x) = 3(x + \sqrt{5})(x - \sqrt{5})$

The critical values are those number for which $f'(x) = 0$, namely $-\sqrt{5}$ and $\sqrt{5}$.

b. $f(x) = (x+4)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x+4)^{\frac{-1}{3}}$

$f'(x) = \frac{2}{3(x+4)^{\frac{1}{3}}}$

We observe that $f'(x)$ doesnot exist, when $x = -4$ since -4 is in the domain of f. We conclude it is a critical value.

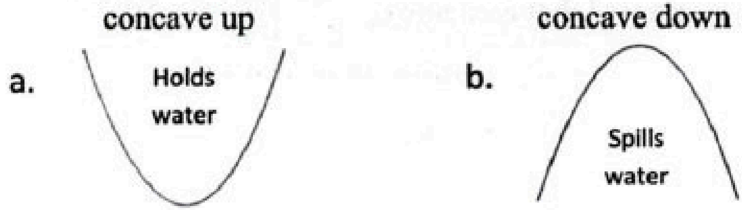
c. $f(x) = \frac{x^2}{x-1}$

$f'(x) = \frac{x(x-2)}{(x-1)^2}$; by quotient rule

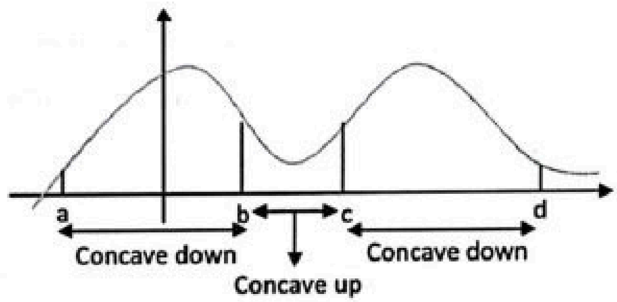
Now $f'(x) = 0$ when $x = 0$ and $x = 2$, whereas $f'(x)$ doesn't exist when $x = 1$. However, inspection of f reveals $x = 1$ is not in its domain and so the any critical values are 0 and 2.

2.20 Second Derivative Test for Relative Extrema

Concavity: We know about the concavity:



The figures (a) and (b) illustrates geometric shapes that are concave upward and concave downward, respectively. Often a shape that is concave upward is said to "hold water" whereas a shape that is concave downward "spills water".



The graph in the fig(c) is concave upward on the interval (b, c) and concave downward on (a, b) and (c, d).

Concavity and The Second Derivative Test

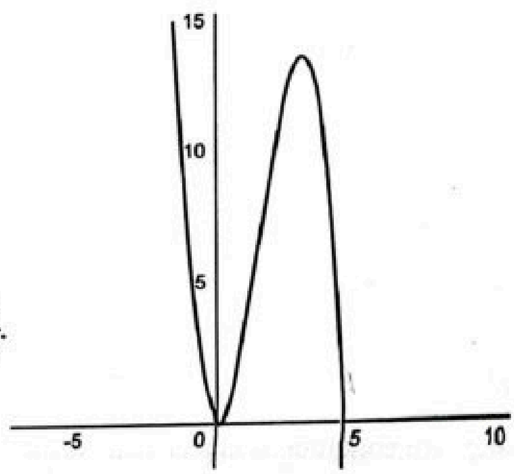
Definition: Test for concavity

Let f be a function for which f''exists on (a, b).

If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward on (a, b).

If $f''(x) < 0$ for all x in (a, b),then the graph of f is concave downward on (a, b).

Example 46: Determine the interval on which the graph of $f(x) = -x^3 + \frac{9}{2}x^2$ is concave upward and the intervals for which the graph is concave downward.



Solution: $f(x) = -x^3 + \frac{9}{2}x^2$

$f'(x) = -3x^2 + 9x$

$f''(x) = -6x + 9 = 6\left(-x + \frac{3}{2}\right)$

We observe that $f''(x) < 0$ when $6\left(-x + \frac{3}{2}\right) > 0$ or

$x < \frac{3}{2}$ and that $f''(x) < 0$ when $6\left(-x + \frac{3}{2}\right) < 0$ or $x > \frac{3}{2}$.

If follows that the graph of f is concave upward on $(-\infty, \frac{3}{2})$ and concave downward on $\left(-\frac{3}{2}, \infty\right)$.

2.21 Point of Inflection

In the example 47 function changes concavity at the point that corresponds to $x = \frac{3}{2}$. As x increases through $\frac{3}{2}$, the graph of f changes from concave upward to concave downward at the point $\left(\frac{3}{2}, \frac{27}{4}\right)$ a point on the graph of a function where the concavity changes from upward or downward or reverse is called a point of inflection.

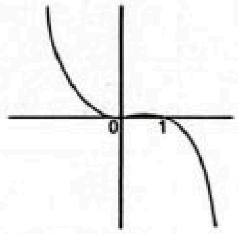
Definition: Point of Inflection

Let f be a continuous at c, a point $(c, f(c))$ is point of inflection if there exists an open interval (a, b) that contains c such that the graph of f is either:

i. Concave upward on (a, c) and concave downward on (c, b) or

ii. Concave downward on (a, c) and concave upward on (c, b).

Example 47: Find points of inflection of $f(x) = -x^3 + x^2$



Solution:

$$f'(x) = -3x^2 + 2x \text{ and } f''(x) = -6x + 2$$

Since $f''(x) = 0$ at $\frac{1}{3}$, the point $\left(\frac{1}{3}, \frac{2}{27}\right)$ is the only possible point of inflection. We have

$$f''(x) = 6\left(-x + \frac{1}{3}\right) > 0 \text{ for } x < \frac{1}{3}$$

$$f''(x) = 6\left(-x + \frac{1}{3}\right) < 0 \text{ for } x > \frac{1}{3}$$

Implies that the graph of f is concave upward on $\left(-\infty, \frac{1}{3}\right)$ and concave downward on $\left(\frac{1}{3}, \infty\right)$.

Thus, $\left(\frac{1}{3}, f\left(\frac{1}{3}\right)\right)$ or $\left(\frac{1}{3}, \frac{2}{27}\right)$ is a point of inflection.

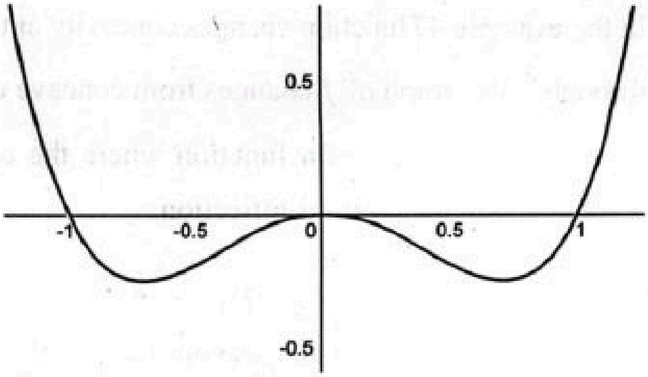
Definition: Second Derivative Test for Relative Extrema

Let f be function for which f'' exists on an interval (a, b) that contains the critical number c.

i. If $f''(c) > 0$, then $f(c)$ is a relative minimum.

ii. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

Example 48: Find the critical point and also relative extrema by second derivative test for $f(x) = x^4 - x^2$.



Solution: $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$

$f''(x) = 12x^2 - 2$

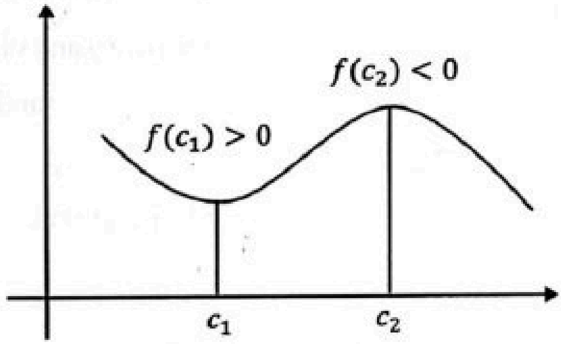
For critical values take $f'(x) = 0$

$$2x(2x^2 - 1) = 0$$

$$x = 0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$$

The second derivative test is summarized as:

x	Sign of $f''(x)$	$f(x)$	Conclusions
0	$f''(x) = -ve$	0	Rel. max
$\frac{\sqrt{2}}{2}$	$f''(x) = +ve$	$\frac{-1}{4}$	Rel. min
$\frac{-\sqrt{2}}{2}$	$f''(x) = +ve$	$\frac{-1}{4}$	Rel. min



Exercise 2.9

1. Find the critical values of the function.
   i. $f(x) = 2x^2 - 6x + 8$
   ii. $f(x) = x^3 + x - 2$
   iii. $f(x) = \frac{x}{x^2+2}$
   iv. $f(x) = \cos 4x$
   v. $f(x) = (4x - 3)^{\frac{1}{3}}$
   vi. $f(x) = x^2(x + 1)^3$
2. Find the absolute extrema of the function on the indicated interval.
   i. $f(x) = -x^2 + 6x : [1, 4]$
   ii. $f(x) = (x - 1)^2 : [2, 5]$
   iii. $f(x) = x^{\frac{2}{3}} : [-1, 8]$
   iv. $f(x) = x^3 - 6x^2 + 2 : [-3, 2]$
   v. $f(x) = 1 + 5\sin 3x : \left[0, \frac{\pi}{2}\right]$
   vi. $f(x) = 2\cos 2x - 4\cos x : [0, 2\pi]$
3. Use the second derivative to determine the intervals on which the function is concave upward and concave downward.
   i. $f(x) = -x^2 + 7x$
   ii. $f(x) = -x^3 + 6x^2 + x - 1$

iii. $f(x) = (x+5)^5$

iv. $f(x) = x(x-4)^3$

v. $f(x) = x^{\frac{1}{2}} + 2x$

vi. $f(x) = x + \frac{9}{x}$

4. Use the second derivative to locate all points of inflection.

i. $f(x) = x^4 - x^3 + 2x^2 + x - 1$

ii. $f(x) = x^{\frac{5}{3}} + 4x$

iii. $f(x) = \sin x$

iv. $f(x) = \cos x$

v. $f(x) = x - \sin x$

vi. $f(x) = \tan x$

5. Use second derivative test to find the relative extrema of the function.

i. $f(x) = -(-2x-5)^2$

ii. $f(x) = x^3 + 3x^2 + 3x + 1$

iii. $f(x) = 6x^5 - 10x^2$

iv. $f(x) = x^2 + \frac{1}{x^2}$

v. $f(x) = \cos 3x, [0, 2\pi]$

vi. $f(x) = \cos x + \sin x, [0, 2\pi]$

6. Determine whether the give function has a relative extremum at the indicated points.

i. $f(x) = \cos x \sin x,\ x = \frac{\pi}{4}$

ii. $f(x) = x \sin x,\ x = 0$

iii. $f(x) = \tan^2 x,\ x = \pi$

iv. $f(x) = (1 + \sin x)^3,\ x = \frac{\pi}{8}$

2.22 Applications of Derivatives

Many real world phenomenon involve changing quantities like the speed of the rocket, the inflation of currency, the number in a bacteria in a culture, the stoke intensity of an earth quake, the voltage of an electrical signal and so forth. In this section we will develop the concept of limits, continuity, derivative and extrema of function for use in real world problems. Another important application of the derivative is to find solution of the optimization problems. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task and if cost is the main consideration we might be interested in finding the least expensive way to perform a task. Mathematically, optimization problem can be reduced to finding the largest or smallest value of a function on some interval and determining where the largest and smallest values occurs. Using derivatives, we will develop the mathematical tools necessary for solving such problems.

Example 49: A side of a cube is measured to be $30cm$ with the possible error of $\pm0.02cm$. What is the approximate maximum possible error in the volume of the cube?

Solution: The volume of a cube is $V = x^3$, where x the length of one side. If Δx represents the error in the length of one side, then the corresponding error in the volume is:

$\Delta V = (x + \Delta x)^3 - x^3$

We use differential: $dv = 3x^2dx = 3x^2\Delta x$

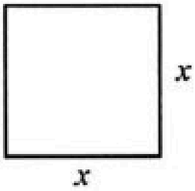
as an approximate to ΔV. Thus, for $x = 30$ and $\Delta x = \pm0.02$, the approximate maximum error is:

$dv = 3(30)^2(\pm0.02) = \pm54cm^3$

Example 50: A square is expanding with time. What is the rate at which the area increases related to the rate at which a side increases?

Solution: At any time the area A of a square is a function of length of one side of x:

$$A = x^2$$



Thus, the related rates are derived from the time derivative.

$$\frac{dA}{dt} = 2x\frac{dx}{dt} \text{ (diff w.r.t "t")}$$

is the same as: $$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

Example 51: Air is being pumped into a spherical baloon at a rate of 20 cubic feet/min. At what rate is the radius changing when the radius is 3ft?

Solution: As shown in fig, we denote the radius of the baloon by r and its volume by V. As per statement, air is being pumped at the rate $20ft^3/min$, means we have: $\frac{dV}{dt} = 20ft^3/min$

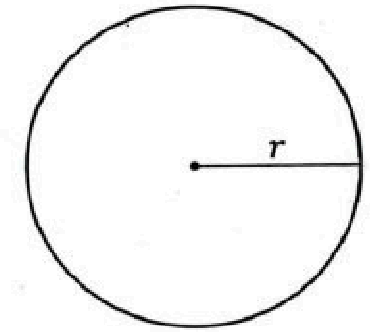
In addition, we require $\frac{dr}{dt}\Big|_{r=3}$

We know the relation between V and r is $V = \frac{4}{3}\pi r^3$

Diff w.r.t "t"

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$



But $\frac{dV}{dt} = 20$, therefore $$20 = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{\pi r^2}$$

Thus, $\frac{dr}{dt}\Big|_{r=3} = \frac{5}{9\pi}\frac{ft}{min} = 0.18\,ft/min$

Example 52: Find two non-negative numbers whose sum is 15 such that the product of one with the square of other is a maximum.

Solution: Let x and y denote the two non-negative numbers (i.e. $x \geq 0$ and $y \geq 0$). It is given that: $x + y = 15$ ....(i)

Let p denote the product: $p = x.y^2$ (Product = one number. square of the other)

We can use $y = 15 - x$ to express p in terms of x: $p(x) = x(15 - x)^2$

The function $p(x)$ defined any for $0 \leq x \leq 15$.

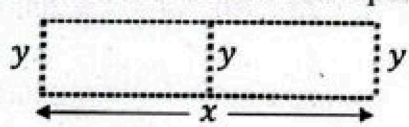
If $x > 15$, then $y = 15 - x$ would be negative.

$p'(x) = x.2(15 - x)(-1) + (15 - x)^2 = (15 - x)(15 - 3x)$

Thus, any critical value is $x = 5$.

Testing the end points of the interval reveal $p(0) = p(15) = 0$, is the minimum value of the product. Hence, $p(5) = 5(10)^2 = 500$ must be the maximum value. The two non-negative numbers are 5 and 10.

Example 53: A rectangular plot of land that contain 1500 m^2 will be fenced and divided into equal portions by any additional fence parallel to two sides. Find the dimensions of the land that require the least amount of fencing.



Solution: Let us introduce variable x and y so that $xy = 1500$. Then the function we wish to minimize is the sum of the lengths of the five portions of the fence.

$L = 2x + 3y$

But $y = \dfrac{1500}{x}$, we have

$L(x) = 2x + \dfrac{4500}{x}$

$L'(x) = 2 - \dfrac{4500}{x^2}$

For critical value, $L'(x) = 0$

$x^2 = 2250$

$x = 15\sqrt{10}$

For 2nd derivative: $L''(x) = \dfrac{13500}{x^3}$

When $x = 15\sqrt{10}$

$L''(15\sqrt{10}) > 0$

Hence $x = 15\sqrt{10}m$, is required minimum amount of fencing.

So,

$L(15\sqrt{10}) = 2(15\sqrt{10}) + \dfrac{4500}{15\sqrt{10}} = 15\sqrt{10}$

$xy = 1500$

$y = \dfrac{1500}{x} = \dfrac{1500}{15\sqrt{10}}$

$y = 10\sqrt{10}m$

Dimension of land:

$xy = 15\sqrt{10} \times 10\sqrt{10}$

Price Growth Model: The price level at time P, considering inflation can be modeled as: $P(t)=P_0e^{rt}$, where, $P(t)$= Price at time t, P_0 = Initial price, continuous annual inflation rate,

t= Time(in years)

To find the rate of change of price with respect to time, take the derivative of $P(t)$ with respect to t. $\frac{d}{dt}P(t)=P_0re^{rt}$, The $\frac{dP(t)}{dt}$, represents the instantaneous rate of change of the price level or how fast prices are increasing at a time t.

Example 54: The price of a product is modeled as: $P(t)=200e^{0.03t}$, where t is the time in years and P (t) is the price at a time t. Find the rate at which the price is increasing after:

a. 0 years b. 5 years c. 10 ears

Solutions: The price inflation is $P(t)=200e^{0.03t}$, the derivative gives the rate of price increase:

$$\frac{dP(t)}{dt}=200(0.03)e^{0.03t}=6e^{0.03t}$$

a. At t = 0, $\frac{dP(0)}{dt}=200(0.03)e^{0.03(0)}=6e^{0}=6\,\text{units/year}$

b. At t = 5, $\frac{dP(5)}{dt}=200(0.03)e^{0.03(5)}=6e^{0.15}=6.92\,\text{units/year}$

c. At t = 10, $\frac{dP(10)}{dt}=200(0.03)e^{0.03(10)}=6e^{0.3}=8.10\,\text{units/year}$

Using Straight Lines: Derivatives help analyze a line relationship in real life scenarios. Straight lines appear in situations, where variables change at a constant rate and derivatives calculate the rate or optimize related process.

Example 55: Economics, Marginal Cost and Revenue: A company's Revenue R(x) from selling x units is given by: $R(x)=50x$ The total cost $C(x)$ for producing x units is: $C(x)=30x+200$

a. Find the marginal revenue and marginal cost.
b. Determine the break-even point (units sold where revenue equals cost).
c. Interpret the meaning of the straight-line equations and slopes.

Solutions: a. Marginal Revenue and Marginal Cost:

- Marginal Revenue $(R'(x))$: $R'(x)=\frac{d}{dx}(50x)=50$, revenue increases by 50/units.
- Marginal Cost $(C'(x))$: $C'(x)=\frac{d}{dx}(30x+200)=30$, cost increase by 30/units.

b. Break-Even Point: At break even, revenue equal costs:

$R(x) = C(x)$

$50x = 30x + 200$, which gives, $x = 10$.

The company breaks even when 10 units are sold.

c. Interpretation of Slopes:

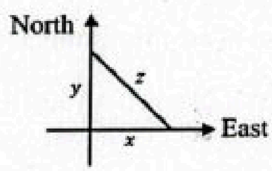
- The slopes of $R(x)$ is 50, showing revenue grows faster than cost.
- The slopes of $C(x)$ is 30, representing slower cost growth.

Exercise 2.10

1. According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$?
2. $f(x) = \begin{cases} kx + 1, & x \leq 3 \\ 2 - kx, & x > 3 \end{cases}$ is continuous at 3. What is K?
3. The volume v of a sphere of radius r is $v = \left(\frac{4\pi}{3}\right) r^3$. Find the surface area s of the sphere if s is the instantaneous rate of change of the volume with respect to the radius.
4. The height S above ground of a projectile time t is given by

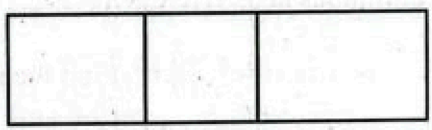
   $S(t) = \frac{1}{2}gt^2 + v_0 t + s_0$.

   Where g, v_0 and s_0 are constants. Find the instantaneous rate of change of S with respect to t at $t = 4$.
5. The side of a square is measured to be 10cm with a possible error of ± 0.3 cm. Use differentials to find an approximation to the maximum error in the area. Find the approximate relative error and the approximate error.
6. A woman jogging at a constant rate of 10km/hr crosses a point A heading north. Ten minutes later a man jogging at a constant rate of 9km/hr crosses the same point heading east. How fast is the difference between the joggers hanging 20 minutes after the man crosses A?



7. A plate in the shape of an equilateral triangle is expanding with time. A side increases at a constant rate of 2cm/hr. At what rate is the area increasing when side in 8cm?

8. A rectangle expands with time. The diagonal of the rectangle increases at a rate of 1in/hr and length increases at a rate of $\frac{1}{4}$ min/hr. How fast is its width increasing when the width is 6 in and length is 8 in?
9. The side of a cube increases at a rate of 5cm/hr. At what rate does the diagonal of the cube increases?
10. A particle moves on a graph of $y^2 = x + 1$ so that $\frac{dx}{dt} = 4x + 4$. What is $\frac{dx}{dt}$ when $x = 8$?
11. At 8:00 am ship S_1 is 20km due north of S_2. Ship S_1 sails south at a rate of 9km/hr and S_2 sails west at a rate of 12 km/hr at 9:20 am. At what rate is the distance between the two ships changing?
12. Find two non-negative numbers whose sum is 60 and whose product is a maximum?
13. If the total fence to be used is 8000 m, find the dimensions of the enclosed land in figure that has the greatest area.



14. An open rectangular box is to be constructed with a square base and a volume of 32,000 cm^3. Find the dimensions of box that require the least amount of material.
15. A company determines that for the production of x units of a commodity its remove and cost functions are, respectively, $R(x) = -3x^2 + 970x$ and $G(x) = 2x^2 + 500$. Find the maximum profit and minimum average cost.
16. If the inflation rate is continuously compounded 4% per year and the price of a commodity is \$50 today.
    a. Derive the function for the price of the commodity over time.
    b. Find the price after 8 years.
    c. Find the instantaneous rate of price at t=8 years.
17. A company models its operational cost as: $C(t) = 500e^{0.04t} - 100t$, where t is the time in years.
    a. Find the rate of change of cost at any time t.
    b. Determining the rate of increase in cost is minimal.
18. The price of commodity $P(t)$ is given by: $P(t) = 150(1 + 0.05t)^2$, where t is measured in years and $P(t)$ is the price level.
    a. Find the instantaneous rate of change of prices at t= 3years.
    b. Calculate the inflation rate at t=3 years.

19. A ship sails in a straight line. Its distance $d(t)$ (in nautical miles) from port is modeled by: $d(t) = 15t$ where t is time in hours.

a. Find the speed of the ship.
b. Calculate the distance after 3 hours.
c. Explain the meaning of the slope.

20: A cyclist is traveling along a straight path, and the distance traveled s(t) (in meters) is given by: $s(t) = 5t^2 + 3t$, where, t is the time in seconds.

a. Find the speed at any time t.
b. Determine the speed at t=4 seconds.
c. Interpret the significance of the slope in this context.

Review Exercise

1. Tick the correct options.

i. If $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$:

a. 3 b. 0 c. Exist d. Doesn't exist

ii. If $f(x) = \begin{cases} 2x-1, & x<0 \\ 2x+1, & x>0 \end{cases}$, then $\lim_{x\to 0^-} f(x) = 0$, is:

a. 1 b. −1 c. 0 d. 2

iii. If f and g are continuous at 2, then $\frac{f}{g}$ is continuous at:

a. 0 b. 1 c. 2 d. 3

iv. The function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, is continuous at:

a. 0 b. 1 c. −1 d. 0.1

v. If f is differentiable for every value of x, then f is:

a. Discontinuous
b. Continuous
c. Finite
d. Infinite

vi. If k is a constant and n is positive integer, then $\frac{d}{dx}k^n$ is:

a. nk^{n-1} b. k^{n-1} c. $\ln n.k^n$ d. 0

vii. If $f(2) = 2, g(x) = x^2$, then $\frac{d}{dx}[\frac{3g(x)}{f(2)}]$ is:

a. $2x$ b. $3x$ c. $\frac{3}{2}x$ d. $\frac{3}{2}x^2$

viii. If $y = f(x)$ is a polynomial function of degree 2, then $\frac{d^3}{dx^3}f(x)$ is:

a. 0 b. 1 c. -1 d. 2

ix. If f is differentiable for every value of x, then f is continuous for:

a. Some value of x b. $[0, \infty]$

c. Every value of x d. $[0, -\infty]$

x. If $f(t) = 2t^3$ is absolute minimum at:

a. 3 b. 0 c. -1 d. 1

2. Evaluate. $\lim_{x\to 0}\frac{x^3}{\sin^2 3x}$

3. Find the derivative. $y = \frac{1+\sin x}{x\cos x}$

4. If $f'(0) = -1$ and $g'(0) = 6$, what is $\frac{d^2}{dx^2}[xf(x) + xg(x)]$ at $x = 0$?

5. Find $\frac{d^2y}{dx^2}$, when $x^3 + y^3 = 27$

6. Use differential to find an approximate of $\sqrt{65}$.

7. An oil storage tank in the form of circular cylinder has a height of 5 m. The radius is measured to be 8 m with a possible error of ± 0.25 m. Use differentials to estimate the maximum error in the volume. Find the approximate relative error and the approximate percentage error.

8. A 15 ft ladder is leaning against a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 ft/min. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 ft from the wall?

9. Find the absolute extrema of $f(x) = x^3 - 3x^2 - 24x + 2$,

a. $[-3, 1]$ b. $[-3, 8]$

10. Graph the function: $f(x) = x + \frac{1}{x}$

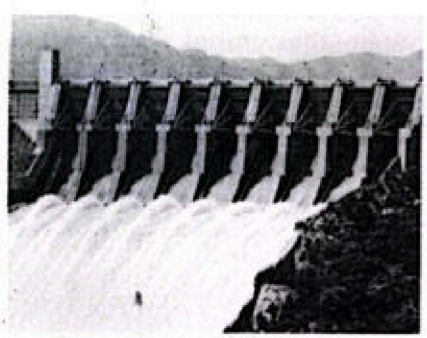
UNIT
03

INTEGRATION

After studying this unit, students will be able to:

- Find the general antiderivative of a given function.
- Recognize and use the terms and notations for antiderivatives.
- State the power rule of integrals.
- State and apply the properties of indefinite integrals.
- Integrate functions involving the exponential and logarithmic functions.
- Identify when to use integration by parts to solve integration problems.
- Apply the integration-by-part formula for definite integrals.
- Solve integration problems involving trigonometric substitution.
- Integrate a rational function using the method of partial fraction.
- State the definition of definite integral.
- Explain the terms integrand, limits of integration and value of integration.
- State and apply the properties of definite integrals.
- State and apply fundamental theorem of calculus to evaluate the definite integrals.
- Describe the relation between the definite integral and net area.
- Find the area of a region bounded by a curve and lines parallel to axes, or between a curve and a line or between two curves.
- Find volume of the revolution about one of the axes.
- Demonstrate trapezium rule to estimate the value of a definite integral.
- Apply concept of integration to real world problems such as volume of a container, consumer and producer surplus, growth rate of a population, investment return time period, drug dosage required by integrating the concentration.

There is a lot of applications of integration in various fields. For example, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us. A view of Tarbela dam is shown below.



3.1 Integration

This unit examines the process by which we determine functions from their derivatives. We are already familiar with inverse operations. For example, addition and subtraction are inverse of each other. Similarly, multiplication and division are inverse of each other. In the same way, the inverse operation of differentiation is anti-differentiation or integration.

This unit provides two processes and their relationship to one another. One step is to find function from their derivatives. In the second step, we can determine things like area and volume through successive approximations. This process is called integration. This is very important area in mathematics and was discovered independently by Leibnitz and Newton.

The process of finding a function from one of its known values and its derivative $f(x)$ has two steps:

The first is to find a formula that gives us all the functions that could possibly have $f(x)$ as a derivative. If $f'(x)$ is defined as derivative, then $f(x)$ is called anti-derivative and the formula that gives them all is called the indefinite integral of $f(x)$. The reverse process of derivative or anti-differentiation is the main topic of this unit.

Definition 3.1:

A function $F'(x)$ is called an anti-derivative of another function $f(x)$ if:

$$F'(x) = f(x)$$

For example:

$$\frac{1}{4}x^4, \quad \frac{1}{4}x^4 + 3, \quad \frac{1}{4}x^4 - \pi, \quad \frac{1}{4}x^4 + c \quad (c \text{ is any constant.})$$

are anti-derivatives of x^3 since the derivative of each is x^3.

Above example shows that a function can have many anti-derivatives. In fact, if $F(x)$ is any anti-derivative of $f(x)$ and c is any constant, then $F(x) + c$ is also an anti-derivative of $f(x)$ since:

$$\frac{d}{dx}[F(x) + c] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[c] = f(x) + 0 = f(x)$$

Therefore, if $F(x)$ is any anti-derivative of $f(x)$ on a given interval, then for any value of c, the function $F(x) + c$ is also an anti-derivative of $f(x)$ on that interval.

Symbolically we write:

$$\int f(x)dx = F(x) + c$$

Where the symbol, "$\int$" is called 'integral sign' and $f(x)$ is called integrand. The symbol dx indicates that the integration is performed with respect to the variable x. The arbitrary constant c is called 'constant of integration'.

For Example,

As, $\quad \frac{d}{dx}(x^4) = 4x^3$

Therefore, $\int 4x^3 dx = x^4 + c$

As mentioned above, the constant c is arbitrary constant. Therefore,

$x^4, x^4+1, x^4-\sqrt{2}, x^4+\pi$ etc. all are anti-derivatives of $4x^3$.

Let us derive some basic and common integral formulae with the help of differentiation.

Key Facts

- The variable other than x, can also be used in indefinite integrals.
- A number of indefinite integral formulae are found by reversing derivative formulas.

Formula 3.1: $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

Derivation: We have,

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right] = \frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] + \frac{d}{dx}[c] = \frac{(n+1)\,x^n}{n+1} + 0 = x^n \qquad \text{(i)}$$

Integrating both sides of (i) with respect to x, we have:

$$\int \frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right]dx = \int x^n dx$$

$$\frac{x^{n+1}}{n+1} + c = \int x^n dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$$

In general,

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$$

Formula 3.2: $\int \frac{1}{x} dx = \ln x + c$

Derivation: We have,

$$\frac{d}{dx}[\ln x + c] = \frac{1}{x} \qquad \text{(ii)}$$

Integrating both sides of (ii) with respect to x, we have:

$$\int \frac{d}{dx}[\ln x + c]dx = \int \frac{1}{x} dx$$

$$\ln x + c = \int \frac{1}{x} dx$$

$$\int \frac{1}{x} dx = \ln x + c$$

In general,

$$\int \frac{f'(x)}{f(x)} dx = \ln[f(x)] + c$$

Formula 3.3: $\int e^x dx = e^x + c$

Derivation: As,

$$\frac{d}{dx}[e^x + c] = e^x \qquad \text{(iii)}$$

Integrating both sides of (iii) with respect to x, we have:

$$\int \frac{d}{dx}[e^x + c]dx = \int e^x dx$$

$$e^x + c = \int e^x dx$$

$$\boxed{\int e^x dx = e^x + c}$$

In general,

$$\boxed{\int e^{f(x)} f'(x)dx = e^{f(x)} + c}$$

Formula 3.4: $\int a^x dx = \frac{1}{\ln a}a^x + c,\ a > 0,\ a \neq 1$

Derivation: As,

$$\frac{d}{dx}\left[\frac{1}{\ln a}a^x + c\right] = a^x \qquad \text{(iv)}$$

Integrating both sides of (iv) with respect to x, we have:

$$\int \frac{d}{dx}\left[\frac{1}{\ln a}a^x + c\right]dx = \int a^x dx$$

$$\frac{1}{\ln a}a^x + c = \int a^x dx$$

$$\boxed{\int a^x dx = \frac{1}{\ln a}a^x + c}$$

In general,

$$\boxed{\int a^{f(x)} f'(x)dx = \frac{1}{\ln a}a^{f(x)} + c}$$

Theorem 3.1:

(i) A constant factor can be moved through an integral sign. That is:

$$\int c\, f(x)dx = c\int f(x)dx$$

(ii) An anti-derivative of a sum is the sum of anti-derivatives. That is:

$$\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$$

(iii) An anti-derivative of a difference is the difference of anti-derivatives. That is:

$$\int [f(x) - g(x)]\, dx = \int f(x)dx - \int g(x)dx$$

(iv) In general, $\int [af(x) \pm bg(x)]\, dx = a\int f(x)dx \pm b\int g(x)dx$

Example 1: Evaluate (i) $\int(4x^7 - 2x^3 + 9x + 3)dx$ (ii) $\int \frac{y^3-2y^6}{y^5}\, dy$

Solution: (i) $\int(4x^7 - 2x^3 + 9x + 3)dx$

$= 4\int x^7 dx - 2\int x^3 dx + 9\int x\, dx + 3\int dx$

Integrating term by term, we get:

$= 4\left(\frac{x^8}{8}\right) - 2\left(\frac{x^4}{4}\right) + 9\left(\frac{x^2}{2}\right) + 3x + c = \frac{x^8}{2} - \frac{x^4}{2} + \frac{9x^2}{2} + 3x + c$

(ii) $\int \frac{y^3-2y^6}{y^5}\, dy = \int\left(\frac{y^3}{y^5} - \frac{2y^6}{y^5}\right) dy = \int\left(\frac{1}{y^2} - 2y\right) dy$

$= \int(y^{-2} - 2y)\, dy = \int y^{-2}\, dy - 2\int y\, dy$

$= \frac{y^{-2+1}}{-2+1} - 2\left(\frac{y^2}{2}\right) + c = -\frac{1}{y} - y^2 + c$

Example 2: Evaluate (i) $\int \frac{ax + \frac{1}{2}b}{ax^2+bx+c}\, dx$ (ii) $\int e^{3x}dx$

Solution: (i) $\int \frac{ax + \frac{1}{2}b}{ax^2+bx+c}\, dx = \frac{1}{2}\int \frac{2ax + b}{ax^2+bx+c}\, dx$

$= \frac{1}{2} ln\,(ax^2 + bx + c) + C$

(ii) $\int e^{3x}dx = \frac{1}{3}\int e^{3x}(3)dx = \frac{1}{3}e^{3x} + c$

Example 3: Evaluate $\int \frac{e^{sin^{-1}x}}{\sqrt{1-x^2}}\, dx$

Solution: Here, $f(x) = sin^{-1}x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

So, by using formula:

$\int e^{f(x)} f'(x) dx = e^{f(x)} + c$

We have:

$\int \frac{e^{sin^{-1}x}}{\sqrt{1-x^2}}\, dx = e^{sin^{-1}x} + c$

Exercise 3.1

Evaluate the following integrals.

1. $\int(x^2 - 3x + 9)dx$
2. $\int(y^2 + 8y + \sqrt{2})dy$
3. $\int\left(\sqrt{y} + \frac{1}{y^2}\right) dy$
4. $\int(4 + x^2)^2 dx$
5. $\int(1 + x)(1 - x^2)dx$
6. $\int\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$
7. $\int(e^{4x} - e^{-1} + 1)dx$
8. $\int(e^{\frac{9}{2}x} + \frac{1}{x})dx$
9. $\int x\, e^{x^2} dx$
10. $\int 5^x dx$
11. $\int 7^{7y} dy$
12. $\int(x^3 + \frac{1}{2x} - \frac{1}{x^3})dx$

13. $\int \frac{2x+1}{x^2+3}\,dx$ 14. $\int \frac{e^{\tan^{-1}z}}{1+z^2}\,dz$ 15. $\int \left(x^{\frac{3}{2}} + e^{3x} + x^0\right) dx$

16. $\int (3x^2+2x)(x^3+x^2+9)^5 dx$ 17. $\int (5e^{5x} - x^{-3} + 3^{2x}) dx$

18. $\int (z^{\frac{-1}{4}} + \sqrt{3z} + \frac{4}{z} - \frac{1}{e^z}) dz$.

3.2 Integration of Trigonometric Functions

While evaluating the integration of trigonometric functions, keep in mind the following formulae.

As, $\frac{d}{dx}(\sin x + c) = \cos x$ therefore, $\int \cos x\, dx = \sin x + c$

Similarly, $\frac{d}{dx}(\cos x + c) = -\sin x$ implies, $\int \sin x\, dx = -\cos x + c$

In the same way, $\frac{d}{dx}(\sin kx + c) = k\cos kx$ implies, $\int \cos kx\, dx = \frac{\sin kx}{k} + c$

And, $\frac{d}{dx}(\cos kx + c) = -k\sin kx$ implies, $\int \sin kx\, dx = -\frac{\cos kx}{k} + c$

Using above pattern, following formulae can be deduced easily.

$\int \sec^2 x\, dx = \tan x + c$ and $\int \text{cosec}^2 x\, dx = -\cot x + c$

$\int \sec x \tan x\, dx = \sec x + c$ and $\int \text{cosec}x \cot x\, dx = -\text{cosec}x + c$

Example 4: Evaluate:

(i) $\int \sin 2x\, dx$ (ii) $\int \cos \frac{3x}{5}\, dx$ (iii) $\int \sec^2 mx\, dx$

(iv) $\int 5\,\text{cosec}^2 \frac{7x}{5}\, dx$ (v) $\int 9 \sec 3x \tan 3x\, dx$.

Solution:

(i) $\int \sin 2x\, dx = \int \frac{1}{2}\sin 2x\,(2)\, dx = \frac{1}{2}\int \sin 2x\,(2)\, dx$

$= \frac{1}{2}(-\cos 2x + c) = \frac{-\cos 2x}{2} + \frac{c}{2} = \frac{-\cos 2x}{2} + C$

(ii) $\int \cos\frac{3x}{5}\, dx = \int \frac{5}{3}\cos\frac{3x}{5}\left(\frac{3}{5}\right) dx = \frac{5}{3}\int \cos\frac{3x}{5}\left(\frac{3}{5}\right) dx$

$= \frac{5}{3}\left(\sin\frac{3x}{5} + c\right) = \frac{5}{3}\sin\frac{3x}{5} + \frac{5}{3}c = \frac{5}{3}\sin\frac{3x}{5} + C$

(iii) $\int \sec^2 mx\, dx = \int \frac{1}{m}\sec^2 mx\,(m)\, dx = \frac{1}{m}\int \sec^2 mx\,(m)\, dx$

$= \frac{1}{m}(\tan mx + c) = \frac{1}{m}(\tan mx) + \frac{c}{m} = \frac{1}{m}(\tan mx) + C$

(iv) $\int 5\,\text{cosec}^2\frac{7x}{5}\, dx = 5 \times \frac{5}{7}\int \text{cosec}^2\frac{7x}{5}\left(\frac{7}{5}\right) dx = \frac{25}{7}\left(-\cot\frac{7x}{5} + c\right)$

$$= -\frac{25}{7}\cot\frac{7x}{5} + \frac{25}{7}c \quad = \quad -\frac{25}{7}\cot\frac{7x}{5} + C$$

(v) $\int 9\sec 3x \tan 3x\, dx \;=\; \int 3\times 3\sec 3x\tan 3x\, dx \;=\; 3\int \sec 3x\tan 3x\,(3)dx$

$= \; 3(\sec 3x + c) \;=\; 3\sec 3x + 3c \;=\; 3\sec 3x + C$

Example 5: Prove that:

(i) $\int \sec x\, dx = \ln|\sec x + \tan x| + c$

(ii) $\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + c$

(iii) $\int \tan x\, dx = -\ln(\cos x) + c = \ln(\sec x) + c$

Solution:

(i) $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$ [Multiplying and dividing by $(\sec x + \tan x)$]

$$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}dx = \int \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}dx$$

$$= \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}dx = \ln|\sec x + \tan x| + c$$

$\int \frac{f'(x)}{f(x)}dx = \ln[f(x)] + c$

(ii) $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}dx$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x\cot x}{\operatorname{cosec} x - \cot x}dx = \int \frac{-\operatorname{cosec} x\cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x - \cot x}dx$$

$$= \int \frac{\frac{d}{dx}(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}dx = \ln|\operatorname{cosec} x - \cot x| + c$$

(iii) $\int \tan x\, dx \;=\; \int \frac{\sin x}{\cos x}dx \;=\; -\int \frac{-\sin x}{\cos x}dx \;=\; -\int \frac{\frac{d}{dx}(\cos x)}{\cos x}dx$

$= \; -\ln(\cos x) + c \;=\; \ln(\cos x)^{-1} + c$

$= \; \ln\frac{1}{\cos x} + c \;=\; \ln(\sec x) + c$

Check Point

Prove that

$\int \cot x\, dx = \ln(\sin x) + c$

3.2.1 Integration of $\sin^2 x$ and $\cos^2 x$

Sometimes it is difficult to evaluate integrals directly. Using trigonometric identities, we can easily evaluate integrals. For example, the integrals of $\sin^2 x$ and $\cos^2 x$ cannot be solved directly and can be handled using following relations.

$\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

Check Point

Evaluate $\int \cos^2 x\, dx$

Example 6: Evaluate $\int \sin^2 x\, dx$

Solution: $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Example 7: Integrate (i) $8\sec 9x - \tan 3x$ (ii) $\cos^2 7x$

Solution:

(i) $\int(8\sec 9x - \tan 3x)dx = \int 8\sec 9x\, dx - \int \tan 3x\, dx$

$= \frac{8}{9}\int \sec 9x\,(9)dx - \frac{1}{3}\int \tan 3x\,(3)dx$

$= \frac{8}{9}\ln|\sec 9x + \tan 9x| - \ln(\sec 3x) + c$

(ii) $\int \cos^2 7x\, dx = \int \frac{1+\cos 14x}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 14x\, dx$

$= \frac{1}{2}x + \frac{1}{2}\left(\frac{\sin 14x}{14}\right) + c = \frac{1}{2}x + \frac{1}{28}\sin 14x + c$

Exercise 3.2

Evaluate the integrals and recheck your answer by differentiating.

1. $\int(\sin\pi x - 3\sin 3x)dx$
2. $\int -\sec^2\left(\frac{3}{2}y\right)dy$
3. $\int[1 - 8\operatorname{cosec}^2(2x)]dx$
4. $\int \frac{1}{2}(\operatorname{cosec}^2 x - \operatorname{cosec}x\cot x)dx$
5. $\int \frac{\cos^2 z}{7}dz$
6. $\int(1+\tan^2\theta)d\theta$
7. $\int \frac{1+\cos 4t}{2}dt$
8. $\int \sec^2(5x-1)\,dx$
9. $\int(\tan 5x + \cos 7x)dx$
10. $\int(\cot 9y - 3)\,dy$

Evaluate the integral.

11. $\int(\tan^2 2\theta + \cot^2 2\theta)d\theta$
12. $\int \sin^2\left(\frac{11}{2}y\right)dy$
13. $\int \operatorname{cosec}11x\tan 11x\,dx$
14. $\int \cos\theta(\tan\theta + \sec\theta)d\theta$
15. $\int \operatorname{cosec}^2\left(\frac{x-1}{3}\right)dx$
16. $\int(\cos x)^{\frac{1}{5}}\sin x\,dx$
17. $\int e^y \sin e^y dy$
18. $\int 9\tan(x+7)dx$

3.3 Integration by Substitution

There are many functions that cannot be integrated by simple techniques and can be integrated easily by using method of substitution. It is an integration technique which involves making a substitution to simplify the integral. In this method any given integral is transformed into a simple form of integral by substituting the independent variable by others. The exact substitution depends on the form of the given integral, as some substitutions are more appropriate for certain problems than others. The choice of substitution is not always immediately obvious. The ability to recognise an appropriate substitution comes from practising many different examples.

Mostly, we substitute trigonometric functions in place of variables to integrate algebraic functions. However, there is no hard and fast rule for selection of trigonometric functions to replace variables as some other substitutions are also used.

Key Facts

Usually, the method of integration by substitution is extremely useful when we make a substitution for a function whose derivative is also present in the integrand. Doing so, the function simplifies and then the basic formulas of integration can be used to integrate the function.

Example 8: Evaluate $\int 3x^2 \cos(x^3)\, dx$

Solution:

In the equation given above the independent variable can be transformed into another variable say t by substituting:

$$x^3 = t \qquad \text{(i)}$$

Differentiation of (i) gives:

$$3x^2\, dx = dt \qquad \text{(ii)}$$

Substituting the values of (i) and (ii) in the given integral.

$$\int 3x^2 \cos(x^3)\, dx = \int \cos t\, dt = \sin t + c$$

Again, substituting back the value of t, we get:

$$\int 3x^2 \cos(x^3)\, dx = \sin(x^3) + c$$

Key Facts

The method of substitution to find an integral is used when it is set up in the special form.

$\int f(g(x)).g'(x).dx = \int f(t).dt$

where $t = g(x)$

Check Point

Integrate:

$x \sin(x^2 - 3)$ with respect to x.

Example 9: Integrate: $\int \frac{e^{\tan^{-1}x}}{1+x^2}\, dx$

Solution: Let $u = \tan^{-1}x$ then $du = \frac{1}{1+x^2}\, dx$

Therefore, $\int \frac{e^{\tan^{-1}x}}{1+x^2}\, dx = \int e^u du = e^u + c = e^{\tan^{-1}x} + c$

Formula 3.5: $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Derivation: Substituting $x = a\sin\theta$, we have $dx = a\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{\sqrt{a^2 - (a\sin\theta)^2}}\, a\cos\theta\, d\theta = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= \int \frac{1}{a\sqrt{1 - \sin^2\theta}}\, a\cos\theta\, d\theta = \int \frac{1}{\cos\theta}\, \cos\theta\, d\theta$$

$$= \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

Note: We can apply the formula directly too.

Formula 3.6: $\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2 - x^2}}{2} + c$

Derivation: Substituting $x = a\sin\theta$, we have $dx = a\cos\theta\, d\theta$

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - (a\sin\theta)^2}\, a\cos\theta\, d\theta = \int \sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta\, d\theta$$

$$= \int a\sqrt{1 - \sin^2\theta}\, a\cos\theta\, d\theta = \int a\cos\theta\, a\cos\theta\, d\theta = a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{\sin 2\theta}{2}\right) + c = \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{2\sin\theta\cos\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\sin\theta\sqrt{1-\sin^2\theta}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + c$$

Example 10: Evaluate: $\frac{1}{\sqrt{5-4x-x^2}}$

Solution: $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx = \int \frac{1}{\sqrt{5+4-4-4x-x^2}}\,dx = \int \frac{1}{\sqrt{9-(4+4x+x^2)}}\,dx$

$$= \int \frac{1}{\sqrt{(3)^2-(2+x)^2}}\,dx = \sin^{-1}\left(\frac{2+x}{3}\right) + c \qquad \text{(Using direct formula)}$$

Note: We can also solve by substituting $x+2 = 3\sin\theta$

Formula 3.7: $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) + C$

Derivation: Substituting $x = a\sec\theta$, we have $dx = a\sec\theta\tan\theta\,d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \int \frac{1}{\sqrt{(a\sec\theta)^2-a^2}}\,a\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{a^2\sec^2\theta-a^2}}\,a\sec\theta\tan\theta\,d\theta = \int \frac{1}{\sqrt{a^2(\sec^2\theta-1)}}\,a\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{a\tan\theta}\,a\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln[\sec\theta+\tan\theta] + c$$

$$= \ln\left[\sec\theta+\sqrt{\sec^2\theta-1}\right] + c = \ln\left[\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right] + c = \ln\left[\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right] + c$$

$$= \ln\left[\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right] + c = \ln\left[\frac{x+\sqrt{x^2-a^2}}{a}\right] + c = \ln\left(x+\sqrt{x^2-a^2}\right) - \ln a + c$$

$$= \ln\left(x+\sqrt{x^2-a^2}\right) + (c-\ln a) = \ln\left(x+\sqrt{x^2-a^2}\right) + C$$

Note: Expression $\frac{1}{\sqrt{x^2-a^2}}$ can also be integrated by making the substitution $x = a\cosh\theta$.

Formula 3.8: $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{a^2+x^2}\right) + C$

Derivation: Substituting $x = a\tan\theta$, we have $dx = a\sec^2\theta\,d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{1}{\sqrt{(a\tan\theta)^2+a^2}}\,a\sec^2\theta\,d\theta = \int \frac{1}{\sqrt{a^2\tan^2\theta+a^2}}\,a\sec^2\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\tan^2\theta+1)}}\,a\sec^2\theta\,d\theta = \int \frac{1}{a\sec\theta}\,a\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

$$= ln[\sec\theta + \tan\theta] + c = ln[\tan\theta + \sec\theta] + c = ln\left[\tan\theta + \sqrt{1+\tan^2\theta}\right] + c$$

$$= ln\left[\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right] + c = ln\left[\frac{x}{a} + \sqrt{\frac{a^2+x^2}{a^2}}\right] + c = ln\left[\frac{x}{a} + \frac{\sqrt{a^2+x^2}}{a}\right] + c$$

$$= ln\left[\frac{x+\sqrt{a^2+x^2}}{a}\right] + c = ln\left(x+\sqrt{a^2+x^2}\right) - lna + c$$

$$= ln\left(x+\sqrt{a^2+x^2}\right) + (c - lna) = ln\left(x+\sqrt{a^2+x^2}\right) + C$$

Formula 3.9: $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

This formula can easily be proved by substituting $x = a\tan\theta$.

Example 11: Evaluate: $\int \frac{1}{x^2+4x+5}\, dx$

Solution: $\int \frac{1}{x^2+4x+5}\, dx = \int \frac{1}{x^2+4x+4+1}\, dx = \int \frac{1}{(x+2)^2+(1)^2}\, dx$

$= \frac{1}{1}\tan^{-1}\left(\frac{x+2}{1}\right) + c = \tan^{-1}(x+2) + c$ (Using direct formula)

Note: We can also solve by substituting $x + 2 = \tan\theta$

Example 12: Evaluate: $\int x(x^2 - a^2)^{\frac{3}{2}}\, dx$

Solution: Putting $x^2 - a^2 = u$

$$\Rightarrow 2xdx = du \Rightarrow xdx = \frac{du}{2}$$

$$\therefore \int x(x^2-a^2)^{\frac{3}{2}}\, dx = \int (u)^{\frac{3}{2}} \frac{du}{2} = \frac{1}{2}\int (u)^{\frac{3}{2}}\, du$$

$$= \frac{1}{2}\frac{(u)^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{1}{2}\times\frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{1}{5}(x^2-a^2)^{\frac{5}{2}} + c$$

Exercise 3.3

Use suitable substitution, to evaluate the integrals.

1. $\int \frac{dx}{x^2+9}$
2. $\int \frac{dx}{\sqrt{5-x^2}}$
3. $\int (2x+7)(x^2+7x+3)^{\frac{4}{5}}\, dx$
4. $\int \frac{x^2}{x^3+1}\, dx$
5. $\int \frac{dy}{y^2+8y+20}$
6. $\int \frac{dx}{\sqrt{20-x^2-4x}}$
7. $\int \frac{x\, dx}{(4x^2+1)^3}$
8. $\int x^4\sqrt{3x^5-5}\, dx$
9. $\int \frac{2ax+b}{ax^2+bx+c}\, dx$
10. $\int \frac{dx}{(1-3x)^2}$
11. $\int \frac{z^3}{1+z^4}\, dz$
12. $\int \frac{\cot^{-1}x}{1+x^2}\, dx$

3.4 Integration by Parts

Integration by parts is a special method of integration that is very helpful technique to evaluate a wide variety of integrals that sometimes do not fit any of the basic integration formula. This method is used to find the integrals by reducing them into standard forms.

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]\,dx \qquad (1)$$

Formula (1) is called the formula for integration by parts. Using this formula, we integrate the product of two functions. The important thing to use this formula is the selection of given functions given in the product as a first or second function. The function whose integration can easily be found is considered as the second function while the first function is chosen whose derivative could be easily found. In formula (1), $f(x)$ is treated as first function while $g(x)$ as a second function.

Key Facts

- Integration by parts is not applicable for functions such as $\int\sqrt{x}\,\sin x\,dx$.
- We do not add any constant while finding the integral of the second function.
- Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, if the other function is an inverse trigonometric function or logarithmic function, then we take them as first function.
- If the product of functions contains exponential and trigonometric functions, then we can select any one of the two as a first function.

Example 13: Evaluate the integral: $\int xe^x\,dx$

Solution: In the integral $\int xe^x\,dx$, we take 'x' as a first function as its derivative will reduce it and 'e^x' as second function.

$$\therefore \quad \int xe^x\,dx = x\int e^x dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx$$

$$= xe^x - \int 1.e^x dx = xe^x - e^x + c$$

Example 14: Evaluate: (i) $\int x^2\,lnx\,dx$ (ii) $\int x\tan^{-1}x\,dx$

Solution:

(i) In the integral $\int x^2\,lnx\,dx$, we take 'lnx' as first function and 'x^2' as second function.

$$\therefore \quad \int x^2\,lnx\,dx = \int(lnx)(x^2)dx = lnx\int x^2dx - \int\left[\frac{d}{dx}(lnx)\int x^2dx\right]dx$$

$$= lnx\,.\frac{x^3}{3} - \int\frac{1}{x}.\frac{x^3}{3}dx = \frac{x^3\,lnx}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3\,lnx}{3} - \frac{1}{3}.\frac{x^3}{3} + c \quad = \frac{x^3\,lnx}{3} - \frac{x^3}{9} + c$$

(ii) In the integral $\int x\tan^{-1}x\,dx$, we take '$\tan^{-1}x$' as first function and 'x' as second function.

$$\therefore \quad \int x\tan^{-1}x\,dx = \int(\tan^{-1}x)(x)\,dx$$

$$= \tan^{-1}x \int x dx - \int \left[\frac{d}{dx}(\tan^{-1}x) \int x dx\right] dx$$

$$= \tan^{-1}x . \frac{x^2}{2} - \int \frac{1}{x^2+1} . \frac{x^2}{2} dx = \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{x^2+1} dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx = \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}(x - \tan^{-1}x) + c$$

Example 15: Apply integration by parts to evaluate:

(i) $\int \sqrt{a^2 - x^2}\, dx$ (ii) $\int \sqrt{a^2 + x^2}\, dx$

Solution:

(i) $\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\, (1)\, dx,$

Here, we take '$\sqrt{a^2 - x^2}$' as first function and '1' as second function.

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\, (1) dx$$

$$= \sqrt{a^2 - x^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{a^2 - x^2}) \int 1 dx\right] dx$$

$$= \sqrt{a^2 - x^2}\,(x) - \int \frac{-2x}{2\sqrt{a^2 - x^2}}(x) dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx + \int \frac{a^2}{\sqrt{a^2 - x^2}} dx$$

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\, dx + \int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$2\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{c}{2} = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

(ii) $\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 + x^2}\, (1)\, dx,$

Here, we take '$\sqrt{a^2 + x^2}$' as first function and '1' as second function.

$$\therefore \int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 + x^2}\, (1) dx$$

$$= \sqrt{a^2 + x^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{a^2 + x^2}) \int 1 dx\right] dx$$

$$= \sqrt{a^2 + x^2}\,(x) - \int \frac{2x}{2\sqrt{a^2 + x^2}}(x) dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}} dx + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$\int \sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\, dx + a^2 \times \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2+x^2}\,dx + \int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$2\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a}{2}\tan^{-1}\left(\frac{x}{a}\right) + \frac{c}{2} = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a}{2}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Check Point

Using integration by parts, prove that:

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$$

Example 16: Apply integration by parts to evaluate:

$$\int e^{ax}\sin bx\,dx$$

Solution: Let, $\mathrm{I} = \int e^{ax}\sin bx\,dx = \int (\sin bx)(e^{ax})\,dx$

$$= \sin bx \int e^{ax}dx - \int \left[\frac{d}{dx}(\sin bx)\int e^{ax}dx\right]dx$$

$$= \sin bx\left(\frac{e^{ax}}{a}\right) - \int \left[(b\cos bx)\left(\frac{e^{ax}}{a}\right)\right]dx$$

$$= \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\int [(\cos bx)(e^{ax})]\,dx$$

$$= \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\left[\cos bx\int e^{ax}\,dx - \int\left\{\frac{d}{dx}(\cos bx)\int e^{ax}dx\right\}dx\right]$$

$$\mathrm{I} = \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\left[\cos bx\left(\frac{e^{ax}}{a}\right) - \int\left\{(-b\sin bx)\left(\frac{e^{ax}}{a}\right)\right\}dx\right]$$

$$\mathrm{I} = \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\cos bx\left(\frac{e^{ax}}{a}\right) - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx + c$$

$$\mathrm{I} = \sin bx\left(\frac{e^{ax}}{a}\right) - \frac{b}{a}\cos bx\left(\frac{e^{ax}}{a}\right) - \frac{b^2}{a^2}\mathrm{I} + c$$

$$\mathrm{I} + \frac{b^2}{a^2}\mathrm{I} = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx + c$$

$$\left(\frac{a^2+b^2}{a^2}\right)\mathrm{I} = e^{ax}\left[\frac{1}{a}\sin bx - \frac{b}{a^2}\cos bx\right] + c$$

$$\mathrm{I} = e^{ax}\left[\frac{1}{a}\times\frac{a^2}{a^2+b^2}\sin bx - \frac{b}{a^2}\times\frac{a^2}{a^2+b^2}\cos bx\right] + c\times\frac{a^2}{a^2+b^2}$$

$$\mathrm{I} = e^{ax}\left[\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\right] + C$$

Exercise 3.4

Evaluate the integrals using integration by parts.

1. $\int \ln x\,dx$
2. $\int (\ln x)^2\,dx$
3. $\int \sin(\ln x)\,dx$
4. $\int x^3\ln x\,dx$
5. $\int y\sin 2y\,dy$
6. $\int e^x\cos x\,dx$
7. $\int x\sec^{-1}x\,dx$
8. $\int \ln(2x+3)\,dx$
9. $\int x^2e^x\,dx$
10. $\int x\cos x\,dx$
11. $\int \cos^{-1}x\,dx$
12. $\int \tan^{-1}x\,dx$
13. $\int x\sec^2x\,dx$
14. $\int x^2\sin^{-1}x\,dx$
15. $\int \ln\left[x+\sqrt{1+x^2}\right]dx$
16. $\int x^3e^{x^2}dx$
17. $\int x^2\sin x\,dx$
18. $\int \frac{\ln x}{\sqrt{x}}\,dx$

3.5 Integration by Partial Fraction

When the terms in the sum:

$$\frac{3}{x+4}+\frac{4}{x+2} \qquad \text{(i)}$$

are combined by means of a common denominator, we obtain a single rational expression:

$$\frac{7x+22}{(x+4)(x+2)} \qquad \text{(ii)}$$

Suppose that we are faced with the problem of evaluating the integral:

$$\int \frac{7x+22}{(x+4)(x+2)}dx$$

From (i) and (ii), we have:

$$\int \frac{7x+22}{(x+4)(x+2)}dx = \int \left[\frac{3}{(x+4)}+\frac{4}{(x+2)}\right]dx = \int \frac{3}{(x+4)}dx + \int \frac{4}{(x+2)}dx$$

$$= 3\int \frac{1}{(x+4)}dx + 4\int \frac{1}{(x+2)}dx = 3\ln(x+4) + 4\ln(x+2) + c$$

This example illustrates a procedure for integrating certain rational fractions $\frac{P(x)}{Q(x)}$, where the degree of $P(x)$ is less than the degree of $Q(x)$. This method, known as partial fractions consists of decomposing such rational fractions into simplest component fractions and then evaluating the integral term by term.

Example 17: Evaluate: $\int \frac{x^3-2x}{x^2+3x+2}dx$

Solution: We observe that degree of numerator is greater than that of denominator.

$$\therefore \int \frac{x^3-2x}{x^2+3x+2}dx = \int \left[x-3+\frac{5x+6}{x^2+3x+2}\right]dx \qquad \text{(i)}$$

Now, $\frac{5x+6}{x^2+3x+2} = \frac{5x+6}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2}$

> **Check Point**
>
> Evaluate $\int \frac{2x+1}{(x-1)(x+3)}dx$

By equating numerator, we get:

$$5x+6 = A(x+2)+B(x+1) \qquad \text{(ii)}$$

If we set $x=-2$ and $x=-1$, we get $B=4$ and $A=1$, repectively.

$$\therefore \int \frac{x^3-2x}{x^2+3x+2}dx = \int \left[x-3+\frac{1}{x+1}+\frac{4}{x+2}\right]dx = \int x\,dx - 3\int dx + \int \frac{1}{x+1}dx + 4\int \frac{1}{x+2}dx$$

$$= \frac{x^2}{2} - 3x + ln(x+1) + 4\,ln(x+2) + c$$

Example 18: Evaluate: $\int \frac{x^2+2x+4}{(x+1)^3}dx$

Solution: Given fraction can be written as:

$$\frac{x^2+2x+4}{(x+1)^3} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

By equating numerator, we get:

$$x^2+2x+4=A(x+1)^2+B(x+1)+C$$
$$x^2+2x+4=Ax^2+(2A+B)x+(A+B+C)$$

Comparing coefficients of like powers of x from both sides, we gat:

$$A=1, 2A+B=2 \text{ and } A+B+C=4$$

Solving the equations, we have:

$$A=1, B=0 \text{ and } C=3$$

$$\therefore \int \frac{x^2+2x+4}{(x+1)^3}dx = \int\left[\frac{1}{x+1}+\frac{0}{(x+1)^2}+\frac{3}{(x+1)^3}\right]dx = \int\frac{1}{x+1}dx+3\int\frac{1}{(x+1)^3}dx$$

$$=\int\frac{1}{x+1}dx+3\int(x+1)^{-3}dx = ln(x+1)-\frac{3}{2}(x+1)^{-2}+c$$

$$= ln(x+1)-\frac{3}{2(x+1)^2}+c$$

Example 19: Evaluate: $\int\frac{3x^2+5x+3}{(x+2)(x^2+1)}dx$

Solution: Given fraction can be written as:

$$\frac{3x^2+5x+3}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$$
$$3x^2+5x+3=A(x^2+1)+(Bx+C)(x+2)$$
$$3x^2+5x+3=(A+B)x^2+(2B+C)x+(A+2C)$$

Equating coefficients:

$$A+B, \quad 2B+C=5, \quad A+2C=3$$

Solving the equations, we have:

$$A=1, B=2, C=1$$

$$\therefore \int\frac{3x^2+5x+3}{(x+2)(x^2+1)}dx=\int\left(\frac{1}{x+2}+\frac{2x+1}{x^2+1}\right)dx$$

$$=\int\frac{1}{x+2}dx+\int\frac{2x}{x^2+1}dx+\int\frac{1}{x^2+1}dx$$

$$= ln(x+1)+ln(x^2+1)+\tan^{-1}x+c$$

Exercise 3.5

Evaluate the integrals using partial fractions.

1. $\int\frac{3x+7}{(x+2)(x+3)}dx$
2. $\int\frac{4x+9}{x^2+x-12}dx$
3. $\int\frac{21-8x}{x^2+x-6}dx$
4. $\int\frac{3x+7}{(x+2)^2}dx$
5. $\int\frac{5x^2-5x+2}{(x+1)(x-1)^2}dx$
6. $\int\frac{9x^2+3x+29}{(x+1)(x^2+4)}dx$
7. $\int\frac{7x^2+7x+4}{(2x+1)(x^2+x+1)}dx$
8. $\int\frac{x^3+4x^2+9x+14}{x^2+4x+3}dx$
9. $\int\frac{1}{x^2-9}dx$
10. $\int\frac{1}{x^3+2x^2+x}dx$
11. $\int\frac{e^x}{(e^x+1)^2(e^x-2)}dx$
12. $\int\frac{x}{(x+1)^2(x^2+1)}dx$

3.6 The Definite Integral

This section introduces the definite integral, a fundamental mathematical tool that establishes relationships between area and other essential quantities including length, volume, density, probability, and work.

3.6.1 Partition of the Interval

A partition of the interval $[a, b]$ is a collection of points:

$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$

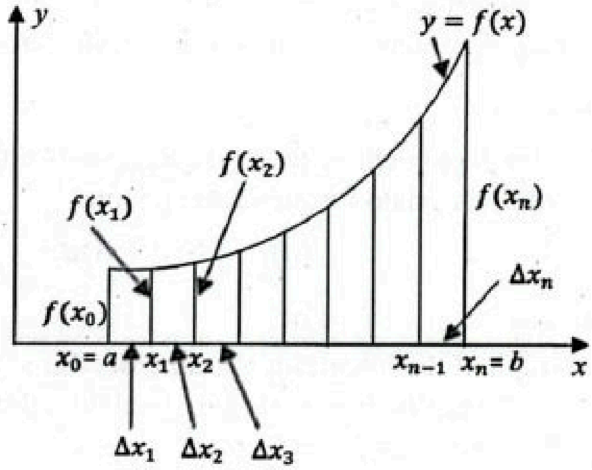
that divides $[a, b]$ into n subintervals of lengths:

$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1,$

$\Delta x_3 = x_3 - x_2, \ldots, \Delta x_n = x_n - x_{n-1}$

The partition is said to be regular provided all subintervals have the same length:

$$\Delta x = \Delta x_1 = \frac{b-a}{n}$$



In the figure, each partition looks like a rectangle. For a regular partition, widths of the rectangles approach to zero as n is made large.

Area of first (left most) rectangle = length × width = $f(x_1) \times \Delta x_1$

Area under the curve = sum of areas of n rectangles

$= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_n)\Delta x_n$

$$= \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots\ldots (i)$$

Expression (i) represents approximation of sum of areas of n rectangles.

Based on our inductive concept, the area under the curve and between the interval $[a, b]$ is:

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots\ldots (ii)$$

Expression (ii) provides the fundamental concept of integral calculus and form the basis of the following definition.

Definition 3.2: A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit:

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

exists and does not depend upon the choice of partitions or on the choice of the points x_k in the subintervals. In the such case, we denote the limit by the symbol:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k \ \ldots\ldots (iii)$$

Expression (iii) is called the definite integral of f from a to b. The numbers a and b are called lower limit and upper limit of integration respectively and $f(x)$ is called the integrand.

Theorem 3.2: If a function f is continuous on an interval $[a, b]$ then f is integrable on $[a, b]$ and the net signed area under the curve between the interval $[a, b]$ is:

$$A = \int_a^b f(x)dx$$

In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute the shaded area.

Example 20:

Sketch the region where area is represented by the definite integral and evaluate the integral using an appropriate formula from geometry.

(i) $\int_1^5 3dx$ (ii) $\int_{-2}^2 (x+3)dx$ (iii) $\int_0^1 \sqrt{1-x^2}dx$

Solution:

(i) Graph of the integral is the horizontal line $y = 3$.
So, the region is a rectangle of height 3 drawn over the interval from1 to 5.
From figure (1), we have:
$\int_1^5 3dx$ = area of rectangle $= 4 \times 3 = 12$ sq. units

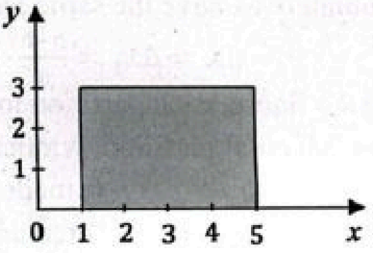


Fig. (1)

(ii) Graph of the integral is the line $y = x + 3$.
When $x = -2,\ y = -2 + 3 = 1$
When $x = 2,\ y = 2 + 3 = 5$
So, the region is trapezoid where base ranges from $x = -2$ to $x = 2$.
From figure (2), we have:
$\int_{-2}^2 (x+3)dx$ = area of trapezoid
$= \frac{1}{2}(1+5)(4) = 12$ sq. units

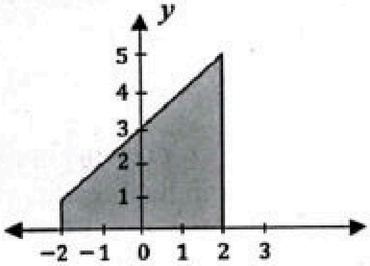


Fig. (2)

(iii) Graph of the function $y = \sqrt{1-x^2}$ is the upper semi-circle of radius 1 centred at the origin.
So, the region is upper right quarter-circle of radius 1 centred at origin.
From figure (3), we have:
$\int_0^1 \sqrt{1-x^2}dx$ = area of quarter circle
$= \frac{1}{4} \times \pi(1)^2 = \frac{\pi}{4}$ sq. units

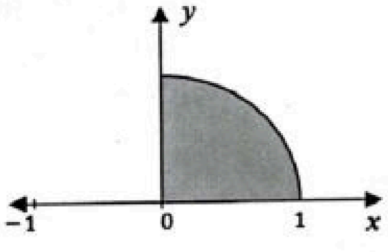


Fig. (3)

Example 21: Evaluate the following.

(i) $\int_0^1 (x-1)dx$ (ii) $\int_0^2 (x-1)dx$

Solution:

(i) The graph of the integral is the line $y = x - 1$.

When $x = 0,\ y = 0 - 1 = -1$

When $x = 1,\ y = 1 - 1 = 0$

The region is a triangle from $x = 0$ to $x = 1$.

From figure (4), we get:

$\int_0^1 (x-1)dx$ = area of triangle $= \frac{1}{2}(1)(1) = \frac{1}{2}$ sq. units

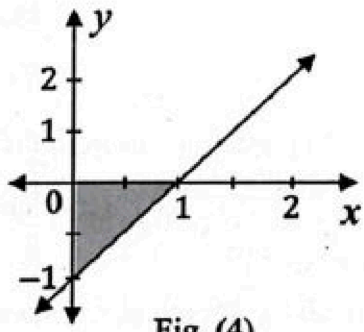


Fig. (4)

(ii) The graph of the integral is the line $y = x - 1$.

When $x = 0,\ y = 0 - 1 = -1$

When $x = 1,\ y = 1 - 1 = 0$

When $x = 2,\ y = 2 - 1 = 1$

The regions are two triangles from $x = -1$ to $x = 0$ and $x = 1$ to $x = 2$. From figure (5), we get:

$\int_0^2 (x-1)dx = \int_0^1 (x-1)dx + \int_1^2 (x-1)dx$

= area of triangle A_1 + Area of triangle A_2

$= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1$ sq. units

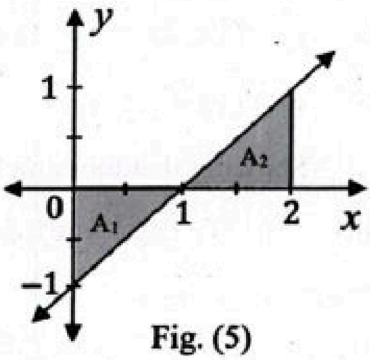


Fig. (5)

Note: In the figure (5), the area of triangle A_1 is below the x-axis and the area of triangle A_2 is above x-axis, therefore:

$A_1 = -\frac{1}{2}$ and $A_2 = \frac{1}{2}$ which implies $A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$

But area cannot be negative, therefore in such cases, we take net area as:

$$A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$$

3.7 Properties of The Definite Integral

In the finite closed interval $[a, b]$, when upper limit of integration in the definite integral is greater than the lower limit of integration $(a < b)$, the following facts are true.

(i) If lower and upper limits of integration are equal, then area is zero. i.e.,

$$\int_a^a f(x)dx = 0$$

For example,

$$\int_2^2 x\,dx = 0$$

(ii) If the lower limit of integration is greater than the upper limit of integration, then:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Which states that interchanging the limits of integral reverses the sign of integral.

For example,

$$\int_1^0 (x-1)dx = -\int_0^1 (x-1)dx = \frac{1}{2}$$

(iii) If c is the point between a and b then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b dx$$

For example, in figure (5), we have:

$$\int_0^2 (x-1)dx = \int_0^1 (x-1)dx + \int_1^2 (x-1)dx$$

Theorem 3.3:

If f and g are integrable on $[a, b]$ and c is a constant, then cf, $f+g$ and $f-g$ are integrable on $[a, b]$ and the following statements are true.

(i) $\int_a^b c\,f(x)dx = c\int_a^b f(x)dx$ (The constant has no effects of limits on it.)

(ii) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Showing that the limit of a sum or difference is the sum or difference of the limits.

(iii) $\int_a^b [c\,f(x) \pm d\,g(x)]dx = c\int_a^b f(x)dx \pm d\int_a^b g(x)dx$

Example 22: Find:

(i) $\int_{-1}^4 [2f(x) + 5g(x)]dx$ if $\int_{-1}^4 f(x)dx = 2$ and $\int_{-1}^4 g(x)dx = 4$

(ii) $\int_{-1}^3 4f(x)dx$ if $\int_{-1}^2 f(x)dx = 3$ and $\int_2^3 f(x)dx = 1$

Solution:

(i) $\int_{-1}^4 [2f(x) + 5g(x)]dx = \int_{-1}^4 2f(x)dx + \int_{-1}^4 5g(x)dx = 2\int_{-1}^4 f(x)dx + 5\int_{-1}^4 g(x)dx$

$= 2(2) + 5(4) = 24$

(ii) $\int_{-1}^3 4f(x)dx = 4\int_{-1}^3 f(x)dx = 4\left[\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx\right]$

$= 4(3+1) = 4 \times 4 = 16$

Exercise 3.6

1. Sketch the region where area is represented by the definite integral and evaluate the integral using an appropriate formula from geometry.

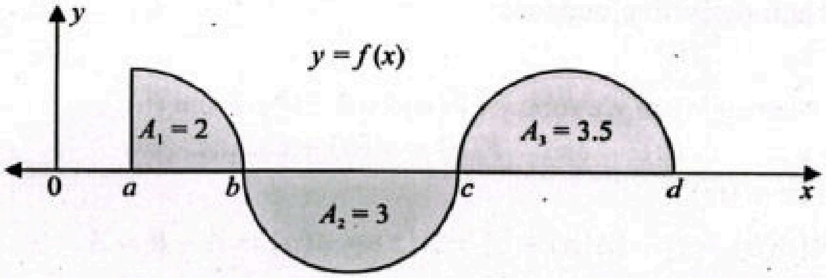
   (i) $\int_0^4 x dx$ (ii) $\int_{-3}^0 x dx$ (iii) $\int_0^2 (x-1)dx$

   (iv) $\int_0^2 (x+1)dx$ (v) $\int_{-3}^3 2dx$

2. Evaluate the integrals in each part when $f(x) = \begin{Bmatrix} x; & x \le 1 \\ 3; & x > 1 \end{Bmatrix}$.

   (i) $\int_0^1 f(x)dx$ (ii) $\int_{-1}^1 f(x)dx$ (iii) $\int_1^4 f(x)dx$ (iv) $\int_{-1}^2 f(x)dx$

3. Using the area shown below in the figure, evaluate the integrals.



(i) $\int_a^b f(x)dx$ (ii) $\int_b^c f(x)dx$ (iii) $\int_c^d f(x)dx$

(iv) $\int_a^c f(x)dx$ (v) $\int_b^d f(x)dx$ (vi) $\int_a^d f(x)dx$

4. Find:

   $\int_1^5[3f(x)-2g(x)]dx$ if $\int_1^5 f(x)dx = 4$ and $\int_1^5 g(x)dx = 5$

5. Find:

   $\int_1^4 f(x)dx$ if $\int_1^2 f(x)dx = 1$ and $\int_2^4 f(x)dx = 2$

6. Find:

   $\int_3^{-2} f(x)dx$ if $\int_{-2}^1 f(x)dx = 1$ and $\int_1^3 f(x)dx = -5$

7. Use appropriate formula from geometry to evaluate integrals.

   (i) $\int_{-1}^4 (3-x)dx$ (ii) $\int_0^1[2+\sqrt{1-x^2}\,]dx$ (iii) $\int_2^2 \sqrt{x^3-4}\,dx$

3.8 Fundamental Theorem of Calculus

In this section, we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the 'Fundamental Theorem of Calculus'. We will provide a powerful method for evaluating definite integrals using anti-derivatives.

We consider a non-negative and continuous function f on an interval $[a, b]$. The area A under the graph f over the interval $[a, b]$ is represented by the definite integral:

$$A = \int_a^b f(x)dx \text{ .... (i)}$$

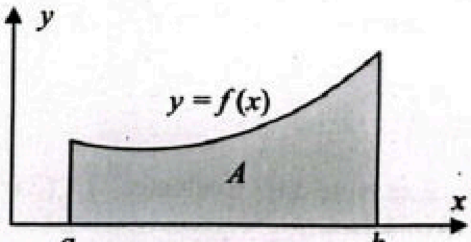


Fig. (6)

From (i), we have:

$A(a) = 0$ [The area under the curve from a to a is the area above the single point a and hence is zero.]

Similarly, $A(b) = A$ [The area under the curve from a to b is A.]

The formula $A'(x) = f(x)$ provides that $A(x)$ is an anti-derivative of $f(x)$ which implies that every other anti-derivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$.

By definition of anti-derivative, suppose:

$F(x) = A(x) + c \dots\dots\dots (ii)$

We check what happens when we subtract $F(a)$ from $F(b)$. From (ii):

$F(a) = A(a) + c \dots\dots\dots (iii)$ and $F(b) = A(b) + c \dots\dots\dots (iv)$

Subtracting (iii) from (iv):

$$F(b) - F(a) = [A(b) + c] - [A(a) + c] = A(b) - A(a) = A - 0 = A$$

Therefore, from (i), we have:

$$A = \int_a^b f(x)dx = F(b) - F(a) \dots\dots\dots (v)$$

Statement: The Fundamental Theorem of Calculus states that if f is continuous on $[a, b]$ ad F is antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

This can be written as:

$$\int_a^b f(x)dx = |F(x)|_{x=a}^{x=b} = F(b) - F(a)$$

We can emphasise that a and b are values for the variable x.

Thus, the definite integral can be evaluated by finding any anti-derivative of the integral and then subtracting the value of this anti-derivative at the lower limit of integration from its value at the upper limit of integration.

Example 23: Evaluate: $\int_1^3 x\,dx$

Solution: $\int_1^3 x\,dx = \left|\frac{x^2}{2}\right|_1^3 = \frac{3^2}{2} - \frac{1^2}{2}$ ⟶ First, we apply upper limit and then lower limit.

$$= \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$$

Example 24: Evaluate: $\int_{-2}^2 (3x^2 - x + 1)dx$

Solution: $\int_{-2}^2 (3x^2 - x + 1)dx = \left|x^3 - \frac{x^2}{2} + x\right|_{-2}^2 = \frac{3^2}{2} - \frac{1^2}{2}$

$$= \left(2^3 - \frac{2^2}{2} + 2\right) - \left((-2)^3 - \frac{(-2)^2}{2} + (-2)\right)$$

$$= (8 - 2 + 2) - (-8 - 2 - 2) = 8 + 12 = 20$$

Example 25: Evaluate: $\int_0^2 \sqrt{2x^2+1}\, xdx$

Check Point

Evaluate: $\int_{\frac{\pi}{6}}^{\pi} \cos x\, dx$

Solution: We can apply two methods.

Method-1: By substitution but without changing the limits.

Let $u = 2x^2+1$ which implies $du = 4x\, dx$

Thus, $\int_0^2 \sqrt{2x^2+1}\, xdx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1} \times 4xdx$

$= \frac{1}{4}\int_0^2 \sqrt{u} \times du = \frac{1}{4} \times \left|\frac{2}{3}u^{\frac{3}{2}}\right|_0^2$ (Substituting for u)

$= \left|\frac{1}{6}(2x^2+1)^{\frac{3}{2}}\right|_0^2$ (Resubstituting for x)

Applying limits, we get:

$= \frac{1}{6}[2(2)^2+1]^{\frac{3}{2}} - \frac{1}{6}[2(0)^2+1]^{\frac{3}{2}} = \frac{1}{6}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$

Method-2: By substitution with changing the limits.

Let $u = 2x^2+1$ which implies $du = 4x\, dx$

When $x = 0, u = 2(0)^2+1 = 1$ and when $x = 2, u = 2(2)^2+1 = 9$

Thus, $\int_0^2 \sqrt{2x^2+1}\, xdx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1} \times 4xdx$

$= \frac{1}{4}\int_1^9 \sqrt{u} \times du = \frac{1}{4} \times \left|\frac{2}{3}u^{\frac{3}{2}}\right|_1^9$ (Substituting for u)

$= \frac{1}{6}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$

Example 26: Evaluate: $\int_a^b \frac{1}{1-\cos x}\, dx$ when $a = \frac{\pi}{4}, b = \frac{\pi}{3}$

Solution: $\int_a^b \frac{1}{1-\cos x}\, dx = \int_a^b \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x}\, dx = \int_a^b \frac{1+\cos x}{1-\cos^2 x}\, dx$

$= \int_a^b \frac{1+\cos x}{\sin^2 x}\, dx = \int_a^b \left[\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right] dx$

$= \int_a^b [\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x] dx$

$= |-\cot x|_a^b + |-\operatorname{cosec} x|_a^b$

Check Point

Evaluate:

$\int_0^1 \sin^{-1} x\, dx$

Applying limits and substituting values of a and b, we get:

$\int_a^b \frac{1}{1-\cos x}\, dx = -\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right) - \left(\operatorname{cosec}\frac{\pi}{3} - \operatorname{cosec}\frac{\pi}{4}\right)$

$= -\left(\frac{1}{\sqrt{3}} - 1\right) - \left(\frac{2}{\sqrt{3}} - \sqrt{2}\right) = \frac{1}{\sqrt{3}} + 1 - \frac{2}{\sqrt{3}} + \sqrt{2}$

$= 1 + \sqrt{2} - \sqrt{3}$

Example 27: Evaluate: $\int_1^e x\, lnx\, dx$

Solution: Taking lnx as first function and integrating by parts, we get:

$$\int_1^e x\, lnx\, dx = \int_1^e (lnx)(x)\, dx = \left| lnx \times \frac{x^2}{2} \right|_1^e - \int_1^e \frac{1}{x} \times \frac{x^2}{2}\, dx$$

$$= \left| lnx \times \frac{x^2}{2} \right|_1^e - \frac{1}{2}\int_1^e x\, dx = \left| lnx \times \frac{x^2}{2} \right|_1^e - \frac{1}{2} \times \left| \frac{x^2}{2} \right|_1^e$$

$$= \left(lne \times \frac{e^2}{2} - ln1 \times \frac{1^2}{2} \right) - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1^2}{2} \right) = \left(1 \times \frac{e^2}{2} - 0 \times \frac{1}{2} \right) - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{2} - 0 - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

Exercise 3.7

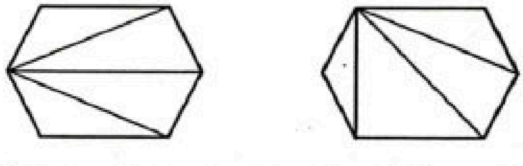
Evaluate the definite integrals.

1. $\int_{-1}^{2} (2x+3)\, dx$
2. $\int_{-4}^{12} \sqrt{y+4}\, dy$
3. $\int_0^{\frac{1}{2}} (2x+1)^{-\frac{1}{3}}\, dx$
4. $\int_0^3 (6x^2 - 4x + 5)\, dx$
5. $\int_{-2}^{1} (12x^5 - 36)\, dx$
6. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \cos\theta\, d\theta$
7. $\int_0^{\frac{\pi}{4}} \sec^2 2\theta\, d\theta$
8. $\int_2^4 \frac{x^2+8}{x^2}\, dx$
9. $\int_{-\frac{1}{2}}^{\frac{3}{2}} x - \cos\pi x\, dx$
10. $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx$
11. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \cos x\, dx$
12. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos\theta}{(\theta + \sin\theta)^2}\, d\theta$
13. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec x + \tan x)^2\, dx$
14. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx$
15. $\int_1^3 lnx\, dx$
16. $\int_2^4 \left(e^{\frac{x}{2}} - e^{\frac{x}{4}} \right) dx$
17. $\int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x}\, dx$
18. $\int_0^{\frac{\pi}{4}} \tan^{-1} y\, dy$
19. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(2+\cos x)(5+\cos x)}\, dx$
20. $\int_2^5 \frac{1}{x(x+1)}\, dx$

3.9 Area and Volume

The definite integrals have applications that extend far beyond the area problems. In this section, we will also apply definite integrals for finding the volume. We have an inductive idea of what is meant by the area of certain geometrical figures. It is a number that in same way measures the size of the region enclosed by the figure. The area of a rectangle is the product of its length and width likewise the area of a triangle is half the product of lengths of the base and the altitude.

The area of a polygon may be defined as the sum of the areas of triangles into which it is decomposed and it can be proved that the area thus obtained is independent of how the polygon is decomposed into triangles.



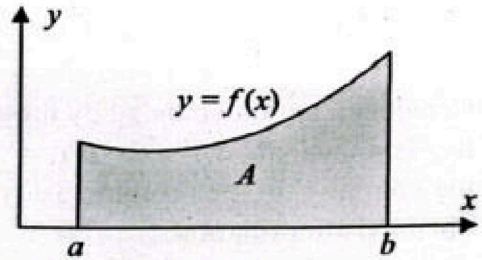
However, how do we define the area of a region in a plane if the region is bounded by a curve? We even certain that such a region has an area? In the same way volume of solids can be found by using definite integration.

3.10 Area of Bounded Region

3.10.1 Area Between a Curve and the X-axis

If f is a non-negative continuous function on $[a, b]$, then the area under the graph of f from a to b is:

$$A = \int_a^b f(x)dx$$



Example 28: Find the area of the region bounded by the line $2y + x = 8$, the x-axis and, the lines $x = 2$ and $x = 4$.

Solution: In the graph, CD is the given line.

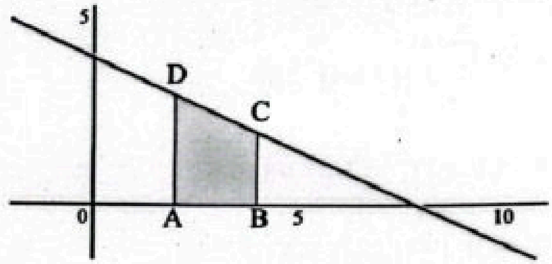
$2y + x = 8 \Rightarrow y = \frac{8-x}{2} \Rightarrow y = 4 - \frac{x}{2}$

Required area = area of trapezium ABCD

= area between line CD and x-axis from $x = 2$ to $x = 4$

$= \int_2^4 ydx = \int_2^4 \left(4 - \frac{x}{2}\right) dx = \left|4x - \frac{x^2}{4}\right|_2^4$

$= \left[4(4) - \frac{4^2}{4}\right] - \left[4(2) - \frac{2^2}{4}\right] = (16 - 4) - (8 - 1) = 5$ sq. units



3.10.2 Area Between Curves

If the function $f(x)$ is greater than the function $g(x)$ for all x between a and b, then the area under the graph of $f(x)$ minus the area under the graph of $g(x)$ is the area between the curves. Thus, the area between the curves $f(x)$ and $g(x)$ is:

$$A = \int_a^b [f(x) - g(x)]dx \; ; \; f(x) > g(x)$$

Example 29: Find the area of the region bounded by graphs of:

$$f(x) = (x - 1)^2 \text{ and } g(x) = 3 - x$$

Solution: To find the limits of integration, we find common points of both functions by solving

$f(x) = g(x) \Rightarrow (x - 1)^2 = 3 - x$

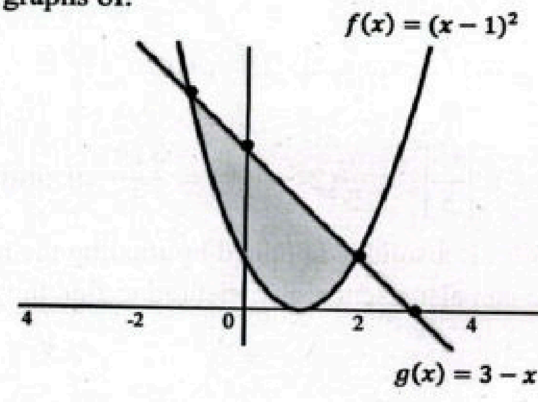
$\Rightarrow x^2 - x - 2 = 0$

After solving, we get:

$x = -1$ and $x = 2$

For $-1 < x < 2$, $g(x) > f(x)$

(Also clear from the graph of both curves.)



Thus, the area of region bounded is:

$$A = \int_{-1}^{2}[g(x) - f(x)]dx = \int_{-1}^{2}[(3-x) - (x-1)^2]dx \int_{-1}^{2}(2 + x - x^2)dx$$

$$= \left|2x + \frac{x^2}{2} - \frac{x^3}{3}\right|_{-1}^{2} = \left[2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right] - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right]$$

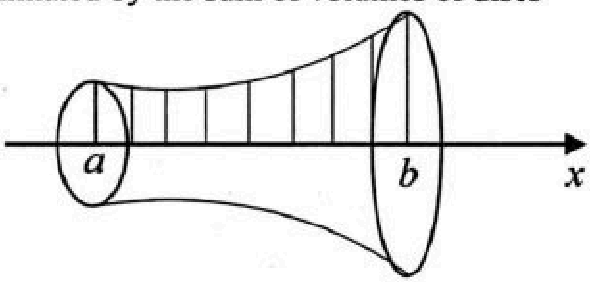
$$= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = 8 - 3 - 0.5 = 4.5 \text{ sq. units}$$

3.11 Volume of Solids of Revolution

3.11.1 Disc Method

Consider a region bounded by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ that is rotated about x-axis. If $a = x_0 < x_1 < x_2 \dots < x_n = b$ is partition of the interval $[a, b]$, the volume V of the resulting 3-D region can approximated by the sum of volumes of discs obtained after rotation.

The radius and height of discs D_i are $f(x)_i$ and Δx_i Respectively. Thus:



$$V = \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x_i$$

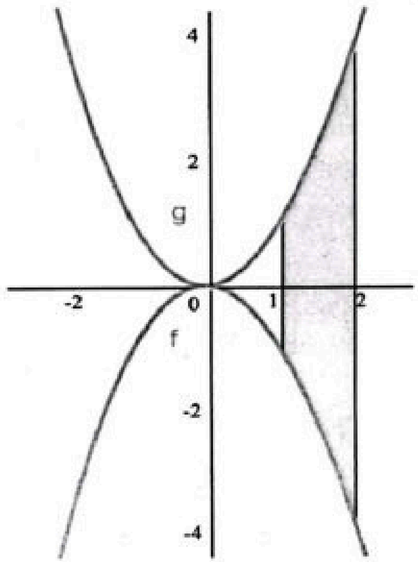
Volume of disc = area of base × height = $(\pi r^2)(h)$

Letting $\Delta x_i \to 0$, we have:

$$V = \pi \int_a^b [f(x)]^2 dx$$

Example 30:

Find the volume of the solid obtained by rotating the graph $y = x^2$ between $x = 1$ and $x = 2$ about x-axis.



Solution:

$$V = \pi \int_1^2 [f(x)]^2 dx$$

$$V = \pi \int_1^2 (x^2)^2 dx = \pi \int_1^2 x^4 dx$$

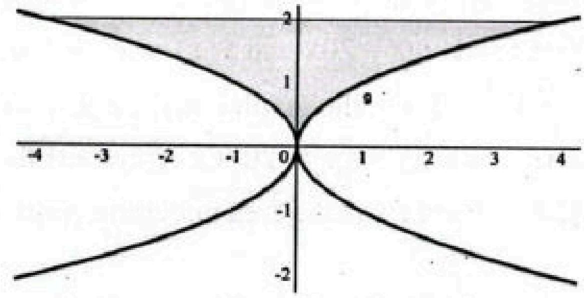
$$V = \pi \left|\frac{x^5}{5}\right|_1^2 = \frac{\pi}{5}(2^5 - 1^5) = \frac{31\pi}{5} \text{ cu. units}$$

Note: If a solid is obtained by rotating the regions bounded by the graph $x = g(y)$ about y-axis, we can also use the disc method to find the volume as follows.

$$V = \pi \int_a^b [g(x)]^2 dx$$

Example 31:

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}, y = 0$ and $y = 2$ is revolved about the y-axis.



Solution:

First sketch the region and the solid. The cross section taken perpendicular to the y-axis and disk suggests that we can rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, $g(y) = y^2$ and the volume is:

$$V = \pi\int_a^b [g(y)]^2 dy = \pi\int_0^2 (y^2)^2 dy = \pi\int_0^2 y^4 dy$$

$$V = \pi\left|\frac{y^5}{5}\right|_0^2 = \frac{\pi}{5}(2^5 - 0^5) = \frac{32\pi}{5} \text{ cu. units}$$

3.12 Applications

3.12.1 Consumer and Producer Surpluses

Economists use the definite integral to define the concept of consumer and producer surpluses.

The demand for a commodity by consumers as well as the amount supplied to the market by the manufacturers can often be expressed as a function of the per unit price. Let $D(x)$ and $S(x)$ be the number of units demanded and the number of units supplied, respectively, when the commodity sells at a price x per unit.

If the demand equals the supply:

$$D(x) = S(x)$$

The market is said to be in equilibrium and the corresponding price of the commodity is called the equilibrium price. If p is the equilibrium price and b is the price at which the demand of the commodity is zero ($b(s) = 0$), the integral:

$$Cs = \int_p^b D(x)dx$$

is called the consumer surplus. Similarly, the integral:

$$Ps = \int_c^P S(x)dx$$

where $S(c) = 0$, is called the producer surplus.

Example 32:

Suppose the demand and supply of a commodity selling for x dollars a unit and $D(x) = 1000 - 20x$ and $S(x) = x^2 + 10x$, respectively. Find the consumer and producer surplus.

Solution: From the graph it is clear that $D(x) = 0$ when $b = 50$, $S(x) = 0$ when $c = 0$ and $D(s) = S(x)$ for $p = 20$. Cs represents the area under the graph of $D(x)$ on the interval [20, 50] and Ps is the area under the graph of $S(x)$ on [0, 20]. We have:

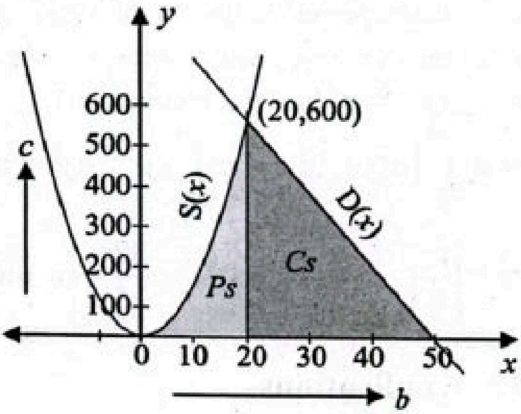
$$Cs = \int_p^b D(x)dx = \int_{20}^{50} (1000 - 20x)dx$$

$$Cs = \left|1000x - \frac{20}{2}x^2\right|_{20}^{50} = \$\,9000$$

And,

$$Ps = \int_c^p S(x)dx = \int_0^{20} (x^2 + 10x)dx$$

$$Ps = \left|\frac{1}{3}x^3 + 5x^2\right|_0^{20} = \$\,4666.67$$



3.12.2 Rectilinear Motion

If $f(t)$ is the position function of an object moving in the straight line, then we have:

velocity = $v(t) = \frac{dS}{dt}$ and acceleration = $a(t) = \frac{dv}{dt}$

By using the definition of anti-derivative, the quantities S and v can be written as indefinite integrals.

$S(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$

By knowing the initial position $S(0)$ and the initial velocity $v(0)$, we can find specific values of the constants of integration.

Key Facts

(i) For upward motion:
$S(0) = 0$, $v(0) > 0$, $a = g = -98m/s^2 = -32ft/s^2$

(ii) For downward motion:
$S(0) = h$, $v(0) = 0$, $a = g = 98m/s^2 = 32ft/s^2$

Example 33:

The position function of an object that moves on a coordinate line is $S(t) = t^2 - 6t$. Where S is measured in centimetres and t in seconds. Find the distance travelled in the time interval [3, 9].

Solution: The velocity function:
$$v(t) = \frac{dS}{dt} = 2t - 6$$

implies that $v \geq 0$ for $3 \leq t \leq 9$. Hence the distance travelled is:

$$S(t) = \int_3^9 v(t)dt = \int_3^9 (2t - 6)dt$$

$$= |t^2 - 6t|_3^9 = (81 - 54) - (9 - 18) = 4\ cm$$

3.12.3 Work

In physics when a constant force F moves an object a distance d in the same direction, the work done is defined as $W = Fd$.

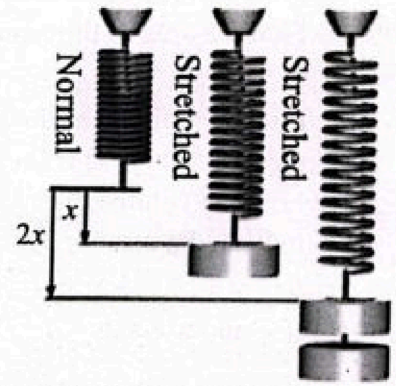
Definition: Let $F(x)$ be a continuous force acting at a point in the interval $[a, b]$, then the work done W by the force on moving an object from a to b is:

$$W = \int_a^b F(x)dx$$

3.12.4 Motion of Spring

Hook's law states that "when a spring is stretched (or compressed) beyond its natural length, the restoring force exerted by the spring is directly proportional to the amount of elongation (or compression)". Thus, in order to stretch a spring, x units beyond its natural length, we need to apply the force:

$F(x) = kx$; k is spring constant.



Example 34:

A force of 130 N is required to stretch a spring 50 cm. Find the work done in stretching the spring 20 cm beyond its natural (unstretched) length.

Solution:

$x = 50\ cm = 0.5m$ and $F = 130N$

Substituting values of x and F in $F = kx$, we have:

$130 = k \times 0.5 \quad \Rightarrow \quad k = 260\ N/m$

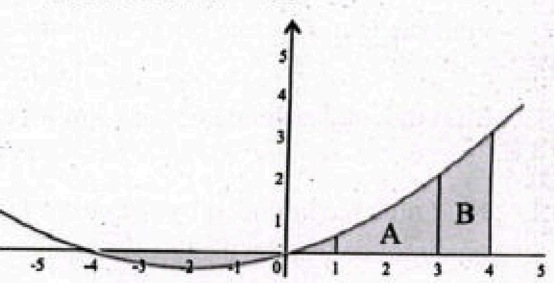
Thus, $F = kx \quad \Rightarrow \quad F = 260\,x$

Now, $x = 20\ cm = 0.2m$, so that the work done in stretching the spring by this amount is:

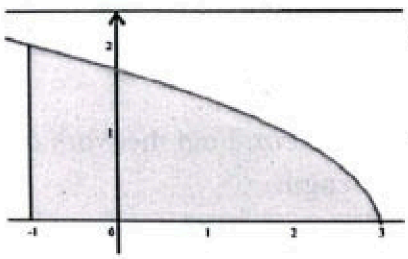
$$W = \int_0^{\frac{1}{5}} 260x\,dx = |130x^2|_0^{\frac{1}{5}} = \frac{26}{5} = 5.2\,J$$

Exercise 3.8

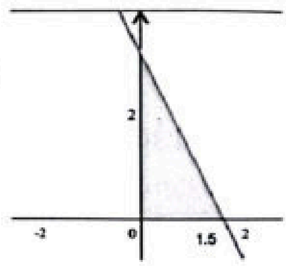
1. Find the area of region bounded by the curve $y = x^2$, the x-axis, lines $x = 1$ and $x = 3$.
2. Find the area under the curve $y = \sqrt{6x+4}$ (above x-axis) from $x = 0$ to $x = 2$.
3. Find the area of region bounded by the curve $y^2 = 4x$ and line $x = 3$.
4. In the figure, a sketch of the function $y = \frac{1}{2}(0.2x^2 + x)$ is shown. Find:
   (i) the area of region A.
   (ii) the area of region B.
   (iii) area of the region from $x = 1$ to $x = 4$.
   (iv) area of the region from $x = -1$ to $x = -4$.



5. Find the area bunded by the graph:
   (i) $y = 1 + \cos x$; $[0, 3\pi]$ (ii) $y = -1 + \sin x$; $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$
6. Find the area of the region bounded by the graphs of $y = x, y = -2x$ and $x = 3$.
7. Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$ and bounded on the sides by the lines $x = 0$ and $x = 2$.
8. Find the area bounded by the curve $y = x^3 + 1$, the x-axis and the line $x = 1$.
9. Find the area of the region enclosed by $x = y^2$ and $y = x - 2$ integrating with respect to y.
10. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis.
11. Find the volume of the sloid that results when the shaded region is revolved about the indicated axis.
   (i) $y = \sqrt{3 - x}$ about x-axis



   (ii) $y = 3 - 2x$ about y-axis



12. An object moves in a straight line according to the position function given below. If f is measured in centimetres, find the distance travelled by the object in the indicated time interval:
   (i) $S(t) = t^2 - 2t$; $[0, 5]$ (ii) $S(t) = t^3 - 3t^2 - 9t$; $[0, 4]$
   (iii) $S(t) = 6 \sin \pi t$; $[1, 3]$
13. It takes a force of 50 N to stretch a spring of 0.5m. Find the work done in stretching the spring 0.6m beyond its natural length.
14. A force $F = \frac{3}{2}x\ lb$ is needed to stretch a 10 inch spring an additional x inch. Find the work done in stretching the spring 16 inch.
15. Find the consumer and producer surpluses, when:
   (i) $S(x) = 24,\ D(x) = 100 - 2x$
   (ii) $S(x) = x^2 - 4,\ D(x) = -x + 8$
   (iii) $S(x) = 2x^2 + 3x,\ D(x) = 36 - x^2$
16. Find the total revenue obtained in 4 years if the rate of increase in dollars per year is:
$$f(t) = 200(t - 5)^2$$
17. Find the total revenue obtained in 8 years if the rate of increase in dollars per year is:
$$f(t) = 600\sqrt{1 + 3t}$$
18. Find the area bounded by the curve $f(x) = x^3 - 2x^2 + 1$ and the x-axis in the first quadrant bonded by the line $x = 1.5$.

Review Exercise

1. Select the correct option in the following.
   (i) If f is integrable, then it is:
   (a) discontinuous (b) unbounded (c) continuous (d) linear
   (ii) If $f'(x) = 3x^2 + 2x$, then $f(x)$ is:
   (a) $6x + 2 + c$ (b) $x^3 + x^2 + c$ (c) $3x^3 + 2x^2 + c$ (d) $1.5x^3 + x^2 + c$
   (iii) $\int \frac{d}{dx}(x^2)dx$ is equal to:
   (a) $x^2 + c$ (b) $2x + c$ (c) $\frac{x^3}{3} + c$ (d) $2x + c$
   (iv) $\int \sin 2x dx$ is:
   (a) $\frac{\cos 2x}{2} + c$ (b) $2\cos 2x + c$ (c) $-\frac{\sin 2x}{2} + c$ (d) $-\frac{\cos 2x}{2} + c$
   (v) $\int_3^7 dx$ is:
   (a) 3 (b) 4 (c) 5 (d) 6
   (vi) $\int_{\frac{\pi}{6}}^{\pi} \cos x dx$ is:
   (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$
   (vii) $\frac{d}{dx}\int_{-2}^{x} t^3 dt$ is equal to:
   (a) t^4 (b) t^3 (c) x^3 (d) $x^3 - 16$
   (viii) What is relation between $\int_1^2 x dx$ and $\int_1^2 t dt$?
   (a) $\int_1^2 x dx < \int_1^2 t dt$ (b) $\int_1^2 x dx > \int_1^2 t dt$
   (c) $\int_1^2 x dx \neq \int_1^2 t dt$ (d) $\int_1^2 x dx = \int_1^2 t dt$
   (ix) Area under the graph of $f(x) = 4$; [2, 5] is:
   (a) 2 (b) 4 (c) 5 (d) 12
   (x) $\int \sqrt{x} dx$ is:
   (a) $x^{\frac{3}{2}} + c$ (b) $\frac{2}{3}x^{\frac{3}{2}} + c$ (c) $\frac{3}{2}x^{\frac{3}{2}} + c$ (d) $x^{\frac{1}{2}} + c$
2. Evaluate:
   (i) $\int \frac{4x+2}{x^2+x+1} dx$ (ii) $\int x(x^2+1)^4 dx$ (iii) $\int \cos^2 3x \, dx$
   (iv) $\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)} dx$ (v) $\int \sin^{-1} x dx$ (vi) $\int 2x \sin 3x \, dx$
   (vii) $\int x^2 e^x dx$ (viii) $\int_0^{\frac{\pi}{4}} (\sin 2x - 5\cos 4x) dx$ (ix) $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}} dx$
3. Use the substitution $u = 2x + 1$ to evaluate $\int_0^1 \frac{x^2}{\sqrt{2x+1}} dx$.
4. A model rocket is launched upward from ground level with an initial speed of 60m/s.
   (a) How long does it take for the rocket to reach its highest point?
   (b) How high does the rocket go?
5. Suppose that a parachute moves with a velocity $V(t) = \cos \pi t$ m/s along a coordinate line. Assuming that the parachute has the coordinate $S = 4m$ at time $t = 0$ sec, find its position.

DIFFERENTIAL EQUATIONS

After studying this unit, students will be able to:

- Identify and construct first order differential equations from practical situations.
- Give the concept of solution of differential equation.
- Solve differential equations of first order and first degree of the form:
  - separable variables equations,
  - homogeneous equations,
- Solve real life problems related to differential equations such as population growth and decay, cooling/warming law, flow of electricity, series circuit, economics and finance, radioactive decay etc.

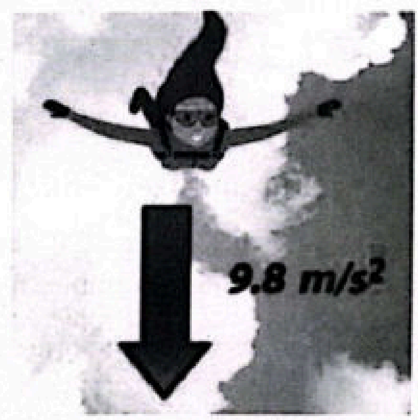
Differential equations are mathematical equations that describe relationships between a function and its derivatives. They are essential for modeling real-world phenomena in various fields. These equations are used to describe how quantities change over time and how they relate to each other. Differential equations are essential tools for modeling dynamic systems in real life.

These equations have wide-ranging applications across various fields of science and engineering, as they are fundamental in modeling dynamic systems and natural phenomena. In physics, they describe the motion of objects under forces, such as in Newton's laws and wave propagation. In biology, they are used to model population dynamics, the spread of diseases, and the interaction between species in ecosystems. In engineering, differential equations are crucial for analyzing electrical circuits, control systems, and mechanical vibrations. In chemistry, they help in understanding reaction rates and diffusion processes. In economics, they are used to model investment growth, market behavior, and optimization problems. Overall, differential equations serve as powerful tools for understanding and predicting the behavior of systems that change over time or space.

Introduction

Many problems in engineering and science can be formulated in terms of differential equations. The formulation of mathematical models is basically to address real-world problems which have been one of the most important aspects of applied mathematics. Differential equations arise in many areas of science and technology, specifically whenever a relation involving some continuously varying quantities and their rates of change in space and/or time (expressed as derivatives). This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's laws allow relating the position, velocity, acceleration and various forces acting on a body and state this relation as a differential equation for the unknown position of the body as a function of time. Such equations are called differential equations.

A mathematical model is a mathematical construction such as a differential equation, that simulates a natural engineering phenomenon. Most applications of differential equations take the form of mathematical models. For example, consider the problem of determining the velocity v of a falling object.



Newton's second law of motion tells us that the net force on the object is equal to the product of the mass, m and its acceleration, $\frac{dv}{dt}$

$$m\frac{dv}{dt} = F$$

This law is a differential equation as it contains derivative.

Ignoring air resistance, for an object falling close to the Earth's surface the force is $F = mg$, directed downward, where g is approximately 9.80 meters per second per second. Thus, the differential equation:

$$m\frac{dv}{dt} = mg$$

is a mathematical model corresponding to the free-falling object.

Key Facts

A good mathematical model has two important properties:

- It is sufficiently simple that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the behavior of the real problem.

4.1 Differential Equation

A differential equation is an equation containing one or more derivatives of an unknown function. A differential equation is an ordinary differential equation if it involves an unknown function of only one variable. For now, we will consider in this unit only ordinary differential equation just call them differential equation (DE). The word differential equation means involvement of derivative and equation are must.

Suppose we have function:

$$f(x) = y = x^4 + c, \text{ where } c \text{ is arbitrary constant.} \qquad \text{(i)}$$

Differentiating (i) with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 \qquad \text{(ii)}$$

This is called a differential equation.

The simplest differential equations are of the form:

$$\frac{dy}{dx} = f(x) \quad \text{or} \quad y' = f(x)$$

Where f is known function of x.

A mathematical equation containing the derivatives of one dependent variable, with respect to one independent variable, is said to be a differential equation (DE).

Some more examples of differential equations are:

$$\frac{dy}{dx} = 2x^2 - 1, \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3, \quad \frac{dy}{dx} = x + y, \quad xy\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx} + x^2$$

Where, y is a dependent and x is an independent variable.

4.2 Order and Degree of Differential Equation

The order of the differential equation is the order of the highest order derivative present in the equation. Here some examples for different orders of the differential equation are given.

- $\frac{dy}{dx} - 2x = 1,$ the order of the DE is 1.
- $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 5 = 0,$ the order of DE is 2.

4.2.1 Types of DE w.r.t. Order

(i) First Order Differential Equation

A differential equation containing first order derivatives is called first order differential equation.

$$\frac{dy}{dx} = f(x, y)$$

For example, $\frac{dy}{dx} = 3x$ is a first order differential equation.

(ii) Second Order Differential Equation

The equation which includes the second-order derivative is called the second-order differential equation. It is represented as:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f(x, y)$$

For example, $y'' + xy' + y = 0$ is a second order differential equation.

4.2.2 Degree of Differential Equation

The degree of the differential equation is the power of the highest order derivative in the equation.

Examples:

- $\frac{dy}{dx} - 3 = 0$, degree is 1.
- $(y'')^2 + 6y' = 9$, degree is 2.
- $(y'')^3 - xy' + y = 0$, degree is 3.

order degree

$$\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right) + y = x + 1$$

Key Facts

- Order and degree (if defined) of a differential equation are always positive integers.
- $y' - \log(y') + 3 = 0$, is not a polynomial equation in y' and the degree of such differential equation cannot be defined.

Example 1:

Determine the order and degree of the following differential equations.

(i) $\frac{dy}{dx} = -\frac{x}{y}$ (ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + x = 0$ (iii) $\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 1$

Solution:

(i) order: 1, degree: 1 (ii) order: 2, degree: 1 (iii) order: 3, degree: 2

Some more types of differential equations are as follows.

- Ordinary differential equations (ODE)
- Partial differential equations (PDE)
- Linear differential equations
- Nonlinear differential equations
- Homogeneous differential equations
- Non-homogeneous differential equations

We will here discuss only ordinary differential equations.

4.2.3 Ordinary Differential Equation (ODE)

If in a differential equation, only one independent variable is involved, the equation is called an ordinary differential equation. It contains one or more of its derivatives with respect to the independent variable.

The general form of nth order ODE is a function F of x, y and derivatives of y.

$$F(x, y, y', \ldots, y^{(n-1)}) = y^{(n)}$$

Which is called an explicit ODE of order n.

Examples of ODE are:

(i) $y' - 4y + 2 = 0$

(ii) $(2x + 3y)dy = (x - 2y)dx = 0$

(iii) $xy'' - 5y' + 11 = y$

(iv) $(y'')^3 + 4y' + 2y = 1$

Key Facts

If a differential equation is not ODE, it is then PDE.

4.2.4 Linear and Non-Linear Differential Equations

Differential equations are classified into linear DEs or nonlinear DEs.

An n^{th} order differential equation is said to be linear if it can be written in the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

that is, it satisfies the following two conditions:

(a) the dependent variable (y) and all its derivatives in the equation are linear (i.e. of power one).

(b) all the coefficients $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ and the function $f(x)$ are either constants or depend only on the independent variable (x).

Note: If any one of these two conditions is not satisfied, then the DE is said to be nonlinear DE.

Example 2:

Identify linear and non-linear differential equations in the following.

(i) $\frac{dy}{dx} = a$ (ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 1$ (iii) $\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} - 2 = 0$

(iv) $\frac{dy}{dx} = \frac{x}{y}$ (v) $\frac{d^2y}{dx^2} + 12xy = 0$ (vi) $\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2x = 5$

Solution:

(i) linear (ii) linear (iii) non-linear

(iv) non-linear (v) linear (vi) non-linear

Note:

The differential equation $y^2(x-3)\frac{dy}{dx} = 2xy^2$ is not linear, but it can be reduced to linear differential equation if we divide both sides of it by y^2 as follows:

$$\frac{y^2}{y^2}(x-3)\frac{dy}{dx} = \frac{2xy^2}{y^2}$$

or $(x-3)\frac{dy}{dx} = 2x$ which is linear differential equation.

4.3 Concept of Solution of Differential Equation

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions as the set of functions that satisfy the equation.

A solution of a differential equation is any function f defined on some interval I that reduces the equation to an identity.

To solve a differential equation such as $\frac{dy}{dx} - x = 0$, we mean to find an unknown function $y = f(x)$ or $y = f(x,y)$.

Consider a simple first order differential equation:

$$\frac{dy}{dx} = f(x) \qquad \text{(i)}$$

Equation (i) can be solved by integration. If $f(x)$ is continuous function, then integrating both sides of (i) gives:

$$y = \int f(x)dx = F(x) + c$$

Where $F(x)$ is an anti-derivative of $f(x)$.

For example, the solution of differential equation $\frac{dy}{dx} = 1 + e^{2x}$ implies:

$$y = \int (1 + e^{2x})dx = x + \frac{1}{2}e^{2x} + c$$

Key Facts

Only the simplest differential equations admit solutions given by explicit formulas. However, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

Example 3:

Show that $x^2 + y^2 = c$ is a solution of the differential equation $y\frac{dy}{dx} + x = 0$. Also plot the graph of solution.

Solution: We have $x^2 + y^2 = c$

Differentiating w.r.t. x, we get:

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad y\frac{dy}{dx} + x = 0$$

Hence, $x^2 + y^2 = c$ is the solution of differential equation $y\frac{dy}{dx} + x = 0$.

We note that the solution, $x^2 + y^2 = c$ depends upon an arbitrary constant c.

By choosing different values of c, we get different solutions.

Let us take $c = 1, 4, 9, 16, \dots$ then we get different solutions as:

$$x^2 + y^2 = 1 = 1^2$$
$$x^2 + y^2 = 4 = 2^2$$
$$x^2 + y^2 = 9 = 3^2$$
$$x^2 + y^2 = 16 = 4^2 \text{ and so on.}$$

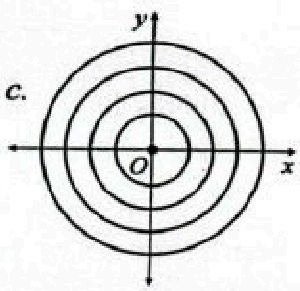


Fig. (i)

These solutions represent a family of circles with radii 1, 2, 3, 4, ...with centre (0, 0) in Fig. (i).

Thus, the solution, $x^2 + y^2 = c$ represents a family of infinite number of circles.

From this example, it has been observed that there are two types of solutions of a differential equation.

(i) general solution (ii) particular solution

4.3.1 General Solution of DE

The solution that contains as many arbitrary constants as the order of the differential equation is called a general solution. It is the relation between the independent variables x and dependent variable y which is obtained after removing the derivatives (by integration) where the relation contains arbitrary constant to denote the order of an equation. In the above example, $x^2 + y^2 = c$ is the general solution.

4.3.2 Particular Solution of DE

The solution free from arbitrary constants is called a particular solution.

If particular values are given to the arbitrary constant, the particular solution of the differential equations is obtained. In the above example, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$ etc. are the particular solutions.

Key Facts

- The solution of a first-order differential equation contains one arbitrary constant whereas the second-order differential equation contains two arbitrary constants.
- The general solution of a differential equation represents a family of curves.
- The particular solution of a differential equation represents a particular curve for a particular value of constant from the family of curves.

Example 4:

Verify that $y = \frac{x^4}{16}$ is a solution of the differential equation $\frac{dy}{dx} - xy^{\frac{1}{2}} = 0$.

Solution: Given solution is:

$$y = \frac{x^4}{16} \qquad \text{(i)}$$

Differentiating (i) with respect to x, we get:

$$\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$$

Substituting for y and $\frac{dy}{dx}$ in the left side of given differential equation:

$$\frac{dy}{dx} - xy^{\frac{1}{2}} = \frac{x^3}{4} - x\left(\frac{x^4}{16}\right)^{\frac{1}{2}} = \frac{x^3}{4} - x\left(\frac{x^2}{4}\right) = \frac{x^3}{4} - \frac{x^3}{4} = 0$$

Which is true $\forall x \in \mathbb{R}$.

Thus, $y = \frac{x^4}{16}$ is a solution of the differential equation $\frac{dy}{dx} - xy^{\frac{1}{2}} = 0$.

Check Point

Find the DE corresponding to the equation $y = 3x^2 + c$. Of which type the family of curves does the solution represent?

Example 5:

Is the function $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$ on the interval $(-\infty, +\infty)$?

Solution: Given solution is:

$$y = xe^x \qquad \text{(i)}$$

Differentiating (i) with respect to x, we get:

$$y' = xe^x + e^x \qquad \text{(ii)}$$

Differentiating (ii) with respect to x, we get:

$$y'' = xe^x + 2e^x \qquad \text{(iii)}$$

Substituting the values in the left side of given differential equation:

$$y'' - 2y' + y = xe^x + 2e^x - 2(xe^x + e^x) + xe^x$$

$$= xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

Which is true $\forall x \in \mathbb{R}$. Thus, $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$ on the interval $(-\infty, +\infty)$.

Note: In examples (4) and (5), we notice that the constant function $y = 0$ for $(-\infty < x < +\infty)$ also satisfies the given differential equation.

Key Facts

- A solution of differential equation that is identically zero on any interval is often called a trivial solution.
- Every differential equation that we write necessarily has a solution either real or imaginary. For example, the differential equation $(y')^2 + 1 = 0$ has no real solution.

4.4 Formation of Differential Equation

We can form a differential equation by eliminating the constants appearing in an algebraic equation; the solution of differential equation.

Let us find the differential equation corresponding to the equation $y = e^x$.

Now, $y = e^x$ gives $y' = e^x$ and solving both equations, we get:

$$y' - y = 0,$$

which is a differential equation.

Example 6:

Find the DE corresponding to the equation $y = a\cos x + b\sin x$

Solution: Given that

$$y = a\cos x + b\sin x$$

$$y' = -a\sin x + b\cos x$$

$$y'' = -a\cos x - b\sin x = -(a\cos x + b\sin x)$$

$y'' = -y$ or $y'' + y = 0$ is required differential equation.

Note: $y = a\cos x + b\sin x$ is a solution of differential equation $y'' + y = 0$.

Example 6:

Eliminate the arbitrary constants from the following equation and form a differential equation of the lowest order: $y = A\sin(2x - B)$

Solution: Given that:

$$y = A\sin(2x - B) \qquad \text{(i)}$$

Taking derivative of equation (i) with respect to x.

$$\frac{dy}{dx} = 2A\cos(2x - B)$$

Again, differentiating w.r.t x, we get:

$$\frac{d^2y}{dx^2} = -4A\sin(2x - B) = -4[A\sin(2x - B)]$$

$$\frac{d^2y}{dx^2} = -4y \quad \Rightarrow \quad \frac{d^2y}{dx^2} + 4y = 0$$

Which is a second order differential equation (ODE). Its physical interpretation is that the acceleration varies as the distance varies. This essentially illustrates the differential equation governing the simple harmonic motion.

Check Point

Check whether equation of parabola

$$y^2 = 4a(x - b)$$

where a and b are arbitrary constants, is a solution of differential equation:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

4.4.1 Explicit and Implicit Solution

A solution of a differential equation that can be written in the form $y = f(x)$ is said to be an explicit solution while a solution of the form $f(x, y) = 0$ is said to be an implicit solution.

Example 7:

Prove that, for $-2 < x < 2$, the relation $x^2 + y^2 - 4 = 0$ is an implicit solution of differential equation:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solution: Given equation is:

$$x^2 + y^2 - 4 = 0 \qquad \text{(i)}$$

By implicit differentiation, we have:

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

Note: The relation $x^2 + y^2 - 4 = 0$ in this example defines two explicit differentiable functions $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$ in the interval $(-2, 2)$.

4.4.2 Number of Solutions

A given differential equation usually possesses an infinite number of solutions. e.g.,

(i) For any value of c, the function $y = \frac{c}{x} + 1$ is a solution of the first order differential equation:

$$x\frac{dy}{dx} + y = 1 \quad \text{on the interval } (0, \infty)$$

The solution $y = \frac{c}{x} + 1$ represents infinite number of solutions for various values of c. In particular, for $c = 0$, we obtain a constant solution $y = 1$.

(ii) The functions $y = c_1 \cos 4x$ and $y = c_2 \sin 4x$ where c_1 and c_2 are arbitrary constants, are solutions of the differential equation:

$$y'' + 16y = 0$$

It is to be noted that the sum of solutions $y = c_1 \cos 4x$ and $y = c_2 \sin 4x$:

$$y = c_1 \cos 4x + c_2 \sin 4x$$

is also a solution of differential equation $y'' + 16y = 0$.

Exercise 4.1

1. Find the order and degree of each of differential equations.

(i) $(1 - x)y'' - 4xy' + 5y = \cos x$ (ii) $yy' + 2y = 1 + x^2$

(iii) $(y'')^3 - 3y' + 2y = x$ (iv) $(y')^2 - yy'' + 2 = 0$

(v) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ (vi) $\frac{dy}{dx} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$

(vii) $x^3\frac{d^4y}{dx^4} - x^2\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^5 + 4xy - 3y = 0$

2. Eliminate the arbitrary constants from the equations.

(i) $y = ae^x + be^{-x} + c$ (ii) $y = \cos(x + b)$

(iii) $y = mx + c$ (iv) $y = bx^2 + 2ax$

3. Verify that the indicated function is a solution of the given differential equation.

(i) $2y' + y = 0$; $y = e^{-\frac{x}{2}}$

(ii) $\frac{dy}{dx} - 2y = e^{3x}$; $y = e^{3x} + 10e^{2x}$

(iii) $y' = 25 + y^2$; $y = 5\tan 5x$

(iv) $y' + y = \sin x$; $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + 10e^{-x}$

(v) $x^3 dy - 2dx = 0$; $y = -\frac{1}{x^2} + 6$

(vi) $y' - \frac{1}{x}y = 1$; $y = x\,lnx, x > 0$

4. Find the order and degree, if defined, for the differential equation $dy - \sin x\, dx = 0$.
5. Verify that the function $y = a\cos x + b\sin x$, where, $a, b \in R$, is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.
6. Show that $y_1 = x^2$ and $y_2 = x^3$ are both solutions of:

$$x^2y'' - 4xy' + 6y = 0$$

(i) Are c_1y_1 and c_2y_2 also a solution? (ii) Check if $y_1 + y_2$ is also a solution.

4.5 Solution of Differential Equation

We know that the solution of an algebraic equation say, $x^2 - 5x + 6 = 0$, is the value of x that satisfies the given equation.

But the solution of a differential equation say, $y\,dy = x\,dx$, means to remove derivative from the given differential equation by using some technique/procedure. All these techniques involve integration. After removing derivative, the dependent variable y in terms of independent variable x (explicit: $y = f(x)$ or implicit: $g(x, y) = 0$) will be the solution of differential equation.

4.5.1 Solution of Differential Equations of First Order and First Degree

To solve the first-order differential equation of first degree, some standard forms are available to get the general solution. Some of them are:

- Variable separable differential equations
- Reducible into the variable separable differential equations
- Homogeneous differential equations
- Equations reducible to homogeneous differential equations

(i) Variable Separable Differential Equations

A first order differential equation of the form:

$$\frac{dy}{dx} = f(x)g(y)$$

where f is the function of x only and g is the function of y only, is said to be separable or to have separable variables.

For example, the equation $\frac{dy}{dx} = y^3xe^{2x+y}$ is separable. We can write it as:

$$\frac{dy}{dx} = (xe^{2x})(y^3e^y) = f(x)g(y)$$

The equation $\frac{dy}{dx} = y + \sin x$ is not separable. We cannot write $y + \sin x$ in the form of $f(x)g(y)$.

Example 8: Solve $\frac{dy}{dx} = \frac{x^2}{y}$

Solution: We first separate the variables of given equation as follows:

$$ydy = x^2dx$$

Integrating both sides, we have:

$$\int ydy = \int x^2dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + c_1$$

$$3y^2 = 2x^2 + 6c_1$$

$$\Rightarrow \quad 3y^2 = 2x^2 + c \qquad (c = 6c_1)$$

Note: To avoid lengthy process, we write constant at one side only.

Example 9: Solve $(1 + x)dy - y\,dx = 0$

Solution: The given equation can be written as:

$$(1 + x)dy = y\,dx$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrating both sides, we have:

$$\int \frac{dy}{y} = \int \frac{dx}{1+x} \quad \Rightarrow \quad \ln y = \ln(x + 1) + \ln c$$

$$\Rightarrow \quad \ln y = \ln[c(x + 1)]$$

Taking antilog on both sides, we have:

$$y = c(x + 1)$$

Example 10: Solve $\frac{dy}{dx} = \frac{1}{x \tan y}$

Solution: The given equation is:

$$\frac{dy}{dx} = \frac{1}{x \tan y} \quad \Rightarrow \quad \tan y\,dy = \frac{1}{x}dx$$

$$\int \tan y\,dy = \int \frac{1}{x}dx \quad \Rightarrow \quad -\ln(\cos y) = \ln x + \ln c$$

$$\Rightarrow \quad 0 = \ln(\cos y) + \ln x + \ln c \quad \Rightarrow \quad \ln(cx \cos y) = 0$$

$$\Rightarrow \quad e^{\ln(cx \cos y)} = e^0 \quad \Rightarrow \quad cx \cos y = 1$$

$$\Rightarrow \quad x \cos y \quad \Rightarrow \quad x \cos y = C \qquad \left(\frac{1}{c} = C\right)$$

(ii) Initial Condition and Initial Value Problem (IVP)

We have observed that general solution of differential equation contains the same number of arbitrary constants as is the order of differential equation. Sometimes we need to find the solution of DE subject to the supplementary conditions.

Suppose we want to find the solution of differential equation $\frac{dy}{dx} = f(x,y)$ subject to conditions $y = y_0$ at $x = x_0$. If we substitute $x = x_0$ and $y = y_0$, in the solution of $\frac{dy}{dx} = f(x,y)$ then we get a particular value of constant obtained in the general solution. Thus, a particular solution is obtained with the choice of some values of variables given in the differential equation. We call $y(x_0) = y_0$ as initial condition and the differential equation of $\frac{dy}{dx} = f(x,y)$ becomes an initial value problem as follows:

$$\frac{dy}{dx} = f(x,y), \; y(x_0) = y_0$$

Example 11: Solve the initial value problem: $\frac{dy}{dx} = -\frac{x}{y}$, $y(1) = 3$

Solution: The given equation can be written as:

$$y dy = -x\, dx$$

Integrating both sides, we get:

$$\int y dy = \int -x\, dx \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + c \quad \Rightarrow \quad x^2 + y^2 = 2c \qquad \text{(i)}$$

Using the initial condition i.e. $x = 1, y = 3$ in equation (i), we get:

$$1^2 + 3^2 = 2c \quad \Rightarrow \quad 2c = 10 \quad \Rightarrow \quad c = 5$$

Substituting the value of c in equation (i), we get:

$$x^2 + y^2 = 2 \times 5 \quad \Rightarrow \quad x^2 + y^2 = 10$$

Which is a solution of initial value problem representing a circle with centre (0, 0) and radius $\sqrt{10}$.

Example 12: Solve $\frac{dy}{dx} = 2x$ such that $y(2) = 4$.

Solution: Given equation can be written as:

$$dy = 2x\, dx$$

Integrating both sides, we get:

$$\int dy = \int 2x\, dx \quad \Rightarrow \quad y = 2\left(\frac{x^2}{2}\right) + c \quad \Rightarrow \quad y - x^2 = c \qquad \text{(i)}$$

Using $x = 2, y = 4$ in equation (i), we get:

$$4 - 2^2 = c \quad \Rightarrow \quad c = 0$$

Substituting the value of c in equation (i), we get:

$$y - x^2 = 0 \qquad \text{(ii)}$$

Which is a solution of initial value problem.

Note: The general solution (i) represents a family of parabolas for different values of c whereas the particular solution (ii) represents a member of family that passes through (2, 4).

Check Point

Show that the solution of differential equation $\frac{dy}{dx} = 2$ represnts a family of parallel lines. Draw some of parallel lines. Also solve differential equation for initial condition $y(0) = 1$.

Exercise 4.2

Slove the differential equations by separating the variables.

1. $\frac{dy}{dx}=-\frac{1}{e^{3x}}$
2. $x\frac{dy}{dx}=4y$
3. $\frac{dy}{dx}=\frac{y^3}{x^2}$
4. $\frac{dy}{dx}=e^{2x+3y}$
5. $\frac{dy}{dx}=\frac{x^2y^2}{1+x}$
6. $2y(x+1)dy=xdx$
7. $\frac{dy}{dx}+y^2\sin x=0$
8. $(\sin x+\cos x)dx=\cot y\cos x\,dy$

Solve the initial value problems.

9. $\frac{dy}{dx}=\cos x$; $y(0)=1$
10. $2\frac{dy}{dx}=4x\,e^{-x}$; $y(0)=2$
11. $\frac{dy}{dx}+\left(\frac{1+x}{x}\right)y=0$; $y(1)=1$
12. $\frac{dy}{dx}+y\tan 2x=0$; $y(0)=2$
13. $\frac{dy}{dx}=y^2+4$; $y(0)=-2$
14. $(1-x)dy+y^{-1}dx=0$; $y(0)=2$
15. $2(y-1)dy=(3x^2+4x+2)dx$; $y(0)=-1$

4.6 Homogeneous First order Differential Equations

Before considering a homogeneous differential equation of first order, we need to recall a homogeneous function.

4.6.1 Homogeneous Function

If a function f has the property that:

$$f(tx,ty)=t^n f(x,y)$$

where $t\in R^+$, $n\in R$. Then f is said to be a homogenous function of degree n.

Example 13: Check whether the function

(i) $f(x,y)=\sqrt{x^3+y^3}$ (ii) $f(x,y)=x^2+y^2+2$ (iii) $(x,y)=\frac{x}{2y}+4$

are homogeneous or not. If homogeneous then find degree.

Solution:

(i) $f(x,y)=\sqrt{x^3+y^3}$

Replacing x with tx and y with ty, we have:

$$f(tx,ty)=\sqrt{(tx)^3+(ty)^3}=\sqrt{t^3x^3+t^3y^3}$$

$$=t^{\frac{3}{2}}\sqrt{x^3+y^3}=t^{\frac{3}{2}}f(x,y)$$

$\therefore f(x,y)$ is homogeneous function of degree $\frac{3}{2}\in R$

Check Point

Check whether the functions are homogeneous or not. If homogeneous then find degree.

(a) $f(x,y)=x^2-3xy+y^2$

(b) $f(x,y)=x-\sqrt{xy}+5y$

(ii) $f(x,y) = x^2 + y^2 + 2$

$$f(tx,ty) = (tx)^2 + (ty)^2 + 2 = t^2x^2 + t^2y^2 + 2 \neq t^2 f(x,y)$$

$\therefore f(x,y)$ is not homogeneous function.

(iii) $f(x,y) = \frac{x}{2y} + 4$

$$f(tx,ty) = \frac{tx}{2ty} + 4 = \frac{x}{2y} + 4 = t^0 f(x,y)$$

$\therefore f(x,y)$ is homogeneous function of degree 0.

4.6.2 Homogeneous Differential Equations

A differential equation of the form:

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

is said to be homogeneous if both M and N are homogeneous functions of the same degree. In other words, differential equation (1) is homogeneous if

$$M(tx,ty) = t^n M(x,y) \quad \text{and} \quad N(tx,ty) = t^n N(x,y)$$

have the same degree n. The differential equation can be reduced to separable variables by either substituting $y = ux$ or $x = vy$, where u and v are new dependent variables. In particular, if we choose $y = ux$, then:

$$\frac{dy}{dx} = u + x\frac{du}{dx} \quad \text{or} \quad dy = udx + xdu$$

Hence the differential equation becomes:

$$P(x,ux)dx + Q(x,ux)[udx + xdu] = 0$$

$$\Rightarrow \quad x^n P(1,u)dx + x^n Q(1,u)[udx + xdu] = 0$$

$$\Rightarrow \quad [P(1,u)dx + uQ(1,u)]dx + xQ(1,u)du = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{Q(1,u)du}{P(1,u)dx + uQ(1,u)} = 0 \qquad (2)$$

Key Facts

- To solve homogeneous differential equations, we have to write out whole procedure for each problem. Therefore, it is not recommended to follow the equation (2) as a formula.
- The substitution $x = vy$ also leads to a separable differential equation.

Example 14: Solve $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

Solution: Putting $y=ux$ in the given differential equation, we have $\frac{dy}{dx}=u+x\frac{du}{dx}$

Therefore, given differential equation leads to:

$$u+x\frac{du}{dx}=\frac{x^2+u^2x^2}{2x\,ux} \quad \Rightarrow \quad u+x\frac{du}{dx}=\frac{1+u^2}{2u}$$

$$\Rightarrow \quad x\frac{du}{dx}=\frac{1+u^2}{2u}-u \quad \Rightarrow \quad x\frac{du}{dx}=\frac{1-u^2}{2u} \quad \Rightarrow \quad \frac{2u}{1-u^2}du=\frac{dx}{x}$$

Integrating, we get:

$$\int\frac{2u}{1-u^2}du=\int\frac{dx}{x}$$

$$-ln(1-u^2)=lnx+lnc \quad \Rightarrow \quad ln(1-u^2)+lnx+lnc=0$$

$$\Rightarrow \quad ln[cx(1-u^2)]=0$$

Taking antilog, we have:

$$cx(1-u^2)=1 \quad \Rightarrow \quad cx\left(1-\frac{y^2}{x^2}\right)=1 \quad \text{..... (replacing } u \text{ by } \tfrac{y}{x})$$

$$\Rightarrow \quad cx\left(\frac{x^2-y^2}{x^2}\right)=1 \quad \Rightarrow \quad x^2-y^2=\frac{x}{c} \quad \Rightarrow \quad x^2-y^2=Cx \qquad \left(\frac{1}{c}=C\right)$$

Example 15: Solve the initial value problem:

$$x\frac{dy}{dx}=y+x\,e^{\frac{y}{x}}\ ; \quad y(1)=1$$

Solution: Given equation is homogeneous of degree zero and can be rewritten as:

$$\frac{dy}{dx}=\frac{y}{x}+e^{\frac{y}{x}}\ ; \quad y(1)=1$$

Substituting $y=ux$ in the given differential equation, we have $\frac{dy}{dx}=u+x\frac{du}{dx}$

Therefore, given differential equation becomes:

$$u+x\frac{du}{dx}=\frac{ux}{x}+e^{\frac{ux}{x}} \quad \Rightarrow \quad u+x\frac{du}{dx}=u+e^{u}$$

$$\Rightarrow \quad e^{-u}\,du=\frac{dx}{x} \quad \Rightarrow \quad \int e^{-u}\,du=\int\frac{dx}{x}$$

$$\Rightarrow \quad -e^{-u}=lnx+c \quad \Rightarrow \quad -e^{-\frac{y}{x}}=lnx+c \qquad \text{(i)}$$

Substituting, $x=1, y=1$ in equation (i), we have:

$$-e^{-1}=ln1+c \quad \Rightarrow \quad c=-e^{-1}$$

Therefore, (i) leads to:

$$-e^{-\frac{y}{x}}=lnx-e^{-1} \quad \Rightarrow \quad e^{-1}-e^{-\frac{y}{x}}=lnx$$

Exercise 4.3

Check whether the functions are homogeneous or not. If homogeneous then find degree.

1. $f(x,y) = 6xy^3 - x^2y^2$
2. $f(x,y) = x^2 - y$
3. $f(x,y) = \frac{2y^3}{x^2y} - 7$

Solve the homogeneous differential equations.

4. $(x-y)dx + xdy = 0$
5. $ydx - (y-x)dy = 0$
6. $\frac{dy}{dx} = \frac{y-x}{x+y}$
7. $\frac{dy}{dx} = \frac{y^2+yx}{x^2}$
8. $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$
9. $\frac{dy}{dx} = \frac{3x^3+y^3}{xy^2}$
10. $\frac{dy}{dx} = \frac{x+3y}{3x+y}$
11. $\left[y + x\cot\left(\frac{y}{x}\right)\right]dx - xdy = 0$

Solve the initial value problems.

12. $xy^2\frac{dy}{dx} = y^3 - x^3$; $y(1) = 2$
13. $(x^2+2y^2)dx = xydy$; $y(1) = 1$
14. $2x^2\frac{dy}{dx} = 3xy + y^2$; $y(1) = -2$
15. $\left(x + ye^{\frac{y}{x}}\right)dx - x\,e^{\frac{y}{x}}dy = 0$; $y(1) = 0$

4.7 Applications of Differential Equations

Differential equations can be applied in solving problems arising in engineering and physical sciences such as physics, chemistry, biology and economics etc.

First order ordinary differential equations are used to calculate the movement or flow of electricity, motion of a pendulum and a falling object, to explain thermodynamics concepts and population growth etc. Also, in medical terms, they are used to check the growth of diseases in graphical representation.

Example 16:

A ball is thrown downward from a tower having height 20m. Develop a differential equation representing the flow phenomenon and find the velocity of the ball after 1 second. Also find the velocity with which the ball hits the ground. Neglect the air resistance.

Solution: For downward motion, acceleration $a = \frac{dv}{dt}$ can be written as:

$$\frac{dv}{dt} = g \quad \text{or} \quad dv = gdt$$

That is required differential equation.

Now, integrating both sides, we get

$$v = gt + c_1$$

At $t = 0, v = 0$, the constant $c_1 = 0$.

Thus, $v = gt$ ......(i)

Substituting the values,

$$v = 9.8 \times 1 = 9.8m/s$$

Which is velocity of the ball after 1 second.

Now from (i)

$$\frac{dS}{dt} = gt \text{ where } S \text{ is the distance covered by the ball.}$$

$$dS = gtdt$$

Integrating both sides, se have:

$$S = g\frac{t^2}{2} + c_2 \quad \text{....... (ii)}$$

At $t = 0, S = 0$, therefore from (ii), $c_2 = 0$.

and, $S = g\frac{t^2}{2}$ implies

$$20 = 9.8 \times \frac{t^2}{2} \quad \text{or} \quad t = 2.02sec$$

Now from (i)

$$v = 9.8 \times 2.02 = 19.8m/s$$

Thus, the velocity with which the ball hits the ground is $19.8m/s$.

Check Point

A thermometer showing the temperature of 20°C indoors is placed outdoors. After 8 minutes it reads 25°C and after another 8 minutes it reads 30°C. Using Newton's law of cooling, find the outdoors temperature.

Example 17:

According to Newton, cooling of a hot body is proportional to the temperature difference between its temperature T and the temperature T_0 of its surrounding medium. If a body at 90°C is allowed to cool in air with temperature 30°C and if it is observed after 5 min the body has cooled to 70°C, find the temperature of the body as a function of time.

Solution: The mathematical formulation of Newton's law of cooling in this problem is:

$$\frac{dT}{dt} \propto (T - T_0) \text{ ....(1)}$$

Introducing a proportionality constant $k > 0$, the above equation can be written as:

$$\frac{dT}{dt} = k(T - T_0) \text{ ....(2)}$$

Here, T is the temperature of the body and t is the time, T_0 is the temperature of the surrounding and $\frac{dT}{dt}$ is the rate of cooling of the body. Substituting, $T_0 = 30°$ in equation (2), we get:

$$\frac{dT}{dt} = k(T - 30) \quad \Rightarrow \quad \frac{dT}{T-30} = kdt$$

Integrating both sides, we get:

$$ln(T-30) = kt + lnc \quad \text{where } c \text{ is the constant of integration.}$$

$$ln(T-30) = lne^{kt} + lnc \quad \Rightarrow \quad ln(T-30) = ln(ce^{kt})$$

Taking antilog, we get:

$$T - 30 = ce^{kt} \qquad \text{.......... (3)}$$

Imposing the initial condition $T(0) = 90^o$, we find:

$$90 - 30 = c \quad \Rightarrow \quad c = 60$$

Therefore (3) implies:

$$T - 30 = 60e^{kt} \quad \Rightarrow \quad T = 60e^{kt} + 30 \qquad \text{.......... (4)}$$

To find the value of constant k, we use the second condition $T(5) = 70^o$ in (4).

$$70 = 60e^{5k} + 30 \Rightarrow \quad 60e^{5k} = 40 \quad \Rightarrow \quad e^{5k} = \frac{2}{3}$$

$$5k = ln\left(\frac{2}{3}\right) \quad \Rightarrow \quad k = \frac{1}{5}ln\left(\frac{2}{3}\right) = -0.081$$

Substituting the value of k in relation (4), we have:

$$T = 60e^{-0.081t} + 30$$

Which shows the temperature of the body as a function of time.

Exercise 4.4

1. Thomas Malthus in 1798 proved that increase in population of a country or a city at a certain time is proportional to the total population of the country at that time $\left(\frac{dP}{dt} \propto P\right)$. If at present the population of city A is 20 million and after 4 years, it is expected to be 25 million, what would be the population of that city after 12 years?
2. Ayesha was preparing a pizza in a baking oven. She observed that temperature of the cooked pizza was 150°C. Four minutes after removing from the oven, the temperature of pizza was 90°C. How long will it take to cool off to a temperature of 40°C if room temperature is 20°C?
3. In a culture, the rate of growth of bacteria is proportional to the population present. If the population of bacteria becomes four times in two days, how much the population would be after ten days at the same rate if the initial population was 20?
4. Most of the radioactive substances disintegrate at the rate proportional to the amount present. If the amount of a radioactive substances is 50 grams and its half life is 1000 years, find the amount of substance present after 800 years.
5. A thermometer showing room temperature of 80°F is placed on a block of ice with a temperature of 30°F. After one minute the temperature of thermometer is 40°F. How long will it take for the thermometer to have a temperature of 70°F?
6. A ball is thrown upward with a velocity of 40m/s. Develop a differential equation representing the flow phenomenon and find the velocity of the ball after 1 second. Also find the maximum height attained by the ball. Neglect the air resistance.

Review Exercise

1. Select the correct option in the following.

(i) The order of differential equation $x\frac{d^3y}{dx^3} - 2\left(\frac{dy}{dx}\right)^4 + y = 0$, is:

(a) 1 (b) 2 (c) 3 (d) 4

(ii) The degree of differential equation $\frac{d^2y}{dx^2} + 9y^3 = \sin x$, is:

(a) 0 (b) 1 (c) 2 (d) 3

(iii) $y = 8$ is a solution of the differential equation:

(a) $\frac{dy}{dx} + 8y = 32$ (b) $\frac{dy}{dx} + 6y = 32$

(c) $\frac{dy}{dx} + 5y = 32$ (d) $\frac{dy}{dx} + 4y = 32$

(iv) $f(x,y) = \frac{x^3 - y^3}{x - y}$ is a homogeneous function of degree:

(a) 1 (b) 2 (c) 3 (d) 4

(v) The solution of the differential equation $dy = dx$ is:

(a) $y = x + c$ (b) $y = x^2 + c$ (c) $y^2 = x^2 + c$ (d) $y^2 = x + c$

(vi) The number of arbitrary constants present in the general solution of a differential equation of first order is:

(a) 1 (b) 2 (c) 3 (d) 0

(vii) The differential equation $\frac{dy}{dx} = e^{x+y}$ has solution:

(a) $e^{-x-y} = c$ (b) $e^{-x} + e^{y} = c$ (c) $e^{x} + e^{y} = c$ (d) $e^{x} + e^{-y} = c$

(viii) The general solution of $y^2dy - x^2dx = 0$ is:

(a) $x^2 - y^2 = c$ (b) $x^3 + y^3 = c$ (c) $x^3 - y^3 = c$ (d) $x^2 + y^2 = c$

(ix) The solution of $\cos x \sin y\, dx + \sin x \cos y\, dy = 0$ is:

(a) $\sin x \cos y = c$ (b) $\cos x \cos y = c$ (c) $\cos x \sin y = c$ (d) $\sin x \sin y = c$

(x) Which of the following cannot be the order of differential equation?

(a) -1 (b) 1 (c) 10 (d) 100

2. Find the order and degree of the differential equations.

(i) $x\left(\frac{dy}{dx}\right)^2 + 2\sqrt{xy}\frac{dy}{dx} + y = 0$ (ii) $\frac{dy}{dx} = \sqrt{1 + \left(\frac{d^2y}{d^2x}\right)^4}$

3. Verify that the indicated function is a solution of the given differential equation.

(i) $2\frac{dy}{dx} + y = 0$; $y = e^{-\frac{x}{2}}$ (ii) $\frac{dy}{dx} + 20y = 24$; $y = \frac{6}{5} - \frac{6}{5}e^{-20x}$

(iii) $\frac{dy}{dx} = 25 + y^2$; $y = 5\tan 5x$ (iv) $x^2dy + 2xydx = 0$; $y = \frac{1}{x^2}$

4. Solve the differential equations.

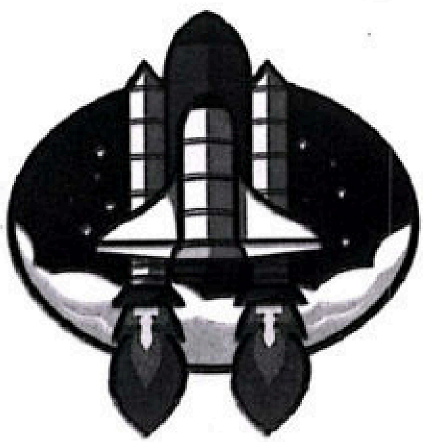
(i) $\frac{dy}{dx} = x\ln x$ (ii) $(y+1)\frac{dy}{dx} + x\sin x$ (iii) $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$; $y(1) = 2$

KINEMATICS OF MOTION IN A STRAIGHT LINE

After studying this unit, students will be able to:

- Recognize distance and speed as scalar quantities, displacement, velocity and acceleration as vector quantities.
- Sketch and interpret displacement-time graph and velocity time graph.
- Use differentiation and integration with respect to time to solve simple problems concerning displacement, velocity and acceleration.
- Use appropriate formula for motion with constant acceleration in a straight line.
- Apply the concept of mechanics to real life problems such as motion of vehicle on roads, projectile motion, free fall motion, relative motion animation.
- Explain the need for a vector valued function.
- Construct vector valued function.
- Identify domain and range of vector valued functions.
- Identify difference between scalar and vector valued functions.
- Explain derivative of a vector function of a single variable and elaborate the result. Apply vector differentiation to calculate velocity and acceleration of a position vector.
- Apply concepts of vector valued functions to real life word problems (such as engineering and transportation).

We see leaves falling from trees, movement of rockets and water flowing from the river. We walk, we run, we drive a car these are the activities that we carry out in our day-to-day life. These activities can be defined as motion. Motion is the change in position or orientation of a body. For example, a body on the surface of the Earth may appear to be at rest, but that is only because the observer is also on the surface of the Earth. The Earth itself, together with both the body and the observer, is moving in its orbit around the Sun and rotating on its own axis at all times in the same way.



5.1 Scalar and Vector Quantities

Physical quantities are divided into two categories, scalars and vectors. If only a magnitude is required to express a quantity, then the quantity is known as scalar and if both magnitude and direction are required to express a quantity then the quantity is known as vector quantity. e.g. time, distance, mass are scalars and velocity, acceleration, weight are vectors.

5.1.1 Distance

Distance is a scalar quantity since it only describes how much path is covered by a moving object regardless of direction. For example, you travel 3Km due north and then 2km due west then the total path covered is 5km.

5.1.2 Speed

Speed is also a scalar quantity since it represents how fast an object is moving regardless of its direction. For example, if you drive a car at the speed of 50km/hr it represents how fast you are moving no matter you are moving in which direction.

5.1.3 Displacement

Change in position of an object is known as displacement. It is a vector quantity because it includes both the magnitude i.e. distance between the initial position and final position and the direction i.e. straight path from the initial position to the final position. For example if you move along a curved path going from A to B then the straight path $\overrightarrow{AB}$ is the displacement.

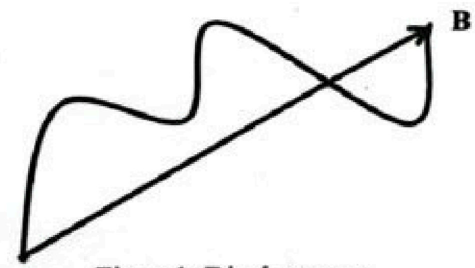


Figure1: Displacement

5.1.4 Velocity

Velocity is also a vector quantity, since it describes both how fast and in which direction an object is moving. In other words, rate of change of displacement with respect to time is called velocity. For example, a car moving 50km/h due west then its velocity is 50km/h westward.

5.2 Displacement-Time Graph

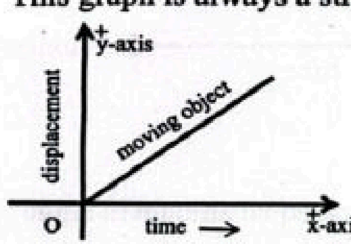
A moving object has different positions with respect to time. At any time 't' what is the position of object is represented on a graph known as displacement-time graph. While drawing the graph:

- Displacement values are drawn along y-axis (vertical axis)(time dependent quantity).
- Time is taken along x-axis (horizontal axis)(independant quantity).

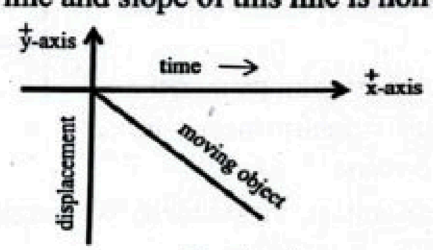
Note that independent values are taken along x-axis and the dependent values along y-axis.

5.2.1 Displacement-Time Graph for Uniform Motion

This is the graph between the displacement and the time when the motion of object is uniform. This graph is always a straight line and slope of this line is non-zero.



Positive slope



Negative slope

Here are some key points about the displacement-time graph.

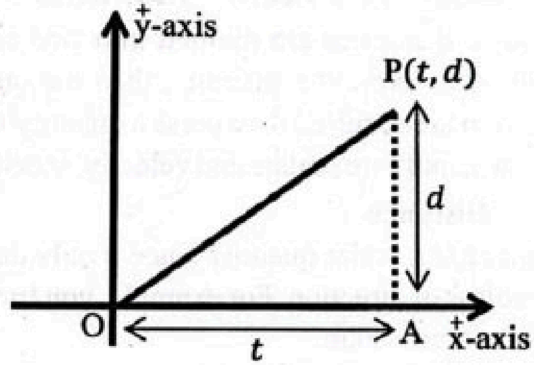
5.2.2 Slope of Displacement-time Graph

Let after time 't' the object moving with uniform speed is at point 'P' where its displacement is 'd'.

From figure slope of the graph $= \frac{d}{t}$.

By definition, ratio of displacement d with time 't' is velocity of the object. Therefore:

Slope of graph = velocity of the object

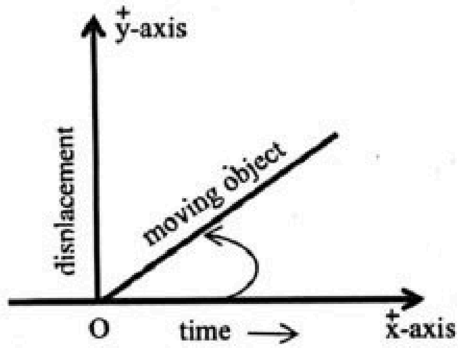


Since time is always non-negative and the displacement may be positive or negative.

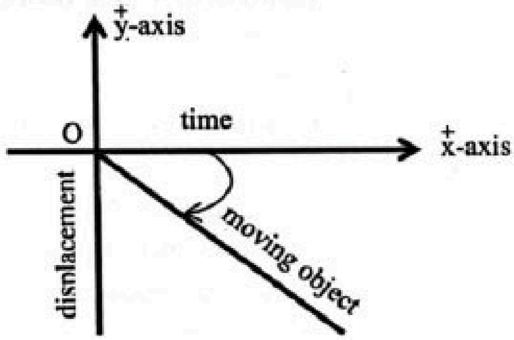
Therefore, slope may be positive or negative. Greater the value of slope means higher the velocity (+ve or -ve) of the object and smaller the value of slope means lower the velocity of object. If slope of the graph is zero (line is horizontal), then velocity of the object is zero.

5.2.3 Direction of Slope

If slope is positive, it indicates that particle is moving in the positive (anti-clockwise) direction and the negative slope indicates that particle is moving in the negative (clockwise) direction.



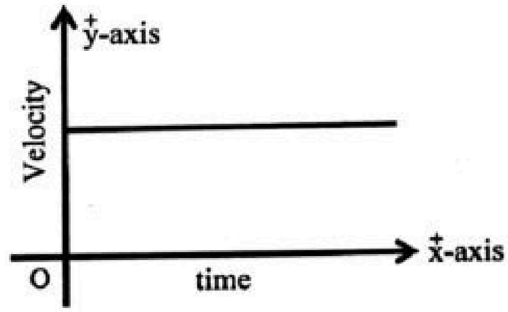
Positive slope



Negative slope

5.2.4 Velocity-Time Graph

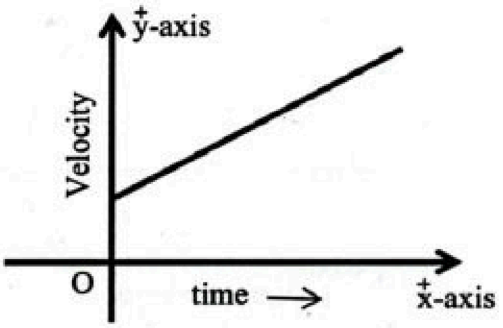
The velocity time graph is the graph plotted between the velocity of the particle with the passage of time. From the graph we can extract various kinds of information e.g. velocity and speed of particle at any time t, distance travelled by the particle, acceleration, average speed and average velocity etc. Velocity is plotted along y-axis and time along x-axis.



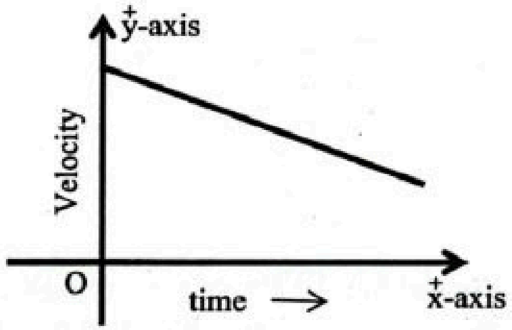
When the particle is moving with constant velocity then the graph is a horizontal line. Constant velocity means for different values of time the velocity is same.

A body is said to have a uniform acceleration if its velocity is changing by an equal amount in equal intervals of time.

In the case when body is moving with uniform acceleration the velocity-time graph is along a non-horizontal and non-vertical line.

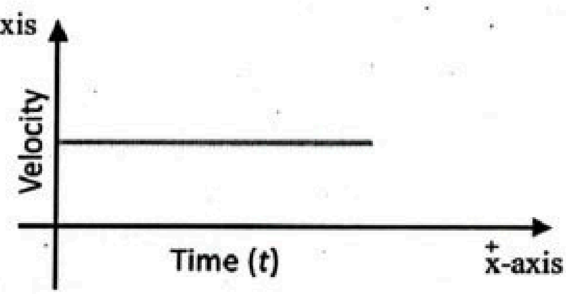


Positive acceleration



Negative acceleration

5.2.5 Slope of Velocity-Time Graph



Case I: When velocity is constant.

As we know that when velocity is constant then the graph is along a horizontal line so its slope is zero.

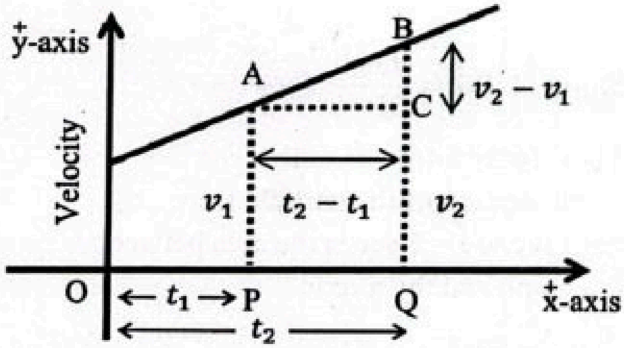
Case II: When acceleration is uniform.

For the particle which is moving with the uniform acceleration the graph is non-horizontal and non-vertical line. Consider any of the case with positive or negative acceleration.

Let at time t_1 the velocity is v_1 and at time t_2 velocity is v_2

From figure slope of line $= \frac{|BC|}{|AC|}$

$$= \frac{v_2 - v_1}{t_2 - t_1} = \frac{\text{Change in velocity}}{\text{Change in time}} = \text{acceleration}$$



Thus, slope of the graph of velocity-time graph is the acceleration of the particle.

Area between the Graph and the ẋ-axis

The area between the graph and the x-axis gives the displacement of the particle in that interval of the time.

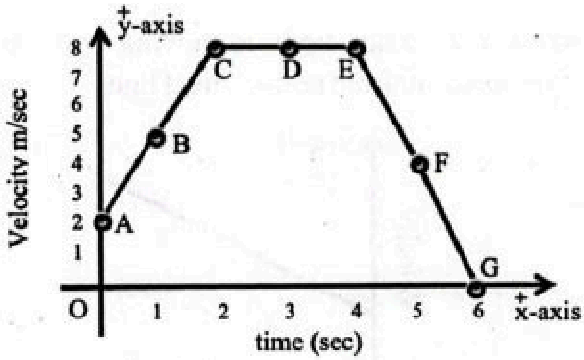
Example 1:

Draw the velocity-time graph of a moving particle with given data. Find its acceleration and retardation. Also find its displacement in the time interval 2 sec to 5 sec.

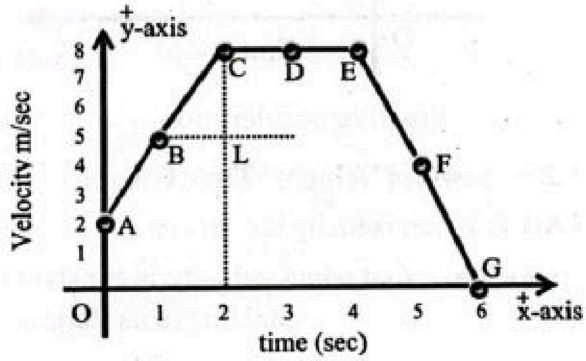
Time = t sec	0	1	2	3	4	5	6
Velocity = v m/s	2	5	8	8	8	4	0

Solution:

From the graph it is clear that particle is accelerating from point A to C as its velocity is increasing G with the passage of time.



For the slope of line $|\overline{AC}|$ take any two points on the line say B and C and draw horizontal line through B and vertical line through C intersecting at point L.



$$\text{acceleration} = \text{slope of } |\overline{AC}|$$

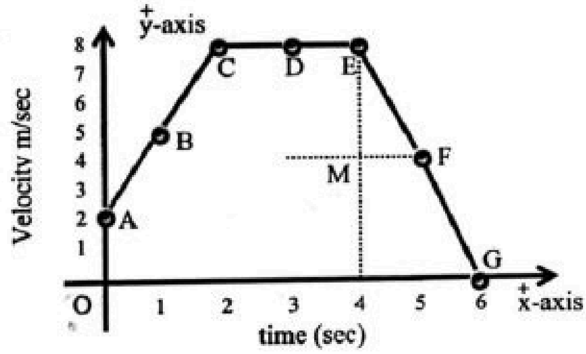
$$= \frac{|CL|}{|BL|} = \frac{8-5}{2-1} = 3m/sec^2$$

From the graph it is clear that particle motion is retarding from E to G as the velocity of the particle is decreasing with the passage of time. Draw a horizontal line through F and a vertical line through E meeting at M.

Slope of $\overline{EG} = \frac{0-8}{6-4} = \frac{-8}{2} = -4$

Thus, retardation $= -4m/sec^2$

Displacement of the particle from t = 2 sec to t = 5 sec is the area between the graph and the x-axis.



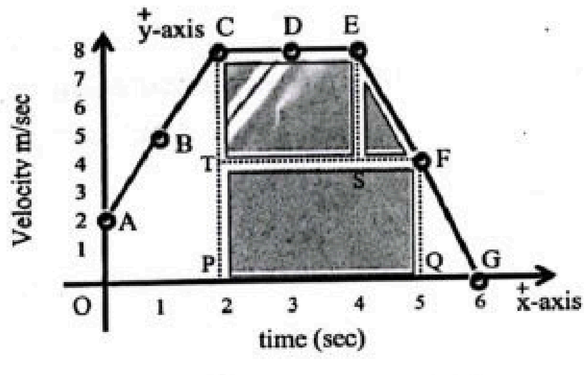
This is area of the region PCEFQ.

Displacement = area of the region PCEFQ

= Area of rectangle PQFT + Area of rectangle TSEC + Area of triangle FSE

$= (|PQ| \times |PT|) + (|TS| \times |TC|) + \frac{1}{2}(|SF| \times |SE|)$

$= (3 \times 4) + (2 \times 4) + \frac{1}{2}(1 \times 4) = 12 + 8 + 2 = 22$

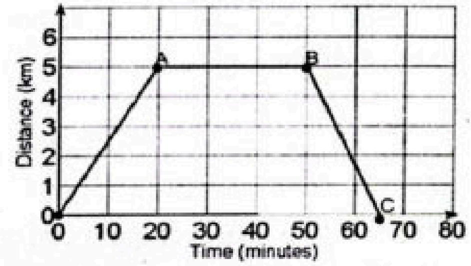


Displacement Function

A function which gives the displacement of moving particle at any time 't' is known as displacement function e.g. $S(t) = 3t^2 + 6t + 1$. At $t = 0$; $S(0) = 0 + 0 + 1 = 1$ is the position of the particle and at $t = 3$; $S(3) = 3(3)^2 + 6(3) + 1 = 46$ so the displacement of the particle at $t = 3$ is 46 - 1 = 45.

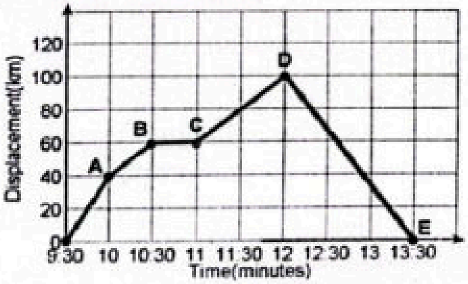
Exercise 5.1

1. A cyclist rides in a straight line for 20 minutes. He waits for half an hour, then returns in a straight line to his starting point in 15 minutes. This is a displacement-time graph for his journey.



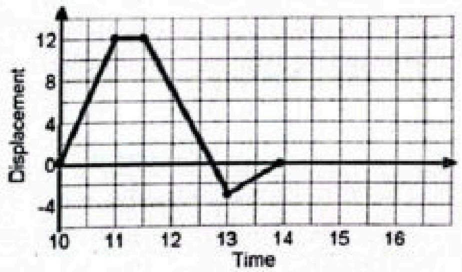
   (i) Work out the average velocity for each stage of the journey in km/h.
   (ii) Write down the average velocity for the whole journey.
   (iii) Work out the average speed for the whole journey.

2. This is a displacement-time graph for a car travelling along a straight road. The journey is divided into 5 stages labelled A to E.



   (i) Workout the average velocity for each stage of the journey.
   (ii) State the average velocity for the whole journey.
   (iii) Workout the average speed for the whole journey.

3. Fatima left home at 10:00 and cycled north-east in a straight line. The diagram shows a displacement time graph for her journey.



   (i) Find the Fatima's velocity between 10:00 and 11:00.

   On her return journey, Fatima continued passed her home before returning.

   (ii) Estimate the time that Fatima passed her home.
   (iii) Find Fatima's velocity for each of the last two stages for her journey.
   (iv) Calculate Fatima's average speed for her entire journey.

4. An electric train starts from the rest at a station A and moves along a straight level track. The train accelerates uniformly at 0.4 m/s^2 to a speed of 16 m/s. The speed is then maintained for a displacement of 2000 m. Finally the train retards uniformly for 20 s before coming to rest at a station B. For this journey from A to B,

   (i) Find the total time taken
   (ii) Find the displacement from A to B
   (iii) Sketch the displacement-time graph, showing clearly the shape of the graph for each stage of the journey.

5. Using the following data, draw time-displacement graph for a moving object.

Time(s)	0	2	4	6	8	10	12	14	16
Displacement(m)	0	2	4	4	4	6	4	2	0

Use the graph to find average velocity for first 4 *s*, for next 4 *s* and for last 6 *s* and the total displacement.

6. Ahmed leaves home at 11 *am*. He cycles at a speed of 16 *km/h* for 90 *minutes*. He stops for half an hour. Ahmed then cycles home and arrives at 3 *pm*.

(i) Draw a displacement-time graph to show Ahmed's journey.

(ii) What is Ahmed's average speed on the return part of his cycle.

7. Dabeer leaves at 14:00. He drives at an average speed of 14 *km/h* for $3\frac{1}{2}$ *hours*.
Dabeer stops the journey for 30 *minutes*. He then drives home at 70 *km/h*.
Draw a displacement-time graph to show Dabeer's journey.

8. A helicopter leaves Islamabad at 09:00. It flies for 45 *minutes* at 80 *km/h*. It lands for 20 *minutes*. The helicopter then returns to its base in Islamabad, flying at 100 *km/h*. Draw a displacement-time graph to show the journey.

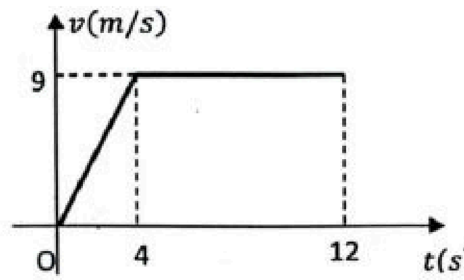
9. The diagram shows the velocity-time graph of the motion of an athlete running along a straight track.

For the first 4 *s*, he accelerates uniformly from rest to a velocity of 9 *m/s*.

This velocity is then maintained for a further 8*s*.

Find: (i) the rate at which the athlete accelerates.

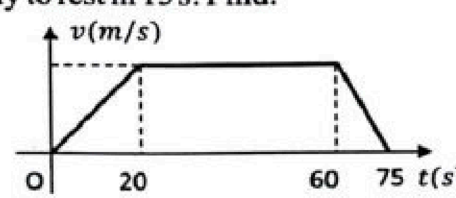
(ii) the displacement from the starting point of the athlete after 12 *s*.



10. The diagram shows the velocity-time of the motion of a cyclist riding along a straight road. He accelerates uniformly from rest to 8 *m/s* in 20 *s*. He then travels at a constant velocity of 8 *m/s* for 40 *s*. Then he decelerates uniformly to rest in 15 *s*. Find:

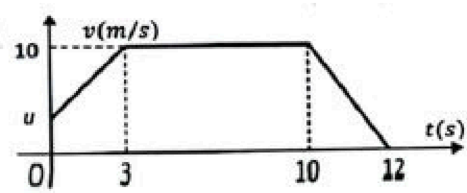
(i) The acceleration of the cyclist in the first 20 *s* of motion.

(ii) The deceleration of the cyclist in the last 15 *s* of motion.



(iii) The displacement from the starting point of the cyclist after 75 *s*.

11. A particle moves 100 *m* in a straight line. The diagram is a sketch of velocity-time graph of the motion of particle. The particle starts with velocity *u m/s* and accelerates to a velocity of 10 *m/s*. The velocity of 10 *m/s* is maintained for 7*s* and then



the particle decelerates to rest in a further 2 s. Find:

(i) The value of u.

(ii) The acceleration of the particle in the first part of the motion.

12. A car is moving along a straight road. When $t = 0\ s$, the car passes a point A with velocity $10\ m/s$ and this velocity is maintained until $t = 30\ s$. The driver then applies the brakes and the car decelerates uniformly, coming to rest at point B when $t = 42\ s$.

    (i) Sketch a velocity-time graph to illustrate the motion of the car.

    (ii) Find the distance from A to B.

13. A particle moves along a straight line. The particle accelerates from rest to a velocity of $20\ m/s$ in $15s$. The particle then moves at a constant velocity of $20\ m/s$ for a period of time. The particle then decelerates uniformly to rest. The period of time for which the particle is travelling at a constant velocity is 4 times the period of time for which it decelerating.

    (i) Sketch a velocity-time graph to illustrate the motion of the particle.

    Given that the displacement from the starting point of the particle after it comes to rest is $480\ m$.

    (ii) Find the total time for which the particle is moving.

14. A motorcyclist M leaves a road junction at time $t = 0\ s$. He accelerates from rest at a rate of $3\ m/s^2$ for $8\ s$ and then maintains the velocity he has reached. A car C leaves the same road junction as M at time $t = 0\ s$. The car accelerates from rest to $30\ m/s$ in $20\ s$ and then maintains a velocity of $30\ m/s$. C passes M as they both pass a pedestrian.

    (i) On the same diagram, sketch velocity-time graphs to illustrate the motion of M and C.

    (ii) Find the distance of the pedestrian from the road junction.

5.3 Velocity as Derivative of Displacement Function

Let a particle be moving and its position can be determined by the displacement function S. By definition velocity is the time rate of change of displacement. So we may write:

$$v = \text{velocity} = \frac{dS}{dt}$$

This gives us the instantaneous velocity of the particle at time 't'. For a particular value of 't' we will get the velocity of the particle at that particular time.

Example 2: A particle is moving such that its position can be determined by the function $S = t^2 + 2\sin t$. Find its velocity at any time 't'. Also find its initial velocity and velocity at time $t = \frac{\pi}{3}$ (S is in meters and t is in seconds).

Solution:

The position function (Displacement function) is

$$S = t^2 + 2\sin t$$

Differentiate it w.r.t 't'

$$\frac{dS}{dt} = 2t + 2\cos t$$

$$v(t) = 2t + 2\cos t \qquad \text{........(1)}$$

(1) shows the velocity of the particle at any time 't'. To find the initial velocity put $t = 0$ in (1).

$$v(0) = 2(0) + 2\cos 0 = 2m/sec$$

Now put $t = \frac{\pi}{3}$ in (1).

$$v\left(\frac{\pi}{3}\right) = 2\left(\frac{\pi}{3}\right) + 2\cos\frac{\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{1}{2}\right) = \frac{2\pi}{3} + 1 = \frac{2\pi + 3}{3} m/sec$$

Which is the velocity of the particle at $t = \frac{\pi}{3}$.

5.3.1 Acceleration as Derivative of Velocity and Displacement

Let the position of the moving particle at any time 't' be determined by the function $S(t)$.
By definition the acceleration of a particle is the rate of change of velocity w.r.t time; so

$$a = \text{acceleration} = \frac{dv}{dt}$$

Which is acceleration is a derivative of its velocity.

As we know that $v = \frac{dS}{dt}$. Hence $a = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d^2S}{dt^2}$ is the acceleration as derivative of its displacement.

Example 3: The position function of a moving particle is given by $S = \sqrt{t} + ln\,(t+1)$. Find the velocity and acceleration of the particle at any instant of time 't'. Also find its velocity and acceleration at $t = 1$ and $t = 4$. Here S is measured in meters and time in seconds.

Solution:

Given that $S = \sqrt{t} + ln\,(t+1)$

Differentiate w.r.t 't'

$$\frac{dS}{dt} = \frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{t+1}$$

$$v(t) = \frac{1}{2\sqrt{t}} + \frac{1}{t+1} \qquad \text{.......(1)}$$

Equation (1) shows the velocity of the particle at any time 't'. Differentiate (1) w.r.t 't'.

$$\frac{dv}{dt} = -\frac{1}{4}t^{-\frac{3}{2}} - \frac{1}{(t+1)^2} \qquad \text{........(2)}$$

Which is the acceleration of the particle at any time 't'.

Velocity at $t = 1$

Put $t = 1$ in Eq. (1).

$$v(1) = \frac{1}{2} + \frac{1}{2} = 1m/sec$$

Velocity at $t = 4$

Put $t = 4$ in Eq. (1).

$$v(4) = \frac{1}{2\sqrt{4}} + \frac{1}{4+1} = \frac{1}{4} + \frac{1}{5} = \frac{9}{20} m/sec$$

Acceleration at $t = 1$

Put $t = 1$in Eq. (2).

$$a(1) = -\left[\frac{1}{4(1)^{\frac{3}{2}}} + \frac{1}{(1+1)^2}\right] = -\frac{1}{2} m/sec^2$$

Acceleration at $t = 4$

Put $t = 4$ in Eq. (2).

$$a(4) = -\left[\frac{1}{4(4)^{\frac{3}{2}}} + \frac{1}{(4+1)^2}\right] = -\left(\frac{1}{32} + \frac{1}{25}\right) = -\frac{37}{800} m/sec^2$$

5.3.2 Displacement as an Integral of Velocity

As we know that for a moving particle

$$\frac{dS}{dt} = v$$

Integrating both sides w.r.t 't'

$$\int \frac{dS}{dt} dt = \int v\, dt$$

$$S = \int v\, dt$$

5.4 Velocity as Integral of Acceleration

For a moving particle we have:

$$\frac{dv}{dt} = a$$

Integrating both sides w.r.t 't', we have:

$$\int \frac{dv}{dt} dt = \int a$$

$$v = \int a\, dt$$

Example 4: The acceleration of a moving particle at any time 't' is given by

$$a = 24t + \cos t$$

Find its velocity and position at any time 't'. Given that at $t = 0; v = 0$ and at $t = \frac{\pi}{2}; S = \frac{\pi^3}{2}$.

Solution:

Given that $a = 24t + \cos t$

$$v = \int a\, dt$$

$\Rightarrow \quad v = \int(24t + \cos t)dt + A = 12t^2 + \sin t + A$

Where A is constant of integration.

It is given that at $t = 0$; $v = 0$, so

$$0 = 12(0)^2 + \sin(0) + A$$

$\Rightarrow$ $A = 0$

Hence $v = 12t^2 + \sin t$

Which is the velocity of the particle at any time 't'.

Also $S = \int v(t)dt$

$\Rightarrow$ $S = \int (12t^2 + \sin t)dt$

$$S = 4t^3 - \cos t + B$$

Where B is constant of integration.

Given that at $t = \frac{\pi}{2}; S = \frac{\pi^3}{2}$, therefore

$$\frac{\pi^3}{2} = 4\left(\frac{\pi}{2}\right)^3 - \cos\left(\frac{\pi}{2}\right) + B$$

$\Rightarrow$ $B = 0$

Hence $S = 4t^3 - \cos t$

Which is the position of particle at any time 't'.

5.5 Application of Mechanics in Real Life Situation

Example 5:

A stone is projected vertically upwards with a velocity of $10m/sec$. Find the height attained by the stone at any time 't'. When will it attain its maximum height? Also find the maximum height attained by the stone and the total distance travelled when it hits the ground.

Solution:

Since the stone is thrown vertically upwards. So $a = -g\ m/sec^2$.

Since, $v = \int a\, dt$

$$v = \int -g dt = -gt + A$$

Given that at $t = 0; V = 10m/sec$ therefore

$$10 = -g(0) + A$$

$\Rightarrow$ $A = 10$

Thus, $v = -gt + 10$

Which is the velocity of the stone at any time 't'.

Also $S = \int v\, dt$

$\Rightarrow$ $S = \int(-gt + 10)dt$

$\Rightarrow$ $S = -\frac{1}{2}gt^2 + 10t + B$

At $t = 0; S = 0$, therefore:

$$0 = 0 + 0 + B \Rightarrow B = 0$$

Hence, $S = -\frac{1}{2}gt^2 + 10t$

Which is the height attained by the stone at any time 't'. Now when stone attains its maximum height its velocity becomes zero. i.e.

$$-gt + 10 = 0$$

$$\Rightarrow \quad t = \frac{10}{g}\text{sec}$$

Which is the time after that stone attains its maximum height. Put the value of t in

$$S = -\frac{1}{2}gt^2 + 10t.$$

$$S = -\frac{1}{2}g\left(\frac{10}{g}\right)^2 + 10\left(\frac{10}{g}\right)$$

$$= -\frac{50}{g} + \frac{100}{g} = \frac{50}{g}\,m$$

Thus, stone will attain a maximum height of $\frac{50}{g}$ meters.

Total distance travelled by the stone = distance travelled in upward direction + distance travelled in downwards direction

$$= \frac{50}{g} + \frac{50}{g} = \frac{100}{g}m$$

Exercise 5.2

1. A projectile is launched vertically upward from an initial height of 129 ft with an initial velocity of 87 ft/s.
   (i) What are the position, velocity, and acceleration functions?
   (ii) When will the projectile hit the ground?
   (iii) What is its impact velocity?
   (iv) When will the projectile reach its maximum height?
   (v) What is the maximum height of projectile?

2. An object has its position defined by $S = t^3 - 9t^2 + 24t + 20$ in feet.
   (i) What are the velocity and acceleration functions?
   (ii) What are the position and velocity of the object when its acceleration is $-6.5ft/s^2$?
   (iii) Find the displacement and the total distance travelled by the particle from $t = 1.5s$ to $t = 7s$.

3. A person is standing on top of the Meinar-e-Pakistan and throws a ball directly upward with an initial velocity of 96 ft/s. The Meinar-e-Pakistan is 176 ft high.
   (i) What are the functions for position, velocity, and acceleration of the ball?
   (ii) When does the ball hit the ground and with what velocity?
   (iii) How far does the ball travel during its flight?

4. A particle moves along a line such that its position is:

$$S = 2t^3 - 9t^2 + 12t - 4, \text{ for } t \geq 0.$$

   (i) Find t for which the distance S is increasing
   (ii) Find t for which the velocity is increasing.
   (iii) Find t for which the speed of the particle is increasing.
   (iv) Find the speed when $t = \frac{3}{2}s$.
   (v) Find the total distance travelled in the time interval $[0, 4]$.

5. The position of an object moving on a line is given by $S = 6t^2 - t^3, t \geq 0$, where S is in metres and t is in seconds.
   (i) Determine the velocity and acceleration of the object at $t = 2$.
   (ii) At what time is the object at rest?
   (iii) In which direction is the object moving at $t = 5s$?
   (iv) When is the object moving in a positive direction?
   (v) When does the object return to its initial position?

6. A particle P moves along the x^+-axis. The acceleration of P in time t seconds, when $t \geq 0$, is $a = (3t + 5)m/s^2$ in the positive x-direction. When $t = 0$, the velocity of P is 2 m/s in the positive x-direction. When $t = T$, the velocity of P is 6 m/s. Find the value of T.

7. A particle P moves along the x^+-axis. At time t seconds the velocity of P is $v = (3t^2 - 4t + 3)m/s$. When $t = 0$, P is at the origin O. Find the distance of P from O when P is moving with minimum velocity.

8. A particle P moves along the x-axis in a straight line so that, at time t seconds, the velocity of P is v m/s, where $v = \begin{cases} 10t - 2t^2, & 0 \leq t \leq 6, \\ -\frac{432}{t^2}, & t > 6. \end{cases}$

   At $t = 0$, P is at the origin O. Find the displacement of P from O when:
   (i) $t = 6s$,
   (ii) $t = 10s$.

9. A particle P moves along the x^+-axis. At time t seconds the velocity v of P is increasing in the direction of x-axis given by $v = \begin{cases} 8t - \frac{3}{2}t^2, & 0 \leq t \leq 4, \\ 16 - 2t, & t > 4. \end{cases}$

   When $t = 0$, P is at the origin O. Find:
   (i) the greatest speed of P in the interval $0 \leq t \leq 4$,
   (ii) the distance of P from O when $t = 4$,
   (iii) the time at which P is instantaneously at rest for $t > 4$,
   (iv) the total distance travelled by P in the first 10 s of its motion.

10. A particle P moves along the x^+-axis. The acceleration of P at time t seconds is $a = (4t - 8)\ m/s^2$, measured in the increasing direction of x. The velocity of P at time t seconds is $v\ m/s$. Given that $v = 6$ when $t = 0$, find:
    (i) v in terms of t.
    (ii) the distance between the two points where P is instantaneously at rest.
11. A particle P moves along the x^+-axis and its acceleration after a given instant t is given by $a = (4t - 9) m/s^2$, $t \geq 0$. When $t = 1$, P is moving with velocity of $-3\ m/s$.
    (i) Find the minimum velocity of P.
    (ii) Determine the times when P is instantaneously at rest.
    (iii) Find the distance travelled by P in the first $4\frac{1}{2}$ seconds of its motion.
12. A car moving on a straight road is modelled as a particle moving along the x^+-axis, and its acceleration $a\ m/s^2$, after a given instant t, is given by

$$a = \begin{cases} 4 - \frac{1}{2}t & 0 \leq t \leq 8 \\ 0 & t > 0 \end{cases}$$

The car starts from rest.
    (i) Find a similar expression for the velocity of the car, as that of its acceleration.
    (ii) Find the time it takes for the car to reach its maximum speed.
    (iii) Show that the displacement of P form O is given by:

$$S = \begin{cases} 2t^2 - \frac{1}{12}t^3 & 0 \leq t \leq 8 \\ 16t - \frac{128}{3} & t > 8 \end{cases}$$

    (iv) Calculate the time it takes the car to cover the first $1000\ m$.
13. A particle is moving with constant acceleration a with an initial velocity v_i and after time t it covers a distance S. Prove that:
    (i) $v_f = v_i + at$
    (ii) $S = v_i t + \frac{1}{2}at^2$
    (iii) $2aS = v_f^2 - v_i^2$

5.6 Vector Valued Function

Vector valued functions provide a useful method for studying various curves both in plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length and curvature of an object's trajectory.

Definition: A vector valued function is a function where the domain is the subset of real numbers and the range is a vector. In three dimensions $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, where x, y, and z are the functions of t or may be constants.

The example of the vector valued functions is
$r(t) = 3\hat{\imath} + t\hat{\jmath} + (\sin t)\hat{k}$

5.6.1 Domain and Range of Vector valued Function

The **domain** of a vector-valued function $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is the set of all t's for which all the component functions $x(t), y(t)$ and $z(t)$are defined. If r(t) is defined in terms of component functions and the domain is not specified explicitly, then the domain is the intersection of the domains of the component functions. The **range** of a vector-valued function is the set of all possible output values that the function can produce, which are the vectors.

Example 6: Find the domain of $r(t) = ln|t-1|\hat{\imath} + e^t\hat{\jmath} + \sqrt{t}\hat{k}$

Solution: Here $x(t) = ln|t-1|$, $y(t) = e^t$ and $z(t) = \sqrt{t}$

Domain of $x(t)$ is $(-\infty, 1) \cup (1, \infty)$

Domain of $y(t)$ is $(-\infty, \infty)$

Domain of $z(t)$ is $[0, \infty)$

Thus, the domain of the function is

$$\{(-\infty, 1) \cup (1, \infty)\} \cap \{(-\infty, \infty)\} \cap \{[0, \infty)\} = [0,1) \cup (1, \infty)$$

5.6.2 Construction of Vector Valued Function

Consider a particle is moving in space then its vector function can be constructed by considering it motion along x-, y- and z-axes. Suppose that the motion of particle along x- and y-axes is in circular shape and z-axis is changing linearly 3 times with time, then its vector function is given by

$$r = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + 3t\hat{k}$$

5.6.3 Scalar Valued Function

A function from a vector to some constant is known as scalar valued function.
For example, when we take the magnitude of the vector or the dot product of two same vectors then the vector valued function becomes a scalar valued function. Let us consider the vector:

$$r = 3\hat{\imath} + t\hat{\jmath} + (\sin t)\hat{k}$$

Its magnitude is $|r| = |3\hat{\imath} + t\hat{\jmath} + (\sin t)\hat{k}|$

$$= \sqrt{3^2 + t^2 + \sin^2 t} = \sqrt{9 + t^2 + \sin^2 t}$$

The value the function $\sqrt{9 + t^2 + \sin^2 t}$ also depends on the scalar variable t.

In this case the domain of the function ranges 0 to infinity (as we considering t as time) and the range of the function is from 9 to infinity.

Key Facts

A vector valued function has three coordinates along three different axes whereas scalar valued function gives one scalar value.

5.6.4 Derivative of a Vector Valued Function

An instantaneous rate of change is known as the derivative of vector valued function.

For example, the function represents the position of an object at a point in time t, the derivative of that function represents its velocity at that time on the same point. Consider a function $f(t)$ which has three components that is $f_1(t), f_2(t)$ and $f_3(t)$. The function $f(t)$ is said to be differentiable if all of its three components are differentiable.

$$f(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$$

Then $f(t)$ is differentiable at $t = t_0$ and its derivative is given as

$$f'(t_0) = f'_1(t_0)\hat{\imath} + f'_2(t_0)\hat{\jmath} + f'_3(t_0)\hat{k}$$

Or we may write it as

$$\left.\frac{df}{dt}\right|_{t=t_0} = \frac{d}{dt}f_1(t_0)\hat{\imath} + \frac{d}{dt}f_2(t_0)\hat{\jmath} + \frac{d}{dt}f_3(t_0)\hat{k}$$

Example 7: Find the derivative of the vector function $f(t) = 3t^2\hat{\imath} + 8t\hat{\jmath} - \frac{1}{t^3}\hat{k}$ at t = 5.

Solution: $f(t) = 3t^2\hat{\imath} + 8t\hat{\jmath} - \frac{1}{t^3}\hat{k}$

$$\frac{df}{dt} = \frac{d}{dt}(3t^2)\hat{\imath} + \frac{d}{dt}(8t)\hat{\jmath} - \frac{d}{dt}\left(\frac{1}{t^3}\right)\hat{k} = 3\frac{d}{dt}(t^2)\hat{\imath} + 8\frac{d}{dt}(t)\hat{\jmath} - \frac{d}{dt}(t^{-3})\hat{k}$$

$$\frac{df}{dt} = 3 \times 2t^{2-1}\hat{\imath} + 8 \times 1\hat{\jmath} - (-3t^{-3-1})\hat{k} = 6t\hat{\imath} + 8\hat{\jmath} + -3t^{-4}\hat{k}$$

$$\frac{df}{dt} = 6t\hat{\imath} + 8\hat{\jmath} + \frac{-3}{t^4}\hat{k}$$

$$\left.\frac{df}{dt}\right|_{t=5} \frac{df}{dt} = 6(5)\hat{\imath} + 8\hat{\jmath} + \frac{-3}{(5)^4}\hat{k} = 30\hat{\imath} + 8\hat{\jmath} + \frac{-3}{625}\hat{k}$$

5.6.5 Velocity and Acceleration of a Vector Valued Function

The derivative of the vector valued function gives the velocity of the function at the particular point and if we take again the derivative of the velocity function then it will be acceleration of the function at the particular point.

Consider the function in space

$$f(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$$

Then $f(t)$ is differentiable at $t = t_0$ and its derivative is given as

$$f'(t_0) = v(t_0) = f'_1(t_0)\hat{\imath} + f'_2(t_0)\hat{\jmath} + f'_3(t_0)\hat{k} \qquad \text{where } f'(t_0) = v(t_0)$$

Which gives the velocity of the function at $t = t_0$ and for the acceleration of the function again we differentiate the velocity function to get the acceleration

$$a(t_0) = \frac{d}{dt}v(t_0) = f''_1(t_0)\hat{\imath} + f''_2(t_0)\hat{\jmath} + f''_3(t_0)\hat{k}$$

Example 8: Find the velocity and acceleration of function $f(t) = 2t^2\hat{\imath} + 3t^4\hat{\jmath} - t^3\hat{k}$ at t = 1.

Solution: $v(t)\ \frac{d}{dt}f(t) = \frac{d}{dt}(2t^2)\hat{\imath} + \frac{d}{dt}(3t^4)\hat{\jmath} - \frac{d}{dt}(t^3)$

$$v(t) = 2 \times 2t\hat{\imath} + 3 \times 4t^3\hat{\jmath} - 3t^2\hat{k} = 4t\hat{\imath} + 12t^3\hat{\jmath} - 3t^2\hat{k}$$

$$v(t) = 4t\hat{\imath} + 12t^3\hat{\jmath} - 3t^2\hat{k}$$

$$\therefore\ v(1) = 4(1)\hat{\imath} + 12(1)^3\hat{\jmath} - 3(1)^2\hat{k} = 4\hat{\imath} + 12\hat{\jmath} - 3\hat{k}$$

For acceleration we differentiate the velocity:

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(4t)\hat{\imath} + \frac{d}{dt}(12t^3)\hat{\jmath} - \frac{d}{dt}(3t^2)\hat{k}$$

$$a(t) = 4 \times 1\hat{\imath} + 12 \times 3t^2\hat{\jmath} - 3 \times 2t\hat{k} = 4\hat{\imath} + 36t^2\hat{\jmath} - 6t\hat{k}$$

$$\therefore \quad a(1) = 4\hat{\imath} + 36(1)^2\hat{\jmath} - 6(1)\hat{k} = 4\hat{\imath} + 36\hat{\jmath} - 6\hat{k}$$

Exercise 5.3

1. If $r(t)$ is the position of the particle. Find its domain and the range at given point. Also find its first and second derivative.
   i. $r(t) = (t+1)i + (t^2-1)j + 5tk, \quad t = 2$
   ii. $r(t) = \frac{t}{t+1}i + \frac{1}{t}j + t^3k, \quad t = -\frac{1}{2}$
   iii. $r(t) = e^t i + \frac{2}{9}e^{2t}j + 5e^{-t}k, \quad t = ln2$
   iv. $r(t) = (\cos 2t)i + (3\sin 2t)j + 5tk, \quad t = 0$
2. Find the velocity and acceleration of the function at $t = 0$
   i. $r(t) = (3t+1)i + \sqrt{3}tj + t^2k$
   ii. $r(t) = \left(\frac{t}{\sqrt{2}}\right)i + \left(\frac{t}{\sqrt{2}} - 16t^2\right)j - 2tk$
   iii. $r(t) = (\ln(t^2+1))i + (\tan^{-1}t)j + \sqrt{t^2+1}k$
   iv. $r(t) = \frac{4}{9}(t+1)^{3/2}i + \frac{4}{9}(1-t)^{3/2}j + \frac{1}{3}tk$
3. Find the scalar valued function in term of the magnitude
   i. $r(t) = (3t-7)i + tj - t^2k$
   ii. $r(t) = \left(\frac{t}{\sqrt{2}}\right)i + \left(\frac{t}{\sqrt{2}} + 16t^2\right)j - 2tk$
   iii. $r(t) = (\ln(t^2+1))i + (\tan t)j + \sec t\, k$
   iv. $r(t) = \frac{4}{9}(t+1)^{3/2}i + \frac{4}{9}(1-t)^{3/2}j + \frac{1}{3}t^{3/2}k$

Review Exercise

1. Choose the right option.
   (i) Which of the following quantities is a vector?
   (a) Charge (b) Mass (c) Momentum (d) Time

(ii) Which of following is scalar quantity?

(a) Displacement (b) Weight (c) Force (d) Work

(iii) Which of following can be used to determine the magnitude of velocity?

(a) Area under acceleration-time graph
(b) Area under velocity-time graph
(c) Gradient of an acceleration-time graph
(d) Gradient of a velocity-time graph

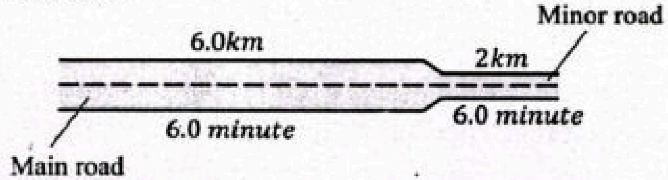
(iv) The winner of 400 metre race must have the greatest:

(a) acceleration (b) average speed
(c) instantaneous speed (d) maximum speed

(v) A car travels $100km$. The journey takes two hours. The highest speed of the car is $80km/h$, and the lowest speed $40km/h$. What is the average speed for the journey?

(a) $40km/h$ (b) $50km/h$ (c) $60km/h$ (d) $120km/h$

(vi) A car travels $6.0km$ along a main road in 6.0 minutes. It then travels $2km$ along minor road in 6.0 minutes. Which calculation of average speed for the whole journey is correct?



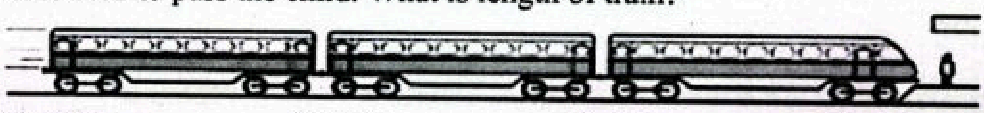
(a) $\frac{8.0}{12.0} = 0.67\frac{km}{min}$ (b) $\frac{12.0}{8.0} = 1.5\frac{km}{min}$

(c) $8.0 + 12.0 = 20\frac{km}{min}$ (d) $8.0 \times 12.0 = 96\frac{km}{min}$

(vii) Which person is experiencing an acceleration?

(a) A driver of a car that is braking to stop at traffic light.
(b) A passenger in a train that is stationary in a railway station.
(c) A shopper in a large store ascending an escalator (moving stairs) at a uniform speed.
(d) A skydiver that is falling at a constant speed towards the Earth.

(viii) A child is standing on the platform of station. A train travelling at $30m/s$ takes $3.0s$ to pass the child. What is length of train?

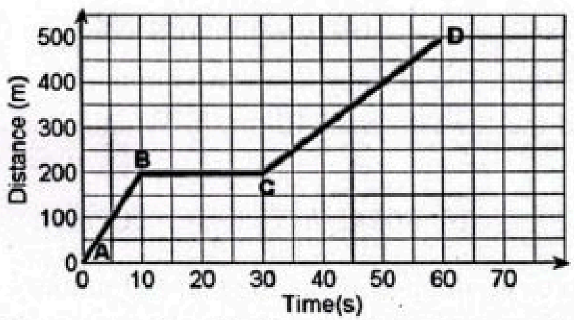


(a) $10m$ (b) $27m$ (c) $30m$ (d) $90m$

(ix) A heavy object is released near the surface of earth and falls freely. Air resistance can be ignored. Which statement about the acceleration of the object due to gravity is correct?

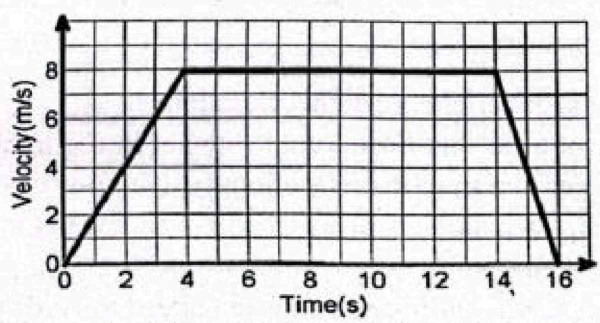
(a) The acceleration depends on mass of the object.
(b) The acceleration depends on volume of the object.
(c) The acceleration is constant.
(d) The acceleration is initially zero and increases as the object falls.

2. The graph shows the movement of a car from point A to point D, D is $500m$ from A, note that there are two slant lines AB and CD. The slant lines indicate that car is moving. The flat line BC indicates that car is stopped or is at rest.



(i) Calculate the speed of car during the first 10 seconds.
(ii) For how long did the car stop?
(iii) What is speed of car on its journey from C to D?
(iv) On which part of the journey did the car travel faster?
(v) What is average speed of the car for whole journey?
(vi) What is the average speed of the car for the time it was moving?

3. The graph below shows how the speed of an athlete varies during a race.



(i) Calculate the acceleration of athlete during the first 4 second.
(ii) What was the athlete doing between the 4^{th} and 14^{th} second?
(iii) Calculate the deceleration of the athlete in final stage of the race.
(iv) How far has the athlete moved in the first 4 seconds?
(v) What is the total distance travelled by the athlete?

4. Let $v(t) = \frac{1}{\pi} + \sin 3t$ represent the velocity of an object moving on a line. At $t = \frac{\pi}{3}$ the position is 4.
   (i) Write the acceleration function.
   (ii) Write the position function.
   (iii) At $t = \frac{\pi}{4}$, is the object speeding up or slowing down? Explain your answer.
   (iv) On the interval $\left[\frac{\pi}{2}, \pi\right]$, what is the velocity of object when acceleration is 3?

5. A particle moves along y-$axis$ so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - tan^{-1}(e^t)$. At time $t = 0$ the particle is at $y = -1$.
   (a) Find acceleration of the particle at time $t = 2$.
   (b) Is the speed of particle increasing or decreasing at time $t = 2$? Give a reason for your answer.
   (c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

UNIT
06

ANALYTICAL GEOMETRY

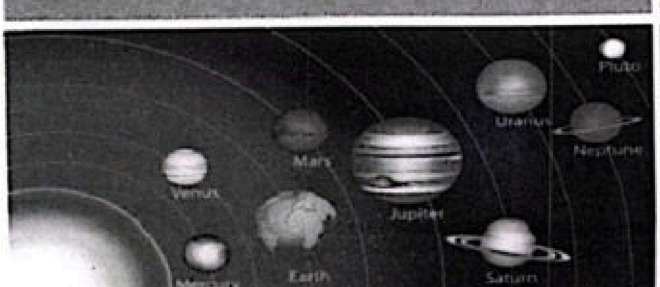
After studying this unit, students will be able to:

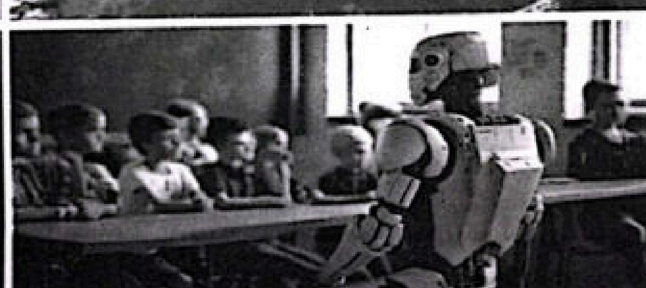
- Find the condition of concurrency of three straight lines.
- Find the equations of altitudes, right bisectors and medians of a triangle.
- Show that altitudes, right bisectors and medians of a triangle are concurrent.
- Find the area of triangular region whose vertices are given.
- Recognize homogeneous linear and quadratic equations in two variables.
- Investigate that second-degree homogeneous equation in two variables represents a pair of straight lines through the origin and find acute angle between them.
- Apply concepts of analytical geometry to real life world problems such as aviation, to track stars, distance between planets and satellites, space science and engineering.

Analytic geometry known as coordinate geometry is a branch of mathematics that combines algebra and geometry. It involves the study of the geometric shapes using the coordinates and equations.

Analytic geometry is used in physics and engineering, as well as in fields like aviation, rocketry, space science, spaceflight, computer graphics, astronomy, cartography and robotics etc.

It serves as the cornerstone for many contemporary geometric disciplines, including algebraic, differential, discrete, and computational geometry.



6.1 Point of Intersection of two Straight Lines

We know that two non-parallel lines intersect each other at one and only one point.

Let $l_1 : a_1x + b_1y + c_1 = 0$ .... (1)

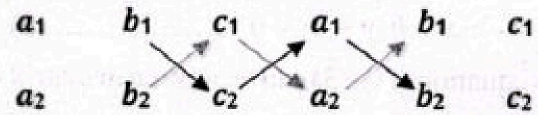
and $l_2 : a_2x + b_2y + c_2 = 0$ .... (2)

be two non-parallel lines and $P(x_1, y_1)$ be the point of intersection of l_1 and l_2. Then:

$a_1x_1 + b_1y_1 + c_1 = 0$ .... (3)

$a_2x_1 + b_2y_1 + c_2 = 0$ .... (4)

For the solution of (3) and (4), we proceed as follow:



$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad x_1 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y_1 = \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}$$

$\therefore P(x_1, y_1) = \left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right)$ is the required point of intersection where $a_1b_2 - a_2b_1 \neq 0$ otherwise $l_1 \parallel l_2$.

Key Facts

- If $l_1 \parallel l_2$, then $a_1b_2 - a_2b_1 = 0 \Rightarrow a_1b_2 = a_2b_1 \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$
- If $l_1 \perp l_2$, then $m_1m_2 = -1 \Rightarrow \left(-\frac{a_1}{a_2}\right)\left(-\frac{b_1}{b_2}\right) = -1 \Rightarrow a_1a_2 = -b_1b_2$

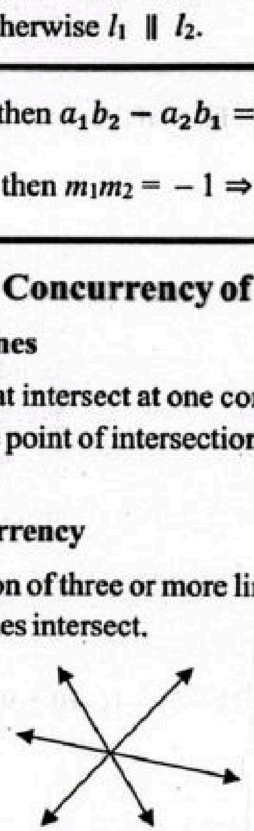
6.2 Condition for Concurrency of Three Lines

6.2.1 Concurrent Lines

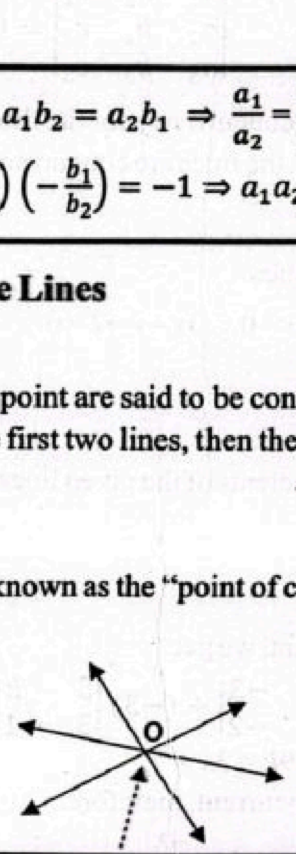
Three or more lines that intersect at one common point are said to be concurrent lines. When a third line passes through the point of intersection of the first two lines, then these three lines are known as the concurrent lines.

6.2.2 Point of Concurrency

The point of intersection of three or more lines is known as the "point of concurrency." It is the point where three or more lines intersect.



Concurrent Lines

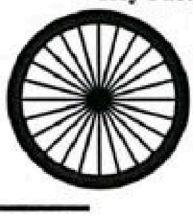


Point of Concurrency

Three lines are concurrent if the point of intersection of two lines, lies on the third line (i.e., satisfies the equation of the third line)

Key Facts

The diameters of a circle are concurrent at the center of the circle.



To check the concurrency of three lines, we use the following methods.

(a) Determinant Method

Consider three straight lines whose equations are:

$$a_1x + b_1y + c_1 = 0 \quad \dots\dots\dots\dots\dots (1)$$
$$a_2x + b_2y + c_2 = 0 \quad \dots\dots\dots\dots\dots (2)$$
$$a_3x + b_3y + c_3 = 0 \quad \dots\dots\dots\dots\dots (3)$$

The system of homogenous equations (1)-(3) can be written in matrix form as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dots\dots\dots\dots (4)$$

If the lines (1) to (3) are concurrent then they must intersect at a point O(x, y) which can be found by solving equations (1) to (3) simultaneously. The system (4) has a non-trivial solution if the determinant of coefficients of the three lines is zero. i.e.,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Which is the condition of concurrency for the three lines. Thus, if the determinant of the coefficients of the given lines is 0, then the lines are concurrent.

Example 1:

Find the value of k if the lines:

$3x+y-3=0, \quad 5x+ky-3=0, \quad 3x-y-2=0$

are concurrent.

Solution:

The determinant of coefficients of the given lines is:

$$D = \begin{vmatrix} 3 & 1 & -3 \\ 5 & k & -3 \\ 3 & -1 & -2 \end{vmatrix}$$

By solving the determinant, we get:

$$D = 3\begin{vmatrix} k & -3 \\ -1 & -2 \end{vmatrix} - 1\begin{vmatrix} 5 & -3 \\ 3 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 5 & k \\ 3 & -1 \end{vmatrix} = 3(-2k-3) - 1(-10+9) - 3(-5-3k)$$
$$= -6k - 9 + 1 + 15 + 9k = 3k + 7$$

As the three lines are concurrent, therefore:

$$3k + 7 = 0 \Rightarrow 3k = -7 \Rightarrow k = \frac{-7}{3}$$

Check Point

Check whether the lines:

$3x+4y-7=0,$

$2x-3y+5=0,$

$3x-5y+8=0$

are concurrent or not.

(b) Direct Method

In this method, we first find the point of intersection of two lines and then check if the point lies on the third line. It ensures that all three lines are concurrent.

> **Check Point**
>
> What are collinear points?
>
> Are concurrent lines coplanar?

Consider equations of three lines as follows:

$$4x-2y-4=0 \quad \text{.........} \ (1)$$
$$y=x+2 \quad \text{.........} \ (2)$$
$$2x+3y=26 \quad \text{.........} \ (3)$$

Step 1: To find the point of intersection of line (1) and line (2), substituting the value of 'y' from equation (2) in equation (1) we get:

$$4x-2(x+2)-4=0$$
$$4x-2x-4-4=0$$
$$2x-8=0 \Rightarrow 2x=8 \Rightarrow x=4$$

Substituting the value of x in equation (2), we get:

$$y=4+2=6$$

Therefore, line (1) and line (2) intersect at the point (4, 6).

Step 2: Substituting the point of intersection of the first two lines in the equation of the third line, we get:

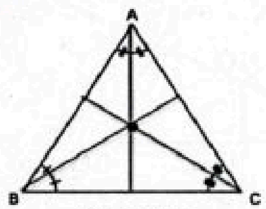
$$2(4)+3(6)=26 \Rightarrow 8+18=26 \Rightarrow 26=26$$

This implies that the line (3) also passes through the point (4, 6). Hence the three lines are concurrent.

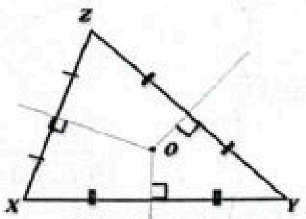
6.3 Concurrent Lines in Triangles

In a triangle, the three angle bisectors, perpendicular bisectors, medians and altitudes are examples of concurrent lines.

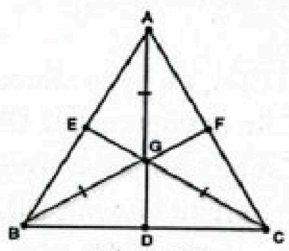
- **Incenter:** The point of intersection of three angle bisectors of a triangle is known as the incenter of a triangle.
- **Circumcenter:** The point of intersection of three perpendicular bisectors of a triangle is known as the circumcenter of a triangle.
- **Centroid:** The point of intersection of three medians of a triangle is known as the centroid of a triangle.
- **Orthocenter:** The point of intersection of three altitudes of a triangle is known as the orthocenter of a triangle.



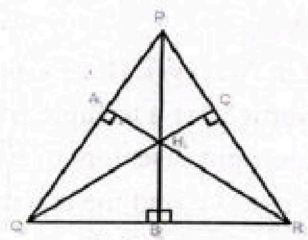
Incenter



Circumcenter



Centroid

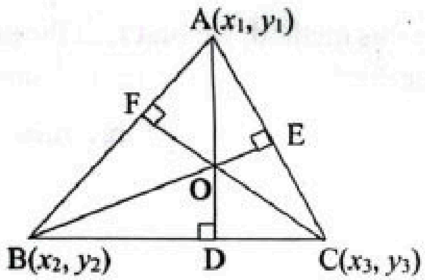


Orthocenter

Theorem 6.1:

Altitudes of a triangle are concurrent.

Proof:



Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the three vertices of a triangle ABC. In the figure, $\overline{AD}, \overline{BE}$ and $\overline{CF}$ are altitudes of $\overline{BC}, \overline{CA}$ and $\overline{AB}$ respectively.

First, we find the equation of altitude $\overline{AD}$.

Slope of $\overline{BC} = \dfrac{y_3 - y_2}{x_3 - x_2}$

Slope of altitude of $\overline{AD} = -\dfrac{x_3 - x_2}{y_3 - y_2}$

Equation of altitude $\overline{AD}$ is:

$$y - y_1 = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1) \quad \text{....... (Point-slope form)}$$

$$\Rightarrow (y - y_1)(y_3 - y_2) = -(x_3 - x_2)(x - x_1)$$

$$\Rightarrow (x_3 - x_2)x + (y_3 - y_2)y - x_1(x_3 - x_2) - y_1(y_3 - y_2) = 0 \quad \text{....... (1)}$$

By symmetry, equations of altitudes $\overline{BE}$ and $\overline{CF}$ are respectively as follows.

$$(x_1 - x_3)x + (y_1 - y_3)y - x_2(x_1 - x_3) - y_2(y_1 - y_3) = 0 \quad \text{....... (2)}$$

$$(x_2 - x_1)x + (y_2 - y_1)y - x_3(x_2 - x_1) - y_3(y_2 - y_1) = 0 \quad \text{....... (3)}$$

The determinant of coefficients of the three lines is:

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -x_1(x_3 - x_2) - y_1(y_3 - y_2) \\ x_1 - x_3 & y_1 - y_3 & -x_2(x_1 - x_3) - y_2(y_1 - y_3) \\ x_2 - x_1 & y_2 - y_1 & -x_3(x_2 - x_1) - y_3(y_2 - y_1) \end{vmatrix}$$

Adding R_2 and R_3 in R_1, we get:

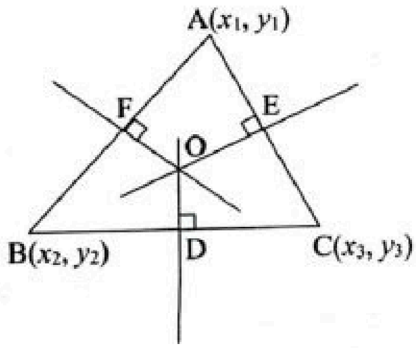
$$\begin{vmatrix} 0 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & -x_2(x_1 - x_3) - y_2(y_1 - y_3) \\ x_2 - x_1 & y_2 - y_1 & -x_3(x_2 - x_1) - y_3(y_2 - y_1) \end{vmatrix} = 0$$

Thus, the altitudes of a triangle are concurrent.

Theorem 6.2:

Right bisectors of a triangle are concurrent.

Proof:



Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the three vertices of a triangle ABC. In the figure, $\overline{OD}, \overline{OE}$ and $\overline{OF}$ are right bisectors of $\overline{BC}, \overline{CA}$ and $\overline{AB}$ respectively.

First, we find the equation of right bisector $\overline{OD}$.

Coordinates of mid point D of $\overline{BC}$ are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

Slope of $\overline{BC} = \frac{y_3 - y_2}{x_3 - x_2}$

Slope of right bisector $\overline{OD} = -\frac{x_3 - x_2}{y_3 - y_2}$

Equation of right bisector $\overline{OD}$ is:

$$y - \frac{y_2 + y_3}{2} = -\frac{x_3 - x_2}{y_3 - y_2}\left(x - \frac{x_2 + x_3}{2}\right) \quad \text{....... (Point-slope form)}$$

$$\Rightarrow \left(y - \frac{y_2 + y_3}{2}\right)(y_3 - y_2) = -(x_3 - x_2)\left(x - \frac{x_2 + x_3}{2}\right)$$

After simplification, we get:

$$(x_3 - x_2)x + (y_3 - y_2)y - \frac{1}{2}(x_3^2 - x_2^2) - \frac{1}{2}(y_3^2 - y_2^2) = 0 \quad \text{....... (1)}$$

By symmetry, equations of right bisectors $\overline{OE}$ and $\overline{OF}$ are respectively as follows.

$$(x_1 - x_3)x + (y_1 - y_3)y - \frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) = 0 \quad \text{....... (2)}$$

$$(x_2 - x_1)x + (y_2 - y_1)y - \frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) = 0 \quad \text{....... (3)}$$

The determinant of coefficients of the three lines is:

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -\frac{1}{2}(x_3^2 - x_2^2) - \frac{1}{2}(y_3^2 - y_2^2) \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) \end{vmatrix}$$

Adding R_2 and R_3 in R_1, we get:

$$\begin{vmatrix} 0 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) \end{vmatrix} = 0$$

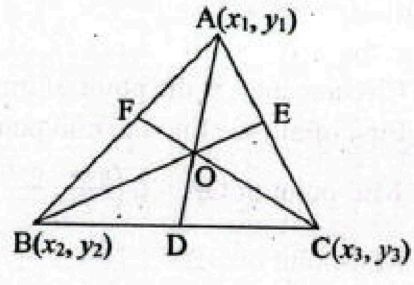
Thus, the right bisectors of a triangle are concurrent.

Theorem 6.3:

Medians of a triangle are concurrent.

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the three vertices of a triangle ABC. In the figure, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are medians of the triangle.



First, we find the equation of median $\overline{AD}$.

Coordinates of mid point D of $\overline{BC}$ are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

Using two-point formula, equation of median $\overline{AD}$ is:

$$\frac{y-y_1}{\frac{y_2+y_3}{2}-y_1} = \frac{x-x_1}{\frac{x_2+x_3}{2}-x_1}$$

$$\Rightarrow \left(\frac{x_2+x_3}{2}-x_1\right)(y-y_1) = \left(\frac{y_2+y_3}{2}-y_1\right)(x-x_1)$$

$$\Rightarrow \left(\frac{x_2+x_3}{2}-x_1\right)y - \left(\frac{x_2+x_3}{2}-x_1\right)y_1 = \left(\frac{y_2+y_3}{2}-y_1\right)x - \left(\frac{y_2+y_3}{2}-y_1\right)x_1$$

$$\Rightarrow \left(\frac{y_2+y_3}{2}-y_1\right)x - \left(\frac{x_2+x_3}{2}-x_1\right)y - \left(\frac{y_2+y_3}{2}-y_1\right)x_1 + \left(\frac{x_2+x_3}{2}-x_1\right)y_1 = 0 \qquad (1)$$

By symmetry, equations of medians $\overline{BE}$ and $\overline{CF}$ are respectively as follows.

$$\left(\frac{y_1+y_3}{2}-y_2\right)x - \left(\frac{x_1+x_3}{2}-x_2\right)y - \left(\frac{y_1+y_3}{2}-y_2\right)x_2 + \left(\frac{x_1+x_3}{2}-x_2\right)y_2 = 0 \qquad (2)$$

$$\left(\frac{y_1+y_2}{2}-y_3\right)x - \left(\frac{x_1+x_2}{2}-x_3\right)y - \left(\frac{y_1+y_2}{2}-y_3\right)x_3 + \left(\frac{x_1+x_2}{2}-x_3\right)y_3 = 0 \qquad (3)$$

The determinant of coefficients of the three lines is:

$$\begin{vmatrix} \left(\frac{y_2+y_3}{2}-y_1\right) & -\left(\frac{x_2+x_3}{2}-x_1\right) & -\left(\frac{y_2+y_3}{2}-y_1\right)x_1 + \left(\frac{x_2+x_3}{2}-x_1\right)y_1 \\ \left(\frac{y_1+y_3}{2}-y_2\right) & -\left(\frac{x_1+x_3}{2}-x_2\right) & -\left(\frac{y_1+y_3}{2}-y_2\right)x_2 + \left(\frac{x_1+x_3}{2}-x_2\right)y_2 \\ \left(\frac{y_1+y_2}{2}-y_3\right) & -\left(\frac{x_1+x_2}{2}-x_3\right) & -\left(\frac{y_1+y_2}{2}-y_3\right)x_3 + \left(\frac{x_1+x_2}{2}-x_3\right)y_3 \end{vmatrix}$$

Adding R_2 and R_3 in R_1, we get:

$$\begin{vmatrix} 0 & 0 & 0 \\ \left(\frac{y_1+y_3}{2}-y_2\right) & -\left(\frac{x_1+x_3}{2}-x_2\right) & -\left(\frac{y_1+y_3}{2}-y_2\right)x_2 + \left(\frac{x_1+x_3}{2}-x_2\right)y_2 \\ \left(\frac{y_1+y_2}{2}-y_3\right) & -\left(\frac{x_1+x_2}{2}-x_3\right) & -\left(\frac{y_1+y_2}{2}-y_3\right)x_3 + \left(\frac{x_1+x_2}{2}-x_3\right)y_3 \end{vmatrix}$$

$= 0$

Thus, the medians of a triangle are concurrent.

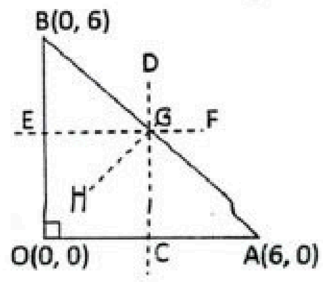
Example 2:

Find (i) circumcenter (ii) centroid and (iii) orthocenter of a triangle ABC with A(0, 0), B(6, 0) and C(0, 6). Also prove that circumcenter, centroid and orthocenter are collinear.

Solution:

Given that A(0, 0), B(6, 0) and C(0, 6) are vertices of triangle ABC.

(i) Circumcenter is the point of intersection of three perpendicular bisectors of triangle.
First of all, we find the mid points of three sides.



Mid-point of OA = $C\left(\frac{0+6}{2}, \frac{0+0}{2}\right) = C(3, 0)$

Mid-point of OB = $E\left(\frac{0+0}{2}, \frac{0+6}{2}\right) = E(0, 3)$

Mid-point of AB = $G\left(\frac{6+0}{2}, \frac{0+6}{2}\right) = G(3, 3)$

Slope of OA $= \frac{0-0}{6-0} = 0$, Slope of OB $= \frac{6-0}{0-0} = \infty$ (undefined)

Slope of AB $= \frac{6-0}{0-6} = -1$

Now, we find slopes of right bisectors.

Slope of right bisector CD $= \frac{-1}{\text{Slope of OA}} = \frac{-1}{0} = \infty$

Slope of right bisector EF $= \frac{-1}{\text{Slope of OB}} = \frac{-1}{\infty} = 0$

Slope of right bisector GH $= \frac{-1}{\text{Slope of AB}} = \frac{-1}{-1} = 1$

Equation of right bisector CD is:

$y - 0 = \infty\,(x-3) \quad \Rightarrow \quad x - 3 = 0$ .... (1)

Equation of right bisector EF is:

$y - 3 = 0\,(x-0) \quad \Rightarrow \quad y - 3 = 0$ .... (2)

Equation of right bisector GH is:

$y - 3 = 1\,(x-3) \quad \Rightarrow \quad y - x = 0$ .... (3)

Equations (1), (2) and (3) are equations of right bisectors of sides of triangle ABC.

Solving (1) and (2), we see that: $x = 3, y = 3$

$\therefore$ Circumcenter = G(3, 3) .... (A)

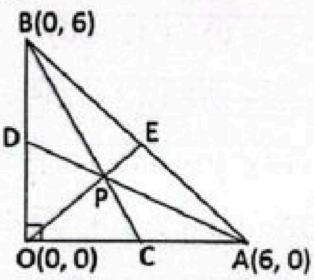
(ii) Centroid is the point of intersection of three medians of a triangle.

First of all, we find the mid points of three sides.

Mid-point of OA = C(3, 0)

Mid-point of OB = D(0, 3)

Mid-point of AB = E(3, 3)



Equation of median OE is:

$\frac{y-0}{3-0} = \frac{x-0}{3-0}$ (two-point formula)

$\Rightarrow y = x$ .... (4)

Equation of median AD is:

$\frac{y-0}{3-0} = \frac{x-6}{0-6}$ (two-point formula)

$\Rightarrow -6y = 3x - 18 \Rightarrow 3x + 6y - 18 = 0$

$\Rightarrow x + 2y - 6 = 0$ .... (5)

Key Facts

Medians of a triangle trisect each other (i.e., bisect each other in the ratio 2 : 1)

Equation of median BC is:

$\frac{y-6}{0-6} = \frac{x-0}{3-0}$ (two-point formula)

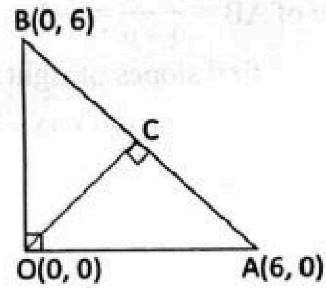
$\Rightarrow 3y - 18 = -6x \Rightarrow 6x + 3y - 18 = 0$

$\Rightarrow 2x + y - 6 = 0$ .... (6)

Solving (4) and (5), we see that: $x = 2, y = 2$

$\therefore$ Centroid = P(2, 2) .... (B)

(iii) Orthocenter is the point of intersection of three altitudes of a triangle.
As the triangle is right angled, therefore two of its sides OA and OB are also altitudes.
The third altitude is OC.



Slope of OA $= 0$, Slope of OB $= \infty$ (undefined)

Slope of AB $= \dfrac{6-0}{0-6} = -1$

Slope of altitude OC $= \dfrac{-1}{\text{Slope of AB}} = \dfrac{-1}{-1} = 1$

Equation of altitude OA is:

$y - 0 = 0\,(x - 0) \quad \Rightarrow \quad y = 0$ .... (7)

Equation of altitude OB is:

$y - 0 = \infty\,(x - 0) \quad \Rightarrow \quad x = 0$ .... (8)

Equation of altitude OC is:

$y - 0 = 1\,(x - 0) \quad \Rightarrow \quad y - x = 0$ .... (9)

Key Facts

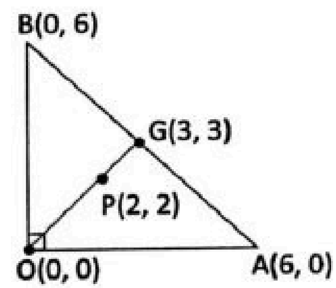
In a right triangle, the orthocentre is the vertex containing right angle.

Equations (1), (2) and (3) are equations of right bisectors of sides of triangle ABC.

Solving (7) and (8), we see that: $x = 0, y = 0$

$\therefore$ Orthocenter = O(0, 0) .... (C)

Now we prove that circumcenter G(3, 3), centroid P(2, 2) and orthocenter O(0, 0) are collinear.



$OP = \sqrt{(2-0)^2 + (2-0)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$PG = \sqrt{(3-2)^2 + (3-2)^2} = \sqrt{1+1} = \sqrt{2}$

$OG = \sqrt{(3-0)^2 + (3-0)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

Now, $OP + PG = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2} = OG$

Which shows that circumcenter, centroid and orthocenter are collinear in any triangle.

Example 3:
The points P(− 1, 2), Q(3, − 2) and R(6, 3) are vertices of a triangle PQR. Show that altitudes, right bisectors and medians of the triangle are concurrent.

Solution:
Let $(x_1, y_1) = (-1, 2)$, $(x_2, y_2) = (3, -2)$ and $(x_3, y_3) = (6, 3)$, then:
We know that the determinant of coefficients of the three altitudes is:

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -x_1(x_3 - x_2) - y_1(y_3 - y_2) \\ x_1 - x_3 & y_1 - y_3 & -x_2(x_1 - x_3) - y_2(y_1 - y_3) \\ x_2 - x_1 & y_2 - y_1 & -x_3(x_2 - x_1) - y_3(y_2 - y_1) \end{vmatrix}$$

Substituting the values $x_1 = -1, y_1 = 2, x_2 = 3, y_2 = -2, x_3 = 6$ and $y_3 = 3$ in above determinant, we get:

$$\begin{vmatrix} 6-3 & 3+2 & 1(6-3)-2(3+2) \\ -1-6 & 2-3 & -3(-1-6)+2(2-3) \\ 3+1 & -2-2 & -6(3+1)-3(-2-2) \end{vmatrix} = \begin{vmatrix} 3 & 5 & -7 \\ -7 & -1 & 19 \\ 4 & -4 & -12 \end{vmatrix}$$

Adding R_3 in R_2, we get:

$$\begin{vmatrix} 3 & 5 & -7 \\ -3 & -5 & 7 \\ 4 & -4 & -12 \end{vmatrix} = \begin{vmatrix} 3 & 5 & -7 \\ 3 & 5 & -7 \\ 4 & -4 & -12 \end{vmatrix} \text{ (Multiplying } R_2 \text{ by } -1)$$

$$= 0 \text{ (} R_1 \text{ and } R_2 \text{ are identical.)}$$

Hence altitudes of triangle are concurrent.

Now, the determinant of coefficients of the three right bisectors is:

$$\begin{vmatrix} x_3-x_2 & y_3-y_2 & -\frac{1}{2}(x_3^2-x_2^2)-\frac{1}{2}(y_3^2-y_2^2) \\ x_1-x_3 & y_1-y_3 & -\frac{1}{2}(x_1^2-x_3^2)-\frac{1}{2}(y_1^2-y_3^2) \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}(x_2^2-x_1^2)-\frac{1}{2}(y_2^2-y_1^2) \end{vmatrix}$$

Substituting the values $x_1 = -1, y_1 = 2, x_2 = 3, y_2 = -2, x_3 = 6$ and $y_3 = 3$ in above determinant, we get:

$$\begin{vmatrix} 6-3 & 3+2 & -\frac{1}{2}(36-9)-\frac{1}{2}(9-4) \\ -1-6 & 2-3 & -\frac{1}{2}(1-36)-\frac{1}{2}(4-9) \\ 3+1 & -2-2 & -\frac{1}{2}(9-1)-\frac{1}{2}(4-4) \end{vmatrix} = \begin{vmatrix} 3 & 5 & -16 \\ -7 & -1 & 20 \\ 4 & -4 & -4 \end{vmatrix}$$

Adding R_3 in R_2, we get:

$$\begin{vmatrix} 3 & 5 & -16 \\ -3 & -5 & 16 \\ 4 & -4 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ -3 & -5 & 16 \\ 4 & -4 & -4 \end{vmatrix} \text{ (Adding } R_2 \text{ in } R_1)$$

$$= 0$$

Hence right bisectors of triangle are concurrent.

Again, the determinant of coefficients of the three medians is:

$$\begin{vmatrix} \left(\frac{y_2+y_3}{2}-y_1\right) & -\left(\frac{x_2+x_3}{2}-x_1\right) & -\left(\frac{y_2+y_3}{2}-y_1\right)x_1+\left(\frac{x_2+x_3}{2}-x_1\right)y_1 \\ \left(\frac{y_1+y_3}{2}-y_2\right) & -\left(\frac{x_1+x_3}{2}-x_2\right) & -\left(\frac{y_1+y_3}{2}-y_2\right)x_2+\left(\frac{x_1+x_3}{2}-x_2\right)y_2 \\ \left(\frac{y_1+y_2}{2}-y_3\right) & -\left(\frac{x_1+x_2}{2}-x_3\right) & -\left(\frac{y_1+y_2}{2}-y_3\right)x_3+\left(\frac{x_1+x_2}{2}-x_3\right)y_3 \end{vmatrix}$$

Substituting the values $x_1 = -1, y_1 = 2, x_2 = 3, y_2 = -2, x_3 = 6$ and $y_3 = 3$ in above determinant, we get:

$$\begin{vmatrix} \left(\frac{-2+3}{2}-2\right) & -\left(\frac{3+6}{2}+1\right) & -\left(\frac{-2+3}{2}-2\right)(-1)+\left(\frac{3+6}{2}+1\right)(2) \\ \left(\frac{2+3}{2}+2\right) & -\left(\frac{-1+6}{2}-3\right) & -\left(\frac{2+3}{2}+2\right)(3)+\left(\frac{-1+6}{2}-3\right)(-2) \\ \left(\frac{2-2}{2}-3\right) & -\left(\frac{-1+3}{2}-6\right) & -\left(\frac{2-2}{2}-3\right)(6)+\left(\frac{-1+3}{2}-6\right)(3) \end{vmatrix}$$

$$\begin{vmatrix} -1.5 & -5.5 & -1.5+11 \\ 4.5 & 0.5 & -13.5+1 \\ -3 & 5 & 18-15 \end{vmatrix} = \begin{vmatrix} -1.5 & -5.5 & 9.5 \\ 4.5 & 0.5 & -12.5 \\ -3 & 5 & 3 \end{vmatrix}$$

Adding R_3 in R_2, we get:

$$\begin{vmatrix} -1.5 & -5.5 & 9.5 \\ 1.5 & 5.5 & -9.5 \\ -3 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ -3 & -5 & 16 \\ 4 & -4 & -4 \end{vmatrix} \quad \text{(Adding } R_2 \text{ in } R_1\text{)}$$

$$= 0$$

Hence medians of triangle are concurrent.

6.4 Area of Triangular Region

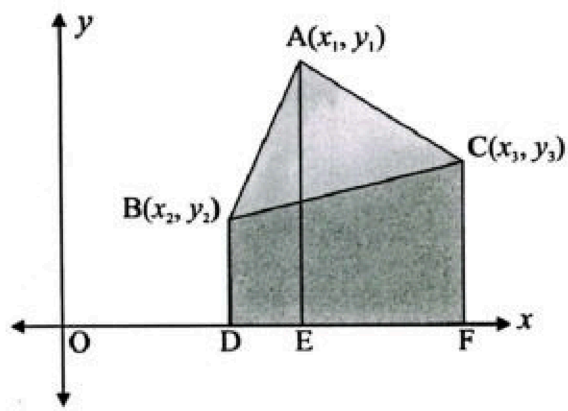
The area of a plane figure is the space covered by it.

Consider ΔABC as given in the adjoining figure with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

In the figure, we have drawn perpendiculars BD, AE and CF from the vertices of the triangle to the x-axis.

Notice that three trapeziums are formed: ABDE, AEFC and BCFD.

We can express the area of triangle ABC in terms of the areas of these three trapeziums as follows.



Area of ΔABC = Area of Trap.ABDE + Area of Trap.AEFC − Area of Trap.BCFD .... (1)

Now, Area of Trap.ABDE $= \frac{1}{2} \times (BD + AE) \times DE$

$= \frac{1}{2} \times (y_2 + y_1) \times (x_1 - x_2)$

Area of Trap.AEFC $= \frac{1}{2} \times (AE + CF) \times EF$

$= \frac{1}{2} \times (y_1 + y_3) \times (x_3 - x_1)$

Area of Trap.BCFD $= \frac{1}{2} \times (BD + CF) \times DF$

$= \frac{1}{2} \times (y_2 + y_3) \times (x_3 - x_2)$

Recall

Area of trapezium is:

$A = \frac{1}{2} \times$ (sum of lengths of parallel sides) × distance between parallel sides (altitude)

Substituting these values in equation (1), we can find area A of triangle ABC as follows.

$$A = \frac{1}{2} \times (y_2 + y_1) \times (x_1 - x_2) + \frac{1}{2} \times (y_1 + y_3) \times (x_3 - x_1) - \frac{1}{2} \times (y_2 + y_3) \times (x_3 - x_2)$$

$$= \frac{1}{2} \times [(y_2 + y_1) \times (x_1 - x_2) + (y_1 + y_3) \times (x_3 - x_1) - (y_2 + y_3) \times (x_3 - x_2)]$$

$$= \frac{1}{2} \times [x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3 y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3]$$

$$= \frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2} \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Check Point

Find the area of a triangle whose vertices are given as (1, -1), (-4, 6) and (-3, -5).

Key Facts

- If points A, B and C are collinear, then area is zero.
- If the sign of value of area obtained is negative, ignore it as the area cannot be negative.
- Area of a triangle can also be found by finding the length of three sides of a triangle using the distance formula and then applying Heron's formula.

Example 4:

Find the area of triangle if points (4, – 2), (– 2, 4) and (5, 5) are vertices of a triangle.

Solution:

Here, $x_1 = 4, y_1 = -2, x_2 = -2, y_2 = 4, x_3 = 5$ and $y_3 = 5$

$$\text{Area of triangle} = \frac{1}{2} \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \times \begin{vmatrix} 4 & -2 & 1 \\ -2 & 4 & 1 \\ 5 & 5 & 1 \end{vmatrix}$$

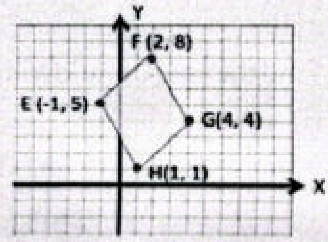
$$= \frac{1}{2} \times [4(4-5) + 2(-2-5) + 1(-10-20)]$$

$$= \frac{1}{2} \times [-4 - 14 - 30] = \frac{1}{2} \times [-48] = -24$$

$\therefore$ Area of triangle = 24 square units

Challenge

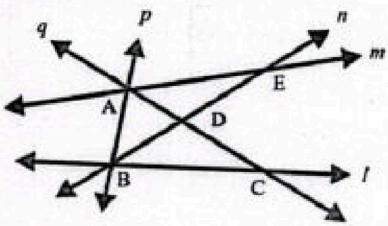
Find the area of parallelogram shown.



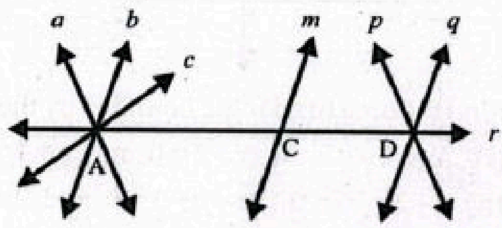
Exercise 6.1

1. Which sets of lines are concurrent in the given figure? Also, tell the point of concurrency.

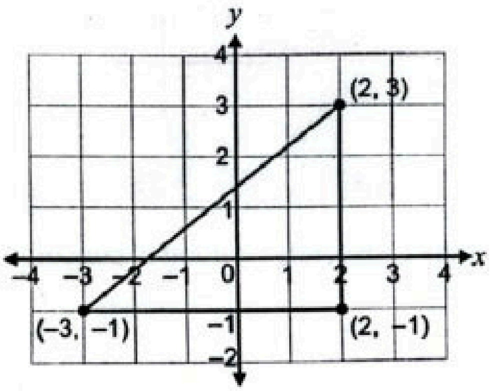
(i)



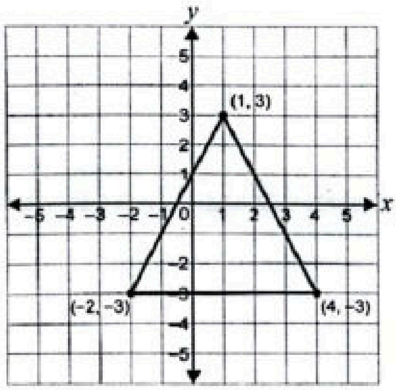
(ii)



2. Check whether the lines are concurrent or not.
   (i) $3x-4y-13=0,\ 8x-11y-33=0,\ 2x-3y-7=0$
   (ii) $x+2y-4=0,\ x-y-1=0,\ 4x+5y-13=0$
3. Determine the value of a if the line $2x-y+3=0,\ x-y=0,\ 3x+ay+1=0$ are concurrent.
4. Show that the lines $x+y+1=0, x-y+1=0$ and x-axis are concurrent. Also prove that x-axis bisects the angle between $x+y+1=0$ and $x-y+1=0$.
5. Find the equations of altitudes and their point of concurrency in the triangle ABC when A(4, –2), B(5, 5) and C(–1, 3). What is the name of point of concurrency?
6. Find the equations of right bisectors and their point of concurrency in the triangle XYZ when X(0, 0), Y(7, 0) and Z(7, 4). What is the name of point of concurrency?
7. Find the equations of medians and their point of concurrency in the triangle DEF when D(–6, –4), E(6, –4) and F(–2, 4). What is the name of point of concurrency?
8. Prove that (a) altitudes, (b) right bisectors and (c) medians of the following triangles are concurrent.
   (i) A(4, 6), B(7, 2), C(2, 3) (ii) P(–4, 0), Q(2, 0), R(0, 3)
9. Find the area of triangles.
   (i) A(1, 1), B(4, 5), C(12, – 1) (ii) D(3, 1), E(2, 3), F(2, 2)
10. By finding area of triangle, show that points A(6, 0), B(– 3, 6) and C(3, 2) are collinear.
11. Using the formula of area, find x if the points P(3, 2), Q(–1, x), R(7, 3) are collinear.
12. Vertices of a triangle are (3, – 4), (4, h) and (2, 6). Find h if area of the triangle is 10 square units.
13. Find the area of following figures.
   (i)



   (ii)



14. Find the area of triangle bounded by the lines:
   (i) $4x-5y+7=0,\ x-2=0$ and $y+1=0$
   (ii) $x-2y-6=0,\ 3x-y+3=0$ and $2x+y-4=0$

6.5 Homogeneous Equations in two Variables

Let $f(x, y) = 0$ be any equation in two variables x and y. The equation $f(x, y) = 0$ is called a homogeneous equation of degree n if

$$f(kx, ky) = k^n f(x, y) \quad \text{.......... (1)}$$

where k is any real number and n is a positive integer.

6.5.1 Homogeneous Linear Equations in two Variables

An equation of the form $ax + by = 0$ is called a homogeneous linear equation in two variables and always passes through origin.

If we take $n = 1$ in equation (1), then we get:

$$f(kx, ky) = kf(x, y) \quad \text{............ (2)}$$

Equation (2) is called homogeneous linear equation in two variables. Consider:

$$f(x, y) = ax + by = 0 \quad \text{..........(3)}$$

Replacing x by kx and y by ky, we have:

$$f(kx, ky) = akx + bky = k(ax + by) = kf(x, y) \quad \text{....... (4)}$$

From (4), it is clear that equation (3) is homogeneous linear equation in two variables.

For example, $x - 4y = 0$ is a homogeneous linear equation in two variables x and y.

Key Facts

In the homogeneous system of equations, no equation has a constant term. A homogeneous linear system may have one or infinitely many solutions. But it has at least one trivial solution always.

6.5.2 Non-Homogeneous Linear Equation

An equation of the form $ax + by + c = 0$ is called a non-homogeneous linear equation in two variables.

For example, $2x - 5y + 7 = 0$ is a non-homogeneous linear equation in two variables x and y.

6.5.3 Joint Equation

Consider two straight lines represented by:

$$a_1x + b_1y + c_1 = 0 \quad \text{....... (5)}$$

$$a_2x + b_2y + c_2 = 0 \quad \text{....... (6)}$$

Multiplying equations (5) and (6), we have:

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0 \quad \text{........ (7)}$$

Equation (7) is called joint equation and can be re-written as:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{........(8)}$$

Equation (8) represents a pair of lines and can be resolved back into two linear equations.

If we put $c_1 = c_2 = 0$ in equations (5) and (6), then:

$$a_1x + b_1y = 0 \quad \text{....... (9)}$$

$$a_2x + b_2y = 0 \quad \text{....... (10)}$$

Equations (9) and (10) represent a system of homogeneous linear equations in two variables.

Multiplying (9) and (10), we get:

$$(a_1x + b_1y)(a_2x + b_2y) = 0 \quad \text{........ (11)}$$

Check Point

Show that $2x + 3y = 0$ is homogeneous linear equation in two variables.

Equation (11) is special joint equation and can be re-written as:

$$a_1a_2x^2 + a_1b_2xy + a_2b_1xy + b_1b_2y^2 = a_1a_2x^2 + (a_1b_2 + a_2b_1)\,xy + b_1b_2y^2$$

If we put $a_1a_2 = a$, $a_1b_2 + a_2b_1 = 2h$ and $b_1b_2 = b$, then we get:

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{........(12)}$$

where a, h and b are not simultaneously zero.

Equation (12) represents a special pair of lines passing through origin and can be resolved back into two homogeneous linear equations. This equation is called general second-degree homogeneous equation.

Any point P(x, y) that satisfies $a_1x + b_1y = 0$ or $a_2x + b_2y = 0$ will also satisfy equation (12).
For example, $6x^2 - 4xy + 8y^2 = 0$ is a homogeneous quadratic equation in two variables x and y.
Equations (9) and (10) can also be written as:
$y = m_1x$ and $y = m_2x$ where m_1 and m_2 are slopes of lines passing through origin.
Their joint equation is:

$$(y - m_1x)(y - m_1x) = 0 \;\Rightarrow\; y^2 - (m_1 + m_2)\,xy + m_1\,m_2\,x^2 \qquad \text{....... (13)}$$

Equation (13) is another special type of second-degree homogeneous equation.
Comparing equations (12) and (13), we have:

$$\frac{m_1m_2}{a} = \frac{-(m_1 + m_2)}{2h} = \frac{1}{b} \qquad \Rightarrow \qquad m_1m_2 = \frac{a}{b} \text{ and } m_1 + m_2 = \frac{-2h}{b}$$

6.5.4 Homogeneous Quadratic Equations in two Variables

If we take $n = 2$ in equation (1), then we get:

$$f(kx, ky) = k^2 f(x, y) \qquad \text{........... (14)}$$

Equation (14) is called homogeneous quadratic equation in two variables. Consider:

$$f(x, y) = ax^2 + 2hxy + by^2 = 0 \text{.......... (15)}$$

Replacing x by kx and y by ky, we have:

$$f(kx, ky) = a(kx)^2 + 2h(kx)(ky) + b(ky)^2 = k^2(ax^2 + 2hxy + by^2) = k^2 f(x, y) \text{ .... (16)}$$

From (16), it is clear that equation (15) is homogeneous quadratic equation in two variables.

Key Facts

The most general equation of second degree: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
Represents a pair of lines if:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Theorem 6.4:

Every homogeneous second-degree equation:

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{.......... (1)}$$

represents a pair of lines passing through the origin. The lines are:

(i) Real and distinct if $h^2 > ab$

(ii) Real and coincident if $h^2 = ab$

(iii) Imaginary if $h^2 < ab$

Proof: Multiplying equation (1) by a, we have:

$$a^2x^2 + 2ahxy + aby^2 = 0$$
$$a^2x^2 + 2ahxy + h^2y^2 - h^2y^2 + aby^2 = 0$$
$$(ax + hy)^2 - y^2(h^2 - ab) = 0$$
$$(ax + hy)^2 - (y\sqrt{h^2 - ab})^2 = 0$$
$$(ax + hy + y\sqrt{h^2 - ab})(ax + hy - y\sqrt{h^2 - ab}) = 0 \quad \text{.... (2)}$$

Which shows that equation (1) represents a pair of lines through origin. From equation (2):

$$ax + hy + y\sqrt{h^2 - ab} = 0 \quad \text{and} \quad ax + hy - y\sqrt{h^2 - ab} = 0$$

or
$$ax + y(h + \sqrt{h^2 - ab}) = 0 \quad \text{...... (3)}$$
$$ax + y(h - \sqrt{h^2 - ab}) = 0 \quad \text{...... (4)}$$

From Equations (3) and (4), it is clear that the lines are:

(i) Real and distinct if $h^2 > ab$ (ii) Real and coincident if $h^2 = ab$

(iii) Imaginary if $h^2 < ab$

Note: It is interesting to note that even the lines are imaginary, they pass through the real point (0, 0) as this point lies on the joint equation.

Example 5:

Find the straight lines represented by $x^2 - 7xy + 12y^2 = 0$

Solution:

$x^2 - 7xy + 12y^2 = 0 \quad \Rightarrow \quad x^2 - 3xy - 4xy + 12y^2 = 0$

$\Rightarrow \quad x(x - 3y) - 4y(x - 3y) = 0 \quad \Rightarrow \quad (x - 3y)(x - 4y) = 0$

$\Rightarrow \quad x - 3y = 0 \quad$ or $\quad x - 4y = 0$

Which are required straight lines.

6.5.5 Angle between Lines Represented by $ax^2 + 2hxy + by^2 = 0$

We have already proved that $ax^2 + 2hxy + by^2 = 0$ represents two straight lines:

$$ax + y(h + \sqrt{h^2 - ab}) = 0 \quad \text{...... (1)}$$
$$ax + y(h - \sqrt{h^2 - ab}) = 0 \quad \text{...... (2)}$$

Slopes of (1) and (2) respectively are:

$$m_1 = \frac{-(h + \sqrt{h^2 - ab})}{b} \quad \text{and} \quad m_2 = \frac{-(h - \sqrt{h^2 - ab})}{b}$$

$$\Rightarrow \quad m_1 + m_2 = \frac{-2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b} \quad \text{and}$$

If θ is the measure of acute angle between (1) and (2), then:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{2\sqrt{h^2 - ab}}{a + b} \quad \text{........ (3)}$$

Special cases:

(i) When $\theta = 0$, then $\tan\theta = 0$ and equation (3) implies $h^2 - ab = 0$ or $h^2 = ab$ which is the condition for the lines to be coincident.

(ii) When $\theta = 90°$, then $\tan\theta =$ undefined and equation (3) implies $a + b = 0$ which is condition for the lines to be orthogonal. i.e., sum of coefficients of x^2 and y^2 is 0.

Example 6:

Find measure of acute angle between lines represented by $6x^2 - xy - y^2 = 0$.

Solution:

Given equation is $6x^2 - xy - y^2 = 0$

Here, $a = 6$, $h = \frac{-1}{2}$, $b = -1$

If θ is the measure of acute angle between lines, then:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{\frac{1}{4} + 6}}{6 - 1} = \frac{5}{5} = 1 \Rightarrow \theta = \tan^{-1}(1) = 45°$$

Example 7:

Find a joint equation of straight lines through the origin and perpendicular to the lines represented by $3x^2 + 5xy + 2y^2 = 0$

Solution:

$$3x^2 + 5xy + 2y^2 = 0 \quad \Rightarrow \quad 3x^2 + 3xy + 2xy + 2y^2 = 0$$

$$\Rightarrow \quad 3x(x + y) + 2y(x + y) = 0 \quad \Rightarrow \quad (x + y)(3x + 2y) = 0$$

Thus, the lines represented by the given equation are:

$$x + y = 0 \quad \text{and} \quad 3x + 2y = 0$$

$$\Rightarrow \quad y = -x \quad \ldots.(1) \quad \text{and} \quad y = \frac{-3x}{2} \quad \ldots..(2)$$

The line through (0, 0) and perpendicular to (1) is:

$$y = x \quad \text{or} \quad y - x = 0 \quad \ldots\ldots(3)$$

The line through (0, 0) and perpendicular to (2) is:

$$y = \frac{2x}{3} \quad \text{or} \quad 3y - 2x = 0 \quad \ldots\ldots(4)$$

Joint equation of lines (3) and (4) is:

$$(y - x)(3y - 2x) = 0 \quad \Rightarrow \quad 3y^2 - 2xy - 3xy + 2x^2 = 0$$

$$\Rightarrow \quad 3y^2 - 5xy + 2x^2 = 0 \quad \text{or} \quad 2x^2 - 5xy + 3y^2 = 0$$

Key Facts

If $y = mx$ ....(i) is equation of line passing through origin, then the line perpendicular to (i) through origin is $y = -\frac{1}{m}x$.

Exercise 6.2

1. Find the lines represented by each of the following joint equations.

(i) $x^2 + 5xy + 6y^2 = 0$ (ii) $7x^2 - 2xy - 9y^2 = 0$

(iii) $x^2 + 6xy = 0$ (iv) $x^2 - 8xy + 12y^2 = 0$

(v) $5x^2 + 3xy - 2y^2 = 0$ (vi) $x^2 - 3xy - y^2 = 0$

2. Find measure of acute angle between lines represented by the following equations.
   (i) $x^2 - y^2 = 0$ (ii) $x^2 + 5xy + 4y^2 = 0$
   (iii) $15x^2 - 19xy + 6y^2 = 0$ (iv) $10x^2 - xy - 9y^2 = 0$
   (v) $5x^2 - 3xy - 2y^2 = 0$ (vi) $7x^2 + 2xy - 9y^2 = 0$
3. Show that the lines represented by following equations are real coincident or real distinct or imaginary.
   (i) $x^2 + 4xy - 21y^2 = 0$ (ii) $4x^2 - 12xy + 9y^2 = 0$
   (iii) $x^2 + xy + y^2 = 0$ (iv) $x^2 - 9y^2 = 0$
4. Show that the angle between lines represented by the following equations is right angle.
   (i) $x^2 + 5xy - y^2 = 0$ (ii) $x^2 - 2(\tan\theta)xy - y^2 = 0$
   Explain the reason of right angle between lines.
5. Find a joint equation of the lines through the origin and perpendicular to the lines:
   (i) $2x^2 - 7xy + 6y^2 = 0$ (ii) $x^2 + 17xy + 60y^2 = 0$
   (iii) $3x^2 - 13xy - 10y^2 = 0$ (iv) $x^2 + 3xy - 28y^2 = 0$

6.6 Application of Analytic Geometry

Analytic geometry is used in physics and engineering, and also in aviation, rocketry, space science, and spaceflight. It is the foundation of most modern fields of geometry, including algebraic, differential, discrete and computational geometry. See the following examples for understanding.

Example 8:

An engineer needs to design a traffic intersection where three straight roads meet. The roads are represented by the equations of lines. The roads are given by the following equations:

$2x - 4y + 5 = 0, 7x - 8y + 5 = 0$ and $4x + 5y = 45$

Determine if the roads are concurrent and if so, find the point of intersection of roads.

Solution:

$2x - 4y + 5 = 0$ (1)

$7x - 8y + 7 = 0$ (2)

$4x + 4y - 11 = 0$ (3)

Multiplying equation (1) by -2 and adding with (2), we get:

$-4x + 8y - 10 = 0$ (4)

$7x - 8y + 7 = 0$ (2)

$3x - 3 = 0 \Rightarrow 3x = 3 \Rightarrow x = 1$

Substituting in equation (1), we have:

$2(1) - 4y + 5 = 0 \Rightarrow -4y = -7 \Rightarrow y = \frac{7}{4}$

Showing that two roads meet at $\left(1, \frac{7}{4}\right)$.

Putting $\left(1, \frac{7}{4}\right)$ in equation (3), we get:

$4(1) + 4\left(\frac{7}{4}\right) - 11 = 0 \quad \Rightarrow \quad 4 + 7 - 11 = 0 \quad \Rightarrow \quad 0 = 0$

Therefore, third road also pass through $\left(1, \frac{7}{4}\right)$. Hence the three roads are concurrent.

Example 9:

A triangular park is bounded by three straight roads that meet at points A(2, 3), B(8, 5) and C(5, 10). A city planner needs to determine the area of the park for landscaping purpose. How can he determine the area of the park?

Solution:

Here, $x_1 = 2, y_1 = 3, x_2 = 8, y_2 = 5, x_3 = 5$ and $y_3 = 10$

$$\text{Area of triangular park} = \frac{1}{2} \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \times \begin{vmatrix} 2 & 3 & 1 \\ 8 & 5 & 1 \\ 5 & 10 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times [2(5 - 10) - 3(8 - 5) + 1(80 - 25)]$$

$$= \frac{1}{2} \times [-10 - 9 + 55] = \frac{1}{2} \times [36] = 18$$

$\therefore$ Area of the plot = 18 square units

Example 10:

A railway engineer is designing an intersection where two rail tracks meet at a junction. The joint equation representing the tracks is given by $x^2 - 4xy - 12y^2 = 0$. Determine the angle between the individual lines representing the tracks.

Solution:

Given joint equation of the tracks is:

$$x^2 - 4xy - 12y^2 = 0 \quad \Rightarrow \quad x^2 + 2xy - 6xy - 12y^2 = 0$$

$$\Rightarrow x(x + 2y) - 6y(x + 2y) = 0 \quad \Rightarrow \quad (x + 2y)(x - 6y) = 0$$

The lines representing the tracks are:

$$x + 2y = 0 \qquad (i)$$

$$x - 6y = 0 \qquad (ii)$$

From (i) and (ii), we have:

$$y = -\frac{1}{2}x \qquad (iii)$$

$$y = \frac{1}{6}x \qquad (iv)$$

Slopes of lines (iii) and (iv) are:

$$m_1 = -\frac{1}{2} \quad \text{and} \quad m_2 = \frac{1}{6}$$

If θ is angle between both tracks, then:

$$\text{Tan}\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{1}{6} + \frac{1}{2}}{1 + \frac{1}{6} \times (-\frac{1}{2})} = \frac{8}{13} = 0.727$$

$$\theta = \text{Tan}^{-1}(0.727) = 36.03^{\circ}$$

Therefore, angle between the tracks is 36.03°.

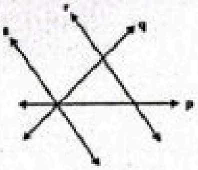
Exercise 6.3

1. Three jet fighters are flying straight in different directions along the lines $2x - y - 14 = 0$, $x + y - 1 = 0$ and $3x + 2y - 7 = 0$
   Check whether the jet fighters will pass through a single point or not. If yes, find that point.
2. A farmer owns a triangular shaped piece of land with corners at points X(3, 7), Y(6, 2) and Z(10, 8). Calculate the cost of planting maize crop @ Rs. 300 per square unit.
3. Hira is designing a triangular section of a roof with vertices at points P(4, 1), Q(9, 5) and R(7,10). She needs to calculate the area of the section to determine how much roofing material is required. Find the area calculated by her.
4. A landscaper is designing a triangular garden bed with vertices A(1, 4), B(5, 1) and C(8, 6). Calculate the cost of planting mango trees in the garden @ Rs.70 per square unit.
5. A civil engineer is tasked with designing a roundabout where three main roads converge. The equations of roads are $2x + y - 11.5 = 0$, $x - 4y + 1 = 0$ and $3x - 2y - 12 = 0$.
   Find the coordinates of the point to design the roundabout.
6. Asad is arranging a flashlight in a marriage ceremony. The position of the flashlight is at the intersection of lines $2x + y - 23 = 0$, $0.5x - y + 3 = 0$ and $x - y = 1$. Find the position of the flashlight.
7. A surveyor is mapping out a triangular park where three straight walkways meet. The walkways are represented by the equations of lines $x + y - 4 = 0$, $2x - y - 2 = 0$ and $x - 2y + 2 = 0$. Find the coordinates of point of intersection of the walkways.
8. An astronomer is studying the behavior of light rays passing through a converging lens. The lens focuses the rays at the origin. The equation of rays is given by $3x^2 - 4xy + y^2 = 0$. Find the path of individual light rays and angel between rays.
9. A welder is designing a support structure for a building. The structure is made up of beams intersecting at origin. The equations of the lines representing the beams are shown by the joint equation $2x^3 - 8xy^2 = 0$. Find the equations of iron beams.

Review Exercise

1. Select the correct option.

(i) *The lines* p, q and s are:



(a) parallel (b) concurrent
(c) perpendicular (d) collinear

(ii) If the determinant of coefficients the three lines is 0, then the lines are:
(a) concurrent (b) intersecting (c) parallel (d) perpendicular

(iii) Which of the following equations is homogeneous?
(a) $x^2+5x=0$ (b) $2x+3y+1=0$
(c) $x^2+5xy=0$ (d) $4y+8=0$

(iv) Which of the following equations is not homogeneous?
(a) $x^2+3xy=0$ (b) $2x-3y=0$
(c) $xy^2+y^3=0$ (d) $5x-5=0$

(v) Equation $ax+by+c=0$ passes through origin if:
(a) $a=0$ (b) $c=0$ (c) $b=0$ (d) $a=b=c=0$

(vi) The lines represented by $ax^2+2hxy+by^2=0$ are real and distinct if:
(a) $h^2>ab$ (b) $h^2=ab$ (c) $h^2<ab$ (d) $h^2\le ab$

(vii) The lines represented by $ax^2+2hxy+by^2=0$ are coincident if:
(a) $h^2>ab$ (b) $h^2=ab$ (c) $h^2<ab$ (d) $h^2\le ab$

(viii) The lines represented by $ax^2+2hxy+by^2=0$ are imaginary if:
(a) $h^2>ab$ (b) $h^2=ab$ (c) $h^2<ab$ (d) $h^2\ge ab$

(ix) The lines represented by $ax^2+2hxy+by^2=0$ are perpendicular if:
(a) $a-b=0$ (b) $b-a=0$ (c) $a+b=1$ (d) $a+b=0$

(x) Half the determinant of vertices of triangle gives its:
(a) perimeter (b) area (c) volume (d) both (a) and (b)

(xi) If the determinant of three points is zero then the points are:
(a) collinear (b) non-collinear (c) imaginary (d) concurrent

(xii) The point of intersection of three angle bisectors of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

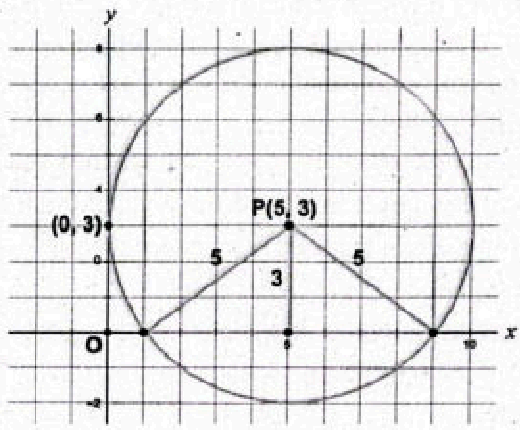
(xiii) The point of intersection of right bisectors of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

(xiv) The point of intersection of three medians of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

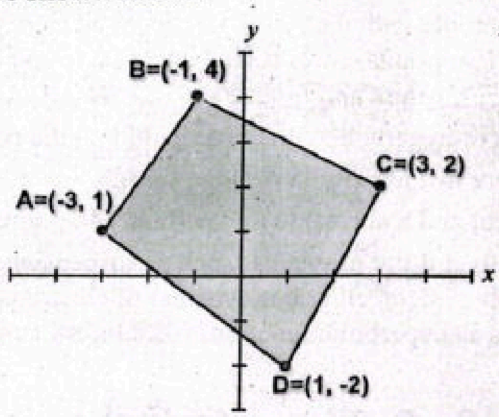
(xv) The point of intersection of three altitudes of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

(xvi) Which of the following is perpendicular to $y=-\frac{1}{2}x$?
(a) $y=2x$ (b) $y=-2x$ (c) $2y=x$ (d) $2y=-x$

2. Prove that altitudes, right bisector and medians of triangle ABC are concurrent when: A(0, 0), B(a, 0), C(b, c) where a, b and c are not equal.
3. Find the value of m if the points (6, 1), (–2, –3) and (8, 2m) are collinear.
4. Find a relation between x and y if the points A(x, y), B(– 4, 6) and C(– 2, 3) are collinear.
5. The points A (0, 3), B (a, 0) and C (0, –3) are the vertices of a triangle ABC right angled at B. Find the values of a and hence the area of ΔABC.
6. A circle has a centre at point P(5, 3) and radius r = 5. This circle intersects the y-axis at one intercept and the x-axis at two intercepts. What is the area of the triangle formed by these three intercepts?



7. Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, – 1), (2, 1) and (0, 3). Find the ratio of this area to the area of the given triangle ABC.
8. Find the value of m, such that l_1, l_2, l_3 intersect each other at one point.
   $l_1: x - y = 1;\quad l_2: 2x + y = 5;\quad l_3: (2m - 5)x - my = 3$
9. Find the area of triangle bounded by the line $2x - y + 10 = 0$ and the coordinate axes.
10. Find the area of triangle bounded by the lines:
    $$x^2 - xy - 6y^2 = 0,\ x - y + 3 = 0$$
11. Find the area of the figure shown below.

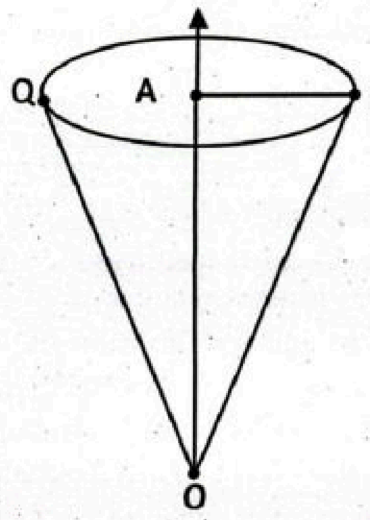


CONIC SECTION

After studying this unit, students will be able to:

- Demonstrate conics and members of its family i.e. circle, parabola, ellipse and hyperbola.
- Derive and apply equation of a circle in standard form i.e. $(x-h)^2+(y-k)^2=r^2$.
- Find the equation of circle passing through: three non collinear points, two points having its centre on a given line, two points and equation of tangent at one of these points is known, two points and touching a given line.
- Find the condition when a line intersects the circle and when it touches the circle.
- Find the equation of tangent to a circle in slope form and a normal to a circle at a point.
- Find the length of tangent to a circle from a given external point.
- Derive and apply the standard equation of a parabola.
- Sketch graphs of parabolas and find their elements.
- Find the equation of a parabola with the following elements: focus and vertex, focus and directrix, vertex and directrix.
- Find the condition when a line is tangent to a parabola at a point and hence write the equation of a tangent line in slope form.
- Find the equation of tangent and normal to a parabola at a point.
- Derive and apply the standard form of equation of an ellipse and identify its elements.
- Convert a given equation to the standard form of equation of an ellipse, find its elements and draw the graph.
- Find points of intersection of an ellipse with a line including the condition of tangency.
- Find the equation of a tangent to an ellipse in slope form.
- Find the equation of a tangent and normal to an ellipse at a point.
- Derive and apply the standard form of equation of a hyperbola and identify its elements.
- Find the equation of a hyperbola with the following given elements: transverse and conjugate axes with centre at origin, two points, eccentricity, latera recta and transverse axes, focus eccentricity and centre, focus, centre and directrix.
- Find points of intersection of hyperbola with a line including the condition of tangency.
- Find the equation of tangent to a hyperbola in slope form.
- Find the equation of tangent and a normal to a hyperbola at a point.
- Apply concepts of conics to real life problems (such as suspension and reflection problems related to parabola, satellite system, elliptic movement of electrons in the atom around the nucleus, radio system uses as hyperbolic functions, flashlights, conics in architecture).

7.1 Conics and Its Family Members



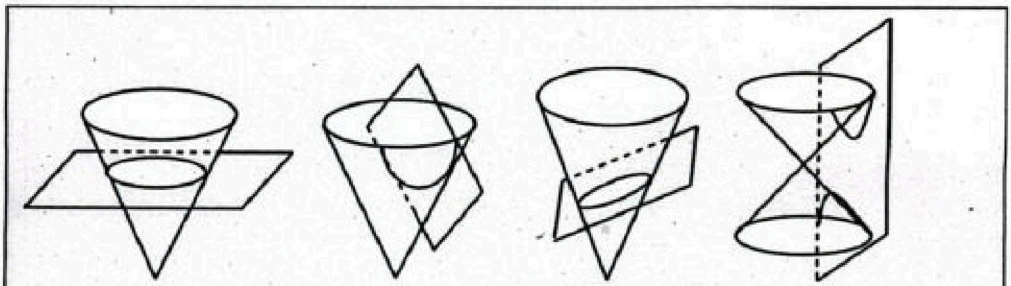
The plane shapes like circle, ellipse, parabola and hyperbola, which are formed with the intersection of a right circular cone and a plane are known as conic section. First of all, we will discuss that what is a right circular cone and a plane. When the line segment $\overline{OP}$ rotates about the circumference of a circle of any radius greater than zero with O as fixed point the shape formed is called a **right circular cone**. The fixed point O is called the **vertex** of the cone. The ray $\overrightarrow{OA}$ is called the axis of the cone, as $\overrightarrow{OA}$ is perpendicular to the radius $\overline{AP}$ of the circle that's why this cone is called **right** circular cone. The line segment $\overline{OP}$ (or any line segment) which join the point **O** with the circumference of the circle is called generator (ruling). Note that a flat or two-dimensional surface that extends indefinitely is called a plane.

Circle: When we cut a cone with a plane so that plane is perpendicular to the axis of cone and not passing through vertex then the intersection is a circle.

Parabola: When we cut a cone with the plane such that plane is parallel to any generator or ruling not passing through the vertex then the intersection is a parabola.

Ellipse: When we cut a cone with the plane such that plane is slightly tilted not passing through the vertex then the intersection is an ellipse.

Hyperbola: When a cone is inverted and is joined with an erected cone such that the vertices of the two cones lie at the same point and their axes of are also same then a shape is formed shown in the figure. When plane intersects the cone parallel to the axis of the cone such that the plane does not pass through the vertex O, then the shape formed on the plane is called hyperbola.



Circle | Parabola | Ellipse | Hyperbola

Note: In case when the plane passes through the vertex then the intersection of plane and the cone is a point or a pair of intersecting lines. These conic sections are known as degenerate conics.

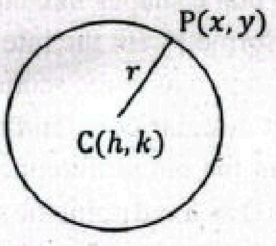
7.2 Circle

It is set of all points in the plane which are equidistant from a fixed point in the plane.

The fixed point is called the centre of the circle and distance of any point on the circle from the centre is called the radius of the circle.

7.2.1 Equation of Circle in Standard Form

Consider a circle with centre at $C(h, k)$ and radius r. Take any point $P(x, y)$ on the circle. Then by definition of the circle.



$|CP| = r$

$\Rightarrow \sqrt{(x-h)^2 + (y-k)^2} = r$

Squaring both sides

$\Rightarrow (x-h)^2 + (y-k)^2 = r^2$

This is the standard form of equation of circle with centre at $C(h, k)$ and radius r.

In particular if the centre of the circle is at origin, i.e. $(h, k) = (0, 0)$ then equation of circle is

$$(x-0)^2 + (y-0)^2 = r^2 \quad \Rightarrow \quad x^2 + y^2 = r^2$$

Example 1: Find the equation of circle with centre at $(2, -5)$ and radius 3 units.

Solution: Given that the centre $(h, k) = (2, -5)$ and radius $r = 3$.

Equation of circle is $(x-h)^2 + (y-k)^2 = r^2$

Putting values of h, k and r, we get required equation of circle as follows:

$$(x-2)^2 + (y-(-5))^2 = 3^2 \quad \Rightarrow \quad (x-2)^2 + (y+5)^2 = 9$$

7.2.2 General Form of Equation of Circle

As we know that the standard form of the equation of circle is:

$(x-h)^2 + (y-k)^2 = r^2$

$\Rightarrow x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$

$\Rightarrow x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \qquad$ (i)

Letting $h^2 + k^2 - r^2 = c$, $-2h = 2g$ and $-2k = 2f$, the equation (i) becomes:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Which is the general form of the equation of circle.

Theorem 7.1: Prove that $x^2 + y^2 + 2gx + 2fy + c = 0$, represents a circle in general.

Proof: Given equation is:

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow (x^2 + 2gx) + (y^2 + 2fy) + c = 0$

$\Rightarrow (x^2 + 2gx + g^2 - g^2) + (y^2 + 2fy + f^2 - f^2) + c = 0$

$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) + (-g^2 - f^2 + c) = 0$

$\Rightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

$\Rightarrow [x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

Comparing it with $(x-h)^2 + (y-k)^2 = r^2$, we have $h = -g$; $k = -f$; $r = \sqrt{g^2 + f^2 - c}$.

Thus, the given equation represents a circle with centre $= (h, k) = (-g, -f)$ and radius $= r = \sqrt{g^2 + f^2 - c}$

Note: As $r = \sqrt{g^2 + f^2 - c}$, so

i. r is a real if $g^2 + f^2 - c > 0$ i.e. $g^2 + f^2 > c$

ii. r is imaginary if $g^2 + f^2 - c < 0$ i.e. $g^2 + f^2 < c$

Thus, condition for a real circle is $g^2 + f^2 > c$ and for an imaginary circle is $g^2 + f^2 < c$

Example 2: Find the centre and radius of the circle $x^2 + y^2 - 6x - 10y + 18 = 0$.

Solution:

To find the centre and radius we convert the given equation of circle into its standard form

$(x^2 - 6x) + (y^2 - 10y) + 18 = 0$

$(x^2 - 6x + 9 - 9) + (y^2 - 10y + 25 - 25) + 18 = 0$

$(x^2 - 6x + 9) + (y^2 - 10y + 25) - 9 - 25 + 18 = 0$

$(x - 3)^2 + (y - 5)^2 - 16 = 0 \Rightarrow \quad (x - 3)^2 + (y - 5)^2 = 16$

$(x - 3)^2 + (y - 5)^2 = 4^2$

Comparing it with $(x - h)^2 + (y - k)^2 = r^2$, we have:

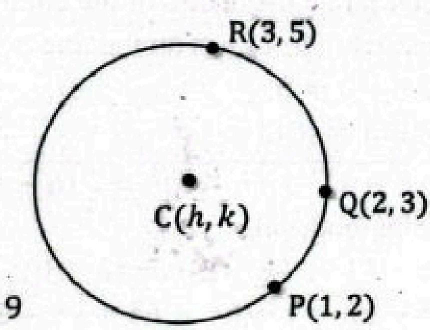
Centre $= (h, k) = (3, 5)$ and radius $= r = 4$

7.2.3 Equation of a Circle with Different Conditions

Example 3: Find the equation of the circle passing through the points (1, 2), (2, 3) and (3, 5).

Solution:

Consider a circle which passes through the points P(1, 2), Q(2, 3) and R(3, 5). Let $C(h, k)$ be the centre of the circle



Then, $|PC| = |QC|$

$\Rightarrow \sqrt{(h-1)^2 + (k-2)^2} = \sqrt{(h-2)^2 + (k-3)^2}$

Squaring both sides:

$(h-1)^2 + (k-2)^2 = (h-2)^2 + (k-3)^2$

$\Rightarrow h^2 - 2h + 1 + k^2 - 4k + 4 = h^2 - 4h + 4 + k^2 - 6k + 9$

$\Rightarrow 2h + 2k = 8$

$\Rightarrow h + k = 4 \qquad$ (i)

Also $\quad |PC| = |RC|$

$\Rightarrow \sqrt{(h-1)^2 + (k-2)^2} = \sqrt{(h-3)^2 + (k-5)^2}$

Squaring both sides:

$(h-1)^2 + (k-2)^2 = (h-3)^2 + (k-5)^2$

$\Rightarrow \quad h^2 - 2h + 1 + k^2 - 4k + 4 = h^2 - 6h + 9 + k^2 - 10k + 25$

$\Rightarrow \quad 4h + 6k = 29 \qquad$ (ii)

Multiplying (i) with 4 and subtracting (ii) from it, we get:

$\Rightarrow 4h + 4k = 16$

$\underline{-4h \pm 6k = -29}$

$-2k = -13 \qquad \Rightarrow \quad k = \frac{13}{2}$

Putting in (i), we get:

$\Rightarrow h + \frac{13}{2} = 4 \Rightarrow h = 4 - \frac{13}{2} = -\frac{5}{2}$

Thus, the centre of circle $= C\left(\frac{-5}{2}, \frac{13}{2}\right)$

The radius of circle is:

$r = |CP| = \sqrt{\left(1 + \frac{5}{2}\right)^2 + \left(2 - \frac{13}{2}\right)^2} = \sqrt{\left(\frac{7}{2}\right)^2 + \left(-\frac{9}{2}\right)^2} = \sqrt{\frac{49}{4} + \frac{81}{4}} = \sqrt{\frac{130}{4}}$

Equation of the circle with centre $C\left(-\frac{5}{2}, \frac{13}{2}\right)$ and radius $r = \sqrt{\frac{130}{4}}$ is:

$$\left[x-\left(-\frac{5}{2}\right)\right]^2 + \left[y-\frac{13}{2}\right]^2 = \left(\sqrt{\frac{130}{4}}\right)^2 \Rightarrow \left(x+\frac{5}{2}\right)^2 + \left(y-\frac{13}{2}\right)^2 = \frac{130}{4}$$

$$\Rightarrow x^2 + 5x + \frac{25}{4} + y^2 - 13y + \frac{169}{4} = \frac{130}{4}$$

$$\Rightarrow x^2 + y^2 + 5x - 13y + \frac{25}{4} + \frac{169}{4} - \frac{130}{4} = 0$$

$$\Rightarrow x^2 + y^2 + 5x - 13y + 16 = 0$$

Example 4: Find the equation of the circle passing through the points (1, 0), (0, 1) and having its centre on the line $x - 2y + 3 = 0$.

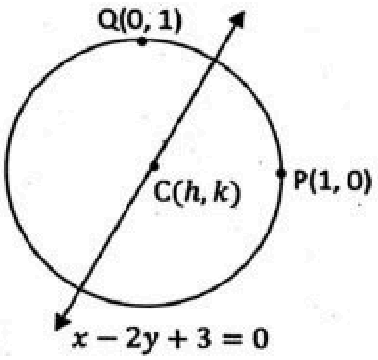
Solution: Consider a circle which is passing through the two given points P(1, 0) and Q(0, 1). Let $C(h, k)$ be the centre of the circle.

Given that $C(h, k)$ lies on the line $x - 2y + 3 = 0$. Thus:

$h - 2k + 3 = 0$ (i)

Also $|PC| = |QC|$

$\Rightarrow \sqrt{(h-1)^2 + (k-0)^2} = \sqrt{(h-0)^2 + (k-1)^2}$



Squaring both sides

$\Rightarrow (h-1)^2 + (k-0)^2 = (h-0)^2 + (k-1)^2$

$\Rightarrow h^2 - 2h + 1 + k^2 = h^2 + k^2 - 2k + 1$

$\Rightarrow -2h + 2k = 0 \Rightarrow -h + k = 0$

$\Rightarrow h = k$ (ii)

Using equation (ii) in equation (i), we get:

$k - 2k + 3 = 0 \Rightarrow -k = -3 \Rightarrow k = 3$

Putting the value of k in (ii), we get $h = 3$.

Therefore, the centre of the circle is $C(h, k) = C(3, 3)$.

The radius of the circle is:

$r = |CP| = \sqrt{(3-1)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$

The equation of the circle with centre at (3, 3) and radius $\sqrt{13}$ is:

$$(x-3)^2 + (y-3)^2 = (\sqrt{13})^2$$

$$\Rightarrow x^2 - 6x + 9 + y^2 - 6y + 9 = 13 \Rightarrow x^2 + y^2 - 6x - 6y + 5 = 0$$

Example 5: Find the equation of the circle passing through the point (2, 3) and the line $x + y - 4 = 0$ is the tangent to the circle at (3, 1).

Solution: Consider a circle passing through the point Q(2, 3) and having tangent line $x + y + 3 = 0$ touching the circle at point P(3, 1). Let $C(h, k)$ be the centre of the circle. Now

$|CP| = |CQ|$

$\Rightarrow \sqrt{(h-3)^2 + (k-1)^2} = \sqrt{(h-2)^2 + (k-3)^2}$

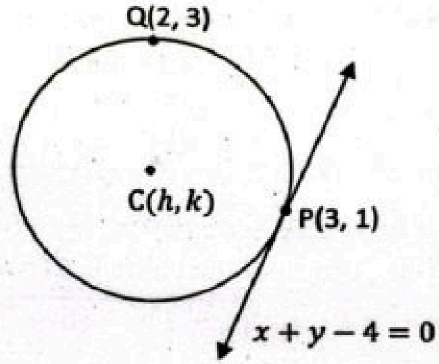
Squaring both sides;

$$(h-3)^2+(k-1)^2=(h-2)^2+(k-3)^2$$

$\Rightarrow h^2-6h+9+k^2-2k+1=h^2-4h+4+k^2-6k+9$

$\Rightarrow -2h+4k=3$

$\Rightarrow 2h-4k=-3$ (i)



Since $\overline{CP}$ is perpendicular to the tangent line $x+y-4=0$.

Thus, (slope of $\overline{CP}$) × (slope of tangent line) = −1

$\Rightarrow \left(\frac{k-1}{h-3}\right)(-1)=-1 \Rightarrow k-1=h-3$

$\Rightarrow h-k=2$ (ii)

Solving equation (i) and (ii)), we have: $h=\frac{11}{2}$ and $k=\frac{7}{2}$

Thus, the centre of the circle is $C\left(\frac{11}{2},\frac{7}{2}\right)$. The radius of the circle is:

$$\Rightarrow r=|CP|=\sqrt{\left(\frac{11}{2}-3\right)^2+\left(\frac{7}{2}-1\right)^2}=\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{5}{2}\right)^2}=\sqrt{\frac{25}{4}+\frac{25}{4}}=\sqrt{\frac{50}{4}}=\frac{5\sqrt{2}}{2}$$

Therefore, the equation of the circle with centre at $\left(\frac{11}{2},\frac{7}{2}\right)$ and radius $\frac{5\sqrt{2}}{2}$ is:

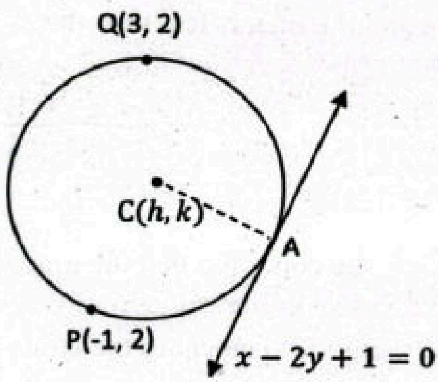
$$\left(x-\frac{11}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\left(\frac{5\sqrt{2}}{2}\right)^2 \Rightarrow x^2-11x+\frac{121}{4}+y^2-7y+\frac{49}{4}=\frac{50}{4}$$

$$\Rightarrow x^2+y^2-11x-7y+\frac{121}{4}+\frac{49}{4}-\frac{50}{4}=0 \Rightarrow x^2+y^2-11x-7y+30=0$$

Example 6: Find the equation of the circle passing through the point (−1, 2), (3, 2) and touching the line $x-2y+1=0$.

Solution:

Consider a circle passing through the given points P(−1, 2) and Q(3, 2). Also, the circle touches the line $x-2y+1=0$ at point A. Let $C(h,k)$ be the centre of the circle. From figure:



$$|CP|=|CQ|$$

$\Rightarrow \sqrt{(h+1)^2+(k-2)^2}=\sqrt{(h-3)^2+(k-2)^2}$

Squaring both sides;

$$(h+1)^2+(k-2)^2=(h-3)^2+(k-2)^2$$

$\Rightarrow h^2+2h+1+k^2-4k+4=h^2-6h+9+k^2-4k+4$

$\Rightarrow 8h=8 \quad \Rightarrow h=1$ (i)

Also, $|CP|=|CA|$, where $|CA|$ is the distance of point C from the line $x-2y+1=0$.

Therefore,

$\sqrt{(h+1)^2+(k-2)^2}=\frac{|h-2k+1|}{\sqrt{1^2+(-2)^2}}$ (ii)

Putting $h=1$ in above equation (ii).

$$\Rightarrow \sqrt{(1+1)^2+(k-2)^2}=\frac{|1-2k+1|}{\sqrt{1+4}} \Rightarrow \sqrt{4+(k-2)^2}=\frac{|2-2k|}{\sqrt{5}}$$

Squaring both sides:

$$\Rightarrow 4+(k-2)^2=\frac{(2-2k)^2}{5}\Rightarrow 4+k^2-4k+4=\frac{4k^2-8k+4}{5}$$

$$\Rightarrow k^2-4k+8=\frac{4k^2-8k+4}{5}\Rightarrow 5k^2-20k+40=4k^2-8k+4$$

$$\Rightarrow k^2-12k+36=0\Rightarrow (k-6)^2=0\Rightarrow k-6=0\ \Rightarrow\ k=6$$

Thus, centre of the circle is $C(h,k)=(1,6)$ and radius of the circle is:

$$r=|CP|=\sqrt{(1+1)^2+(6-2)^2}=\sqrt{4+16}=\sqrt{20}$$

Equation of the circle with centre at $(1,6)$ and radius $\sqrt{20}$ is:

$$(x-1)^2+(y-6)^2=\left(\sqrt{20}\right)^2$$

$$\Rightarrow x^2-2x+1+y^2-12y+36=20$$

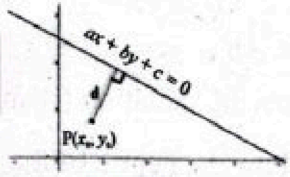
$$\Rightarrow x^2+y^2-2x-12y+17=0$$

Which is required equation of circle.

Key Facts

The distance 'd' of a point $P(x_0, y_0)$ from a line $ax+by+c=0$, is the length of the perpendicular drawn from the point to the given line as shown below:

$$d=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$



7.3 Line and a Circle

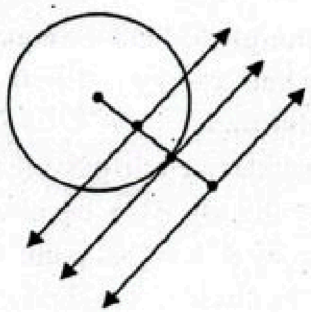
Consider a circle $x^2+y^2=r^2$ (i)

and a line $y=mx+c$ which implies:

$$mx-y+c=0 \qquad \text{(ii)}$$

The centre of the circle is at $(0,0)$ and its radius is r.

i. When line intersects the circle at two distinct points.



When the line intersects the circle then the distance between the centre and the line, is less than the radius of the circle. i.e.,

$$\frac{|m(0)-0+c|}{\sqrt{m^2+(-1)^2}}<r$$

$$\Rightarrow\ \frac{|c|}{\sqrt{m^2+1}}<r\ \text{ or }\ |c|<r\sqrt{m^2+1}$$

$$\Rightarrow\ c^2<r^2(m^2+1)$$

Which is the condition that the line will intersect the circle at two distinct points.

ii. When the line is tangent to circle.

When the line is tangent to the circle then distance between the centre and the line is equal to the radius of the circle. i.e.,

$$\frac{|m(0)-0+c|}{\sqrt{m^2+1}}=r\ \Rightarrow\ \frac{|c|}{\sqrt{m^2+1}}=r\ \Rightarrow\ |c|=r\sqrt{m^2+1}$$

$$\Rightarrow\ c=\pm r\sqrt{m^2+1}$$

Putting the value of c in $y=mx+c$, we get:

$$y=mx\pm r\sqrt{m^2+1}$$

Which are the equations of the tangent lines to the circle $x^2+y^2=r^2$.

iii. When the line neither touches nor intersects the circle

When the line neither touches nor intersects the circle then distance of the centre from the line is greater than the radius of the circle i.e.;

$\frac{|m(0)-0+c|}{\sqrt{m^2+1}} > r \Rightarrow \frac{|c|}{\sqrt{m^2+1}} > r \quad \Rightarrow |c| > r\sqrt{m^2+1}$

$\Rightarrow c^2 > r^2(m^2+1)$

Example 7: Check whether the line $x + 2y - 3 = 0$ is tangent to circle $x^2 + y^2 = r^2$ or not.

Solution: Equation of circle is $x^2 + y^2 = 16$ or $x^2 + y^2 = 4^2$

$\Rightarrow$ radius of the circle $= r = 4$

Equation the line is $x + 2y - 3 = 0$.

$\Rightarrow 2y = -x + 3 \quad \Rightarrow \quad y = -\frac{1}{2}x + \frac{3}{2} \Rightarrow m = -\frac{1}{2}$ and $c = \frac{3}{2}$

The line is tangent to the circle if $c = r\sqrt{m^2+1}$

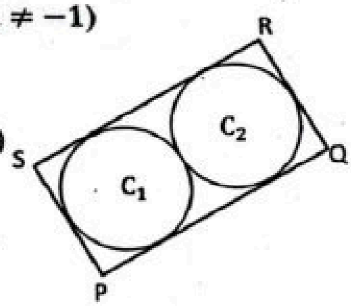
$\Rightarrow \left|\frac{3}{2}\right| = 4\sqrt{\left(-\frac{1}{2}\right)^2 + 1} = 4\sqrt{\frac{5}{4}} = 4 \times \frac{\sqrt{5}}{2} = 2\sqrt{5} \Rightarrow \left|\frac{3}{2}\right| = 2\sqrt{5}$

Which is not true. Thus, the given line is not a tangent to the circle.

Exercise 7.1

1. Find the equation of circle when its centre and radius is given:
   (i) Centre at $(3, -1)$ and radius 2 (ii) Centre at $\left(-\frac{1}{2}, -\frac{1}{3}\right)$ and radius $\frac{1}{5}$
   (iii) Centre at $\left(\frac{1}{a}, a\right)$ and radius is $a; a \neq 0$
2. Convert the following equations of circle into standard form and hence find their centre and radius:
   (i) $x^2 + y^2 - 4x + 6y - 36 = 0$ (ii) $5x^2 + 5y^2 - 2x + 4y - 27 = 0$
   (iii) $4x^2 + 4y^2 + 2ax + by - a^2 = 0$
3. Find the equation of circle passing through given three non-collinear points.
   (i) $(0,2), (2,0), (1,3)$ (ii) $(1,3), (3,6), (5,7)$
   (iii) $(0,0), (a,0), (0,b); a \neq 0, b \neq 0$
4. Find the equation of circle with centre lying on the line $x + y = 2$ and passing through the points $(2,-2)$ and $(3,4)$.
5. Find the equation of circle passing through the points $(1,-2)$ and $(4,-3)$ and whose centre lies on the line $3x + 4y = 7$.
6. Find the equation of the circle passing through the points $(2,1)$ and touching the line $x + 2y - 1 = 0$ at the point $(3,-1)$.
7. A circle touches the line $2x - 3y + 1 = 0$ at point $(1,1)$ and passes through the point of intersection of the lines $x + y + 1 = 0$ and $x - 3y + 5 = 0$. Find the equation of circle.
8. Equations of two diameters of a circle are $x - y = 3$ and $3x + y = 5$ and its radius is 5. Find the equation of circle.
9. A circle touches both the axes in the first quadrant and area of the circle is 13π square units. Find the equation of the circle.
10. The two points $A(4,3)$ and $B(2,5)$ lie on the circle and the centre of the circle lies on the perpendicular bisector of the chord $\overline{AB}$. The distance between the centre and the chord $\overline{AB}$ is $\sqrt{7}$. Find the equation of the circle.

11. Find the equation of the circle passing through the intersection of the circles $C_1: x^2 + y^2 - 8x - 2y + 7 = 0$ and $C_2: x^2 + y^2 - 4x + 10y + 8 = 0$ and passes through $(-1, -2)$. (Hint: equation of the required circle is $C_1 + \lambda C_2 = 0; \lambda \neq -1$)
12. The diagram shows a rectangle PQRS and the circles C_1 and C_2. Both the circles touch each other and three sides of the rectangle. The coordinates of the points P, Q, R and S are $(0,4), (1,1), (7,3)$ and $(6,6)$. Find the equation of the circles C_1 and C_2.



13. A circle has its centre at the point C(0,1) and a line touches the circle. The point P(3,5) lies on the line touching the circle. The distance between P and C is five times the radius of the circle. Find the equation of the circle and the point where the line touches the circle.
14. The three lines $2x - y + 1 = 0; 2x + y - 3 = 0$ and $x - 2y + 4 = 0$ touch the circle. Find the centre of the circle. Also find the equation of circle.

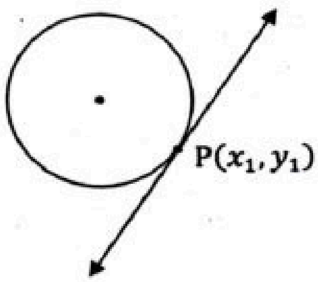
7.4 Equation of Tangent and Normal to a Circle at a Point on the Circle

7.4.1 Equation of Tangent at $P(x_1, y_1)$ on a Circle

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

Let $P(x_1, y_1)$ be a given point on the circle.

Differentiating equation (i) w.r.t x, we get:



$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow (2x + 2y)\frac{dy}{dx} = -(2x + 2g) \Rightarrow \frac{dy}{dx} = -\frac{2x + 2g}{2y + 2f} = -\frac{x + g}{y + f}$$

$$\frac{dy}{dx} \text{ at } P(x_1, y_1) = m = -\frac{x_1 + g}{y_1 + f}$$

Which is the slope of the tangent line at the point $P(x_1, y_1)$.

By point-slope formula, equation of the tangent line at point P is:

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow (y_1 + f)(y - y_1) = -(x_1 + g)(x - x_1)$$

$$\Rightarrow (y_1 + f)y - (y_1 + f)y_1 = -(x_1 + g)x + (x_1 + g)x_1$$

$$\Rightarrow (y_1 + f)y - (y_1 + f)y_1 + (x_1 + g)x - (x_1 + g)x_1 = 0$$

$$\Rightarrow (x_1 + g)x + (y_1 + f)y - y_1^2 - fy_1 - x_1^2 - gx_1 = 0$$

$$\Rightarrow (x_1 + g)x + (y_1 + f)y - (x_1^2 + y_1^2 + gx_1 + fy_1) = 0$$

Since the point $P(x_1, y_1)$ lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

So, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

or $x_1^2 + y_1^2 + gx_1 + fy_1 + (gx_1 + fy_1 + c) = 0$

or $x_1^2 + y_1^2 + gx_1 + fy_1 = -(gx_1 + fy_1 + c) = 0$ (ii)

Putting in equation (ii), we get:

$$(x_1 + g)x + (y_1 + f)y - [-(gx_1 + fy_1 + c)] = 0$$

$$\Rightarrow (x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$$

Which is the required equation of tangent line at $P(x_1, y_1)$.

Key Facts

Derivative at point of the curve is the slope of the tangent line to the curve at that point.

7.4.2 Equation of Normal at $P(x_1, y_1)$ on a Circle

Since normal line is perpendicular to the tangent line, so its slope is:

$$m_1 = -\frac{1}{m} = \frac{y_1 + f}{x_1 + g}$$

By point-slope formula the equation of normal line is:

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

$$\Rightarrow (x_1 + g)(y - y_1) = (y_1 + f)(x - x_1)$$

$$\Rightarrow (x_1 + g)y - (x_1 + g)y_1 = (y_1 + f)x - (y_1 + f)x_1$$

$$\Rightarrow (x_1 + g)y - (x_1 + g)y_1 - (y_1 + f)x + (y_1 + f)x_1 = 0$$

$$\Rightarrow -(y_1 + f)x + (x_1 + g)y - x_1y_1 - gy_1 + x_1y_1 + fx_1 = 0$$

$$\Rightarrow -(y_1 + f)x + (x_1 + g)y - gy_1 + fx_1 = 0$$

$$\Rightarrow (y_1 + f)x - (x_1 + g)y + gy_1 - fx_1 = 0 \quad \text{(Multiplying both sides of equation by } -1\text{)}$$

$$\Rightarrow (y_1 + f)x - (x_1 + g)y - (fx_1 - gy_1) = 0$$

Which is the equation of the normal line at point $P(x_1, y_1)$.

Key Facts

A line which is perpendicular to the tangent line at the point of tangency is called the normal line at that point.

Example 8: Find the equation of tangent and normal to the circle $x^2 + y^2 - 4x + 2y - 5 = 0$ at point $P(1, 2)$.

Solution:

Given equation of circle is $x^2 + y^2 - 4x + 2y - 5 = 0$

Differentiating w.r.t x, we have:

$$2x + 2y\frac{dy}{dx} - 4 + 2\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow (2y + 2)\frac{dy}{dx} = -2x + 4 \Rightarrow \frac{dy}{dx} = \frac{-2x+4}{2y+2} = \frac{-x+2}{y+1}$$

$$\frac{dy}{dx} \text{at } P(1, 2) = m = \frac{-1+2}{2+1} = \frac{1}{3}$$

By the point-slope formula equation of the tangent line is:

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 2 = \frac{1}{3}(x - 1)$$

$$\Rightarrow 3y - 6 = x - 1 \quad \Rightarrow \quad x - 3y + 5 = 0$$

Since normal is perpendicular to the tangent line. Thus, slope of the normal line is -3 and by the point-slope formula, equation of the normal line is:

$$y - 2 = -3(x - 1) \Rightarrow y - 2 = -3x + 3 \Rightarrow 3x + y - 5 = 0$$

Alternatively

Equation of circle is $x^2 + y^2 - 4x + 2y - 5 = 0$

Comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$, we have:

$$2g = -4 \Rightarrow g = -2$$

$$2f = 2 \Rightarrow f = 1 \quad \text{and} \quad c = -5$$

Given point is $P(1, 2)$. i.e., $x_1 = 1, y_1 = 2$

Equation of tangent line is:

$$(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0 \qquad \text{(i)}$$

Substituting the values of x_1, y_1, g, f and c in equation (i), we have:

$(1-2)x + (2+1)y + ((-2)1 + 1(2) - 5) = 0 \quad \Rightarrow -x + 3y + 5 = 0$

Equation of normal is:

$(y_1 + f)x - (x_1 + g)y - (fx_1 - gy_1) = 0 \qquad$ (ii)

Substituting the values of x_1, y_1, g and f in equation (ii), we have:

$(2+1)x - (1-2)y - (1 \times 1 - (-2 \times 2) = 0 \quad \Rightarrow 3x + y - 5$

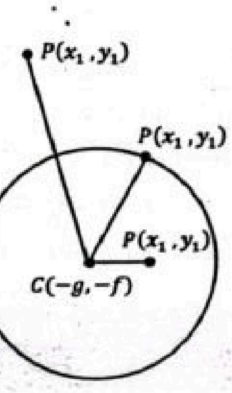
Exercise 7.2

1. Find the points of intersection of the given line and the circle.
   (i) $x - y + 1 = 0;\ x^2 + y^2 - 3x - 8 = 0$
   (ii) $2x + y + 4\sqrt{5} = 0;\ x^2 + y^2 - 2x + 4y - 11 = 0$
   (iii) $3x + 2y + 1 = 0;\ x^2 + y^2 - x + y + 2 = 0$
2. The equation of a circle is $x^2 + y^2 - 4x + y - 7 = 0$ and the equation of line is $2x - y + c = 0$. Find the value(s) of "c" such that the line:
   (i) intersects the circle at two distinct points.
   (ii) is tangent to circle.
   (iii) has no common point with the circle.
3. The tangents to a circle $(x-1)^2 + (y-2)^2 = 3^2$ are perpendicular to the line $x + 3y - 6 = 0$. Find the equations of the tangent lines.
4. Find the equations of tangent and normal to the circle $x^2 + y^2 + 3x + 2y + 3 = 0$ at the point $(2, -1)$.
5. Find the equation of the tangent to the circle $36x^2 + 36y^2 - 72y + 11 = 0$ at the point on the circle with abscissa $\frac{1}{2}$.
6. Find the equation of the normal to the circle $x^2 + y^2 - 2x + 2y - 11 = 0$ at the point on the circle with ordinate -4.
7. Circles with equations $x^2 + y^2 - 2y - 3 = 0$ and $x^2 + y^2 - 8x + 4y + 11 = 0$ touch each other externally. Find the equation of the common tangent to the given circles.
8. Show that the tangent line at any point P on the circle is always perpendicular to the radial line through point P.
9. $A(-5, -1)$ and $B(1, 5)$ are two points on the circle $x^2 + y^2 = 26$. Find the point of intersection of the tangents to the circle at A and B. Show that the point of intersection of the tangent lines, the midpoint of chord $\overline{AB}$ and the centre of the circle are collinear.
10. A normal line cuts the circle with centre at $(1, 2)$ at the point $(3, 5)$. Find the other point of intersection of the normal and the circle. Also find the equation of the circle.

7.5 Position of a Point with respect to a Circle

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of a circle and $P(x_1, y_1)$ be any point in the plane. We want to check that the given point lies outside, on or inside the circle.

The centre of the circle is $C(-g, -f)$ and the radius of the circle is $r = \sqrt{g^2 + f^2 - c}$. Observe that P lies outside the circle if $|CP| > r$, P lies on the circle if $|CP| = r$ and P lies inside the circle if $|CP| < r$. Combining all these; we may write $|CP| \gtreqless r$



$\Rightarrow \sqrt{(x_1+g)^2+(y_1+f)^2} \lesseqqgtr \sqrt{g^2+f^2-c}$

Squaring both sides

$(x_1+g)^2+(y_1+f)^2 \lesseqqgtr g^2+f^2-c$

$\Rightarrow x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2 \lesseqqgtr g^2+f^2-c$

$\Rightarrow x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-g^2-f^2+c \lesseqqgtr 0$

$\Rightarrow x_1^2+2gx_1+g^2+y_1^2+2fy_1+c \lesseqqgtr 0$

Which is the condition that a point lies inside, on or outside the circle.

Example 9: Check whether the point $P(2,4)$ lies outside, on or inside the circle:

$$2x^2+2y^2-6x+8x+1=0$$

Solution: Given equation of circle is $2x^2+2y^2-6x+8x+1=0$.

First make the coefficients of x^2 and y^2 one. Dividing both sides by 2, we have:

$x^2+y^2-3x+4x+\frac{1}{2}=0$

Given point is $(2,4)$. So:

$$x_1^2+y_1^2-3x_1+4x_1+\frac{1}{2}=(2)^2+(4)^2-3(2)+4(4)+\frac{1}{2}$$

$$=4+16-6+16+\frac{1}{2}=\frac{61}{2}>0$$

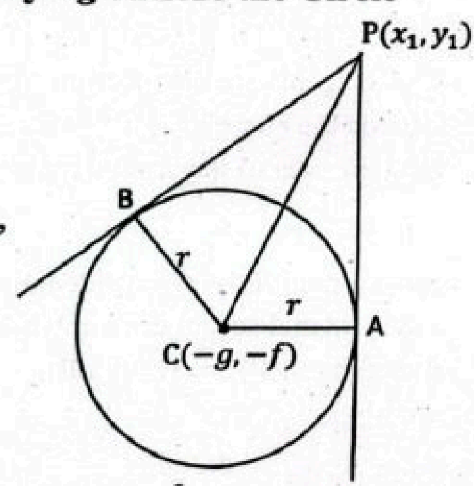
Thus, the point lies outside the circle.

7.6 Length of a Tangent Drawn from a Point Lying outside the Circle

Let $x^2+y^2+2gx+2fy+c=0$ be the equation of a circle and $P(x_1, y_1)$ lies outside the circle.

The centre of the circle is $C(-g,-f)$ and the radius of the circle is $r=\sqrt{g^2+f^2-c}$.

Two tangents can be drawn from the point P to circle. Thus, both tangents have the same length. i.e., $|AP|=|BP|$.



Since PCA is a right-angled triangle, so by Pythagoras theorem:

$|CA|^2+|AP|^2=|CP|^2$

$\Rightarrow r^2+|AP|^2=\left(\sqrt{(x_1+g)^2+(y_1+f)^2}\right)$

$\Rightarrow \left(\sqrt{g^2+f^2-c}\right)^2+|AP|^2=\left(\sqrt{x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2}\right)^2$

$\Rightarrow g^2+f^2-c+|AP|^2=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2$

$\Rightarrow |AP|^2=x_1^2+y_1^2+2gx_1+2fy_1+c$

$\Rightarrow |AP|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

Which is the length of tangent line.

Example 10:

Find the length of the tangent line to the circle $x^2 + y^2 + 3x - 4y + 15 = 0$ from the point $(1,2)$.

Solution:

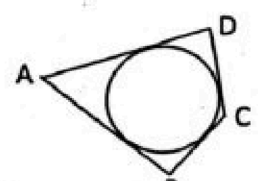
Equation of circle is $x^2 + y^2 + 3x - 4y + 15 = 0$ and the point $(1,2)$ lies outside the circle.

Length of tangent line = $\sqrt{x_1^2 + y_1^2 + 3x_1 - 4y_1 + 15} = \sqrt{1^2 + 2^2 + 3(1) - 4(2) + 15}$

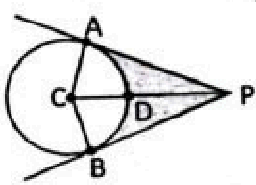
$= \sqrt{1 + 4 + 3 - 8 + 15} = \sqrt{15}$ units

Exercise 7.3

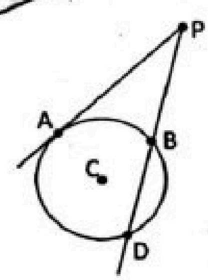
1. Check whether the given point lies inside, outside or on the given circle
   (i) $(3,5)$; $x^2 + y^2 - 6x + 2y - 18 = 0$
   (ii) $(1,6)$; $2x^2 + 2y^2 - 4x - 16y + 5 = 0$
   (iii) $(-1,-2)$; $x^2 + y^2 + 6x + 4y + 9 = 0$
2. Find the length of the tangent to the circle $x^2 + y^2 - 18x + 16y + 10 = 0$ from the point $(-1,-1)$ lying outside the circle.
3. Find the length of tangent to the circle $3x^2 + 3y^2 + 18x - 24y + 50 = 0$ drawn from a point $(-3,1)$ lying outside the circle.
4. The point $P(-11,-10)$ lies outside the circle $x^2 + y^2 + 6x + 8y + 5 = 0$. Find the equations of the tangents to the circle drawn from point P. Also find the points of contact.
5. A quadrilateral ABCD is circumscribing a circle. Prove that $\overline{AB} + \overline{CD} = \overline{AD} + \overline{BC}$



6. Two tangents are drawn from a point $P(6,1)$ lying outside the circle $x^2 + y^2 - 8x - 2y + 14 = 0$. Find the area of the shaded region.



7. From a point P lying outside the circle a tangent and secant lines are drawn. Prove that $|\overline{AP}|^2 = |\overline{PB}| \cdot |\overline{PD}|$



7.7 Parabola

It is the set of all the points in the plane which are equidistant from a fixed point and a fixed line in the plane (fixed point not lying on the fixed line).

The fixed point is called focus of the parabola and the fixed line is called its directrix. The ratio of distance at any point on the parabola from its focus to its directrix is called eccentricity of the parabola and is denoted by "e". Since by the definition, points on the parabola are equidistant from the focus and the directrix thus eccentricity of the parabola is 1.

7.7.1 Standard Equation of Parabola

Consider a parabola with focus $F(a, 0)$ where $a > 0$ and directrix $x = -a$ or $x + a = 0$.

Take a point $P(x, y)$ on the parabola then by definition of the parabola:

$|PF| = |PM|$

Where $|PM|$ is the distance of the point P from the directrix.

$$\Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = \frac{|x+a|}{\sqrt{1^2+0^2}}$$

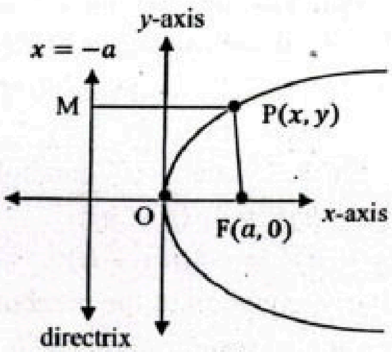
$$\Rightarrow \sqrt{x^2 - 2ax + a^2 + y^2} = |x + a|$$

Squaring both sides, we have:

$$\Rightarrow x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

$$\Rightarrow -2ax + y^2 = 2ax \quad \Rightarrow y^2 = 4ax$$

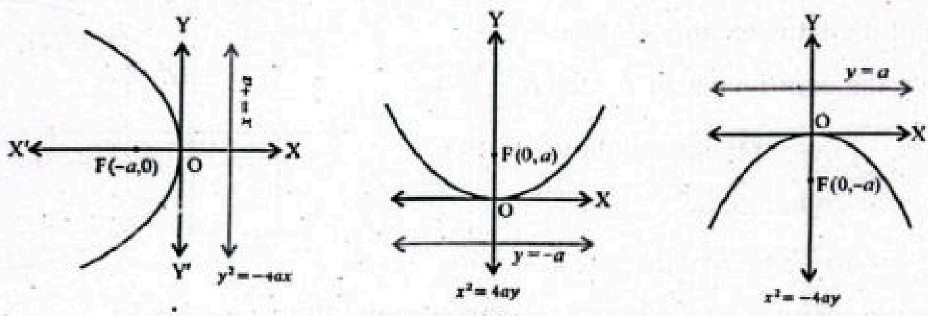
Which is the equation of the parabola.



(a)

Similarly,

- If we take focus at $F(-a, 0)$ the equation of parabola is $y^2 = -4ax$. (b)
- If we take focus at $F(0, a)$ the equation of parabola is $x^2 = 4ay$ (c)
- If we take focus at $F(0, -a)$ the equation of parabola is $x^2 = -4ay$ (d)



(b) (c) (d)

7.7.2 Elements of Parabola

Directrix: The fixed line is called the directrix of parabola.

Focus: The fixed point (not lying on the directrix) is called the focus of the parabola.

Axis of Parabola (Axis of symmetry)

The line which passes through the focus and is perpendicular to the directrix is called axis of parabola and its equation is $y = 0$. Axis of parabola is also known as axis of symmetry.

Vertex of Parabola:

The point where the parabola cuts its axis is called vertex of the parabola. In this case $O(0, 0)$ is the vertex of the parabola which is closest to the focus.

Chord of a Parabola:

A line segment with its end points on the parabola is called chord of parabola.

Focal Chord:

A chord of the parabola which passes through the focus of parabola is called focal chord.

Focal Distance:

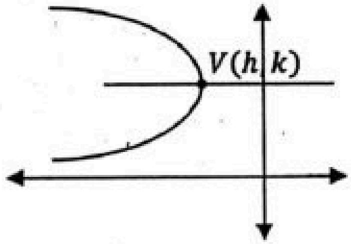
The distance of any point of the parabola from the focus is called focal distance.

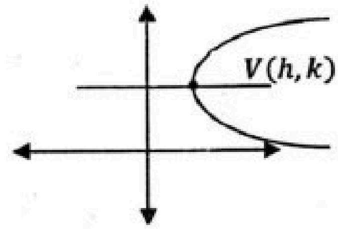
Latus rectum of Parabola:

A focal chord of the parabola which is parallel to the directrix of a parabola is called latus rectum of the parabola and its length is $4a$.

7.7.3 *Standard* Equation of Parabola

When the vertex is at any arbitrary point $V(h, k)$ and the axis of symmetry is parallel to x-axis then equation of parabola then:

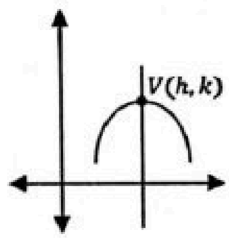


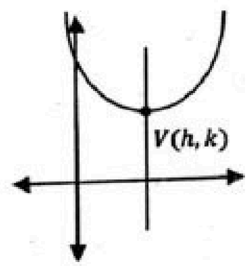


$$(y-k)^2 = 4a(x-h)$$

or $(y-k)^2 = -4a(x-h)$

Similarly, equation of the parabola with vertex at any arbitrary point $V(h, k)$ and the axis of symmetry parallel to y-axis is



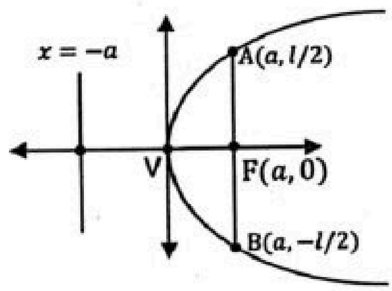


$$(x-h)^2 = 4a(y-k)$$

or $(x-h)^2 = -4a(y-k)$

7.7.4 Length of Latus Rectum of Parabola

Consider the parabola $y^2 = 4ax$. Its focus is at F(a, 0) and vertex is at V(0, 0).



Let l be the length of the latus rectum $\overline{AB}$ then $|\overline{FA}| = \frac{l}{2}$. Therefore, the coordinates of A are $\left(a, \frac{l}{2}\right)$.

Since the point $A\left(a, \frac{l}{2}\right)$ lies on the parabola $y^2 = 4ax$

Thus, it must satisfy its equation. i.e.;

$$\left(\frac{l}{2}\right)^2 = 4a(a) \Rightarrow \frac{l^2}{4} = 4a^2 \Rightarrow l^2 = 16a^2 \Rightarrow l = \pm 4a$$

Since length is always positive, so the length of latus rectum of the parabola is:

$$l = 4a$$

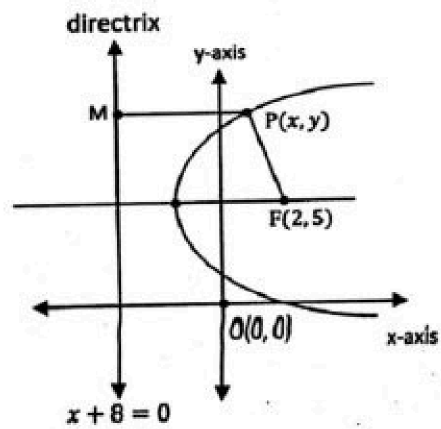
Example 11:

Find the equation of parabola with focus at (2, 5) and the equation of directrix is $x + 8 = 0$.

Solution:

Consider any point $P(x, y)$ on the parabola with Focus at $F(2, 5)$ and equation of directrix $x + 8 = 0$.

By the definition of the parabola



$$|PF| = |PM|$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-5)^2} = \frac{|x+8|}{\sqrt{1^2 + 0^2}}$$

Squaring both sides:

$\Rightarrow (x-2)^2+(y-5)^2=(x+8)^2 \quad \Rightarrow \quad (y-5)^2=(x+8)^2-(x-2)^2$

$\Rightarrow (y-5)^2=x^2+16x+64-x^2+4x-4$

$\Rightarrow (y-5)^2=18x+60 \quad \Rightarrow (y-5)^2=18\left(x+\frac{10}{3}\right)$

Which is the required equation of parabola.

7.7.5 Elements of Parabola and its Graph

To find the elements of parabola when its equation is given; first we convert the given equation into the standard form and then compare it with one of the four standard equations and then find the elements of parabola.

Example 12: Find elements of parabola with equation $y=x^2-3x+7$ and draw its graph.

Solution: Given equation of parabola is:

$$y=x^2-3x+7 \quad \Rightarrow x^2-3x=y-7$$

$$\Rightarrow x^2-3x+\frac{9}{4}=y-7+\frac{9}{4} \quad \Rightarrow \left(x-\frac{3}{2}\right)^2=y-\frac{19}{4}$$

Let $x-\frac{3}{2}=X$ and $y-\frac{19}{4}=Y$ then $X^2=Y$ or $X^2=4\left(\frac{1}{4}\right)Y$

Which is of the form $X^2=4aY$ where $a=\frac{1}{4}$.

Now we write the elements of the parabola.

Vertex: We know that vertex is at $(0,0)$ i.e. $(X,Y)=(0,0)$

$\Rightarrow X=0$ and $Y=0$

$\Rightarrow x-\frac{3}{2}=0$ and $y-\frac{19}{4}=0 \Rightarrow x=\frac{3}{2}$ and $y=\frac{19}{4}$

Thus, the vertex of the given parabola is at $\left(\frac{3}{2},\frac{19}{4}\right)$.

Focus: The focus of the parabola is at $(0,a)$. i.e., $(X,Y)=(0,a)$

$$\Rightarrow X=0 \text{ and } Y=a \Rightarrow x-\frac{3}{2}=0 \text{ and } y-\frac{19}{4}=\frac{1}{4} \quad \Rightarrow x=\frac{3}{2} \text{ and } y=5$$

Thus, the focus of the parabola is at $\left(\frac{3}{2},5\right)$.

Axis of Parabola: Equation of axis of parabola is $X=0$.

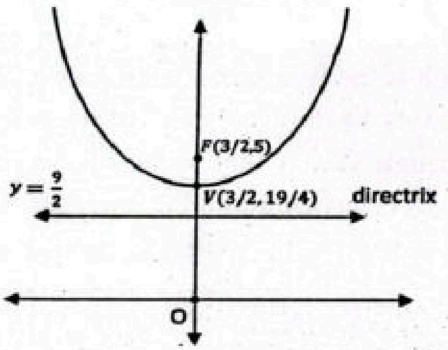
$\Rightarrow x-\frac{3}{2}=0$ or $x=\frac{3}{2}$

Which is the equation of axis of parabola.

Directrix: Equation of directrix of parabola is $Y=-a$ or $Y+a=0$

$\Rightarrow y-\frac{19}{4}+\frac{1}{4}=0 \quad \Rightarrow y-\frac{9}{2}=0.$

Which is the equation of directrix of parabola.



Length of Latus Rectum:

Length of latus rectum is $4a=4\left(\frac{1}{4}\right)=1$ unit

Graph: To draw the graph of parabola, we find its x-intercept and y-intercept. For x-intercept put $y=0$ in the equation $y=x^2-3x+7$. We have:

$x^2-2x+7=0$

$\Rightarrow x = \frac{-(-3)\pm\sqrt{(-3)^2-4(1)(7)}}{2(1)} = \frac{3\pm\sqrt{9-28}}{2} = \frac{3\pm i\sqrt{19}}{2}$

Which are complex numbers thus parabola has no x-intercept. For y-intercept put $x = 0$ in equation $y = x^2 - 3x + 7$. We have:

$y = 0 - 0 + 7 \Rightarrow y = 7$

Thus $(0, 7)$ is the y-intercept of parabola.

7.7.6 General equation of Parabola

Prove that the equation $y = ax^2 + bx + c$ where a, b, c are real numbers with $a \neq 0$ represents a parabola.

Proof: Given equation is

$$y = ax^2 + bx + c \quad \Rightarrow ax^2 + bx = y - c$$

Since $a \neq 0$; dividing both sides by a.

$$x^2 + \frac{b}{a}x = \frac{1}{a}(y - c)$$

Adding $\frac{b^2}{4a^2}$ to both sides, we have:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{1}{a}(y - c) + \frac{b^2}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left[y - c + \frac{b^2}{4a}\right] = \frac{1}{a}\left[y + \frac{b^2-4ac}{4a}\right] = 4\left(\frac{1}{4a}\right)\left[y + \frac{b^2-4ac}{4a}\right]$$

$$\Rightarrow \left(x - \frac{-b}{2a}\right)^2 = 4\left(\frac{1}{4a}\right)\left[y - \frac{-b^2+4ac}{4a}\right] \qquad \text{(i)}$$

Which is of the form $(x - h)^2 = 4p(y - k)$ where $h = -\frac{b}{2a}$; $k = -\frac{b^2-4ac}{4a}$ and $p = \frac{1}{4a}$.

(i) is the equation of parabola with vertex $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$ and its branches open upwards or downwards according as $p > 0$ or $p < 0$.

Example 13: Find elements of parabola with vertex at $(2, 3)$ and focus at $(7, 3)$.

Solution: Given that vertex is $V(2, 3)$ and focus is $F(7, 3)$. Therefore, $h = 2$ and $k = 3$. Observe that y-coordinate of both V and F is same, thus $y = 3$ is the axis of parabola. Since x-coordinate of V is less than x-coordinate of F. Thus branches of parabola open on the right side.

The distance between F and V is "a". i.e.,

$$a = |FV| = \sqrt{(7-2)^2 + (3-3)^2} = \sqrt{25 + 0} = 5$$

So, the equation of parabola is:

$$(y - k)^2 = 4a(x - h)$$

$$\Rightarrow (y - 3)^2 = 4(5)(x - 2) \Rightarrow (y - 3)^2 = 20(x - 2)$$

Which is the required equation of parabola.

Example 14: Find elements of parabola with focus at $(3, -1)$ and its directrix is $3x - 4y + 1 = 0$.

Solution: Given that focus of the parabola is at $F(3, -1)$ and directrix is $3x - 4y + 1 = 0$. If $P(x, y)$ is any point on the parabola then by definition of parabola.

$|PF| = |PM|$; where $|PM|$ is the distance of P from directrix. Therefore:

$$\sqrt{(x-3)^2 + (y+1)^2} = \frac{3x-4y+1}{\sqrt{(3)^2+(-4)^2}}$$

Squaring both sides, we have:

$$(x-3)^2+(y+1)^2=\frac{(3x-4y+1)^2}{25}$$

$$\Rightarrow 25[x^2-6x+9+y^2+2y+1]=9x^2+16y^2+1-24xy+6x-8y$$

$$\Rightarrow 25x^2-150x+25y^2+50y+250=9x^2+16y^2+1-24xy+6x-8y$$

$$\Rightarrow 16x^2+9y^2+24xy-156x+58y+249=0$$

Which is the equation of parabola.

Example 15: Find elements of parabola with vertex at $(2,3)$ and equation of directrix is $y=-4$.

Solution: As we know that the focus of the parabola is on the opposite side of vertex as that of directrix, so in this case it opens upwards. Therefore its equation is:

$$(x-h)^2=4a(y-k) \qquad \text{(i)}$$

Given that the vertex is $(2,3)$, so $h=2$ and $k=3$. Also, the distance between vertex and directrix is $y=-4$ or $y+4=0$. Therefore, $a=\frac{|3+4|}{\sqrt{0^2+1^2}}=7$

Putting this value in equation (i), we get:

$$(x-2)^2=4(7)(y-3) \quad \text{or} \quad (x-2)^2=28(y-3)$$

Which is the required equation of parabola.

Exercise 7.4

1. Find focus, vertex, axis of symmetry, directrix, length of latus rectum, end points of the latus rectum of parabola with the given equations. Also draw the graph of the parabola.
   (i) $y^2=6x$ (ii) $y^2=-\frac{3}{2}x$ (iii) $x^2=24y$ (iv) $x^2=-5y$
   (v) $y^2-2y-12x-71=0$ (vi) $3x^2+42x+y+149=0$
   (vii) $4y^2+4y=15-32x$ (viii) $9x^2-6x=108y+26$
2. Find the equation of the parabola in each of the following.
   (i) Vertex at origin and focus $(0,-\frac{1}{32})$ (ii) Vertex $(-8,-9)$ and focus $(-\frac{31}{4},-9)$
   (iii) Vertex $(-6,-9)$ and directrix: $x=-\frac{47}{8}$ (iv) Vertex $(5,-1)$ and y-intercept: $-\frac{27}{2}$
   (v) Focus $(-\frac{3}{4},-1)$ and directrix $y=\frac{2}{5}$
   (vi) Opens left or right with vertex $(7,6)$ and passes through $(-11,9)$.
   (vii) Opens up or down and passes through the points $(11,15),(7,7)$ and $(4,22)$.
   (viii) Vertex $(10,0)$; axis of symmetry: $y=0$; length of latus rectum $=1$; $a<0$
   (ix) Vertex $(4,2)$; axis of symmetry: $x=4$; length of latus rectum $=\frac{1}{3}$; $a>0$
   (x) Vertex at origin; opens left; distance between focus and vertex is $\frac{1}{8}$ units.
3. Find the equation of the parabola with the focus at $(p\sin\theta, p\cos\theta)$ whose directrix is $x\cos\theta+y\sin\theta=p$.
4. Find the coordinates of the vertex of each parabola by differentiating its equation both sides and then solving for horizontal (or vertical) tangent.
   (i) $y=x^2-4x+10$ (ii) $4x^2+24x+39-3y=0$
   (iii) $y^2-10y+4x+28=0$ (iv) $y^2-14y=-3x-45y$

7.8 Equation of Tangent and Normal of Parabola

7.8.1 Condition for a Line to be Tangent to a Parabola

Consider a parabola $(y-k)^2 = 4a(x-h)$ (1)

and a line $y = mx + c$ (2)

On solving these equations, we will get the points of intersections of the line and the parabola. Using equation (2) in equation (1), we have:

$$[(mx+c)-k]^2 = 4a(x-h)$$
$$\Rightarrow [mx+(c-k)]^2 = 4a(x-h)$$
$$\Rightarrow m^2x^2 + 2m(c-k)x + (c-k)^2 = 4ax - 4ah$$
$$\Rightarrow m^2x^2 + \{2m(c-k)x - 4ax\} + (c-k)^2 + 4ah = 0$$
$$\Rightarrow m^2x^2 + \{2m(c-k) - 4a\}x + (c-k)^2 + 4ah = 0$$

On solving this equation, we will get at most two values of x. But for the line to be tangent to the parabola it must intersect only at a point. i.e., Both values of x should be same, thus discriminant of the above quadratic equation must be zero.

$$\{2m(c-k)-4a\}^2 - 4m^2\{(c-k)^2 + 4ah\} = 0$$
$$\Rightarrow 4m^2(c-k)^2 - 16am(c-k) + 16a^2 - 4m^2(c-k)^2 - 16ahm^2 = 0$$
$$\Rightarrow -16am(c-k) + 16a^2 - 16ahm^2 = 0$$
$$\Rightarrow -16a[m(c-k) - a + hm^2] = 0 \quad \Rightarrow \quad m(c-k) - a + hm^2 = 0$$
$$\Rightarrow m(c-k) = a - m^2h \quad \Rightarrow \quad c - k = \frac{a-m^2h}{m}$$
$$\Rightarrow c = \frac{a-m^2h}{m} + k = \frac{a-m^2h+mk}{m}$$

Putting this value of c in equation (2), we get:

$$y = mx + \frac{a-m^2h+mk}{m} = mx - \frac{m^2h-mk-a}{m}$$

Which is the equation of tangent to the parabola. Here m is the slope of the tangent line.

Particular Case:

When the vertex of the parabola is at $(0,0)$ i.e.; $h = 0$ and $k = 0$ then equation of the tangent line is:

$$y = mx - \frac{m^2(0)-m(0)-a}{m} \quad \Rightarrow \quad y = mx - \frac{-a}{m} \quad \Rightarrow \quad y = mx + \frac{a}{m}$$

Example 16: Find the equation of the tangent to the parabola $y^2 - 6y - 16x + 25 = 0$ with the slope $1/2$.

Solution: Equation of parabola is:

$$y^2 - 6y - 16x + 25 = 0$$
$$\Rightarrow y^2 - 6y = 16x - 25 \Rightarrow y^2 - 6y + 9 = 16x - 25 + 9$$
$$\Rightarrow (y-3)^2 = 16x - 16 = 16(x-1)$$
$$\Rightarrow (y-3)^2 = 4(4)(x-1)$$

Which is of the form $(y-k)^2 = 4a(x-h)$

Here $h = 4$; $k = 3$ and $a = 4$ and given that slope is $m = \frac{1}{2}$, therefore equation of tangent to the parabola is:

$$y = mx - \frac{m^2h-mk-a}{m}$$

Putting values, we get:

$$y = \frac{1}{2}x - \frac{\left(\frac{1}{2}\right)^2 4 - \frac{1}{2}(3) - 4}{\frac{1}{2}} \qquad \Rightarrow \quad y = \frac{1}{2}x - \frac{1 - \frac{3}{2} - 4}{\frac{1}{2}}$$

$$\Rightarrow \quad y = \frac{1}{2}x - (2 - 3 - 8) = \frac{1}{2}x + 9 \qquad \Rightarrow \quad 2y = x + 18$$

$$\Rightarrow \quad x - 2y + 18 = 0$$

7.8.2 Equation of Tangent Line to the Parabola at a Given Point

Consider a parabola $(y - k)^2 = 4a(x - h)$ (1)

and let $P(x_1, y_1)$ be a given point on the parabola. Differentiating equation (1) w. r. t. x

$$2(y - k)\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y-k}$$

Slope of tangent at $P(x_1, y_1) = m = \frac{dy}{dx}$ at $P(x_1, y_1) = \frac{2a}{y_1 - k}$

Thus, equation of the tangent line at $P(x_1, y_1)$: is

$$y - y_1 = m(x - x_1) \qquad \Rightarrow \quad y - y_1 = \frac{2a}{y_1 - k}(x - x_1)$$

$$\Rightarrow \quad (y_1 - k)(y - y_1) = 2a(x - x_1) \quad \Rightarrow \quad (y_1 - k)y - (y_1 - k)y_1 = 2ax - 2ax_1$$

$$\Rightarrow \quad 2ax - (y_1 - k)y - 2ax_1 + (y_1 - k)y_1 = 0$$

Which is the equation of the tangent line at $P(x_1, y_1)$.

In particular if vertex is at $(0, 0)$ then $h = 0$, $k = 0$. Then, the equation of tangent line is:

$$2ax - (y_1 - 0)y - 2ax_1 + (y_1 - 0)y_1 = 0 \quad \Rightarrow \quad 2ax - y_1y - 2ax_1 + y_1^2 = 0$$

Since (x_1, y_1) lies on the parabola, so $y_1^2 = 4ax_1$ and we have:

$$2ax - y_1y - 2ax_1 + 4ax_1 = 0 \qquad \Rightarrow \quad 2ax - y_1y + 2ax_1 = 0$$

7.8.3 Equation of Normal Line to the Parabola at a Given Point

As, we know that normal line is perpendicular to the tangent line.

Thus, slope of the normal line is

$$m = \frac{-1}{\text{slope of tangent line}} = -\frac{(y_1 - k)}{2a}$$

Equation of the normal line at point $P(x_1, y_1)$ is:

$$y - y_1 = -\frac{(y_1 - k)}{2a}(x - x_1)$$

$$\Rightarrow \quad 2a(y - y_1) = -(y_1 - k)(x - x_1) \quad \Rightarrow \quad (y_1 - k)(x - x_1) + 2a(y - y_1) = 0$$

Which is the equation of normal line at point P.

In particular if vertex of the parabola is at $(0, 0)$, the equation of normal becomes:

$$(y_1 - 0)(x - x_1) + 2a(y - y_1) = 0 \quad \text{or} \quad y_1(x - x_1) + 2a(y - y_1) = 0$$

Example 17: Find the equations of the tangent and normal to the parabola $y^2 - 6y + 8x - 9 = 0$ at point $P(\frac{1}{4}, -1)$.

Solution: Equation of parabola is $y^2 - 6y + 8x - 9 = 0$

Diff. w. r. t. x, we have:

$$2y\frac{dy}{dx} - 6\frac{dy}{dx} + 8 = 0 \quad \Rightarrow \quad (2y - 6)\frac{dy}{dx} = -8 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-8}{2y-8} = \frac{-4}{y-3}$$

Slope of tangent line $= m = \frac{dy}{dx}$ at $P = -\frac{-4}{-1-3} = 1$

Thus, equation of tangent line at P is:

$$y-(-1)=1\left(x-\frac{1}{4}\right) \quad \Rightarrow \quad y+1=x-\frac{1}{4}$$

$$\Rightarrow \quad 4y+4=4x-1 \quad \Rightarrow \quad 4x-4y-5=0$$

Since normal line is perpendicular to the tangent line, so slope of the normal line is $\frac{-1}{1}=-1$.

Equation of normal line is:

$$y-(-1)=-1\left(x-\frac{1}{4}\right) \quad \Rightarrow \quad y+1=-x+\frac{1}{4}$$

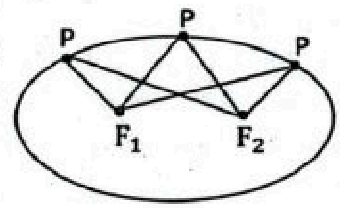
$$\Rightarrow \quad 4y+4=-4x+1 \quad \Rightarrow \quad 4x+4y+3=0$$

Which is the equation of normal line.

Exercise 7.5

1. Find the points of intersections of the line $x-2y+3=0$ and the parabola $y^2=8x+1$. Also find the chord intercepted. Is the chord a focal chord?
2. For what value(s) of c the line $2x+2y-c=0$ never touches or intersects at different points of the parabola $x^2-x+2y+3=0$.
3. Find the value of a so that the line $ax-2y+3=0$ is tangent to the parabola:
$$y^2-2y+3x+7=0\,;\ a\neq 0$$
4. Prove that the line $3x+y-5=0$ is tangent to the parabola $y^2-2y+6x-6=0$. Also find the point of contact.
5. Find the equation of tangent and normal to the parabola $2y^2-3y+11x-16=0$ at the point $(1,-1)$.
6. Find the equation of tangent and normal to the parabola $x^2-5x+2y+6=0$ at the point where abscissa is 1.
7. Find the equation of tangent and normal to the parabola $y^2=18x$ at the end points of its latus rectum.
8. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2=4ax$. Prove that the chord joining P and Q is a focal chord if $t_1t_2=-1$.
9. A tangent line is drawn to the parabola at any point P. Prove that the line segment of the tangent cut off between P and directrix subtends a right angle at the focus.
10. Prove that tangents at the end points of any focal chord intersect at right angles on the directrix.
11. Prove that tangents to the parabola at any point P on the parabola make equal angle with line joining P and its focus and the line through P parallel to the axis of parabola. (Reflecting property of parabola).
12. Prove that the semi latus rectum is a harmonic mean between the segments of any focal chord.

7.9 Ellipse



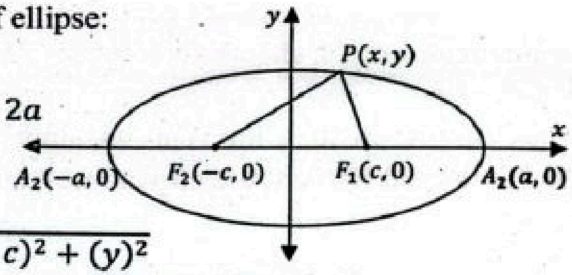
It is the set of all the points in the plane such that the sum of the distances of each point from two fixed points in the plane remains same. The two fixed points are known as foci (plural of focus) of the ellipse.

The midpoint of the foci is called the centre of ellipse.

7.9.1 Standard Equation of an Ellipse

Consider an ellipse with centre at origin and the foci on x-axis. Let the foci be $F_1(-c, 0)$ and $F_2(c, 0)$. Also suppose that sum of the distance of each pint of ellipse from foci is $2a$ which is constant.

Take any point $P(x, y)$ on ellipse then by definition of ellipse:



$$|PF_1| + |PF_2| = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + (y)^2} = 2a - \sqrt{(x-c)^2 + (y)^2}$$

Squaring both sides, we get:

$$(x+c)^2 + y^2 = 4a^2 + ((x-c)^2 + y^2) - 4a\sqrt{(x-c)^2 + (y)^2}$$

$$\Rightarrow \cancel{x^2} + \cancel{c^2} + 2cx + \cancel{y^2} = 4a^2 + \cancel{x^2} + \cancel{c^2} - 2cx + \cancel{y^2} - 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx \quad \Rightarrow \quad a\sqrt{(x-c)^2 + y^2} = a^2 - cx$$

Again, squaring both sides, we have:

$$a^2[(x-c)^2 + y^2] = a^4 + c^2x^2 - 2a^2cx$$

$$\Rightarrow a^2[x^2 + c^2 - 2cx + y^2] = a^4 + c^2x^2 - 2a^2cx$$

$$\Rightarrow a^2x^2 + a^2c^2 - 2a^2cx + a^2y^2 = a^4 + c^2x^2 - 2a^2cx$$

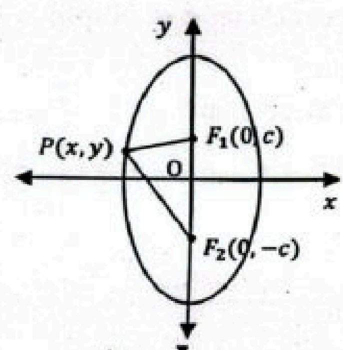
$$\Rightarrow (a^2x^2 - c^2x^2) + a^2y^2 = a^4 - a^2c^2 \quad \Rightarrow \quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \qquad \text{(i)}$$

Since $a > c \Rightarrow a^2 > c^2 \Rightarrow a^2 - c^2 > 0$

Let $a^2 - c^2 = b$ (say), thus, equation (i) becomes:

$b^2x^2 + a^2y^2 = a^2b^2$

Dividing both sides by a^2b^2, we have: $\dfrac{b^2x^2}{a^2b^2} + \dfrac{a^2y^2}{a^2b^2} = \dfrac{a^2b^2}{a^2b^2}$



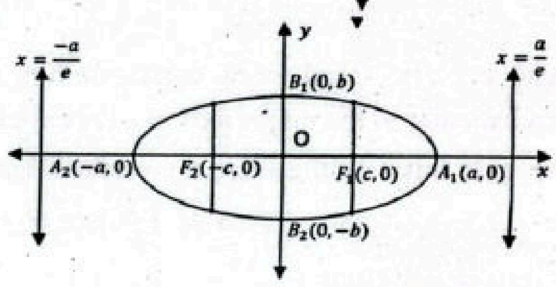
$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of ellipse in standard form.

Note: If we take foci on y-axis i.e. $F_1(0, -c)$ and $F_2(0, c)$ then equation of ellipse will be: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

Elements of Ellipse

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Following are its elements.



Foci

The two fixed points $F_1(-c, 0)$ and $F_2(c, 0)$ are known as foci of ellipse.

Major Axis

The line which passes through both the foci of ellipse is called major axis of the ellipse and its length is $2a$. In this case major axis is x-axis.

Vertices

The points where the ellipse cuts its major axis are known as the vertices of ellipse. In this case vertices are $A_1(-a, 0)$ and $A_2(a, 0)$.

Centre

The mid-point of both the foci (or both the vertices) is called the centre of the ellipse. In this case $(0, 0)$ is the centre of ellipse.

Minor Axis

The line passing through the centre of the ellipse and perpendicular to the major axis is known as minor axis of the ellipse. Its length is $2b$. In this case y-axis is the minor axis.

Co-Vertices

The points where the ellipse cuts its minor axis are known as co-vertices of the ellipse. In this case $B_1(0, -b)$ and $B_2(0, b)$ are co-vertices of the ellipse. Note that the mid point of the co-vertices is also the centre of ellipse.

Chord

A line segment with its end points on the ellipse is called chord of the ellipse.

Focal Chord

A chord which passes through any of the foci is called focal chord. e.g., B_1B_2 is focal chord.

Latus Rectum

A focal chord which is perpendicular to the major axis is called latus rectum of the ellipse. There are two latus rectums (latera recta) of an ellipse through each of the foci. The length of both the latus rectums is same and is $\frac{2b^2}{a}$.

Eccentricity

The eccentricity e of ellipse is $\frac{c}{a}$. i.e., $e = \frac{c}{a}$ since $c < a$ thus eccentricity of the ellipse is always less than 1.

As we know that:

$$b^2 = a^2 - c^2 \quad \Rightarrow \quad \frac{b^2}{a^2} = \frac{a^2}{a^2} - \frac{c^2}{a^2} \quad \Rightarrow \quad \left(\frac{b}{a}\right)^2 = 1 - \left(\frac{c}{a}\right)^2$$

$$\Rightarrow \quad \left(\frac{b}{a}\right)^2 = 1 - e^2 \quad \Rightarrow \quad e^2 = 1 - \left(\frac{b}{a}\right)^2 = \frac{a^2 - b^2}{a^2}$$

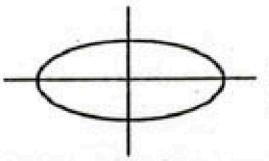
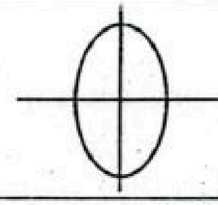
or $\quad e = \frac{\sqrt{a^2 - b^2}}{a}$

Directrices

There are two fixed lines corresponding to each foci lying outside of the ellipse and are perpendicular to the major axis at a specific distance $\frac{a}{e}$. These lines are known as directrices of the ellipse. In this case these are the vertical lines and their equations are:

$$x = \pm\frac{a}{e} \text{ or } x = \pm\frac{a}{e^2} \text{ or } x = \pm\frac{a^2}{c}$$

The table shows summary of the elements of ellipse.

	 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	 $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
Centre	$O(0,0)$	$O(0,0)$
Foci	$(-c,0)$ & $(c,0)$	$(0,-c)$ & $(0,c)$
Major Axis	x-axis with equation $y=0$	y-axis with equation $x=0$
Minor Axis	y-axis with equation $x=0$	x-axis with equation $y=0$
Vertices	$(-a,0)$ & $(a,0)$	$(0,-a)$ & $(0,a)$
Co-Vertices	$(0,-b)$ & $(0,b)$	$(-b,0)$ & $(b,0)$
Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$

Note: Equation of the ellipse with centre at arbitrary point (h,k) is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{or} \quad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

Example 18: Find the equation of an ellipse with foci $(-3,0)$ and $(3,0)$ and the sum of the distance of any point from the foci is 10.

Solution: Given that foci of the ellipse are $F_1(-3,0)$ and $F_2(3,0)$. Take any point $P(x,y)$ on the ellipse, then by definition of ellipse:

$$|PF_1|+|PF_2|=10$$
$$\Rightarrow \sqrt{(x+3)^2+(y-0)^2}+\sqrt{(x-3)^2+(y-0)^2}=10$$
$$\Rightarrow \sqrt{(x+3)^2+y^2}=10-\sqrt{(x-3)^2+y^2}$$

Squaring both sides:

$$(x+3)^2+y^2=100+((x-3)^2+y^2)-20\sqrt{(x-3)^2+y^2}$$
$$\Rightarrow \cancel{x^2}+6x+\cancel{9}+\cancel{y^2}=100+\cancel{x^2}-6x+\cancel{9}+\cancel{y^2}-20\sqrt{(x-3)^2+y^2}$$
$$\Rightarrow 20\sqrt{(x-3)^2+y^2}=100-12x \quad \text{or} \quad 5\sqrt{(x-3)^2+y^2}=25-3x$$

Again, squaring both sides:

$$25((x-3)^2+y^2)=625+9x^2-150x$$
$$\Rightarrow 25(x^2-6x+9+y^2)=625+9x^2-150x$$
$$\Rightarrow 25x^2-150x+225+25y^2)=625+9x^2-150x$$
$$\Rightarrow 16x^2+25y^2=400$$

Divide both sides by 400, we get the required equation of ellipse as:

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

Example 19: Convert the equation of ellipse $4x^2 + 9y^2 + 8x - 36y + 4 = 0$ in standard form. Also find its elements and draw the graph.

Solution: Given equation of ellipse is:

$$4x^2 + 9y^2 + 8x - 36y + 4 = 0 \qquad \Rightarrow \quad (4x^2 + 8x) + (9y^2 - 36y) = -4$$

$$\Rightarrow 4(x^2 + 2x) + 9(y^2 - 4y) = -4$$

$$\Rightarrow 4[x^2 + 2x + 1 - 1] + 9[y^2 - 4y + 4 - 4] = -4$$

$$\Rightarrow 4(x+1)^2 - 4 + 9(y-2)^2 - 36 = -4 \quad \text{or} \quad 4(x+1)^2 + 9(y-2)^2 = 36$$

Dividing both sides by 36, we get:

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1 \qquad \text{or} \qquad \frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1$$

Which is the standard form of the equation of ellipse.

Letting $x + 1 = X$ and $y - 2 = Y$, the equation becomes:

$$\frac{X^2}{3^2} + \frac{Y^2}{2^2} = 1$$

Its major axis is along x-axis. Hence $a = 3$ and $b = 2$.

Now we find its elements.

Centre: As the centre of ellipse is $(0, 0)$, therefore $(X, Y) = (0, 0)$

$\Rightarrow X = 0$	$Y = 0$
$\Rightarrow x + 1 = 0$	$y - 2 = 0$
$\Rightarrow x = -1$	$y = 2$

Thus, the centre of ellipse is at $(-1, 2)$.

Foci: As we know that $c = \pm\sqrt{a^2 - b^2}$ hence for the given equation $c = \pm\sqrt{9-4} = \pm\sqrt{5}$ therefore foci are $F_1(-c, 0) = (-\sqrt{5}, 0)$ and $F_2(-c, 0) = (-\sqrt{5}, 0)$.

Major Axis: As major axis of ellipse is along $x\text{-}axis$ hence its equation is $Y = 0$ $\Rightarrow y - 2 = 0 \qquad \Rightarrow y = 2$ is equation of major axis.

Minor Axis: Equation of minor axis is $X = 0 \qquad \Rightarrow x + 1 = 0 \qquad \Rightarrow x = -1$

Vertices: The vertices of given ellipse are $(-a, 0) = (-3, 0)$ and $(a, 0) = (3, 0)$.

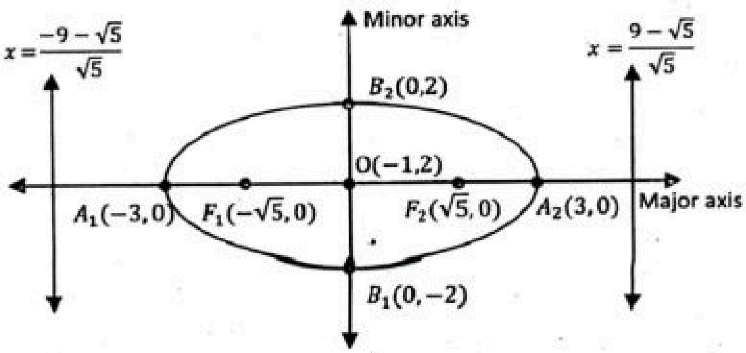
Co-Vertices: Co-vertices of given ellipse are $(0, -b) = (0, -2)$ and $(0, b) = (0, 2)$.

Directrices: The equation of directrices is $X = \pm\frac{a^2}{c}$ i.e. $x + 1 = \pm\frac{9}{\sqrt{5}}$

$$\text{or } x = \pm\frac{9}{\sqrt{5}} - 1 = \frac{9-\sqrt{5}}{\sqrt{5}} = \frac{-9-\sqrt{5}}{\sqrt{5}}$$

$$x = \frac{-9-\sqrt{5}}{\sqrt{5}}$$

The graph is shown in the adjoining figure.



Exercise 7.6

1. Find the centre, vertices, co-vertices, foci, eccentricity, length and equation of major axis, length and equation of minor axis, directrices, and length of latus rectums for the following equations of ellipse. Also draw the ellipse in each case
   (i) $\frac{x^2}{9}+\frac{y^2}{16}=1$ (ii) $2x^2+3y^2=30$
   (iii) $\frac{x^2}{49}+\frac{(y-3)^2}{64}=1$ (iv) $x^2+9y^2+6x-90y+225=0$
   (v) $16x^2+9y^2-32x+36y-92=0$
2. From the given information, find the equation of ellipse in each of the following.
   (i) Vertices $(-4,6),(-16,6)$ and co-vertices $(-10,2),(-10,10)$
   (ii) Centre at $(-8,5)$; vertex $(-8,15)$; Focus $(-8,5+\sqrt{5})$
   (iii) Eccentricity is $\frac{\sqrt{15}}{4}$; centre $(-5,5)$; co-vertex $(-8,5)$
   (iv) Eccentricity is $\frac{3}{4}$ and passes through the point $(3, 1)$ with centre at $(0,0)$; major axis along y-axis.
   (v) Focus at $(3,4)$; directrix $x=5$ and eccentricity $\frac{2}{3}$.
   (vi) Centre at $(1,-1)$; horizontal tangents are $y=7$ and $y=-9$ and vertical tangents are $x=6$ and $x=-4$.
   (vii) Length of latus rectum is 4; centre at $(0,0)$ and eccentricity is $\frac{1}{3}$; major axis along y-axis.
   (viii) Centre at $(1,1)$; eccentricity is $\frac{2}{5}$ and one of the directrices is $x=5$.
3. Find the coordinates of the vertices and co-vertices of the following ellipse by differentiating both sides of equations and solving for horizontal and vertical tangent lines
   (i) $2x^2+3y^2+2x-3y-5=0$ (ii) $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{16}=1$
   (iii) $\frac{1}{2}x^2+\frac{3}{4}y^2+x-y-5=0$ (iv) $x^2+2y^2-6x-4=0$

7.10 Equation of Tangent and Normal to Ellipse

7.10.1 Condition of Tangency

Consider the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ ______ (1)

and the line $y=mx+c$ ______ (2)

on solving (1) and (2) simultaneously we will get the common points, from (2) put the value of y in equation (1)

$$\frac{(x-h)^2}{a^2}+\frac{(mx+c-k)^2}{b^2}=1$$

$$b^2(x-h)^2+a^2(mx+c-k)^2=a^2b^2$$

$$b^2(x^2-2hx+h^2)+a^2\{m^2x^2+2m(c-k)+(c-k)^2\}=a^2b^2$$

$$(b^2+a^2m^2)x^2+\left(-2b^2h+2a^2m(c-k)\right)x+b^2h^2+a^2(c-k)^2-a^2b^2=0$$

which is quadratic equation in x.

If Disc > 0 then equation will have two distinct real roots so, line will cut the ellipse at two different points.

If Disc < 0 then equation will have no real roots, i.e. there is no common point.

If Disc = 0 then equation will have one real root i.e. there is only one common point. Thus line will be tangent to the ellipse if Disc = 0.

$$\Rightarrow \{-2b^2h + 2a^2m(c-k)\}^2 - 4(b^2 + a^2m^2)[b^2h^2 + a^2(c-k)^2 - a^2b^2] = 0$$

$$\Rightarrow 4\{-b^2h + a^2m(c-k)\}^2 - 4(b^2 + a^2m^2)[b^2h^2 + a^2(c-k)^2 - a^2b^2] = 0$$

$$\Rightarrow b^4h^2 + a^4m^2(c-k)^2 - 2a^2b^2hm(c-k) - b^4h^2 - a^2b^2(c-k)^2 + a^2b^4 - a^2b^2m^2h^2$$
$$- a^4b^2m^2h^2 - a^4m^2(c-k)^2 - a^4b^2m^2 = 0$$

Taking common $-a^2b^2$

$$-a^2b^2[2hm(c-k) + (c-k)^2 - b^2 + m^2h^2 + a^2m^2] = 0$$
$$\Rightarrow 2hm(c-k) + (c-k)^2 - b^2 + m^2(a^2 + h^2) = 0$$
$$\Rightarrow b^2 = 2hm(c-k) + (c-k)^2 + m^2(a^2 + h^2)$$

is the condition for the line to be tangent to the ellipse. In particular when centre of the ellipse is at (0,0) then $h = 0$ and $k = 0$. In this case the condition reduces to

$$b^2 = 2(0)m(c-0) + (c-0)^2 + m^2(a^2 + 0)$$
$$\Rightarrow b^2 = c^2 + m^2a^2$$
$$\Rightarrow c^2 = b^2 - m^2a^2$$
$$\Rightarrow c = \sqrt{b^2 - m^2a^2} \qquad \text{provided that } b^2 - m^2a^2 \geq 0$$

Put value in equation (2)

$$y = mx \pm \sqrt{b^2 - m^2a^2}$$

Are the equations of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the slope form.

Example 20: Find the equations of the tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{16} = 1$ with slope $\frac{1}{2}$.

Solution: Equation of the ellipse is $\frac{x^2}{2} + \frac{y^2}{16} = 1$.

Here $a^2 = 2$; $b^2 = 16$ and given that $m = \frac{1}{2}$.

Equations of the tangent to ellipse are $y = mx \pm \sqrt{b^2 - m^2a^2}$

Putting the values; we have:

$$y = \frac{1}{2}x \pm \sqrt{16 - \frac{1}{4}(2)} \Rightarrow y = \frac{1}{2}x \pm \frac{\sqrt{62}}{2} \Rightarrow 2y = x \pm \sqrt{62}$$

are the required equations of the tangent lines.

Example 21: prove that the line $\sqrt{3}x - 2y + 8 = 0$ is tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Solution:

Equation of line is $\sqrt{3}x - 2y + 8 = 0$ (1)

and equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$$x^2 + 4y^2 = 16 \qquad (2)$$

From equation (1) $2y = \sqrt{3}x + 8 \Rightarrow y = \frac{\sqrt{3}x+8}{2}$

Put in equation (2)

$$x^2 + 4\left(\frac{\sqrt{3}x + 8}{2}\right)^2 = 16$$

$$\Rightarrow x^2 + 4\left(\frac{3x^2 + 16\sqrt{3}x + 64}{4}\right) = 16$$

$$\Rightarrow x^2 + 3x^2 + 16\sqrt{3}x + 64 = 16$$

$$\Rightarrow 4x^2 + 16\sqrt{3}x + 48 = 0$$

Dividing both sides by 4

$$\Rightarrow x^2 + 4\sqrt{3}x + 12 = 0$$

Taking its discriminant

$$\Rightarrow Disc. = \left(4\sqrt{3}\right)^2 - 4(1)(12) = 48 - 48$$

$\Rightarrow Disc. = 0$ this shows that line is tangent to the ellipse.

7.11 Equations of Tangent and Normal to an Ellipse at a Given Point

Consider the equation of ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow b^2(x-h)^2 + a^2(y-k)^2 = a^2b^2 \qquad (1)$$

Let $P(x_1, y_1)$ be any point on this ellipse. Differentiating equation (1) w. r. t. x

$$\Rightarrow 2b^2(x-h) + 2a^2(y-k)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{(x-h)}{(y-k)}$$

At point $P(x_1, y_1)$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{(x_1-h)}{(y_1-k)}$$

is the slope of the tangent line at point $P(x_1, y_1)$. Thus, equation of tangent line at this point P is

$$y - y_1 = -\frac{b^2}{a^2}\frac{(x_1-h)}{(y_1-k)}(x - x_1)$$

$$\Rightarrow a^2(y_1-k)(y-y_1) = -b^2(x_1-h)(x-x_1)$$

$$\Rightarrow b^2(x_1-h)(x-h+h-x_1) + a^2(y_1-k)(y-k+k-y_1) = 0$$

$$\Rightarrow b^2(x_1-h)[(x-h)-(x_1-h)] + a^2(y_1-k)[(y-k)-(y_1-k)] = 0$$

$$\Rightarrow b^2(x_1-h)(x-h) - b^2(x_1-h)^2 + a^2(y_1-k)(y-k) - a^2(y_1-k)^2 = 0$$

$$\Rightarrow b^2(x_1-h)(x-h) + a^2(y_1-k)(y-k) = b^2(x_1-h)^2 + a^2(y_1-k)^2$$

Since, the point P lies on the ellipse, thus from equation (1) we have

$$b^2(x_1-h)^2+a^2(y_1-k)^2=a^2b^2$$

$$\Rightarrow b^2(x_1-h)(x-h)+a^2(y_1-k)(y-k)=a^2b^2$$

Dividing both sides by a^2b^2

$$\Rightarrow \frac{(x_1-h)(x-h)}{a^2}+\frac{(y_1-k)(y-k)}{b^2}=1$$

is the required equation of the tangent line.

In particular case if the centre of the ellipse lies at origin, then $h=0, k=0$. In this case equation of tangent is

$$\frac{(x_1-0)(x-0)}{a^2}+\frac{(y_1-0)(y-0)}{b^2}=1$$

$$\Rightarrow \frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

7.11.1 Equation of Normal

As the normal line is perpendicular to the tangent line thus,

slope of the normal line $=\dfrac{-1}{\text{slope of tangent line}}$

$$=\frac{-1}{-\frac{b^2}{a^2}\frac{(x_1-h)}{(y_1-k)}}=\frac{a^2(y_1-k)}{b^2(x_1-h)}$$

Thus, equation of normal line at point P is

$$y-y_1=\frac{a^2(y_1-k)}{b^2(x_1-h)}(x-x_1)$$

$$\Rightarrow b^2(x_1-h)(y-y_1)=a^2(y_1-k)(x-x_1)$$

$$\Rightarrow a^2(y_1-k)(x-x_1)-b^2(x_1-h)(y-y_1)=0$$

is the equation of the normal line.

In particular if the centre of the ellipse is at origin, then $h=0$ and $k=0$. So, equation of normal becomes

$$a^2(y_1-0)(x-x_1)-b^2(x_1-0)(y-y_1)=0$$

$$\Rightarrow a^2y_1(x-x_1)-b^2x_1(y-y_1)=0$$

Example 22: Find the equations of tangent and normal to the ellipse $x^2+2y^2-2x+4y=0$ at point $P(0,0)$.

Solution:

Equation of ellipse is $x^2+2y^2-2x+4y=0$

Differentiating w. r. t. x

$$2x+4y\frac{dy}{dx}-2+4\frac{dy}{dx}=0$$

$$\Rightarrow (4y+4)\frac{dy}{dx}=-(2x-2)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2x-2}{4y+4}=-\frac{x-1}{2y+2}$$

At point (0,0)

$$\Rightarrow \frac{dy}{dx} = -\frac{0-1}{2(0)+2} = \frac{1}{2}$$

is the slope of the tangent line at $P(0,0)$. Thus, equation of tangent line is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 0 = \frac{1}{2}(x - 0) \Rightarrow 2y = x \quad \Rightarrow x - 2y = 0$$

Now slope of the normal line is $-\frac{1}{1/2}$. Thus, equation of normal line at point $P(0,0)$ is

$$y - 0 = -2(x - 0) \Rightarrow y = -2x$$

$$2x + y = 0$$

Alternatively

Equation of ellipse is $x^2 + 2y^2 - 2x + 4y = 0$, convert it into standard form, i.e.;

$$(x^2 - 2x) + 2(y^2 + 2y) = 0$$

$$\Rightarrow (x^2 - 2x + 1 - 1) + 2(y^2 + 2y + 1 - 1) = 0$$

$$\Rightarrow [(x-1)^2 - 1] + 2[(y+1)^2 - 1] = 0$$

$$\Rightarrow (x-1)^2 - 1 + 2(y+1)^2 - 2 = 0$$

$$\Rightarrow (x-1)^2 + 2(y+1)^2 = 3$$

Dividing both sides by 3

$$\Rightarrow \frac{(x-1)^2}{3} + \frac{2(y+1)^2}{3} = 1$$

$$\Rightarrow \frac{(x-1)^2}{3} + \frac{(y-(-1))^2}{3/2} = 1$$

Compare it with standard equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; h = 1; k = -1; a^2 = 3; b^2 = 3/2$

Equation of tangent line is

$$\frac{(x_1 - h)(x - h)}{a^2} + \frac{(y_1 - k)(y - k)}{b^2} = 1$$

Putting the values

$$\frac{(0-1)(x-1)}{3} + \frac{(0-(-1))(y-(-1))}{3/2} = 1$$

$$\Rightarrow \frac{-x+1}{3} + \frac{2(y+1)}{3} = 1 \Rightarrow -x + 1 + 2y + 2 = 3 \Rightarrow -x + 2y = 0$$

$$\Rightarrow x - 2y = 0$$

And the equation of normal line is

$$a^2(y_1 - k)(x - x_1) - b^2(x_1 - h)(y - y_1) = 0$$

Putting the values

$$\Rightarrow 3(0-(-1))(x-0) - 3/2(0-1)(y-0) = 0$$

$$\Rightarrow 3x + \frac{3}{2}y = 0$$

$\Rightarrow 2x + y = 0$, is the equation of normal line.

Exercise 7.7

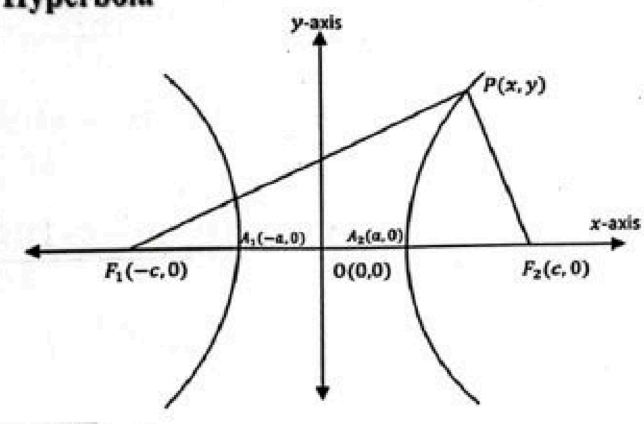
1. Find the points of intersection of the line and ellipse and also find the length of chord intercepted $x - y + 1 = 0$; $x^2 + 2y^2 + 3x - 7y - 11 = 0$.
2. Find the value of m so that the line $5x + y - 20 = 0, m \neq 0$ touches the ellipse $\frac{x^2}{4} + \frac{y^2}{5} = 1$. Also find the point where it touches the ellipse.
3. Find the equation of tangent to the ellipse $x^2 + 2y^2 - 3x + 5y - 3 = 0$ with slope 1.
4. Find the equations of tangent to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.
5. Find the equations of the tangent and normal to the ellipse $2x^2 + 3y^2 - 5x - 10y + 5 = 0$ at the point $(3,2)$.
6. Find the equations of the tangents and normal to the ellipse $15x^2 + 6y^2 - 8x - 5y + 2 = 0$ at the point with ordinate $\frac{1}{3}$.
7. Prove that tangent at any point to the ellipse make equal angles with the line joining the point with the foci (reflecting property of ellipse).

7.12 Hyperbola

It is the set of all points in a plane such that the difference of the distances of each point from two fixed points in the plane is same. The two fixed points are known as foci of the hyperbola. The midpoint of foci is known centre of the hyperbola.

7.12.1 Standard Form of the Equation of Hyperbola

Consider a hyperbola with centre at origin and foci on $x - axis$. Let the foci be $F_1(-c, 0)$ and $F_2(c, 0)$ and the difference of the distances of each point from the foci is $2a$.



Take any points $P(x, y)$ on hyperbola then by the definition of the hyperbola

$|PF_1| - |PF_2| = 2a$

$$\Rightarrow \sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

squaring both sides

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + (y)^2} + ((x-c)^2 + y^2)$$

$$\Rightarrow \cancel{x^2} + \cancel{c^2} + 2cx + \cancel{y^2} = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + \cancel{x^2} + \cancel{c^2} - 2cx + \cancel{y^2}$$

$\Rightarrow 4a\sqrt{(x-c)^2+y^2} = 4cx - 4a^2$

$\Rightarrow a\sqrt{(x-c)^2+y^2} = cx - a^2$

Again, squaring both sides

$a^2[(x-c)^2+y^2] = c^2x^2 + a^4 - 2a^2cx$

$\Rightarrow a^2[x^2+c^2-2cx+y^2] = c^2x^2+a^4-2a^2cx$

$\Rightarrow a^2x^2 + a^2c^2 - \cancel{2a^2cx} + a^2y^2 = c^2x^2 + a^4 - \cancel{2a^2cx}$

$\Rightarrow (c^2x^2 - a^2x^2) - a^2y^2 = a^2c^2 - a^4$

$\Rightarrow (c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2)$

since $c > a \Rightarrow \; c^2 > a^2$

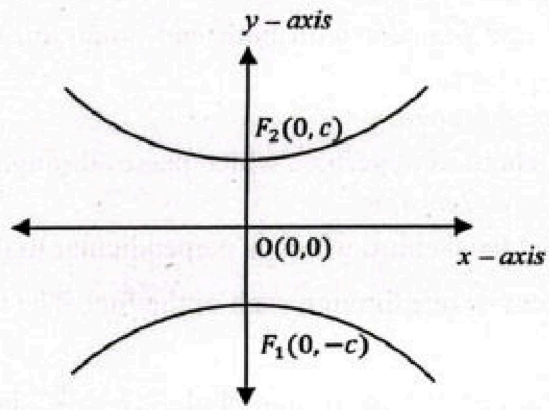
$\Rightarrow \; c^2 - a^2 > 0$

Let $c^2 - a^2 = b^2$

Therefore $b^2x^2 - a^2y^2 = a^2b^2$

Dividing both sides by a^2b^2, we have:

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



is the standard form of the equation of hyperbola. Similarly, if we take the foci on $y-axis$ the equation of hyperbola will be

$\Rightarrow \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Elements of Hyperbola

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci

The two fixed points $F_1(-c, 0)$ and $F_1(c, 0)$ are the foci of the hyperbola.

Centre

The midpoint of both the foci is called the centre of the hyperbola.

Focal Axis or Transverse Axis or Real Axis

The line which passes through both the foci of hyperbola is called the focal axis. In this case x-axis is the focal axis with equation $y = 0$ and its length is $2a$.

Conjugate Axis or Imaginary Axis

The line passing through the centre of the hyperbola and perpendicular to its focal axis is known as conjugate axis or imaginary axis. In this case y-axis is conjugate axis and its equation is $x = 0$. The length of conjugate axis is $2b$.

Vertices

The points where the hyperbola cuts its focal axis are known as the vertices of the hyperbola. In this case $A_1(-a, 0)$ and $A_2(a, 0)$ are vertices of the hyperbola. Note that the mid point of the vertices is the centre of the hyperbola.

Co-Vertices

The points $(0, -b)$ and $(0, b)$ lying on the conjugate axis and equidistant from the centre are known as co-vertices of the hyperbola.

Focal Length

The distance between the two foci is called focal length and its value is $2c$.

Chord

A line segment with both end points on the same branch of hyperbola is called chord of the hyperbola.

Focal Chord

A chord of hyperbola which passes through the focus of the hyperbola is called focal chord.

Latus Rectum

The focal chord which is perpendicular to the focal axis is called latus rectum. There are two latus rectums one through each of the foci. The length of each latus rectum is $\frac{2b^2}{a}$.

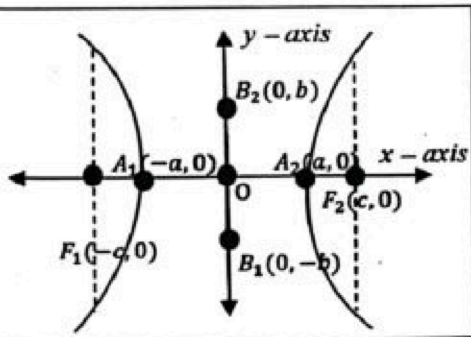
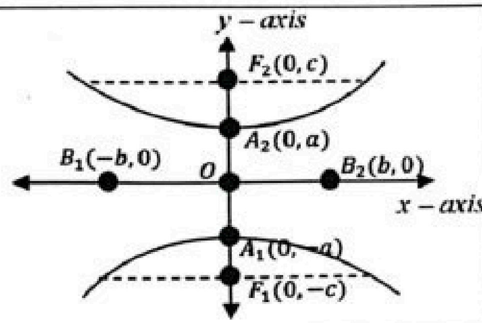
Eccentricity

The eccentricity of hyperbola is $e = \frac{c}{a}$ since $c > a$ thus eccentricity of the hyperbola is always greater than 1.

The equation of hyperbola with centre at (h, k) when the axis of symmetry remains parallel to the coordinate axis is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

The table shows the summary of elements of hyperbola.

Sketch and Equation		
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$
Centre	$O(0, 0)$	$O(0, 0)$
Foci	$(-c, 0)$ & $(c, 0)$	$(0, -c)$ & $(0, c)$
Vertices	$A_1(-a, 0)$ & $A_2(a, 0)$	$A_1(0, -a)$ & $A_2(0, a)$
Co-Vertices	$B_1(0, -b)$ & $B_2(0, b)$	$B_1(-b, 0)$ & $B_2(b, 0)$

Focal Axis	x-axis with equation $y = 0$	y-axis with equation $x = 0$
Conjugate Axis	y-axis with equation $x = 0$	x-axis with equation $y = 0$
Eccentricity	$e = \frac{c}{a} > 1$	$e = \frac{c}{a} > 1$
Directrices	$x = \pm\frac{c}{a}$ or $x = \pm\frac{a}{e}$	$y = \pm\frac{c}{a}$ or $y = \pm\frac{a}{e}$
Latus Rectum	length of latus rectum $2\frac{b^2}{a}$	length of latus rectum $2\frac{b^2}{a}$

Example 23: Convert the equation of hyperbola $4x^2 - 9y^2 - 16x - 18y - 29 = 0$ into standard form and find its elements.

Solution: Given equation is:

$4x^2 - 9y^2 - 16x - 18y - 29 = 0$ $\quad\Rightarrow (4x^2 - 16x) - (9y^2 + 18y) = 29$

$\Rightarrow 4(x^2 - 4x) - 9(y^2 + 2y) = 29$

$\Rightarrow 4(x^2 - 4x + 4 - 4) - 9(y^2 + 2y + 1 - 1) = 29$

$\Rightarrow 4[(x-2)^2 - 4] - 9[(y+1)^2 - 1] = 29$ $\quad\Rightarrow 4(x-2)^2 - 16 - 9(y+1)^2 + 9 = 29$

$\Rightarrow 4(x-2)^2 - 9(y+1)^2 = 36$

Dividing both sides by 36:

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{4} = 1 \quad \text{or} \quad \frac{(x-2)^2}{3^2} - \frac{(y+1)^2}{2^2} = 1$$

is the standard form of the equation of the hyperbola.

Let $x - 21 = \text{X}$ and $y + 1 = \text{Y}$

the equation reduces to $\frac{\text{X}^2}{3^2} - \frac{\text{Y}^2}{2^2} = 1.$

Centre: Centre of hyperbola is at $(0, 0)$. i.e., $(\text{X}, \text{Y}) = (0, 0)$

$\Rightarrow \text{X} = 0$	$\text{Y} = 0$
$\Rightarrow x - 2 = 0$	$y + 1 = 0$
$\Rightarrow x = 2$	$y = -1$

Therefore, $(2, -1)$ is the centre of hyperbola.

Vertices: The vertices of given hyperbola are $(\pm a, 0)$ i.e. $(\text{X}, \text{Y}) = (\pm a, 0)$

$\Rightarrow \text{X} = \pm a$	$\text{Y} = 0$
$\Rightarrow x - 2 = \pm 3$	$y + 1 = 0$
$\Rightarrow x = \pm 3 + 2$	$y = -1$
$\Rightarrow x = 5, x = -1$	

Hence vertices are $(-1, -1)$ and $(5, -1)$.

Covertices: Covertices hyperbola are $(0, \pm b)$ i.e. $(\text{X}, \text{Y}) = (0, \pm b)$

$\Rightarrow \text{X} = 0$	$\Rightarrow \text{Y} = \pm b$
$\Rightarrow x - 2 = 0$	$\Rightarrow y + 1 = \pm 2$
$\Rightarrow x = 2$	$\Rightarrow y = -1 \pm 2$
	$\Rightarrow y = -3, x = 1$

Thus, covertices are $(2, -3)$ and $(2,1)$.

Foci: Foci of the hyperbola are $(\pm c, 0)$ i.e. $(X, Y) = (\pm c, 0)$

$\Rightarrow X = \pm c$	$\Rightarrow Y = 0$
$\Rightarrow x - 2 = \pm c$	$\Rightarrow y + 1 = 0$
$\Rightarrow x = 2 \pm c$	$\Rightarrow y = -1$

As we know that $b^2 = c^2 - a^2$

$\Rightarrow 2^2 = c^2 - 3^2 \quad \Rightarrow c^2 = 13 \quad \Rightarrow c = \sqrt{13}$

Thus, $x = 2 \pm \sqrt{13}$ and $y = 1$

$\Rightarrow (2 \pm \sqrt{13}, -1)$ are foci of the hyperbola.

Eccentricity: Eccentricity of the hyperbola is $e = \frac{c}{a} = \frac{\sqrt{13}}{3}$

Focal Axis: Equation of focal axis is $Y = 0 \Rightarrow y + 1 = 0$ and the length of focal axis is $2a = 2(3) = 6$ units

Conjugate Axis: Equation of conjugate axis is $X = 0 \Rightarrow x - 2 = 0$ and the length of conjugate axis is $2b = 2(2) = 4$ units.

Directrices: The equation of directrices of hyperbola are $X = \pm\frac{c}{a}$

$$\Rightarrow x - 2 = \pm\frac{\sqrt{13}}{3} = 2 \pm \frac{\sqrt{13}}{3}$$

Latus Rectum: The length of each latus rectum of hyperbola is $2\frac{b^2}{a} = 2\frac{2^2}{3} = \frac{8}{3}$ units

Example 24: Find the equation of hyperbola with centre at (2,4) and the length of semi transverse and conjugate axis are 3 and 5 units respectively. Also, the transverse axis is parallel to $x - axis$

Solution: Given that centre of the hyperbola is at (2,4) and length of semi transverse and conjugate axis are 3 and 5 respectively. So $a = 3$ and $b = 5$

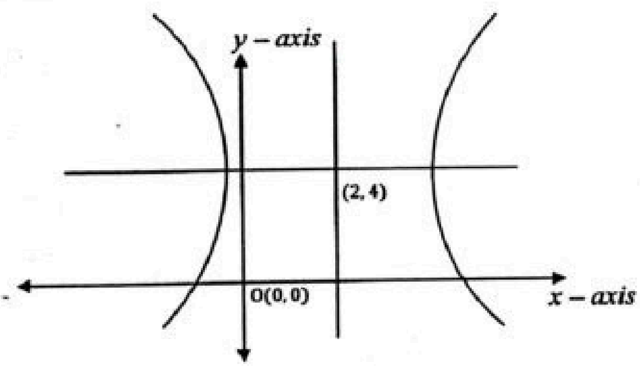
Since transverse axis is parallel to x-axis, so equation of hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Putting the values we have

$$\frac{(x-2)^2}{3^2} - \frac{(y-4)^2}{5^2} = 1$$

$$\Rightarrow \quad \frac{(x-2)^2}{9} - \frac{(y-4)^2}{25} = 1$$



Example 25: Find the equation of hyperbola with centre at $(0, 2)$ and it passes through the points $(1, 1)$ and $(4, 7)$.

Solution: Let the equation of hyperbola be $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$

Given that centre is at $(0, 2)$ thus equation is

$$\frac{(x-0)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{(y-2)^2}{b^2} = 1 \qquad (1)$$

Given that it passes through the point (1,1); so from Eq (1)

$$\frac{1^2}{a^2} - \frac{(1-2)^2}{b^2} = 1 \qquad \Rightarrow \frac{1}{a^2} - \frac{1}{b^2} = 1 \qquad ....(2)$$

Also, it passes through the point (4,7), so from Eq (1)

$$\frac{4^2}{a^2}-\frac{(7-2)^2}{b^2}=1$$

$$\frac{16}{a^2}-\frac{25}{b^2}=1 \qquad \text{........(3)}$$

Multiply Eq (2) by 16 and then subtract Eq (3) from it

$$\frac{16}{a^2}-\frac{16}{b^2}=16$$

$$\pm\frac{16}{a^2}\mp\frac{25}{b^2}=\mp 1$$

$$\frac{9}{b^2}=15$$

$b^2=\frac{9}{15}$ put in Eq (2)

$$\frac{1}{a^2}-\frac{1}{\left(\frac{9}{15}\right)}=1$$

$$\Rightarrow \frac{1}{a^2}-\frac{15}{9}=1 \quad \Rightarrow \frac{1}{a^2}=1+\frac{15}{9}$$

$$\Rightarrow \frac{1}{a^2}=\frac{24}{9} \qquad \Rightarrow a^2=\frac{9}{24}$$

putting the values in Eq (1)

$$\Rightarrow \frac{x^2}{\left(\frac{9}{24}\right)}-\frac{y^2}{\left(\frac{9}{15}\right)}=1$$

or

$$\Rightarrow \frac{24x^2}{9}-\frac{15y^2}{9}=1$$

is the required Equation of hyperbola.

Example 26: Find the equation of hyperbola with centre at origin with eccentricity $\frac{3}{2}$ and length of its latus rectum is 3.

Solution:

Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{........(1)}$$

Given that $e=\frac{3}{2}$ and $2\frac{b^2}{a}=3$ ....(2)

Since $c=ae \ \Rightarrow c=a\left(\frac{3}{2}\right)=\frac{3a}{2}$

also $\quad b^2=c^2-a^2=\left(\frac{3a}{2}\right)^2-a^2$

$$b^2=\frac{9a^2}{4}-a^2=\frac{5a^2}{4} \qquad \text{........(3)}$$

put in Eq (2)

$$\frac{2\left(\frac{5a^2}{4}\right)}{a}=3 \ \Rightarrow a=\frac{6}{5} \qquad \Rightarrow a^2=\frac{36}{25}$$

put in eq (3)

$$b^2=\frac{5}{4}\left(\frac{36}{25}\right)=\frac{9}{5}$$

put the values of a^2 and b^2 in Eq (1)

$$\Rightarrow \frac{x^2}{\left(\frac{36}{25}\right)}-\frac{y^2}{\left(\frac{9}{5}\right)}=1$$

or

$$\Rightarrow \frac{25x^2}{36}-\frac{5y^2}{9}=1$$

is the equation of hyperbola

Example 27: Find the equation of hyperbola in the standard form with centre at origin and eccentricity $\frac{4}{3}$; also, length of its semi conjugate axis is 3, which is along $y-axis$.

Solution:

Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{.......(1)}$$

Given that $e=\frac{4}{3}$ and $b=3$

$$\therefore b^2=c^2-a^2=(ae)^2-a^2=a^2(e^2-1)$$

$$=a^2\left(\frac{16}{9}-1\right)=\frac{7}{9}a^2$$

put the value of b

$$3^2=\frac{7}{9}a^2 \quad \Rightarrow a^2=\frac{81}{7}$$

put the values of a^2 and b^2 in Eq (1)

$$\Rightarrow \frac{x^2}{\left(\frac{81}{7}\right)}-\frac{y^2}{3^2}=1 \Rightarrow \frac{7x^2}{81}-\frac{y^2}{9}=1$$

is the required equation of hyperbola

Example 28: Find the equation of hyperbola with centre at origin and eccentricity $\sqrt{3}$. Its one focus is (3,0) and the directrix is $x=1$.

Solution: As the y coordinate of focus is zero; so its transverse axis is along $x-axis$. Equation of directrix is $x=1$ or $x-1=0$.

Take any point $P(x,y)$ on the hyperbola then by definition

$$e=\frac{\text{distance of point from focus}}{\text{distance of point from line}} \quad \text{........(1)}$$

distance of point P from focus $=\sqrt{(x-3)^2+(y-0)^2}$

$$=\sqrt{x^2-6x+9+y^2}=\sqrt{x^2+y^2-6x+9}$$

distance of point from directrix$=\frac{|x-1|}{\sqrt{1^2+0^2}}=|x-1|$

putting values in Eq (1)

$$\sqrt{3}=\frac{\sqrt{x^2+y^2-6x+9}}{|x-1|}$$

$$\Rightarrow \sqrt{3}|x-1|=\sqrt{x^2+y^2-6x+9}$$

squaring both sides

$$3(x-1)^2=x^2+y^2-6x+9$$

$$3x^2-6x+3=x^2+y^2-6x+9$$

$$\Rightarrow 2x^2-y^2=6$$

Dividing both sides by 6

$$\frac{x^2}{3}-\frac{y^2}{6}=1$$

is the required equation of hyperbola

Alternate Method

Given that $e=\sqrt{3}$ and focus is (3,0) $\Rightarrow c=3$

$$\Rightarrow ae=3 \quad \Rightarrow a\sqrt{3}=3 \Rightarrow a=\sqrt{3} \Rightarrow a^2=3$$

also $b^2=c^2-a^2$

$$\Rightarrow b^2=3^2-(\sqrt{3})^2=9-3=6$$

putting values in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\frac{x^2}{3}-\frac{y^2}{6}=1$ is the required equation of hyperbola.

Exercise 7.8

1. Find the centre, vertices, co-vertices, foci, eccentricity, length and equation of transverse axis, length and equation of conjugate axis, directrices and length of latus rectums for the given equations of hyperbola. Also sketch the hyperbola in each case.

   (i) $\frac{(x-5)^2}{36}-\frac{(y-4)^2}{81}=1$

   (ii) $\frac{(y-10)^2}{10}-\frac{(x-1)^2}{15}=1$

   (iii) $-16x^2+9y^2+32x+144y-16=0$

   (iv) $x^2-4y^2+2x-40y-135=0$

2. From the given information find the equation of hyperbola in each of the following.

   (i) Vertices: $(8,14),(8,-10)$; conjugate axis of length 6 units.

   (ii) Vertices $\left(-2,\frac{5}{2}\right),\left(-16,\frac{5}{2}\right)$ and end points of conjugate axis are $\left(-9,\frac{15}{2}\right);\left(-9,-\frac{5}{2}\right)$.

   (iii) Vertices are $(-5,1),(-5,-7)$ and foci are $\left(-5,-3+\sqrt{97}\right),\left(-5,-3-\sqrt{97}\right)$.

   (iv) Foci are $\left(8,-5+\sqrt{53}\right),\left(8,-5-\sqrt{53}\right)$ and end points of conjugate axis are $(15,5);(1,-5)$.

   (v) Vertices are $(0,9),(0,-9)$ and passes through the point $(8,15)$.

   (vi) Eccentricity is $\frac{5}{4}$; centre at $(1,1)$ and passes through $(-7,2)$, focal axis is horizontal.

   (vii) Focus $(5,3)$; directrix $y=-2$ and eccentricity $\frac{5}{3}$.

   (viii) Centre at $(0,0)$; length of latus rectum is 5, eccentricity $\frac{5}{4}$; conjugate axis along x-axis.

3. Find the vertices of the following hyperbolas by differentiating the given equation and solving for horizontal/vertical tangent lines.

   (i) $\frac{\left(x-\frac{1}{2}\right)^2}{3}-\frac{(y+3)^2}{5}=1$

   (ii) $\frac{(y-1)^2}{2^2}-\frac{(x+1)^2}{19}=1$

   (iii) $5x^2-5y^2+25x-5y+20=0$

   (iv) $-x^2+y^2-10x-4y-28=0$

7.13 Equation of Tangent and Normal to a Hyperbola

Consider the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ ....(1)

and the line $y=mx+c$ ....(2)

on solving Eqs (1) and (2) simultaneously we wil get the points of intersections of the line and hyperbola; for this use Eq (2) in (1)

$$\frac{(x-h)^2}{a^2}-\frac{(mx+c-k)^2}{b^2}=1$$

$$\Rightarrow b^2(x-h)^2-a^2(mx+(c-k))^2=a^2b^2$$

$$\Rightarrow b^2x^2-2b^2hx+b^2h^2-a^2m^2x^2-2a^2m(c-k)x-a^2(c-k)^2-a^2b^2=0$$

$$\Rightarrow (b^2-a^2m^2)x^2-2(b^2h+a^2m(c-k))x+(b^2h^2-a^2(c-k)^2-a^2b^2)=0$$

which is quadratic equation in 'x'.

If discriminant of this equation is greater than zero or positive the equation will have two distinct values of x so line will cut the hyperbola at two distinct points. If discriminant of this equation is negative then the roots of the equation will be imaginary i.e. there is no point of intersection and if value of discriminant is zero then equation has repeated roots i.e. there is only one point of intersection. In this case the line will be tangent to the hyperbola, so condition for the line to be tangent to hyperbola is

$$Disc=0$$

$$\Rightarrow \left(2\left(b^2h+a^2m(c-k)\right)\right)^2-4(b^2-a^2m^2)(b^2h^2-a^2b^2-a^2(c-k)^2)=0$$

$$\Rightarrow 4\left(b^2h+a^2m(c-k)\right)^2-4(b^2-a^2m^2)(b^2h^2-a^2b^2-a^2(c-k)^2)=0$$

Dividing both sides by '4'

$$\left(b^2h+a^2m(c-k)\right)^2-(b^2-a^2m^2)(b^2h^2-a^2b^2-a^2(c-k)^2)=0$$

$$\Rightarrow b^4h^2+a^4m^4(c-k)^2+2a^2b^2mh(c-k)^2-b^4h^2+a^2b^4+a^2b^2(c-k)^2+a^2b^2m^2h^2-a^4b^2m^2-a^4m^2(c-k)^2=0$$

$$\Rightarrow a^2b^2[2mh(c-k)^2+b^2+(c-k)^2+m^2h^2-a^2m^2]=0$$

Dividing both sides by a^2b^2

$$\Rightarrow (2mh+1)(c-k)^2+b^2+m^2h^2-a^2m^2=0$$

is condition for tangency.

In particular when centre of the hyperbola is at (0,0); then $h=0, k=0$;

In this case condition of tangency is

$$(2m(0)+1)(c-0)^2+b^2+m^2(0)^2-a^2m^2=0$$

$$c^2+b^2-a^2m^2=0$$

$$c^2=a^2m^2-b^2$$

from here we have $c=\pm\sqrt{a^2m^2-b^2}$ provided that $a^2m^2-b^2\geq 0$.

Put in $y=mx+c$

$$\boxed{y=mx\pm\sqrt{a^2m^2-b^2}}$$

are the equations of the tangents to the hyperbola with slope m.

Example 29: Find the equations of tangents to the hyperbola $\frac{x^2}{3}-\frac{y^2}{2}=1$ with slope $\frac{7}{3}$.

Solution:

Equation of hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$

here $a^2=3$; $b^2=2$ and the slope is given to be $\frac{7}{3}$ i.e. $m=\frac{7}{3}$

Equation of tangents to the hyperbola are

$y=mx\pm\sqrt{a^2m^2-b^2}$

put the values

$$y=\frac{7}{3}x\pm\sqrt{3\left(\frac{7}{3}\right)^2-2}$$

$y=\frac{7}{3}x\pm\frac{1}{3}\sqrt{129}$ or $3y=7x\pm\sqrt{129}$ are the required equations of the tangent lines.

Example 30: Check whether the line $x-y-1=0$ is the tangent to the hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$

Solution:

The equation of hyperbola is $\frac{x^2}{4}-\frac{y^2}{3}=1$ or $3x^2-4y^2=12$ ....(1)

Equation of line is $x-y-1=0$

$\Rightarrow y=x-1$ put in Eq (1)

$\Rightarrow 3x^2-4(x-1)^2=12$ $\quad\Rightarrow 3x^2-4x^2+8x-4=12$

or $\Rightarrow x^2-8x+16=0$

compute its discriminant

$Disc=(-8)^2-4(1)(16)=64-64=0$

this shows that the line is tangent to the hyperbola.

7.14 Equation of Tangent and Normal to the Hyperbola at a Given Point

Consider the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ and point $P(x_1,y_1)$ be any point on the hyperbola.

Equation of hyperbola may be written as

$b^2(x-h)^2-a^2(y-k)^2=a^2b^2$ ....(1)

Differentiate w.r.t. 'x'

$2b^2(x-h)-2a^2(y-k)\frac{dy}{dx}=0$ $\quad\Rightarrow\frac{dy}{dx}=\frac{b^2}{a^2}\left(\frac{x-h}{y-k}\right)$

at point $P(x_1,y_1)$

$\Rightarrow\frac{dy}{dx}=\frac{b^2}{a^2}\left(\frac{x_1-h}{y_1-k}\right)$ is the slope of the tangent line. So, equation of the tangent line is

$$y-y_1=\frac{b^2}{a^2}\left(\frac{x_1-h}{y_1-k}\right)(x-x_1)$$

$\Rightarrow a^2(y_1-k)(y-y_1)=b^2(x_1-h)(x-x_1)$

$\Rightarrow b^2(x_1-h)x-a^2(y_1-k)y-b^2(x_1-h)x_1+a^2(y_1-k)y_1=0$ ....(2)

Since the point $P(x_1,y_1)$ lies on the hyperbola so from Eq (1)

$b^2(x_1-h)^2-a^2(y_1-k)^2=a^2b^2$

$\Rightarrow b^2(x_1-h)(x_1-h)-a^2(y_1-k)(y_1-k)=a^2b^2$

$\Rightarrow b^2x_1(x_1-h)-b^2h(x_1-h)-a^2y_1(y_1-k)+a^2k(y_1-k)=a^2b^2$

$\Rightarrow -b^2x_1(x_1-h)-a^2y_1(y_1-k)=a^2k(y_1-k)-b^2h(x_1-h)-a^2b^2$

put in Eq (2)

$b^2(x_1-h)x-a^2(y_1-k)y+a^2k(y_1-k)-b^2h(x_1-h)-a^2b^2=0$

is equation of tangent line at point $P(x_1,y_1)$.

In particular if centre of the hyperbola is at (0,0) then $h=0,k=0$ so equation of tangent line reduces to

$b^2(x_1-0)x-a^2(y_1-0)y+a^2(0)(y_1-0)-b^2(0)(x_1-0)-a^2b^2=0$

$\Rightarrow b^2x_1x-a^2y_1y-a^2b^2=0$

or

$b^2x_1x-a^2y_1y=a^2b^2$ dividing both sides by a^2b^2

$\frac{x_1x}{a^2}-\frac{yy_1}{b^2}=1$

Now we find the equation of the normal line.

Since normal line is perpendicular to tangent line, thus

$$slope\ of\ normal\ line=-\frac{1}{slope\ of\ tangent\ line}$$

$$=-\frac{1}{\frac{b^2}{a^2}\left(\frac{x_1-h}{y_1-k}\right)}=-\frac{a^2(y_1-k)}{b^2(x_1-h)}$$

the equation of the normal line at point P is

$y-y_1=-\frac{a^2(y_1-k)}{b^2(x_1-h)}(x-x_1)$

$\Rightarrow b^2(x_1-h)(y-y_1)=-a^2(y_1-k)(x-x_1)$

or

$a^2(y_1-k)(x-x_1)+b^2(x_1-h)(y-y_1)=0$

is equation of normal line.

In particular if centre of the hyperbola is at (0,0) then $h=0,k=0$ so equation of normal line in this case is

$a^2(y_1-0)(x-x_1)+b^2(x_1-0)(y-y_1)=0$

$\Rightarrow a^2y_1(x-x_1)+b^2x_1(y-y_1)=0$

$\Rightarrow a^2y_1x-a^2x_1y_1+b^2x_1y-b^2x_1y_1=0$

$\Rightarrow a^2y_1x+b^2x_1y-(a^2+b^2)x_1y_1=0$

Example 31: Find the equation of tangent and normal to the hyperbola $4x^2 - 9y^2 - 16x - 18y - 29 = 0$ at the point $(5, -1)$.

Solution: Equation of the hyperbola is $4x^2 - 9y^2 - 16x - 18y - 29 = 0$

Differentiating w.r.t. 'x'

$$8x - 18y\frac{dy}{dx} - 16 - 18\frac{dy}{dx} = 0$$

$$\Rightarrow -18(y+1)\frac{dy}{dx} + 8(x-2) = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{8(x-2)}{18(y+1)} = \frac{4(x-2)}{9(y+1)}$$

at point $(5, -1)$

$$\Rightarrow \frac{dy}{dx} = \frac{4(5-2)}{9(-1+1)} = \infty \text{ undefined}$$

it means tangent line is vertical; so its equation is $x = constant$

$\Rightarrow x = 5$ or $x - 5 = 0$

$$Slope\ of\ normal\ line = -\frac{1}{Slope\ of\ tangent\ line} = -\frac{1}{\infty} = 0$$

it means normal line is horizontal; so its equation is $y = constant$

$\Rightarrow y = -1$ or $y + 1 = 0$

Exercise 7.9

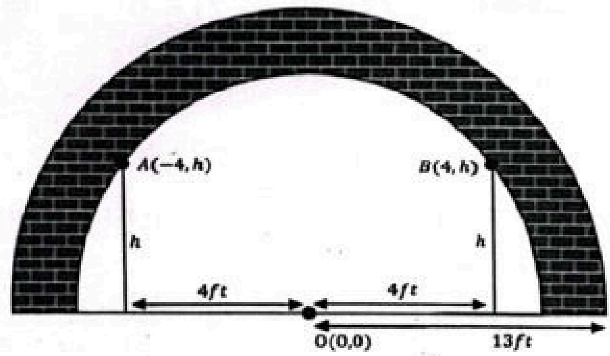
1. Find the length of chord intercepted by the line $2x + y - 1 = 0$ and the hyperbola $2x^2 - 3y^2 + 7x - 4y + 13 = 0$.
2. Find the value of m so that the line $3x + 4y + m = 0$ is tangent to the hyperbola $x^2 - y^2 - 7x - 2y + 13 = 0$.
3. Find the equation of tangent to the hyperbola $x^2 - 2y^2 + 4x - 6y + 11 = 0$ which is parallel to the line $4x - 8y + 7 = 0$.
4. Find the equations of tangent to the hyperbola $2x^2 - 4y^2 + 6x - 8y - 7 = 0$ which is perpendicular to the line $x + 2y + 5 = 0$. Also find the point of tangency.
5. Find the equations of the tangent and normal to the hyperbola $6x^2 - 6y^2 - 14x + 21y - 5 = 0$ at the point $(2, 3)$.
6. Find the equations of the tangents and normal to the hyperbola $\frac{\left(y-\frac{7}{4}\right)^2}{\frac{25}{48}} - \frac{\left(x-\frac{7}{6}\right)^2}{\frac{25}{72}} = 1$ at the point whose ordinate is 3 and abscissa is an integer.
7. Prove that the tangent at any point on the hyperbola makes equal angles with the lines joining the point with the foci of the hyperbola.
8. The line $12x - 5\sqrt{11} - 50 = 0$ is tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$. Find the point where the tangent line intersects the focal axis of hyperbola. Also find the ratio in which this point of intersection divides the distance between the foci.

7.15 Application of Conic Sections

Conic sections are counted as one of the prominent topics in Geometry and possess numerous applications in science and technology, including astronomy, optics, and even architecture. Here we are discussing some of them.

Example 32: A train track is $8ft$ wide and goes through a semi-circular tunnel of radius $13ft$. How high is the tunnel at edge of the track?

Solution:



Let the centre of the tunnel be at origin.

Given that radius of tunnel is $13ft$. So equation of the circle is $(x-0)^2+(y-0)^2=13^2$

$\Rightarrow x^2+y^2=169$ ....(1)

The track is $8ft$ wide. Let $h\ ft$ be the height of the tunnel at the edges of the track so coordinates of the A are $(-4,h)$ and B are $(4,h)$.

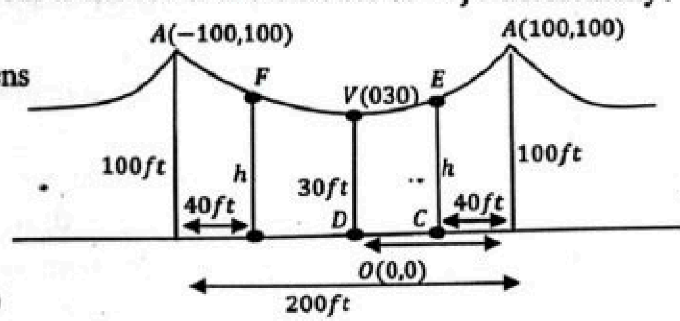
Since the point $B(4,0)$ (also the point $A(-4,0)$) lies on the circle, so from Eq (1)

$4^2+h^2=169$

$\Rightarrow h^2=169-16 \quad \Rightarrow h^2=153 \quad$ or $\quad h=\sqrt{153}$

Example 33: The cable of a suspension bridge hangs in a shape of parabola. The tower supporting the cable are $200ft$ apart and $100ft$ in height. If the lowest point of the cable is $20ft$ above the bridge at its mid point. How high is the cable from either tower at a distance of $40ft$ horizontally?

Solution:



The vertex of the parabola is $(0,30)$ and it opens upwards, so equation of parabola is:

$(x-h)^2=4a(y-k)$

or

$(x-0)^2=4a(y-30)$

$\Rightarrow x^2=4a(y-30)$ ... (1)

The coordinates of the point A are $(-100,100)$ and B are $(100,100)$. As the point $B(100,100)$ (also, the point $A(-100,100)$) lies on parabola

So, from Eq (1)

$(100)^2=4a(100-30)$

$\Rightarrow 10000=280a \Rightarrow a=\frac{10000}{280}=\frac{1000}{28}$

Put in Eq (1).

$\Rightarrow x^2=4\left(\frac{1000}{28}\right)(y-30) \quad \Rightarrow x^2=\frac{1000}{7}(y-30)$ ... (2)

The point which is at a distance of $40ft$ from tower is at a distance of $100 - 40 = 60ft$ from origin. Let h be the height of the cable at this point C.

The coordinates of E are $(60, h)$ and the point lies on the parabola, so from Eq (2):

$$(60)^2 = \frac{1000}{7}(h - 30)$$

$$\Rightarrow (h - 30) = 3600 \times \frac{1000}{7} = 25.2 \qquad \Rightarrow h = 30 + 25.2 = 55.2ft$$

7.16 Use of Conic Section

Example 34: The head light of an automobile is in the shape of parabola. The bulb is placed at the focus which is 2 *inch* from its vertex. The depth of the head light is 3 *inches*. What is the width of the head light at its opening?

Solution:

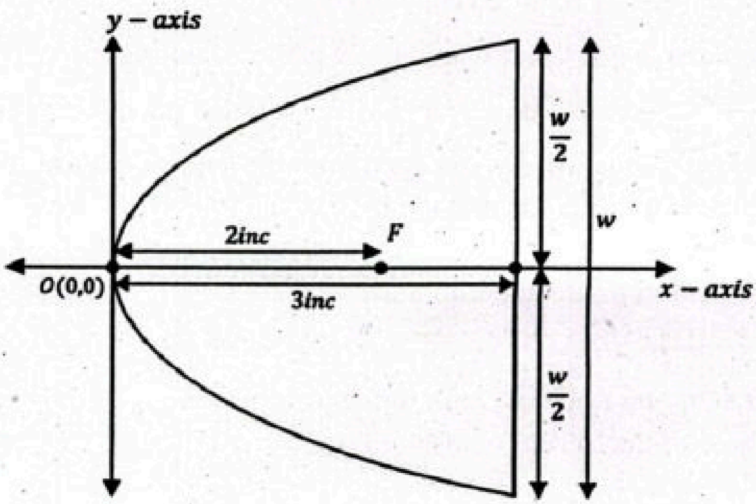
Consider the vertex of the parabola is at origin and it opens on the right side. Given that focus is at 2*inches* from the vertex; so

$(a, 0) = (2,0) \quad \Rightarrow a = 2$

As we know equation of this parabola is $y^2 = 4ax$ so in this case

$y^2 = 8x \qquad \ldots (1)$

Let 'w' be the width of the parabola, so coordinates of the points A and B are $A(3, -\frac{w}{2})$ and $B(3, \frac{w}{2})$.



Since the point $B(3, \frac{w}{2})$ (also point $A(3, -\frac{w}{2})$) lies on the parabola, so from Eq (1)

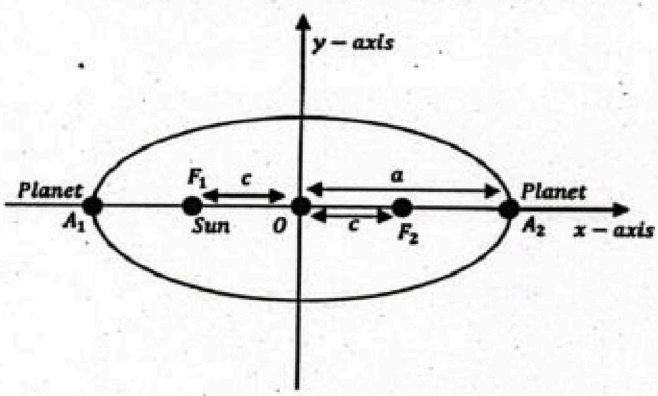
$\frac{w}{2} = 8(3) \quad \Rightarrow w = 48\ inches$

Example 35: According to Kepler's law planets have elliptical orbits with Sun at one of the foci. If the longest distance of the planet from the sun is 4.4 billion km then write the equation of the orbit of the planet taking its centre at origin.

Solution:

Consider the ellipse with centre at origin and its major axis is along $x - axis$. Thus, its equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (1)$

Let the Sun be at its focus F_1. The planet will be at maximum distance from Sun when it is at A_2 and its distance from Sun is minimum when it is at A_1.



Given that $|F_1A_2| = 7.4 \Rightarrow a + c = 7.4$ ... (2)
and $|F_1A_1| = 4.4 \Rightarrow a - c = 4.4$ ... (3)
Adding Eq (2) and Eq (3)
$2a = 11.8 \Rightarrow a = 5.9$
Subtract Eq (3) from Eq (2)
$2c = 3 \Rightarrow c = 1.5$
since $b^2 = a^2 - c^2 = (5.9)^2 - (1.5)^2 \Rightarrow b^2 = 32.56 \Rightarrow b = 5.7$
put the value in Eq (1)

$$\frac{x^2}{(5.9)^2} + \frac{y^2}{(5.7)^2} = 1$$

Which is the required equation of the orbit.

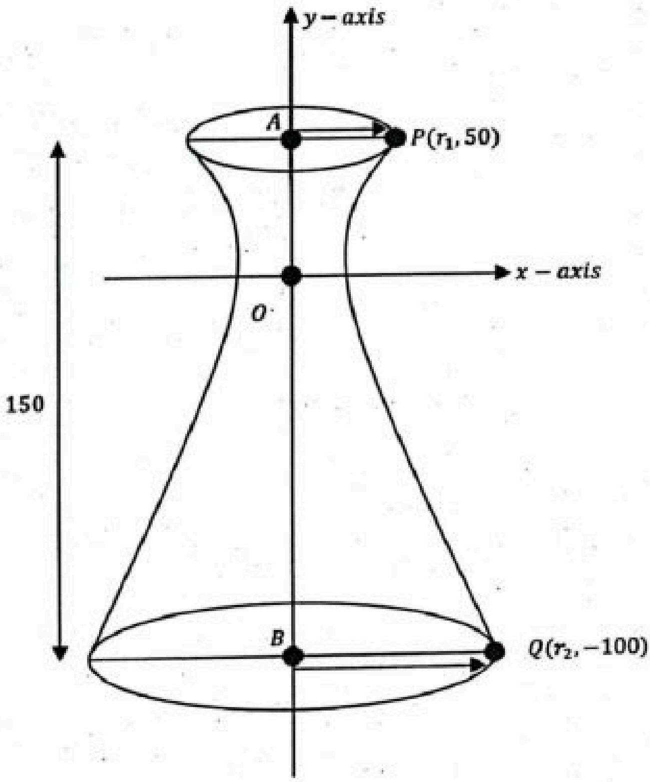
Example 36: Cross section of a nuclear cooling tower is in the shape of hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is $150m$ tall and the distance from the top of the tower to the centre of hyperbola is half the distance from the base of the tower to the centre of hyperbola. Find the diameter of the top and the base of the tower.

Solution:

Equation of the hyperbola is

$$\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1 \quad \dots (1)$$

Let r_1 and r_2 be the radii of the top and the base of the tower.



Given that

$$|OA| = \frac{1}{2}|OB| \Rightarrow 2|OA| = |OB|$$

also $\Rightarrow |OA| + |OB| = |AB|$

$\Rightarrow |OA| + 2|OA| = 150 \Rightarrow |OA| = 50$

$|OB| = 2|OA| = 2(50) = 100$

$\Rightarrow |OB| = 100$

Thus, coordinates of the points P and Q are $(r_1, 50)$ and $(r_2, -100)$. Since the point $P(r_1, 50)$ lies on the hyperbola so from Eq (1)

$$\frac{r_1^2}{30^2} - \frac{(50)^2}{44^2} = 1$$

$$\Rightarrow \frac{r_1^2}{900} - \frac{(50)^2}{44^2} = 1 \Rightarrow \frac{r_1^2}{900} = 1 + \left(\frac{25}{22}\right)^2$$

$$\Rightarrow \frac{r_1^2}{900} = \frac{22^2 + 25^2}{22^2}$$

Taking square root

$$\Rightarrow \frac{r_1}{30} = \frac{\sqrt{22^2+25^2}}{22} = \frac{33.30}{22} \quad \Rightarrow r_1 = 30\left(\frac{33.30}{22}\right) = 45.41$$

The diameter of top of tower $= 2r_1 = 2(45.41)$
$= 90.82m$

Also, the point $Q(r_2, -100)$ lies on the hyperbola,
so, from Eq (1)

$$\frac{r_2^2}{30^2} - \frac{(-100)^2}{44^2} = 1 \quad \Rightarrow \frac{r_2^2}{900} - \frac{(-100)^2}{44^2} = 1 \quad \Rightarrow \frac{r_2^2}{900} = 1 + \left(\frac{25}{11}\right)^2$$

$$\Rightarrow \frac{r_2^2}{900} = \frac{11^2+25^2}{11^2}$$

Taking square root

$$\Rightarrow \frac{r_2}{30} = \frac{\sqrt{11^2+25^2}}{11} = \frac{27.31}{11} \quad \Rightarrow r_2 = 30\left(\frac{27.31}{11}\right) = 74.49$$

The diameter of base of tower $= 2r_2 = 2(74.49) = 148.98m$

Exercise 7.10

1. A cell phone company has installed three signal towers A, B and C at different locations to provide service to their customers. Tower A is located at the position $(-2, 10)$ and covers an area up to $8km$, tower B is located at $(-3, 5)$ and covers an area up to $6km$ and tower C is located at $(10, -1)$ and covers an area up to $5km$. A man is standing at a position of $(12, 3)$, which tower will provide service to the man?
2. The tyre of a car is of radius 15 inches and it revolves 1000 times in a minute. What is the speed of the car in $\frac{km}{h}$?
3. A batsman hits the ball. The ball attains the maximum height of $25m$ and drops on the ground at a distance of $80m$ from the batsman. Assuming the origin at the position of batsman write the equation of path of the ball. Is it possible for a man of height $1.6m$ to catch the ball, standing $70m$ away from the batsman?
4. Playing team coaches use parabolic shaped antennas to listen the conversation between the players in the ground. If the antenna has the cross section of 18 inches and depth of 4 inches then where the microphone should be placed to hear the conversation?
5. Man A is standing at one focus of the whispering gallery which is 16 feet from the nearest wall. The man B is standing at the other focus $100ft$ away. What is the length of the whispering gallery? How high is the elliptical ceiling at the centre?
6. The orbit of a planet is in elliptical shape with Sun setting at one of its two foci. The eccentricity of the ellipse is 0.085 and the minimum distance of the planet from the Sun is 105 million miles. What is the maximum distance of the planet from Sun?

7. The control towers are located at points Q$(-500, 0)$ and R$(500, 0)$ on a straight shore where the x-axis runs through. At same moment both towers send a radio signal to a ship out at sea, each travelling at $300m/\mu s$. The ship received the signal from Q, $3\mu s$ earlier the message from R. Find the equation of hyperbola containing the possible location of the ship.

8. An architect's design for a building includes some large pillars with cross sections in the shape of hyperbolas. The curve can be modelled by the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ where the units are in meters. If the pillars are $4.2m$ tall. Find the width of the top of each pillar. Also find the width at the middle of the pillar.

Review Exercise

1. Choose the correct option.

(i) The eccentricity is the ratio of distance of a point on the conic section from:
(a) Focus to directrix (b) Directrix to focus (c) Vertex to directrix (d) Directrix to vertex

(ii) Eccentricity of circle is:
(a) $e > 1$ (b) $e < 1$ (c) $e = 1$ (d) $e = 0$

(iii) The focus of the parabola $x^2 = -16y$ is:
(a) $(4, 0)$ (b) $(-4, 0)$ (c) $(0, 4)$ (d) $(0, -4)$

(iv) Length of the latus rectum of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is:
(a) $\frac{32}{9}$ (b) $\frac{9}{32}$ (c) $\frac{9}{2}$ (d) $\frac{8}{9}$

(v) Equation of the directrices of ellipse $\frac{x^2}{16} + \frac{y^2}{36} = 1$ is:
(a) $y = \pm\frac{18}{\sqrt{5}}$ (b) $y = \pm\frac{\sqrt{5}}{18}$ (c) $x = \frac{18}{\sqrt{5}}$ (d) $x = \frac{\sqrt{5}}{18}$

(vi) Eccentricity of the hyperbola $\frac{x^2}{25} - \frac{y^2}{81} = 1$ is:
(a) $\frac{5}{\sqrt{106}}$ (b) $\frac{\sqrt{106}}{5}$ (c) $\frac{\sqrt{106}}{9}$ (d) $\frac{9}{\sqrt{106}}$

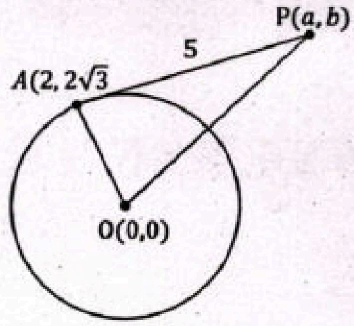
(vii) Equation of conjugate axis of the hyperbola $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{12} = 1$ is:
(a) $x = 1$ (b) $x = -1$ (c) $y = 3$ (d) $y = -3$

(viii) Length of the tangent drawn from the point $(1, 2)$ to the circle $2x^2 + 2y^2 + 3x + 2y - 6 = 0$ is
(a) 11 (b) $\sqrt{11}$ (c) $\frac{11}{2}$ (d) $\sqrt{\frac{11}{2}}$

(ix) The chord joining the two points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is focal chord if:
(a) $t_1 + t_2 = 1$ (b) $t_1 + t_2 = -1$ (c) $t_1 t_2 = 1$ (d) $t_1 t_2 = -1$

(x) Exactly one tangent can be drawn to a circle if point lies;
(a) outside the circle (b) on the circle (c) inside circle (d) centre of circle

2. A point $P(a, b)$ lies outside a circle with centre at origin. From P a tangent is drawn to the circle at point $A(2, 2\sqrt{3})$. If the area of the triangle POA is 10 sq units then find equation of the circle and the coordinates of the point P.



3. On the Siri Nagar highway overhead steel bridges are constructed to cross the road. They are supported by the parabolic arc over them. The highway is $300ft$ wide and walkway is $15ft$ above the highway. The centre of the parabolic arc is $10ft$ high from the walkway. Find the equation of the parabolic arc take origin at the mid of the road.

4. If r_1 and r_2 are the minimum and maximum distance from a focus of an ellipse then prove that semi-major axis is the arithmetic mean between r_1 and r_2 and semi-minor axis is the geometric mean between r_1 and r_2.

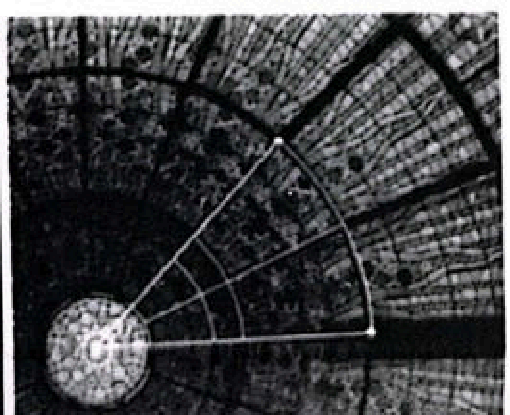
5. A hyperbolic mirror has the property that the light directed at the focus will be reflected to the other focus. If one of the foci has coordinates $(24, 0)$ and the top mount point of mirror has coordinates $(24, 24)$. Find the vertex of the mirror.

INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS

After studying this unit, students will be able to:

- Find domain and range of the principal trigonometric functions and inverse trigonometric functions.
- Draw graphs of the inverse trigonometric functions of cosine, sine, tangent, secant, cosecant and cotangent within the domain from -2π to 2π.
- State, prove and apply the addition and subtraction formulae of inverse trigonometric functions.
- Apply concepts of inverse trigonometric functions to real life word problems, such as mechanical engineering, architecture to find height of the building, angle of elevation and depression, identifying the angle of bridges to build scale models.

Inverse Trigonometric Functions play a crucial role in mathematics, particularly in calculus, geometry, and engineering, by enabling the determination of angles from known trigonometric function values. These functions are also essential in fields such as physics, engineering, and computer graphics. They are widely used to solve equations involving trigonometric functions, analyze periodic phenomena, and develop algorithms for 3D rendering and modeling. A solid understanding of these functions and their properties allows us to address complex problems in both theoretical and applied mathematics, making them a powerful tool in any mathematician's toolkit.

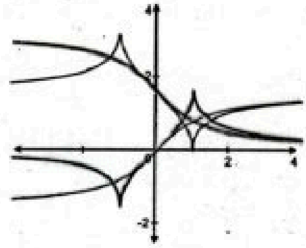


8.1 The Inverse Trigonometric Functions

Inverse trigonometric functions are the inverse operations of basic trigonometric functions cosine, sine, tangent, secant, cosecant, and cotangent.

The primary inverse trigonometric functions include:

- $Arccos$ or Cos^{-1} the inverse of cosine function.
- $Arcsin$ or Sin^{-1} the inverse of sine function.
- $Arctan$ or Tan^{-1} the inverse of tangent function.
- $Arcsec$ or Sec^{-1} the inverse of secant function.
- $Arccsc$ or Csc^{-1} the inverse of cosecant function.
- $Arccot$ or Cot^{-1} the inverse of cotangent function.



Overlay of all six trigonometric function on one graph

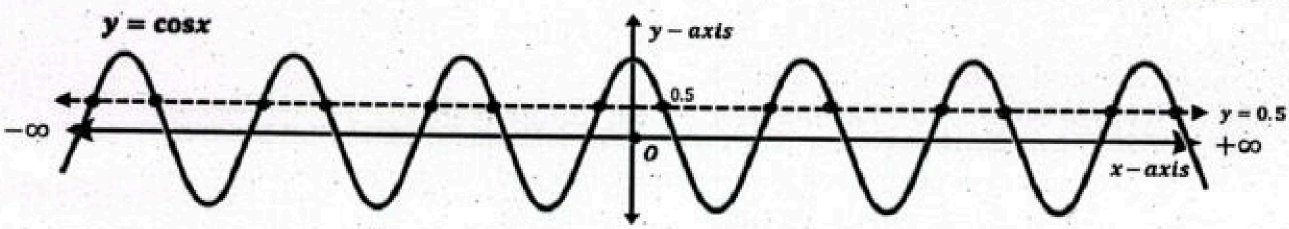
These functions are used to find the angle whose trigonometric function value is given. For example, if we know that the cosine of an angle is 0.5, we can use the arccosine function to find the angle which is 60° or $\frac{\pi}{3}$ radians.

8.1.1 The Principal Cosine Function

Consider the following graph of cosine function. You will observe that an infinite number of x values are found for which $y = \cos x = 0.5$ for each $x \in (-\infty, +\infty)$.

By this, one can record infinitely many solutions to the equation $\cos x = k$ where $k \in [-1, 1]$.

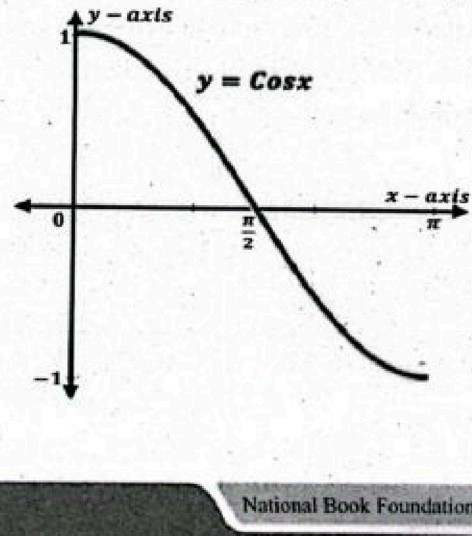
Note that Principal and Inverse Trigonometric functions are written in capital letters; e.g., cosine function:
$y = \cos x$ with $x \epsilon (-\infty, +\infty)$, $y \epsilon (-\infty, +\infty)$
Principal Cosine function:
$y = \text{Cos}\, x$ with $x \epsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \epsilon [-1, 1]$
Cosine inverse function:
$y = \text{Cos}^{-1} x$ with $x \epsilon [-1, 1]$, $y \epsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



If we need an interval for x where there is only one solution to $\cos x = k$ for $k \in [-1, 1]$, we choose the interval from 0 to π. We have an option to consider an interval $x \in [\pi, 2\pi]$ or many others. It is a mathematical convention to choose $x \in [0, \pi]$.

In the interval $[0, \pi]$, we can find a **unique solution** to the equation $\text{Cos}\, x = k$, where $k \in [-1, 1]$.
We write this solution as $x = \text{Cos}^{-1} k$.
In other words, "x is a real number in the interval $[0, \pi]$ whose cosine is k".

The cosine function defined on $x \in [0, \pi]$ for which there is only **one solution** of the equation $\text{Cos}\, x = k$ where $k \in [-1, 1]$ is called the **Principal Cosine Function.**



Example 1: Find the principal value of $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$

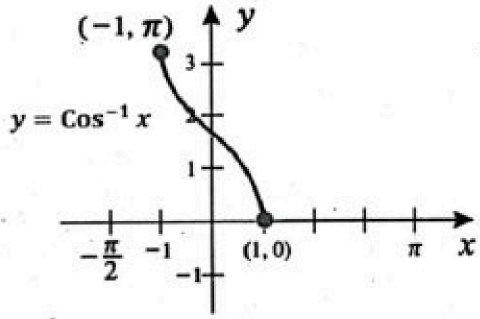
Solution: Let $y = \text{Cos}^{-1}\left(-\frac{1}{2}\right)$ then $\text{Cos}\, y = -\frac{1}{2}$, where $y \in [0, \pi]$

Consider $\text{Cos}\, y = -\frac{1}{2}$ [We need to find y whose cosine value is $-\frac{1}{2}$]

$\Rightarrow \text{Cos}\, y = \text{Cos}\left(\frac{2\pi}{3}\right)$ Since $\text{Cos}\, y < 0 \Rightarrow y$ lies in Quad II.

$\Rightarrow y = \frac{2\pi}{3} \Rightarrow \text{Cos}^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

8.1.2 The Inverse Cosine Function



For the cosine function $y = \text{Cos}\, x$ where $x \in [0, \pi]$ and $y \in [-1, 1]$, we define an inverse cosine function $y = \text{Cos}^{-1}(x)$ where $x \in [-1, 1]$ and $y \in [0, \pi]$.
For $x \in [-1, 1]$, $y = \text{Cos}^{-1} x$ is a real number in the interval $[0, \pi]$ whose cosine is x.
In view of above, we observe that:

$\text{Cos}^{-1}(1) = 0$ since $\text{Cos}(0) = 1$

$\text{Cos}^{-1}(0) = \frac{\pi}{2}$ since $\text{Cos}\left(\frac{\pi}{2}\right) = 0$

$\text{Cos}^{-1}(-1) = \pi$ since $\text{Cos}\, \pi = -1$

8.1.3 The Domain and Range of Inverse Cosine Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

> If $y = \text{Cos}\, x$ with $x \in [0, \pi]; y \in [-1,1]$ then $y = \text{Cos}^{-1}(x)$ with $x \in [-1, 1]; y \in [0, \pi]$

For the cosine function $y = \text{Cos}\, x$

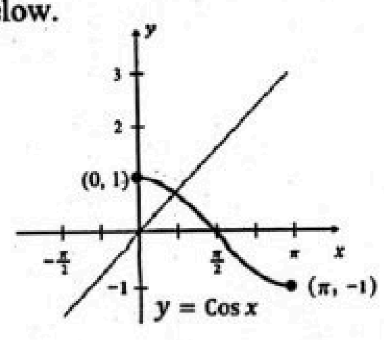
$$\textit{Domain} = [0, \pi] \text{ and } \textit{Range} = [-1,1]$$

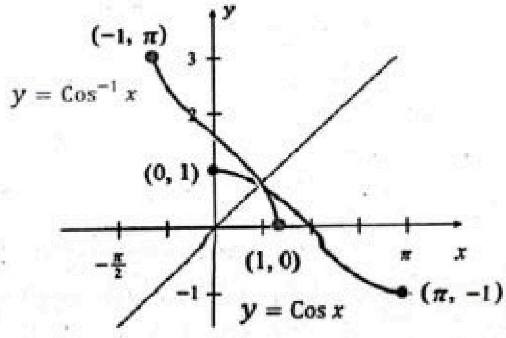
For the inverse cosine function $y = \text{Cos}^{-1} x$

$$\textit{Domain} = [-1, 1] \text{ and } \textit{Range} = [0, \pi]$$

> The inverse trigonometric function's domain is a real number and its range is a real angle measured in degree or radian.

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.
Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown below.





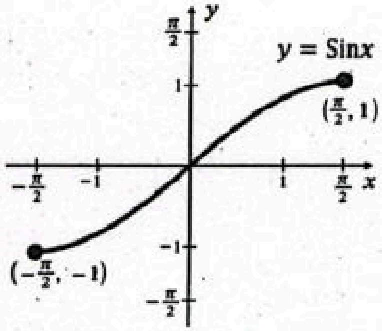
8.1.4 The Principal Sine Function

To define principal sine function, we need an interval for x where there is only one solution to $\sin x = k$ for $k \in [-1, 1]$. Such a solution is possible in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can find a **unique solution** to the equation $\text{Sin}\, x = k$, where $k \in [-1, 1]$. We write this solution as $x = \text{Sin}^{-1} k$.

In other words, "x is a real number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine value is k".

The sine function defined on $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which there is only **one solution** of the equation $\text{Sin}\, x = k$ where $k \in [-1, 1]$ is called the **Principal Sine Function**.



$\text{Sin}x = \sin x$; $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

- Sin^{-1} and **sin** are not inverses of each other (They do not cancel each other)
- **sin** does not have an inverse
- The functions that are inverses are Sin^{-1} and **Sin.**

Example 2: Find the principal value of $\text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution: Let $y = \text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ then $\text{Sin}\, y = -\frac{1}{\sqrt{2}}$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Consider $\text{Sin}\, y = -\frac{1}{\sqrt{2}}$ [We need to find y whose sine value is $-\frac{1}{\sqrt{2}}$.]

$\Rightarrow \text{Sin}\, y = \text{Sin}\left(-\frac{\pi}{4}\right)$ Since $\text{Sin}y < 0 \Rightarrow y$ lies in Quad-IV.

$\Rightarrow y = -\frac{\pi}{4}$

$\Rightarrow \text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

Check Point

Find the principal value of

(i) $\text{Sin}^{-1}\left(\frac{1}{2}\right)$

(ii) $\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$

8.1.5 The Inverse Sine Function

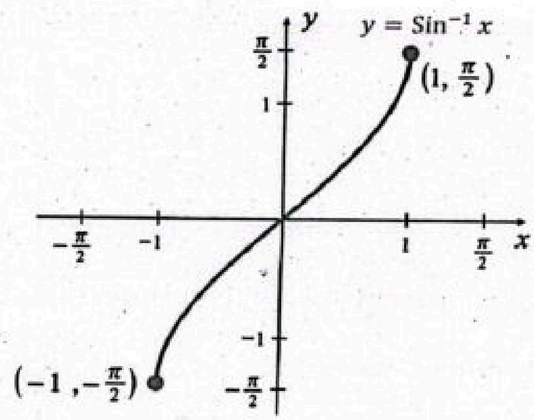
For the sine function $y = \text{Sin}\, x$ where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$ we define an inverse sine function $y = \text{Sin}^{-1}(x)$ where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

In view of above, we observe that:

$\text{Sin}^{-1}(-1) = -\frac{\pi}{2}$ since $\text{Sin}\left(-\frac{\pi}{2}\right) = -1$

$\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ since $\text{Sin}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\text{Sin}^{-1}(1) = \frac{\pi}{2}$ since $\text{Sin}\left(\frac{\pi}{2}\right) = 1$



8.1.6 The Domain and Range of Inverse Sine Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the sine function $y = \text{Sin}\, x$

$Domain = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $Range = [-1,1]$

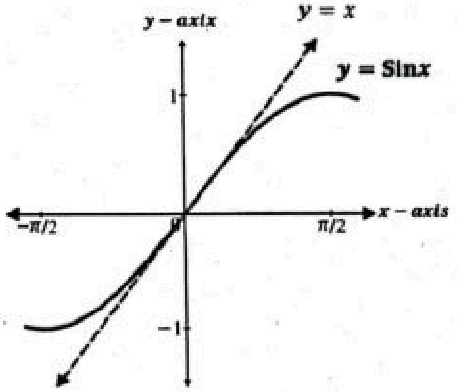
For the inverse sine function $y = \text{Sin}^{-1} x$

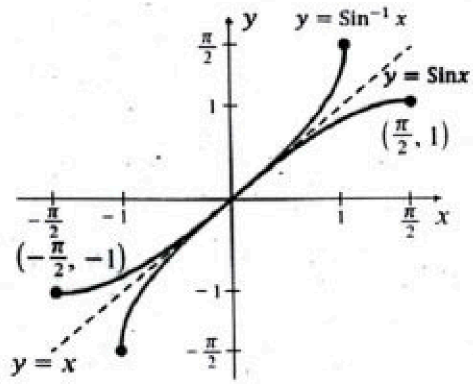
$Domain = [-1, 1]$ and $Range = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

> If $y = \text{Sin}\, x$ with $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; y \in [-1,1]$
> Then $y = \text{Sin}^{-1}(x)$ with
> $x \in [-1, 1]; y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown below.





Key Facts

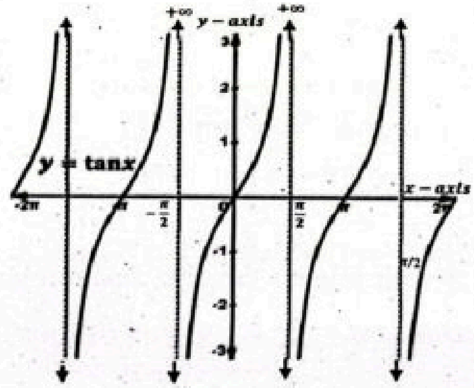
- The expression $\text{Sin}^{-1} x$ is not the same as $\frac{1}{\text{Sin}x}$. In other words, -1 is not an exponent. It simply means the inverse function.
- Remember that there will be no solution if k lies outside the interval $[-1, 1]$
- $\textbf{Cos}x = \textbf{cos}x\ ;\ x \in [0, \pi]$
- $\textbf{Cos}^{-1}$ and $\textbf{cos}$ are not inverses of each other. (They do not cancel each other)
- **cos** does not have an inverse The functions that are inverses are: $\textbf{Cos}^{-1}$ and **Cos**

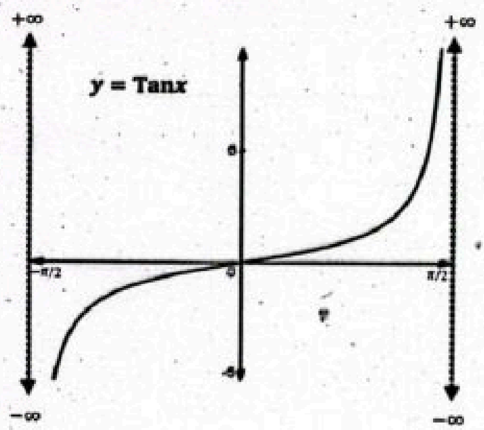
8.1.7 The Principal Tangent Function

We know that an infinite number of x values can be found from $y = \tan x = 1$ for each $x \in R - (2n+1)\frac{\pi}{2}$ where $n \in Z$. By this, one can record infinitely many solutions to the equation $\tan x = k$ where $k \in (-\infty, +\infty)$. If we need an interval for x where there is only one solution to $\tan x = k$ for $k \in (-\infty, +\infty)$, we choose the interval between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

In the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can find a **unique solution** to the equation $\text{Tan}x = k$, where $k \in (-\infty, \infty)$.

We write this solution as $x = \text{Tan}^{-1} k$. In other words, "x is a real number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent value is k". The tangent function defined on $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which there is only **one solution** of the equation $\tan x = k$ where $k \in (-\infty, \infty)$ is called the **Principal Tangent Function.**





$\text{Tan}x = \tan x\,;\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Tan^{-1} and **tan** are not inverses of each other
(They do not cancel each other)

tan does not have an inverse

The functions that are inverses are Tan^{-1} and **Tan**

Example 3: Find the principal value of $\text{Tan}^{-1}(\sqrt{3})$.

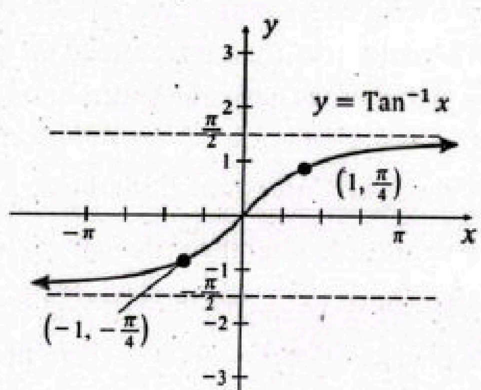
Solution: Let $y = \text{Tan}^{-1}(\sqrt{3})$ if and only if $\text{Tan}\,y = \sqrt{3}$, where $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Consider $\text{Tan}\,y = \sqrt{3}$ [We need to find y whose tangent value is $\sqrt{3}$.]

$\Rightarrow \text{Tan}\,y = \text{Tan}\left(\frac{\pi}{3}\right)$ Since $\tan y > 0 \Rightarrow y$ lies in Quad I.

$\Rightarrow y = \frac{\pi}{3} \quad \Rightarrow \text{Tan}^{-1}(\sqrt{3}) = \frac{\pi}{3}$

8.1.8 The Inverse Tangent Function



For the tangent function $y = \text{Tan}\,x$ where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $y \in (-\infty, +\infty)$, we define an inverse tangent function $y = \text{Tan}^{-1}(x)$ where $x \in (-\infty, +\infty)$ and $y(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
In view of above, we observe that:

$\text{Tan}^{-1}(-\infty) = -\frac{\pi}{2}$ since $\text{Tan}\left(-\frac{\pi}{2}\right) = \infty$

$\text{Tan}^{-1}(1) = \frac{\pi}{4}$ since $\text{Tan}\left(\frac{\pi}{4}\right) = 1$

$\text{Tan}^{-1}(0) = 0$ since $\text{Tan}(0) = 0$

$\text{Tan}^{-1}(\infty) = \frac{\pi}{2}$ since $\text{Tan}\left(\frac{\pi}{2}\right) = \infty$

Check Point

Find the principal value of $\text{Tan}^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

8.1.9 The Domain and Range of Inverse Tangent Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the tangent function $y = \text{Tan}\, x$

Domain $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and *Range* $= (-\infty, \infty)$

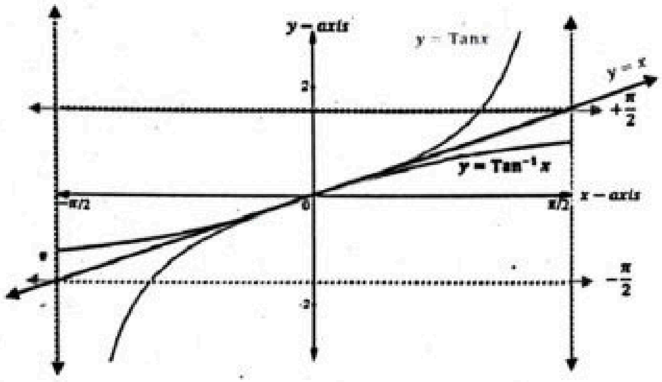
For the inverse tangent function $y = \text{Tan}^{-1} x$

Domain $= (-\infty, \infty)$ and *Range* $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

If $y = \text{Tan}\, x$ with
$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); y \in (-\infty, \infty)$
then
$y = \text{Tan}^{-1}(x)$ with
$x \in (-\infty, \infty); y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in figure.

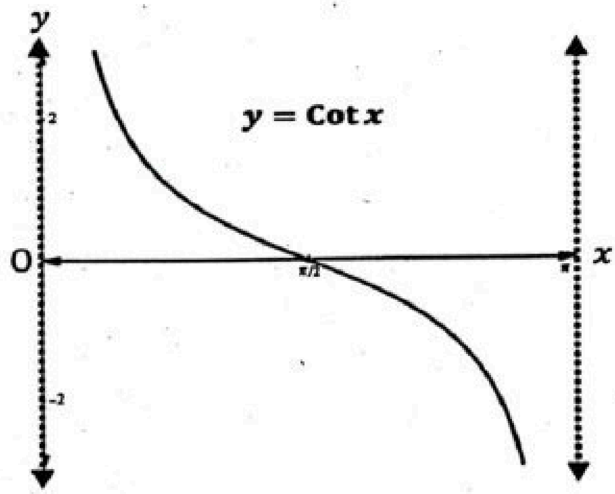


8.1.10 The Principal Cotangent Function

To define principal cotangent function, we need an interval for x where there is only one solution to $\cot x = k$ for $k \in (-\infty, +\infty)$. Such a solution is possible in the interval between 0 and π. In the interval $(0, \pi)$, we can find a **unique solution** to the equation $\text{Cot}\, x = k$, where $k \in (-\infty, \infty)$.

We write this solution as $x = \text{Cot}^{-1} k$.

The cotangent function defined on $x \in (0, \pi)$ for which there is only **one solution** of the equation $\text{Cot}\, x = k$ where $k \in (-\infty, \infty)$ is called the **Principal Cotangent Function.**



Example 4: Find the principal value of $\text{Cot}^{-1}(-\sqrt{3})$

Solution: Let $y = \text{Cot}^{-1}(-\sqrt{3})$ if and only if $\text{Cot}\, y = -\sqrt{3}$, where $y \in (0, \pi)$.

Consider $\text{Cot}\, y = -\sqrt{3}$ [We need to find y whose cotangent value is $-\sqrt{3}$.]

$\text{Tan}\, y = -\frac{1}{\sqrt{3}}$ Since, $\text{Tan}\, y < 0 \Rightarrow \text{Cot} y < 0 \Rightarrow y$ lies in Quad II.

$\Rightarrow \text{Tan}\, y = \text{Tan}\left(-\frac{\pi}{3}\right)$

$\Rightarrow y = -\frac{\pi}{3} \quad \Rightarrow \text{Cot}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Check Point

Find the principal value of $\text{Cot}^{-1}(1)$.

8.1.11 The Inverse Cotangent Function

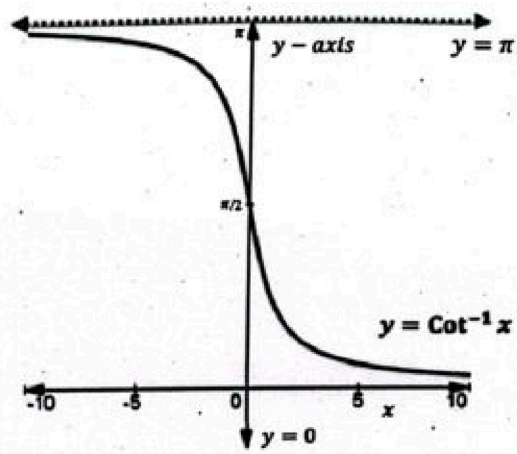
For the cotangent function $y = \text{Cot}\,x$ where $x \in (0, \pi)$ and $f(x) \in (-\infty, +\infty)$, we define an inverse cotangent function $y = \text{Cot}^{-1}(x)$ where $x \in (-\infty, +\infty)$ and $y \in (0, \pi)$.

In view of above, we observe that:

$\text{Cot}^{-1}(\infty) = 0$ since $\text{Cot}(0) = \infty$

$\text{Cot}^{-1}(0) = \frac{\pi}{2}$ since $\text{Cot}\left(\frac{\pi}{2}\right) = 0$

$\text{Cot}^{-1}(-\infty) = \pi$ since $\text{Cot}(\pi) = -\infty$



8.1.12 The Domain and Range of Inverse Cotangent Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the cotangent function $y = \text{Cot}\,x$

$Domain = (0, \pi)$ and $Range = (-\infty, \infty)$

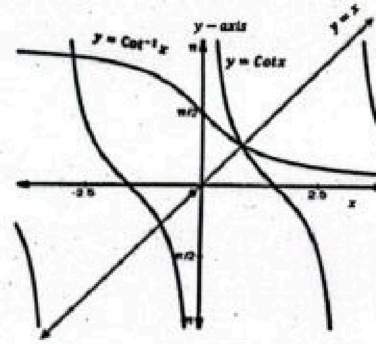
For the inverse tangent function $y = \text{Cot}^{-1}x$

$Domain = (-\infty, \infty)$ and $Range = (0, \pi)$

If $y = \text{Cot}\,x$ with $x \in (0, \pi); y \in (-\infty, \infty)$ then $y = \text{Cot}^{-1}(x)$ with $x \in (-\infty, \infty); y \in (0, \pi)$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in figure.

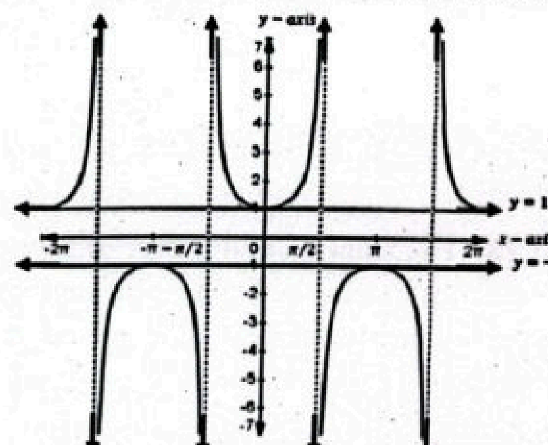


8.1.13 The Principal Secant Function

Consider the following graph of secant function and observe that an infinite number of x values found for which $y = \sec x = 1$ for each $x \in R - (2n+1)\frac{\pi}{2}$ where $n \in Z$.

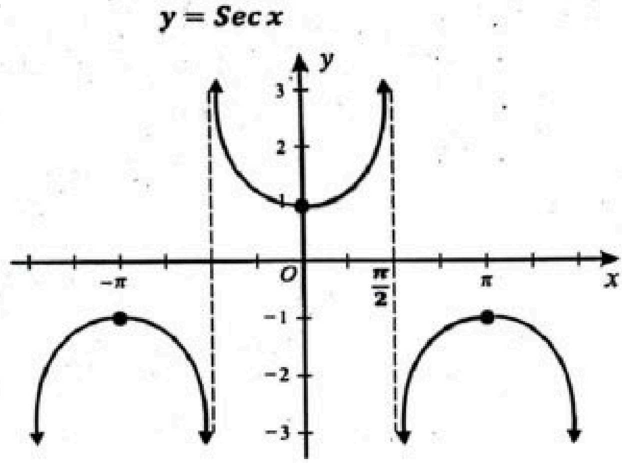
By this, one can record infinitely many solutions to the equation $\sec x = k$ where $k \in R - (-1, 1)$.

If we need an interval for x where there is only one solution to $\sec x = k$ for $k \in R - (-1, 1)$, we choose the interval from 0 to π except $\frac{\pi}{2}$. We have an option to consider an interval $x \in (\pi, 2\pi), x \neq \frac{3\pi}{2}$ or many others. It is a mathematical convention to choose $x \in [0, \pi], x \neq \frac{\pi}{2}$.



$y = \sec x$

In the interval $x \in [0, \pi], x \neq \frac{\pi}{2}$, we can find a **unique solution** to the equation $\text{Sec}\, x = k$, where $k \in R - (-1, 1)$. We write this solution as $x = \text{Sec}^{-1} k$. In other words, "x is a real number in the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ whose secant value is k". The secant function defined on $x \in [0, \pi], x \neq \frac{\pi}{2}$ for which there is only **one solution** of the equation $\sec x = k$ where, $k \in R - (-1, 1)$ is called the **Principal Secant Function.**



Example 5: Find the principal value of $\text{Sec}^{-1}(-2)$.

Solution: Let $y = \text{Sec}^{-1}(-2)$ if and only if $\text{Sec}\, y = (-2)$, where $y \in [0, \pi], y \notin \frac{\pi}{2}$.

Consider $\text{Sec}\, y = (-2)$ [We need to find y whose secant value is (-2).]

$\Rightarrow \text{Cos}\, y = -\frac{1}{2}$ Since $\text{Cos}\, y < 0 \Rightarrow \text{Sec} y < 0 \Rightarrow y$ lies in Quad II.

$\Rightarrow \text{Cos}\, y = \text{Cos}\left(\frac{2\pi}{3}\right)$

$\Rightarrow y = \frac{2\pi}{3}$

$\Rightarrow \text{Sec}^{-1}(-2) = \frac{2\pi}{3}$

Check Point

Find the principal value of $\text{Sec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

8.1.14 The Inverse Secant Function

For the secant function $y = \text{Sec}\, x$

where $x \in [0, \pi], x \neq \frac{\pi}{2}$ and $y \in R - (-1, 1)$,

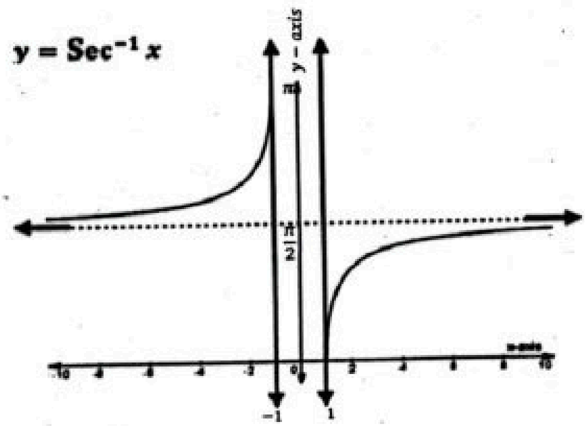
we define an inverse secant function $y = \text{Sec}^{-1}(x)$

where $x \in R - (-1, 1)$ and $y \in [0, \pi], x \neq \frac{\pi}{2}$.

In view of above, we observe that:

$\text{Sec}^{-1}(1) = 0$ since $\text{Sec}(0) = 1$

$\text{Sec}^{-1}(-1) = \pi$ since $\text{Sec}(\pi) = -1$



8.1.15 The Domain and Range of Inverse Secant Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function. For the secant function $y = \text{Sec}\, x$

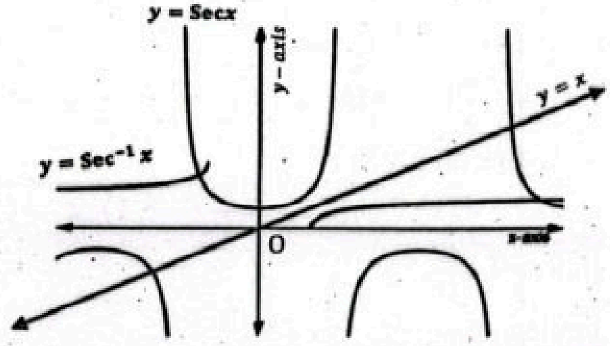
$$Domain = x \in [0, \pi], x \neq \frac{\pi}{2} \text{ and } Range = R - (-1, 1)$$

For the inverse secant function $y = \text{Sec}^{-1} x$

$$Domain = R - (-1, 1) \text{ and } Range = x \in [0, \pi], x \neq \frac{\pi}{2}$$

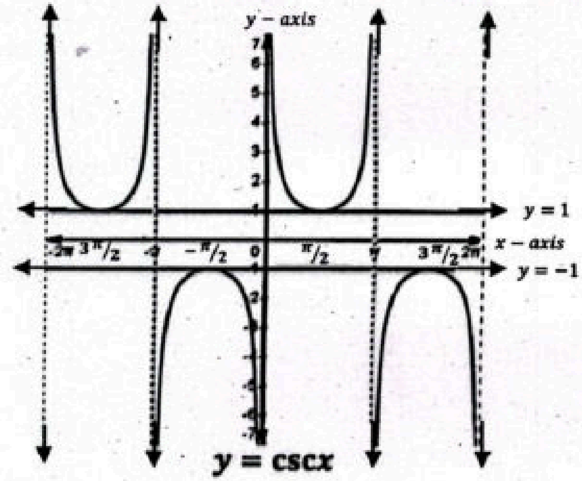
Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in the adjoining figure.



8.1.16 The Principal Cosecant Function

The graph of cosecant function from -2π to $+2\pi$ is shown in the adjoining figure.

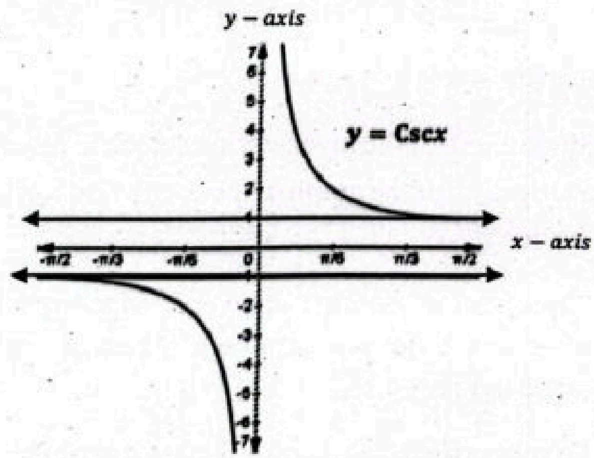


From the graph it is clear that $\csc x = k$ for $k \in R - (-1, 1)$, has many solutions.

To define principal cosecant function, we need an interval for x where there is only one solution to $\csc x = k$ for $k \in R - (-1, 1)$.

Such a solution is possible in the interval from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ except 0. In the interval $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $x \neq 0$, we can find a **unique solution** to the equation $\text{Csc}\, x = k$, where $k \in R - (-1, 1)$.

We write this solution as $x = \text{Csc}^{-1} k$. In other words, "x is a real number in the interval $\left[-\frac{\pi}{2}\ 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant value is k". The cosecant function defined on $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$ for which there is only **one solution** of the equation $\text{Csc}\, x = k$ where $k \in R - (-1, 1)$ is called the **Principal Cosecant Function.**



Example 6: Find the principal value of $\text{Csc}^{-1}(2)$.

Solution: Let $y = \text{Csc}^{-1}(2)$ if and only if $\text{Csc}\, y = (2)$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \notin 0$.

Consider $\text{Csc}\, y = (2)$ [We need to find y whose cosecant value is (2).]

$\Rightarrow \text{Sin}\, y = \frac{1}{2}$ Since, $\text{Sin} y > 0 \Rightarrow \text{Csc}\, y > 0 \Rightarrow y$ lies in Quad I.

$\Rightarrow \text{Sin}\, y = \text{Sin}\left(\frac{\pi}{6}\right)$

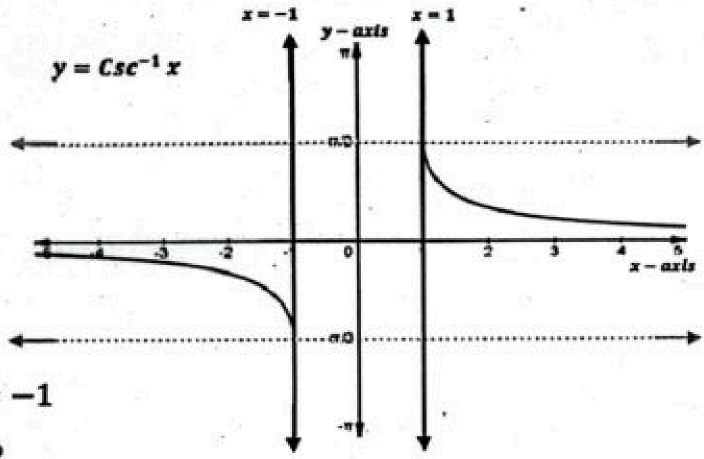
$\Rightarrow y = \frac{\pi}{6}$

$\Rightarrow \text{Csc}^{-1}(2) = \frac{\pi}{6}$

Check Point

Find the principal value of $\text{Csc}^{-1}(-\sqrt{2})$.

8.1.17 The Inverse Cosecant Function



For the cosecant function $y = \text{Csc}\,x$ where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$ and $f(x) \in R - (-1, 1)$, we define an inverse cosecant function $y = \text{Csc}^{-1}(x)$ where $x \in R - (-1,1), x \neq 0$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

In view of above, we observe that:

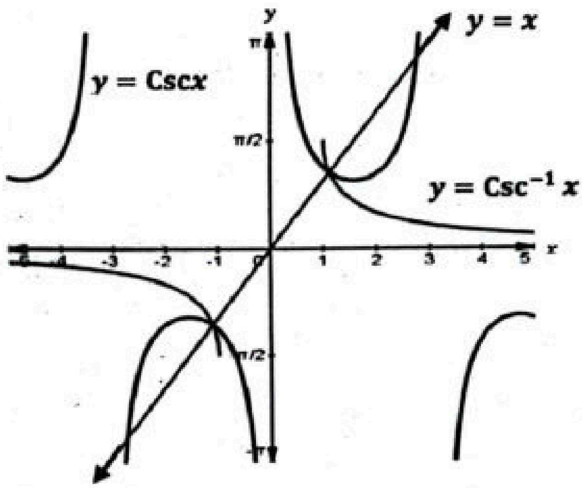
$\text{Csc}^{-1}(-1) = -\frac{\pi}{2}$ since $\text{Csc}\left(-\frac{\pi}{2}\right) = -1$

$\text{Csc}^{-1}(\infty) = 0$ since $\text{Csc}(0) = \infty$

$\text{Csc}^{-1}(\sqrt{2}) = \frac{\pi}{4}$ since $\text{Csc}\left(\frac{\pi}{4}\right) = \sqrt{2}$

8.1.18 The Domain and Range of Inverse Cosecant Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.



For the cosecant function $y = \text{Csc}\,x$

$Domain = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$ and

$Range = R - (-1, 1)$

For the inverse cosecant function

$y = \text{Csc}^{-1}x$

$Domain = R - (-1, 1)$ and

$Range = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in the figure.

Properties of Inverse Trigonometric Functions

Property I

- $\text{Cos}^{-1}(\text{Cos}\,x) = x$ for $x \in [0, \pi]$ and $\text{Sin}^{-1}(\text{Sin}\,x) = x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\text{Tan}^{-1}(\text{Tan}\,x) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\text{Cot}^{-1}(\text{Cot}\,x) = x$ for $x \in (0, \pi)$
- $\text{Sec}^{-1}(\text{Sec}\,x) = x$ for $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
- $\text{Csc}^{-1}(\text{Csc}\,x) = x$ for $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Property II

- $\text{Cos}(\text{Cos}^{-1}x) = x$ for $x \in [-1,1]$ and $\text{Sin}(\text{Sin}^{-1}x) = x$ for $x \in [-1,1]$
- $\text{Tan}(\text{Tan}^{-1}x) = x$ for $x \in (-\infty,+\infty)$ and $\text{Cot}(\text{Cot}^{-1}x) = x$ for $x \in (-\infty,+\infty)$
- $\text{Sec}(\text{Sec}^{-1}x) = x$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Csc}(\text{Csc}^{-1}x) = x$ for $x \in (-\infty,-1] \cup [1,\infty)$

Property III

- $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x$ for $x \in [-1,1]$
- $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$ for $x \in [-1,1]$
- $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x)$ for $x \in (-\infty,+\infty)$
- $\text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}(x)$ for $x \in (-\infty,+\infty)$
- $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Csc}^{-1}(-x) = -\text{Csc}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$

Property IV

- $\text{Sin}^{-1}\left(\frac{1}{x}\right) = \text{Csc}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Sec}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Tan}^{-1}\left(\frac{1}{x}\right) = \begin{cases} \text{Cot}^{-1}(x) & \text{if } x > 0 \\ -\pi + \text{Cot}^{-1}(x) & \text{if } x < 0 \end{cases}$

Property V

- $\text{Sin}^{-1}(x) + \text{Cos}^{-1}(x) = \frac{\pi}{2}$ for $x \in [-1,1]$
- $\text{Tan}^{-1}(x) + \text{Cot}^{-1}(x) = \frac{\pi}{2}$ for $x \in (-\infty,+\infty)$
- $\text{Sec}^{-1}(-x) + \text{Csc}^{-1}(-x) = \frac{\pi}{2}$ for $x \in (-\infty,-1] \cup [1,\infty)$

Property VI

- $\text{Sin}^{-1}(x) = \text{Cos}^{-1}(\sqrt{1-x^2}) = \text{Tan}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \text{Cot}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{Sec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{Csc}^{-1}\left(\frac{1}{x}\right)$
- $\text{Cos}^{-1}(x) = \text{Sin}^{-1}(\sqrt{1-x^2}) = \text{Tan}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{Cot}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \text{Sec}^{-1}\left(\frac{1}{x}\right) = \text{Csc}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$
- $\text{Tan}^{-1}(x) = \text{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \text{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \text{Cot}^{-1}\left(\frac{1}{x}\right) = \text{Sec}^{-1}(\sqrt{1+x^2}) = \text{Csc}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

Exercise 8.1

1. Find the principal values of each of the following without using a calculator.

i. $\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ ii. $\text{Sin}^{-1}(1)$ iii. $\text{Tan}^{-1}(\sqrt{3})$

iv. $\text{Cot}^{-1}\left(\frac{\sqrt{3}}{3}\right)$ v. $\text{Sec}^{-1}\left(\frac{2\sqrt{3}}{3}\right)$ vi. $\text{Csc}^{-1}(-\sqrt{2})$

vii. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$ viii. $\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$ ix. $\text{Csc}^{-1}(-2)$

2. Find sum of principal values of the following inverse trigonometric expressions without using calculator.

   i. $\text{Tan}^{-1}(1) + \text{Cos}^{-1}\left(-\frac{1}{2}\right) + \text{Sin}^{-1}\left(-\frac{1}{2}\right)$ ii. $\text{Cos}^{-1}\left(\frac{1}{2}\right) + 2\,\text{Sin}^{-1}\left(\frac{1}{2}\right)$

   iii. $\text{Tan}^{-1}(\sqrt{3}) - \text{Sec}^{-1}(-2)$ iv. $\text{Cot}^{-1}(-\sqrt{3}) + \text{Csc}^{-1}(-2) - \text{Cos}^{-1}\left(\frac{\sqrt{2}}{2}\right)$

3. Find the exact real number value of each without using a calculator.

   i. $\text{Cos}^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right]$ ii. $\text{Sin}^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$ iii. $\text{Cos}^{-1}\left(\sin\frac{11\pi}{6}\right)$

   iv. $\text{Cos}^{-1}\left(\cos\frac{\pi}{6}\right)$ v. $\text{Sin}\left(\text{Tan}^{-1}\frac{3}{4}\right)$ vi. $\text{Cos}\left(2\,\text{Sin}^{-1}\frac{\sqrt{2}}{2}\right)$

   vii. $\text{Sin}\left[2\,\text{Sin}^{-1}\left(\frac{4}{5}\right)\right]$ viii. $\text{Cos}\left(\text{Sin}^{-1}\frac{5}{13}\right)$ ix. $\text{Sin}\left[\text{Cos}^{-1}\left(-\frac{3}{5}\right)\right]$

   x. $\text{Sin}\left[\text{Sin}^{-1}\frac{2}{3} + \text{Cos}^{-1}\frac{1}{2}\right]$ xi. $\text{Cos}\left(\text{Sin}^{-1}\frac{3}{4} + \text{Cos}^{-1}\frac{5}{13}\right)$

   xii. $\text{Cos}[\text{Sec}^{-1}(3) + \text{Tan}^{-1}(2)]$

4. Find the unknown angles and use a calculator to evaluate the following as real numbers to three decimal places:

   i. $\text{Cos}^{-1}\left(\frac{3}{5}\right) = \text{Sin}^{-1}(\ )$ ii. $\text{Sin}^{-1}\left(\frac{2}{3}\right) = \text{Cos}^{-1}(\ )$ iii. $\text{Sin}^{-1}\left(-\frac{1}{\sqrt{5}}\right) = -\text{Cos}^{-1}(\ )$

   iv. $\text{Tan}^{-1}\left(\frac{1}{4}\right) = \text{Cos}^{-1}(\ )$ v. $\text{Tan}^{-1}(-1.2) = -\text{Cos}^{-1}(\ )$

   vi. $\text{Cot}^{-1}\left(-\frac{3}{4}\right) = -\text{Sin}^{-1}(\ )$ vii. $\text{Sec}^{-1}(2.041) = \text{Tan}^{-1}(\ )$

   viii. $\text{Sec}^{-1}(-\sqrt{5}) = \text{Cot}^{-1}(\ )$ ix. $\text{Csc}^{-1}(1.172) = \text{Sin}^{-1}(\ )$

   x. $\text{Csc}^{-1}\left(-\frac{5}{3}\right) = \text{Tan}^{-1}(\ )$

8.2 Graphs of Inverse Trigonometric Functions

Drawing graphs of trigonometric functions is a fundamental skill. It allows the problem solver to interpret and understand the relationship visually between two variable quantities. Graphs provide an intuitive way to analyze the behavior, identify the trend and solve problems.

Graphing becomes valuable in case of inverse trigonometric functions, where the restricted domain and range were not enough to understand the problem.

Plotting these graphs, we gain insights the characteristics, such as symmetry, continuity, and asymptotic behavior. Whether it is engineering or navigation, the ability to draw accurately and interpret graphs of inverse trigonometric functions is essential for solving real-world problems involving angles and distances.

Following table helps in drawing graphs of the inverse trigonometric functions.

Inverse cosine function	Inverse sine function	Inverse tangent function	Inverse secant function	Inverse cosecant function	Inverse cotangent function
Domain $[-1, 1]$	Domain $[-1, 1]$	Domain R	Domain $(-\infty, -1] \cup [1, \infty)$	Domain $(-\infty, -1] \cup [1, \infty)$	Domain R
Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Range $[0, \pi]$	Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Range $[0, \pi] - \left\{\frac{\pi}{2}\right\}$	Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	Range $(0, \pi)$
non periodic function	non periodic function	non periodic function	non periodic function	non periodic function	non periodic function
odd function	neither even nor odd function	odd function	odd function	neither even nor odd function	neither even nor odd function
strictly increasing function	strictly decreasing function	strictly increasing function	strictly decreasing function with respect to its domain	strictly decreasing function with respect to its domain	strictly decreasing function
one to one function	one to one function	one to one function	one to one function	one to one function	one to one function
graphically $\text{Cos}^{-1}(x)$ is a reflection of $\text{Cos}(x)$ across the line $y = x$	graphically $\text{Sin}^{-1}(x)$ is a reflection of $\text{Sin}(x)$ across the line $y = x$	graphically $\text{Tan}^{-1}(x)$ is a reflection of $\text{Tan}(x)$ across the line $y = x$	graphically $\text{Sec}^{-1}(x)$ is a reflection of $\text{Sec}(x)$ across the line $y = x$	graphically $\text{Csc}^{-1}(x)$ is a reflection of $\text{Csc}(x)$ across the line $y = x$	graphically $\text{Cot}^{-1}(x)$ is a reflection of $\text{Cot}(x)$ across the line $y = x$
		S-shaped curve approaches but never reaches the horizontal asymptotes $y = \pm\frac{\pi}{2}$	Smooth curve approaches but never reaches the vertical asymptote $y = \frac{\pi}{2}$	Smooth curve approaches but never reaches the vertical asymptote $y = 0$	S-shaped curve approaches but never reaches the horizontal asymptotes $y = 0$ and $y = \pi$

key points include $(-1,\pi)$, $\left(0,\frac{\pi}{2}\right)$, $(1,0)$.	key points include $\left(-1,\frac{\pi}{2}\right)$, $(0,0)$, $\left(1,\frac{\pi}{2}\right)$	key points include $\left(-1,-\frac{\pi}{4}\right)$, $(0,0)$, $\left(1,\frac{\pi}{4}\right)$			
Adding or subtracting constants affects the graphs horizontally or vertically (horizontal and vertical shifts).					
Multiplying by constants to stretch or compress the graph (stretching and compressing).					
Multiplying by negative values shows reflection of the graphs across axes (Reflections).					

8.2.1 Graph of $y = \text{Cos}^{-1} x$

The graph of the function $y = \text{Cos}^{-1} x$ represents the angle y whose cosine is x. The domain of this function is $x \in [-1,1]$, and the range is $y \in [0,\pi]$. This means the output values y will always be within this interval, corresponding to angles in radians from 0 to π.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from -1 to 1 and y-axis with values from 0 to π.
- Choose an appropriate scale.
- Mark the points $(1,0)$, $\left(0.9,\frac{\pi}{6}\right)$, $\left(0.5,\frac{\pi}{3}\right)$, $\left(0,\frac{\pi}{2}\right)$, $\left(-0.5,\frac{2\pi}{3}\right)$, $\left(-0.9,\frac{5\pi}{6}\right)$, $(-1,\pi)$ on the graph.
- Connect these points with a smooth curve.

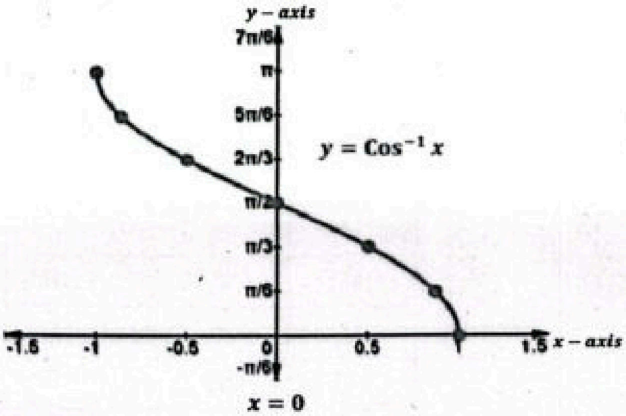
Properties:

- **Domain and Range:** The domain is $[-1,1]$ and the range is $[0,\pi]$.
- **Key Points:** $(1,0)$, $\left(0,\frac{\pi}{2}\right)$, $(-1,\pi)$
- **Appearance:** The graph appears as a smooth curve starting from $(1,0)$, decreasing through the key points, passes through the point $(1,0)$ and ending at $(-1,\pi)$.
- **Monotonicity:** The graph is monotonically decreasing smoothly (it always decreases as x increases).
- **Neither even nor odd function**
- **Reflection:** The graph of $y = \text{Cos}^{-1}x$, is a reflection of $y = \text{Cos}\, x$ across the line $y = x$.
- This means that the x and y coordinates of the points on the graph of $y = \text{Cos}\, x$ are switched to obtain the points on the graph of $y = \text{Cos}^{-1}x$
- **Reflection Symmetry:** $\begin{bmatrix} \text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x \text{ for } x \in [-1,1] \\ \left(\text{Symmetric about the line } y = \frac{\pi}{2}\right) \end{bmatrix}$

  Symmetry along the range $[0,\pi]$ reflects across the vertical axis at $x = 0$.
- **Inverse Function:** $\text{Cos}(\text{Cos}^{-1}x) = x$ for $x \in [-1,1]$ and

  $\text{Cos}^{-1}(\text{Cos}\, x) = x$ for $x \in [0,\pi]$

- **Identities:** $\text{Cos}^{-1}x = \text{Sin}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [0,1]$ and
  $\text{Cos}^{-1}x = \pi - \text{Sin}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [-1,0)$



x	$y = \text{Cos}^{-1}x$
1	0 or 0°
0.9	$\pi/6$ or 30°
0.5	$\pi/3$ or 60°
0	$\pi/2$ or 90°
−0.5	$2\pi/3$ or 120°
−0.9	$5\pi/6$ or 150°
−1	π or 180°

8.2.2 Graph of $y = \text{Sin}^{-1}x$

The graph of the function $y = \text{Sin}^{-1}x$ represents the angle y whose sine is x. The domain of this function is $x \in [-1, 1]$, and the range is $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This means the output values y will always be within this interval, corresponding to angles in radians from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Construction Steps:

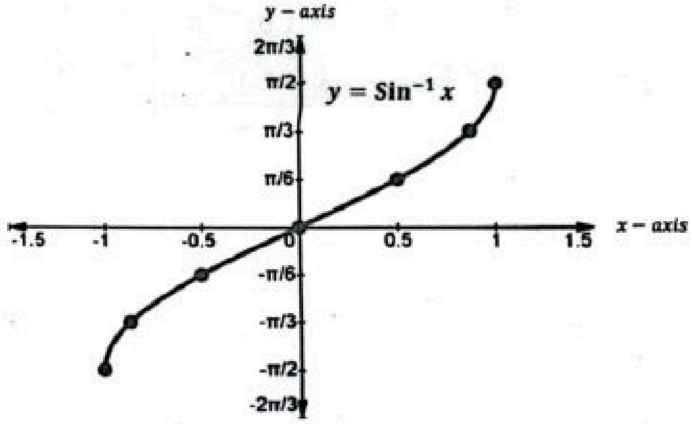
- Draw the coordinate axes. Mark x-axis with values from -1 to 1 and y-axis with values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and choose an appropriate scale.
- Mark the points $\left(1,\frac{\pi}{2}\right)$, $\left(0.9,\frac{\pi}{3}\right)$, $\left(0.5,\frac{\pi}{6}\right)$, $(0,0)$, $\left(-0.5,-\frac{\pi}{6}\right)$, $\left(-0.9,-\frac{\pi}{3}\right)$, $\left(-1,-\frac{\pi}{2}\right)$ on the graph and connect these points with a smooth curve.

Properties:

- **Domain and Range:** The domain is $[-1,1]$ and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
- **Key Points:** $\left(1,\frac{\pi}{2}\right)$, $(0,0)$, $\left(-1,-\frac{\pi}{2}\right)$
- **Appearance:** The graph appears as a smooth curve starting from $\left(-1,-\frac{\pi}{2}\right)$, increasing through the key points, passes through the point $(0, 0)$ and ending at $\left(1,\frac{\pi}{2}\right)$.
- **Monotonicity:** The graph is monotonically increasing smoothly.
- **Odd Inverse Function:** $\left[\begin{matrix}\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x \text{ for } x \in [-1,1] \\ \text{(Symmetric about origin)}\end{matrix}\right]$

  The symmetry along range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ shows that graph is reflected across origin.
- **Reflection:** The graph of $y = \text{Sin}^{-1}x$, is a reflection of $y = \text{Sin}\,x$ across the line $y = x$.
- **Inverse Function:** $\text{Sin}(\text{Sin}^{-1}x) = x$ for $x \in [-1,1]$ and
  $\text{Sin}^{-1}(\text{Sin}\,x) = x$ for $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

- **Identities:** $\text{Sin}^{-1}x = \text{Cos}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [0,1]$ and
  $\text{Sin}^{-1}x = -\text{Cos}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [-1,0)$



x	$y = \text{Sin}^{-1}x$
1	$\pi/2$ or $90°$
0.9	$\pi/3$ or $60°$
0.5	$\pi/6$ or $30°$
0	0 or $0°$
-0.5	$-\pi/6$ or $-30°$
-0.9	$-\pi/3$ or $-60°$
-1	$-\pi/2$ or $-90°$

8.2.3 Graph of $y = \text{Tan}^{-1}x$

The graph of the function $y = \text{Tan}^{-1}x$ represents the angle y whose tangent is x. The domain of this function is $x \in (-\infty, +\infty)$, and the range is $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means the output values y will always be within this interval, corresponding to angles in radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Construction Steps:

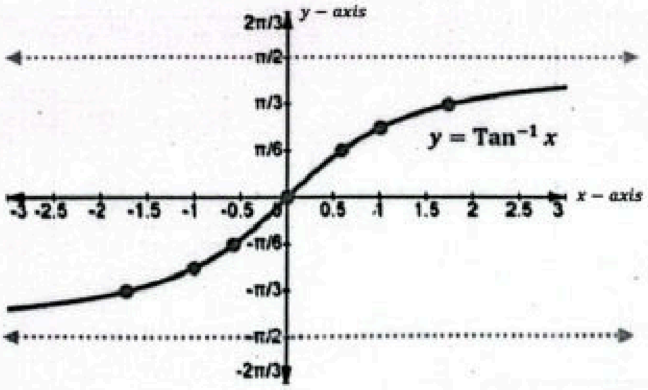
- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and choose an appropriate scale.
- Mark the points $\left(-\infty, -\frac{\pi}{2}\right)$, $\left(-1.7, -\frac{\pi}{3}\right)$, $\left(-1, -\frac{\pi}{4}\right)$, $\left(-0.6, -\frac{\pi}{6}\right)$, $(0,0)$, $\left(1, \frac{\pi}{4}\right)$, $\left(1.7, \frac{\pi}{3}\right)$, $\left(0.6, \frac{\pi}{6}\right)$, $\left(\infty, \frac{\pi}{2}\right)$ on the graph and connect these points with a smooth curve.

Properties:

- **Domain and Range:** The domain is $(-\infty, +\infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- **Key Points:** $\left(-\infty, -\frac{\pi}{2}\right)$, $\left(-1.7, -\frac{\pi}{3}\right)$, $(0,0)$, $\left(1.7, \frac{\pi}{3}\right)$ $\left(\infty, \frac{\pi}{2}\right)$
- **Asymptotes:** Draw two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ approaching the tangent curve at $x = \pm\infty$.
- **Appearance:** The graph appears as a smooth curve starting from $\left(-\infty, -\frac{\pi}{2}\right)$, increasing through the key points, passes through the point $(0,0)$ and ending at $\left(+\infty, \frac{\pi}{2}\right)$.
- **Monotonicity:** The graph is monotonically increasing smoothly between two horizontal asymptotes.
- **Odd Inverse Function:** $\left[\begin{array}{c}\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x) \text{ for } x \in (-\infty, +\infty) \\ \text{(Symmetric about origin)}\end{array}\right]$

  The symmetry along range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ shows that graph is reflected across origin.

- **Reflection:** The graph of $y = \text{Tan}^{-1}x$, is a reflection of $y = \text{Tan}\,x$ across the line $y = x$.
- **Inverse Function:** $\text{Tan}(\text{Tan}^{-1}x) = x$ for $x \in (-\infty, +\infty)$ and
$$\text{Tan}^{-1}(\text{Tan}\,x) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
- **Identities:** $\text{Tan}^{-1}(x) + \text{Tan}^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$ for $x > 0$ and
$$\text{Tan}^{-1}(x) + \text{Tan}^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2} \text{ for } x < 0$$



x	$y = \text{Tan}^{-1}x$
$-\infty$	$-\pi/2$ or $-90°$
-1.7	$-\pi/3$ or $-60°$
-1	$-\pi/4$ or $-45°$
-0.6	$-\pi/6$ or $-30°$
0	0 or $0°$
0.6	$\pi/6$ or $30°$
1	$\pi/4$ or $45°$
1.7	$\pi/3$ or $60°$
∞	$\pi/2$ or $90°$

8.2.4 Graph of $y = \text{Cot}^{-1}x$

The graph of the function $y = \text{Cot}^{-1}x$ represents the angle y whose cotangent is x. The domain of this function is $x \in (-\infty, +\infty)$, and the range is $y \in (0, \pi)$. This means the output values y will always be within this interval, corresponding to angles in radians between 0 and π.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from 0 to π and choose an appropriate scale.
- Mark the points $(-\infty, \pi)$, $\left(-1.7, \frac{5\pi}{6}\right)$, $\left(-1, \frac{3\pi}{4}\right)$, $\left(-0.6, \frac{2\pi}{3}\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(0.6, \frac{\pi}{3}\right)$ $\left(1, \frac{\pi}{4}\right)$, $\left(1.7, \frac{\pi}{6}\right)$, $(+\infty, 0)$ on the graph.
- Connect these points with a smooth curve.

Properties:

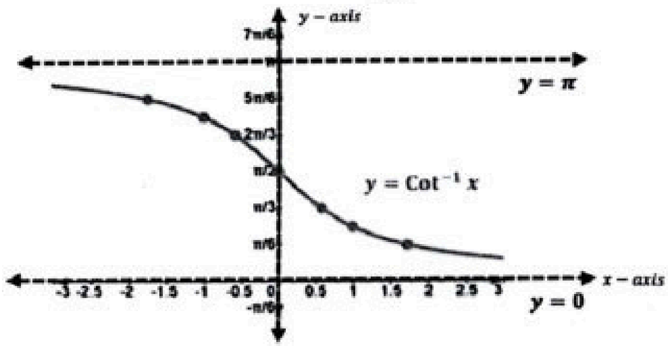
- **Domain and Range:** The domain is $(-\infty, +\infty)$ and the range is $(0, \pi)$.
- **Key Points:** $(-\infty, \pi)$, $\left(-1.7, \frac{5\pi}{6}\right)$, $(0, 0)$, $\left(1.7, \frac{\pi}{6}\right)$ $(+\infty, 0)$
- **Asymptotes:** Draw two horizontal asymptotes $y = 0$ and $y = \pi$ approaching the cotangent curve at $x = \pm\infty$.
- **Appearance:** The graph appears as a smooth curve starting from $(-\infty, \pi)$, increasing through the key points, passes through the point $(0, \frac{\pi}{2})$ and ending at $(+\infty, 0)$.
- **Monotonicity:** The graph is monotonically decreasing smoothly between two horizontal asymptotes.
- **Neither even nor odd function**
- **Reflection:** The graph of $y = \text{Cot}^{-1}x$, is a reflection of $y = \text{Cot}\,x$ across the line $y = x$.

This means that the x and y coordinates of the points on the graph of $y = \text{Cot}\, x$ are switched to obtain points on the graph of $y = \text{Cot}^{-1} x$.

- **Reflection Symmetry:** $\begin{bmatrix} \text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}(x) \text{ for } x \in (-\infty, +\infty) \\ \left(\text{about the line } y = \frac{\pi}{2}\right) \end{bmatrix}$

  The symmetry along range $(0, \pi)$reflects across the vertical axis at $x = 0$.
- **Inverse Function:** $\text{Cot}(\text{Cot}^{-1} x) = x$ for $x \in (-\infty, +\infty)$ and

  $\text{Cot}^{-1}(\text{Cot}\, x) = x$ for $x \in (0, \pi)$
- **Identities:** $\text{Cot}^{-1}(x) = \text{Tan}^{-1}\left(\frac{1}{x}\right)$ for $x \neq 0$



x	$y = \text{Cot}^{-1} x$
$+\infty$	0 or 0°
1.7	$\pi/6$ or 30°
1	$\pi/4$ or 45°
0.6	$\pi/3$ or 60°
0	$\pi/2$ or 90°
−0.6	$2\pi/3$ or 120°
−1	$3\pi/4$ or 135°
−1.7	$5\pi/6$ or 150°
$-\infty$	π or 180°

8.2.5 Graph of $y = \text{Sec}^{-1} x$

The graph of the function $y = \text{Sec}^{-1} x$ represents the angle y whose secant is x. The domain of this function is $x \in (-\infty, -1] \cup [1, \infty)$, and the range is $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. This means the output values y will always be within this interval, corresponding to angles in radians between 0 and π.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from 0 to π.
- Choose an appropriate scale.
- Mark the points $\left(+\infty, \frac{\pi}{2}\right)$, $\left(2, \frac{\pi}{3}\right)$, $\left(1.4, \frac{\pi}{4}\right)$, $\left(1.2, \frac{\pi}{6}\right)$, $(1, 0)$, $(-1, \pi)$, $\left(-1.2, \frac{5\pi}{6}\right)$, $\left(-1.4, \frac{3\pi}{4}\right)$, $\left(-2, \frac{2\pi}{3}\right)$, $\left(-\infty, \frac{\pi}{2}\right)$ on the graph.
- Connect these points with a smooth curve.

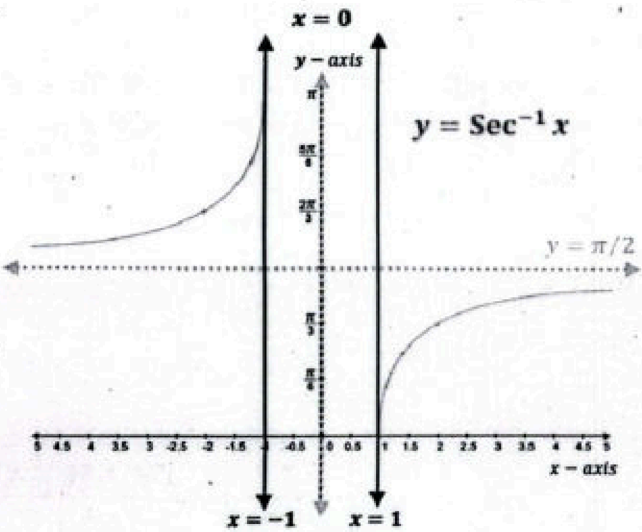
Properties:

- **Domain and Range:** The domain is $(-\infty, -1] \cup [1, \infty)$ and the range is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.
- **Key Points:** $\left(-\infty, \frac{\pi}{2}\right)$, $\left(-2, \frac{2\pi}{3}\right)$, $(-1, \pi)$, $(1, 0)$, $\left(2, \frac{\pi}{3}\right)$, $\left(+\infty, \frac{\pi}{2}\right)$
- **Asymptotes:** Draw a horizontal asymptote $y = \frac{\pi}{2}$ approaching the secant curve at $x = \pm\infty$.
- **Appearance:** The secant graph appears increasing from $(1, 0)$, moves along the horizontal asymptote $y = \frac{\pi}{2}$ and ends at $\left(+\infty, \frac{\pi}{2}\right)$. The secant graph appears decreasing from $(-1, \pi)$, moves along the horizontal asymptote $y = \frac{\pi}{2}$ and ends at $\left(-\infty, \frac{\pi}{2}\right)$.

- **Monotonicity:** The graph is monotonically increasing and decreasing smoothly (it always increases/decreases as x increases/increases) along the horizontal asymptote $y = \frac{\pi}{2}$.
- **Neither even nor odd function**
- **Reflection:** The graph of $y = \text{Sec}^{-1}x$, is a reflection of $y = \text{Sec}\,x$ across the line $y = x$. This means that the x and y coordinates of the points on the graph of $y = \text{Sec}\,x$ are switched to obtain the points on the graph of $y = \text{Sec}^{-1}x$.
- **Reflection Symmetry:** $\begin{bmatrix} \text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}(x) \text{ for } x \in (-\infty, -1] \cup [1, \infty) \\ \left(\text{symmetric about the line } y = \frac{\pi}{2}\right) \end{bmatrix}$

  The symmetry along range $[0, \pi]/\left\{\frac{\pi}{2}\right\}$ reflects across the vertical axis at $x = 0$.
- **Inverse Function:** $\text{Sec}(\text{Sec}^{-1}x) = x$ for $x \in (-\infty, -1] \cup [1, \infty)$ and

  $$\text{Sec}^{-1}(\text{Sec}\,x) = x \text{ for } x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$
- **Identities:** $\text{Sec}^{-1}(x) = \text{Cos}^{-1}\left(\frac{1}{x}\right)$ for $x \in (-\infty, -1] \cup [1, \infty)$



x	$y = Sec^{-1}x$
∞	$\pi/2$ or $90°$
2	$\pi/3$ or $60°$
1.4	$\pi/4$ or $45°$
1.2	$\pi/6$ or $30°$
1	0 or $0°$
-1	π or $180°$
-1.2	$5\pi/6$ or $150°$
-1.4	$3\pi/4$ or $135°$
-2	$2\pi/3$ or $120°$
$-\infty$	$\pi/2$ or $90°$

8.2.6 Graph of $y = \text{Csc}^{-1}x$

The graph of the function $y = \text{Csc}^{-1}x$ represents the angle y whose cosecant is x. The domain of this function is $x \in (-\infty, -1] \cup [1, \infty)$, and the range is $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. This means the output values y will always be within this interval, corresponding to angles in radians between 0 and π.

Construction Steps:

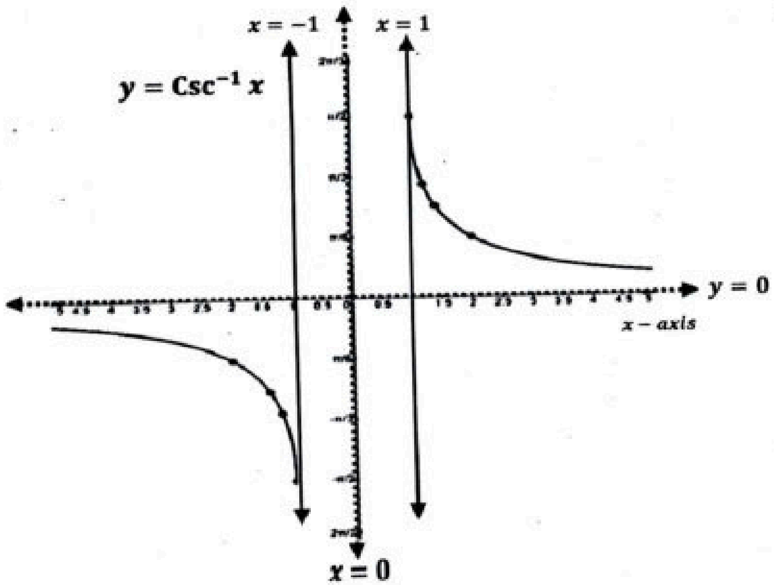
- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from 0 to π and choose an appropriate scale.
- Mark the points $\left(1, \frac{\pi}{2}\right)$, $\left(1.2, \frac{\pi}{3}\right)$, $\left(1.4, \frac{\pi}{4}\right)$, $\left(2, \frac{\pi}{6}\right)$, $(+\infty, +0)$, $\left(-1, -\frac{\pi}{2}\right)$, $\left(-1.2, -\frac{\pi}{3}\right)$, $\left(-1.4, -\frac{\pi}{4}\right)$, $\left(-2, -\frac{\pi}{6}\right)$, $(-\infty, -0)$ on the graph and connect them smoothly.

Properties:

- **Domain and Range:** The domain is $(-\infty,-1] \cup [1,\infty)$ and the range is $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$.
- **Key Points:** $\left(-\infty,\frac{\pi}{2}\right), \left(-2,\frac{2\pi}{3}\right), (-1,\pi), (1,0), \left(2,\frac{\pi}{3}\right), \left(+\infty,\frac{\pi}{2}\right)$
- **Asymptotes:** Draw a horizontal asymptote $y = 0$ approaching the cosecant curve at $x = \pm\infty$.
- **Appearance:** The cosecant graph appears decreasing from $\left(1,\frac{\pi}{2}\right)$, moves along the horizontal asymptote $y = 0$ and ends at $(+\infty, 0)$. The cosecant graph appears increasing from $\left(-1,-\frac{\pi}{2}\right)$, moves along the horizontal asymptote $y = 0$ and ends at $(-\infty,-0)$.
- **Monotonicity:** The graph is monotonically increasing and decreasing smoothly (it always increases/decreases as x increases/increases) along the horizontal asymptote $y = 0$.
- **Odd Inverse Function:** $\begin{bmatrix} \text{Csc}^{-1}(-x) = -\text{Csc}^{-1}(x) \text{ for } x \in (-\infty,-1] \cup [1,\infty) \\ \text{(Symmetric about origin)} \end{bmatrix}$

  The symmetry along range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]/\{0\}$ shows that graph is reflected across origin.
- **Reflection:** The graph of $y = \text{Csc}^{-1}x$, is a reflection of $y = \text{Csc}\, x$ across the line $y = x$. This means that the x and y coordinates of the points on the graph of $y = \text{Csc}\, x$ are switched to obtain the points on the graph of $y = \text{Csc}^{-1}x$.
- **Inverse Function:** $\text{Csc}(\text{Csc}^{-1}x) = x$ for $x \in (-\infty,-1] \cup [1,\infty)$ and

  $$\text{Csc}^{-1}(\text{Csc}\, x) = x \text{ for } x \in \left[-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right]$$
- **Identities:** $\text{Csc}^{-1}(x) = \text{Sin}^{-1}\left(\frac{1}{x}\right)$ for $x \in (-\infty,-1] \cup [1,\infty)$



x	$y = \text{Csc}^{-1} x$
$+\infty$	$+0$ or $+0°$
2	$\pi/6$ or $30°$
1.4	$\pi/4$ or $45°$
1.2	$\pi/3$ or $60°$
1	$\pi/2$ or $90°$
-1	$-\pi/2$ or $-90°$
-1.2	$-\pi/3$ or $-60°$
-1.4	$-\pi/4$ or $-45°$
-2	$-\pi/6$ or $-30°$
$-\infty$	-0 or $-0°$

4. i. $1-i$ ii. $\frac{-5}{2}-\frac{5\sqrt{3}}{2}i$ iii. $-2i$ iv. $-2\sqrt{3}+2i$ v. $\sqrt{3}+i$

vi. $\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ vii. $6.43+7.66i$ viii. $-1+i$

5. i. $x+y=1$ ii. $x^2+y^2=4$ iii. $-\sqrt{3}\le\frac{y}{x-4}\le\sqrt{3}$

iv. $(-\sqrt{3}-1)x+(\sqrt{3}-1)y+4(\sqrt{3}+1)\le 0\le -x+y+4$

v. $x^2+y^2=1$ vi. $\sqrt{3}(x^2-y^2+1)-2xy=0$

6. i. $\frac{\sqrt{2}}{500}(1+i)$ ii. $\frac{1}{500}(1+\sqrt{3}i)$ iii. $\frac{1}{500}(\sqrt{3}+i)$ 7. i. $1+2\sqrt{3}i$ ii. 0.4756

8. i. $\frac{5}{2}-\frac{35}{2}i$ ii. $-\frac{10}{73}(77+38i)$ 9. 0.3 cost 10. 0.8 cost

11. 7.936 cos(t +39.39°) 12. $\sqrt{61}\,cos(t+86.33°)$

REVIEW EXERCISE

1. i. c ii. b iii. a iv. d v b vi d vii c viii d ix b x b
2. i. 0 ii. $\sqrt{2}$ iii. $\sqrt{221}$ iv. $-\frac{9}{34}-\frac{19}{34}i$ 3. i. $3(x-6i)(x+6i)$ ii. $4(x-\sqrt{10}i)(x+\sqrt{10}i)$
4. $z=x$ 5. $z=\frac{-14}{29}+\frac{64}{29}i$ 6. $2+11i$ 7. $z=\frac{11\pm i\sqrt{71}}{4}$ 8. 2

Unit 2: Matrices and Determinants

Exercise 2.1

1. i. 2 x 3 ii. 3x2 iii. 3x1 iv. 1x4 v. 1x1 vi. 2x2
2. i. rectangular ii. Square iii. column iv. square v. row vi. square
3. i. lower triangular ii. scalar iii. diagonal iv. identity v. diagonal vii. upper triangular viii. diagonal ix. scaler
4. i. $\begin{bmatrix}2 & \sqrt{5} & 1\\ 0 & 6 & 9\end{bmatrix}$ neither symmetric nor skew symmetric ii. $\begin{bmatrix}1\\6\\2\\0\end{bmatrix}$ neither symmetric nor skew symmetric iii. $\begin{bmatrix}2 & 6\\ 6 & 2\end{bmatrix}$ symmetric

iv. $\begin{bmatrix}0 & -1 & -9\\ 1 & 0 & 5\\ 9 & 5 & 0\end{bmatrix}$ skew symmetric v. $\begin{bmatrix}3 & -6 & 9\\ -6 & 2 & 0\\ 9 & 0 & 0\end{bmatrix}$ symmetric vi. $\begin{bmatrix}9 & 0 & 0\\ 0 & 6 & 0\\ 1 & 3 & 1\end{bmatrix}$ neither symmetric nor skew symmetric

Exercise 2.2

1. i. $\begin{bmatrix}2 & 7/2\\ 5/2 & 4\end{bmatrix}$ ii. $\begin{bmatrix}1/2 & 1\\ 1 & 2\end{bmatrix}$ iii. $\begin{bmatrix}1 & 1/2\\ 2 & 1\end{bmatrix}$ iv. $\begin{bmatrix}-1/3 & -4/3\\ 1/3 & -2/3\end{bmatrix}$
2. i. $\begin{bmatrix}0 & -1/3 & -2/3\\ 1 & 2/3 & 1/3\\ 8/3 & 7/3 & 2\end{bmatrix}$ ii. $\begin{bmatrix}0 & -3/2 & -4\\ 3/4 & 0 & -5/4\\ 8/9 & 5/9 & 0\end{bmatrix}$ iii. $\begin{bmatrix}2/3 & 1/2 & 2/5\\ 2/5 & 1/3 & 2/7\\ 2/7 & 1/4 & 2/9\end{bmatrix}$

iv. $\begin{bmatrix}1 & 5/3 & 5/2\\ 5/3 & 2 & 13/5\\ 5/2 & 13/5 & 3\end{bmatrix}$ 3. $C=\begin{bmatrix}-5 & 0 & -9\\ 0 & -8 & 0\\ 4 & -4 & 1\end{bmatrix}$

4. i. $A=\begin{bmatrix}-5 & 7/2\\ 8 & -11/2\end{bmatrix}$ ii. $\begin{bmatrix}1/2 & 1 & 0\\ 2 & 4 & 1\end{bmatrix}$ iii. $\begin{bmatrix}7x\\ x\end{bmatrix}$ *where* $x \in \mathbb{R}$

iv. $z = 4, t = 0,\ x^2 + y^2 = 20$ v. $\alpha = -10; \beta = 9$ vi. $-4, 3$

6. $\alpha = -9;\ \ \beta = -1$ 10. skew symmetric

12. $X = \begin{bmatrix}1 & 3 & -1\\ 1 & 0 & 3\end{bmatrix}$, $Y = \begin{bmatrix}1 & 0 & 1\\ 0 & -1 & -1\end{bmatrix}$ 13. $X = \begin{bmatrix}\frac{3}{5} & \frac{4}{5} & \frac{3i}{5}\\ \frac{-3+12i}{5} & 2-i & \frac{7}{5}\\ \frac{1}{5} & \frac{1}{5} & \frac{-6+6i}{5}\end{bmatrix}$, $Y = \begin{bmatrix}\frac{1}{5} & \frac{13}{5} & \frac{6i}{5}\\ \frac{19-i}{5} & -1+3i & \frac{14}{5}\\ \frac{7}{5} & \frac{7}{5} & \frac{18-3i}{5}\end{bmatrix}$

Exercise 2.3

1. i. 15 ii. 1 iii. – 6 iv. 16 + 8i
2. i. – 17 ii. 27 iii. 1 – 16i iv. – 17 + 11i
3. singular ii. Non-singular iii. Non-singular iv. singular
4. i. 16/23 ii. – 4 iii. – 1 + 5i iv. $-\frac{7}{100} - \frac{i}{100}$
5. i. $\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0\\ \frac{-1}{9} & \frac{2}{9} & \frac{-1}{3}\\ \frac{5}{9} & \frac{-1}{9} & \frac{-1}{3}\end{bmatrix}$ ii. $\begin{bmatrix}\frac{-1}{3} & \frac{-4}{9} & \frac{26}{9}\\ \frac{-1}{3} & \frac{-1}{9} & \frac{11}{9}\\ \frac{1}{3} & \frac{4}{9} & \frac{-17}{9}\end{bmatrix}$ iii. $\begin{bmatrix}\frac{-4i}{5} & 0 & \frac{1}{5}\\ \frac{8-i}{5} & -1 & \frac{-1+2i}{5}\\ \frac{1}{5} & 0 & \frac{-i}{5}\end{bmatrix}$ iv. $\begin{bmatrix}\frac{3}{11} & \frac{2+2i}{11} & \frac{-2+i}{22}\\ 0 & \frac{1-i}{2} & \frac{1+i}{4}\\ \frac{-2i}{11} & \frac{-1+i}{22} & \frac{5-i}{44}\end{bmatrix}$ 6. $\begin{bmatrix}\frac{1}{3} & \frac{-1}{2} & \frac{1}{6}\\ 0 & 1 & 0\\ \frac{-1}{9} & 0 & \frac{1}{9}\end{bmatrix}$

Exercise 2.5

1. i. $\begin{bmatrix}1 & 3 & 5\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ii. $\begin{bmatrix}1 & -18\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$ iii. $\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 27\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

iv. $\begin{bmatrix}1 & -2 & 3/2\\ 0 & 1 & -8/9\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ v. $\begin{bmatrix}1 & -8 & -6\\ 0 & 1 & 4/5\end{bmatrix}\begin{bmatrix}1 & 0 & 2/5\\ 0 & 1 & 4/5\end{bmatrix}$ vi. $\begin{bmatrix}0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

2. i. 3 ii. 2 iii. 3 iv. 2
3. $\frac{1}{2}\begin{bmatrix}-12 & -5 & -3\\ -4 & -1 & -1\\ 2 & 1 & 1\end{bmatrix}$ ii. $\frac{1}{6}\begin{bmatrix}-2 & 0 & 2\\ 19 & -15 & -16\\ -6 & 6 & 6\end{bmatrix}$ iii. $\frac{1}{12}\begin{bmatrix}0 & -6 & 6\\ 3 & -9 & 6\\ 2 & -4 & 6\end{bmatrix}$ iv. $\begin{bmatrix}-8 & 5 & 2\\ -18 & 18 & -9\\ 15 & -6 & 3\end{bmatrix}$

Exercise 2.6

1. i. $\begin{bmatrix}x_3\\ 2x_3\\ x_3\end{bmatrix}$ ii. $\begin{bmatrix}\frac{-7}{5}x_3\\ \frac{2}{5}x_3\\ x_3\end{bmatrix}$ iii. $\begin{bmatrix}x_3\\ 2x_3\\ x_3\end{bmatrix}$ iv. does not exist.
2. i. $\lambda = \frac{-7}{11}; \begin{bmatrix}\frac{-10}{13}x_3\\ \frac{11}{13}x_3\\ x_3\end{bmatrix}$ ii. $\lambda = 2; \begin{bmatrix}-x_3\\ \frac{1}{2}x_3\\ x_3\end{bmatrix}$ *or* $\lambda = -7; \begin{bmatrix}17x_3\\ 5x_3\\ x_3\end{bmatrix}$
3. i. $;\frac{66}{19}; -\frac{63}{19}$ ii. No solution iii. $\frac{5}{4}; \frac{5}{4}; \frac{-1}{2}$ iv. $-7; -7; 5$
4. i. 3;1;2 ii. $-\frac{1}{7}; \frac{1}{7}; 0$ iii. solution not possible as A is singular iv. $\frac{6}{11}; -\frac{7}{11}; \frac{2}{11}$
5. $\frac{1}{11}; -\frac{3}{11}; \frac{70}{11}$ ii. $\frac{37}{12}; \frac{7}{3}; \frac{11}{12}$ iii. $\frac{7}{4}; -\frac{23}{2}; -\frac{29}{4}$ iv. 2; 3; 5
6. $\begin{bmatrix}-3/62 & 9/62 & 5/62\\ 13/31 & -8/31 & -1/31\\ 19/62 & 5/62 & -11/62\end{bmatrix}$ 1; 1; 1
7. $\lambda = \pm 4$, no solution; $\lambda \neq \pm 4$ unique solution

10. $\begin{bmatrix}1&0&0\\0&1&-5\\0&0&1\end{bmatrix}$; $x'^2+y'^2+10y'+16=0$ 11. $\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}$; $x'^2+8x'-3y'+4=0$

12. $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$; $2x'^2-5y'^2-4x'-8=0$ 13. $\begin{bmatrix}1&0&3\\0&1&2\\0&0&1\end{bmatrix}$; $2x'-7y'+11=0$

REVIEW EXERCISE

1. i. b ii. d iii. d iv. b v. c vi. b vii. d viii. c ix. c x. d
2. – 11, 3, 10; 87 4. 1/3

Unit 3: Vectors

Exercise 3.1

1. i. $-7\hat{\imath}-5\hat{\jmath}$ ii. $-22\hat{\imath}-16\hat{\jmath}$ iii. $-7\hat{\imath}+28\hat{\jmath}$ iv. $-\frac{35}{2}\hat{\imath}-\frac{13}{2}\hat{\jmath}$ v. $-18\hat{\imath}+155\hat{\jmath}$

3. i. $\frac{2}{3}\hat{\imath}+\frac{2}{3}\hat{\jmath}$ ii. $\vec{u}=-5\hat{\imath}+8\hat{\jmath}$; $\vec{v}=7\hat{\imath}-11\hat{\jmath}$ iii. $8\hat{\imath}-\hat{\jmath}-\hat{k}$

4. i. $m=4/3$ ii. $\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $-2\hat{\imath}+2\hat{\jmath}+6\hat{k}$, $\lambda\hat{\imath}-\lambda\hat{k}$

5. i. $\frac{1}{\sqrt{38}}\hat{\imath}-\frac{1}{\sqrt{38}}\hat{\jmath}+\frac{6}{\sqrt{38}}\hat{k}$ ii. $\sqrt{5}$; 3 7. i. $\lambda=\pm2\sqrt{11}$ ii. $|\vec{a}|=|\vec{b}|$; $\vec{a}\neq\vec{b}$

8. i. $\frac{5}{\sqrt{10}}\hat{\jmath}-\frac{15}{\sqrt{10}}\hat{k}$ ii. $-\frac{3}{\sqrt{257}}\hat{\imath}+\frac{36}{7\sqrt{257}}\hat{\jmath}-\frac{24}{7\sqrt{257}}\hat{k}$ 9. i. $\frac{7}{5}\hat{\imath}+\frac{1}{5}\hat{k}$ ii. $\hat{\imath}-12\hat{\jmath}+5\hat{k}$

10. i. D(– 2, 1) ii. x = 6 and y = 3 14. i. PS = $\vec{s}-\vec{r}$

16. $\overrightarrow{AC}=\vec{a}+\vec{b}$, $\overrightarrow{CD}=\vec{b}-\vec{a}$, $\overrightarrow{EF}=-\vec{b}$, $\overrightarrow{DA}=-2\vec{a}$, $\overrightarrow{EB}=2(\vec{a}-\vec{b})$, $\overrightarrow{FA}=\vec{a}-\vec{b}$, $\overrightarrow{FC}=2\vec{a}$

Exercise 3.2

1. i. 15 ii. 90 iii. – 16 iv. 147 v. 4
2. i. $\theta=\cos^{-1}\left(\frac{-5}{2\sqrt{13}}\right)$ ii. $\theta=\cos^{-1}\left(\frac{57}{\sqrt{6342}}\right)$ iii. $\theta=\cos^{-1}\left(\frac{-30}{\sqrt{1870}}\right)$
   iv. $\theta=\cos^{-1}\left(\frac{-15}{\sqrt{357}}\right)$ v. $\theta=\cos^{-1}\left(\frac{18}{\sqrt{438}}\right)$
3. i. $\cos^{-1}\left(\frac{1}{4}\right)$ ii. 90°
4. i. $\lambda=\frac{29}{44}$
   i. $\cos\alpha=\frac{2}{\sqrt{29}}$; $\cos\beta=\frac{-3}{\sqrt{29}}$; $\cos\gamma=\frac{4}{\sqrt{29}}$ ii. $\frac{2}{\sqrt{114}}$; $\frac{21}{\sqrt{62}}$
6. i. 45°, 45° ii. $\pm\frac{5}{\sqrt{3}}\hat{\imath}\pm\frac{5}{\sqrt{3}}\hat{\jmath}\pm\frac{5}{\sqrt{3}}\hat{k}$
7. i. – 45/2
8. $\vec{r}=\hat{\imath}+2\hat{\jmath}+\hat{k}$
   14. $350/\sqrt{11}\,joules$ 15. 28 units 16. $150\sqrt{3}$

Exercise 3.3

1. i. (4, – 15, – 7) ii. (30, 11, –27) iii. (4, – 6, 2)
2. i. (– 18, - 8, 3) ii. (3, 15, 6)
3. i. $\frac{\sqrt{78}}{\sqrt{29}\sqrt{26}}$ ii. $\frac{3\sqrt{62}}{\sqrt{29}\sqrt{83}}$ 4. i. $\left(\frac{40}{\sqrt{1533}},\frac{185}{\sqrt{1533}},\frac{50}{\sqrt{1533}}\right)$
   ii. parallel $\frac{-1}{5}\hat{\imath}+\frac{1}{10}\hat{\jmath}-\frac{3}{10}\hat{k}$; perpendicular $\frac{26}{5}\hat{\imath}+\frac{19}{10}\hat{\jmath}-\frac{27}{10}\hat{k}$ 5. ii. $\vec{d}=-\frac{2}{5}\hat{\imath}-\frac{1}{5}\hat{\jmath}+\frac{4}{5}\hat{k}$
6. i. $\vec{a}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$ 9. i. $\frac{15\sqrt{15}}{4}$ ii. 6

10. $i.\ 3\sqrt{59}$ ii. $5/\sqrt{2}$ $\alpha = 74.21°, \beta = 60.50°, \gamma = 45.29°$

11. $\sqrt{75}$ 12. Either $\vec{a} = 0$ or $\vec{b} = 0$ or Both are zero

14. i. $\sqrt{181}$ ii. $11\sqrt{6}$ 15. i. $\frac{1}{4}(7-3t)\hat{\imath} + \frac{1}{2}(1-t)\hat{\jmath} + t\hat{k}$ ii. $\left(-\frac{1}{6}(4t+21), -\frac{1}{3}(2t+3), t\right)$

16. $11\hat{\imath} + \hat{\jmath} - 5\hat{k}$ 17. $15\hat{\imath} - 20\hat{\jmath} + 7\hat{k};\ -9\hat{\imath} - 26\hat{\jmath} + 19\hat{k};\ 6\hat{\imath} - 46\hat{\jmath} + 26\hat{k}$

Exercise 3.4

1. i. -14 ii. -20
2. i. 2 ii. -8
3. ii. $\lambda = -1/2$
4. i. $\lambda = 2$ 6. i. zero ii. 68 7. i. 27/6 ii. 3

REVIEW EXERCISE

1. i. d ii. c iii. b iv. b v. c vi. d vii. b viii. b ix. c x. a
2. i. 2/3 ii. - 3/20
3. $\sqrt{2}$ 4. $\sqrt{19}$ 5. $-11/2$

8. Ground speed $\approx 235.492\ km/h$ true course$\approx 64.872°$

9. Speed $\approx 237.816\ km/h$ direction $\approx 107.980°$

Unit 4: Sequences and Series

Exercise 4.1

1. (i) $a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_{10} = 31, a_{15} = 46$
   (ii) $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11, a_{10} = 29, a_{15} = 44$
   (iii) $a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, a_3 = \frac{3}{4}, a_4 = \frac{4}{5}, a_{10} = \frac{10}{11}, a_{15} = \frac{15}{16}$
   (iv) $a_1 = 2, a_2 = 5, a_3 = 10, a_4 = 17, a_{10} = 101, a_{15} = 226$
   (v) $a_1 = -1, a_2 = 0, a_3 = 3, a_4 = 8, a_{10} = 80, a_{15} = 95$
   (vi) $a_1 = 0, a_2 = \frac{3}{5}, a_3 = \frac{4}{5}, a_4 = \frac{15}{17}, a_{10} = \frac{99}{101}, a_{15} = \frac{112}{113}$
   (vii) $a_1 = 1, a_2 = -\frac{1}{2}, a_3 = \frac{1}{4}, a_4 = -\frac{1}{8}, a_{10} = -\frac{1}{512}, a_{15} = \frac{1}{16384}$
   (viii) $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_{10} = 100, a_{15} = 225$
   (ix) $a_1 = -4, a_2 = 5, a_3 = -6, a_4 = 7, a_{10} = 13, a_{15} = -18$
   (x) $a_1 = -2, a_2 = -1, a_3 = 4, a_4 = -7, a_{10} = -25, a_{15} = 40$
2. (i) $a_8 = 29$ (ii) $a_9 = 56$ (iii) $a_7 = 225,$ (iv) $a_{12} = -23.5$
   (v) $a_{22} = 528{,}528,$ (vi) $a_{20} = \frac{441}{400}$ (vii) $a_{43} = 43$ (viii) $a_{67} = 67$
3. (i) $a_n = 2n - 1$ (ii) $a_n = 3^n$ (iii) $a_n = \sqrt{2n}$ (iv) $a_n = n(n+1)$

Exercise 4.2

1. **(i)** $a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13$ **(ii)** $a_1 = 7, a_2 = 12, a_3 = 17, a_4 = 22$
   (iii) $a_1 = 16, a_2 = 14, a_3 = 12, a_4 = 10$ **(iv)** $a_1 = 38, a_2 = 34, a_3 = 30, a_4 = 26$
   (v) $a_1 = \frac{3}{4}, a_2 = 1, a_3 = \frac{5}{4}, a_4 = \frac{3}{2}$ **(vi)** $a_1 = \frac{3}{8}, a_2 = 1, a_3 = \frac{13}{8}, a_4 = \frac{9}{4}$
2. **(i)** The next three terms of the sequence are 17, 21, 25
   (ii) The next three terms of the sequence are 20, 23, 26
   (iii) The next three terms of the sequence are $\frac{7}{2}, \frac{9}{2}, \frac{11}{2}$
   (iv) The next three terms of the sequence are 0.22, 0.27, 0.32 3. $a_{11} = 0.57$
4. $a_1 = 19, a_2 = \frac{33}{2}, a_3 = 14, a_4 = \frac{23}{2}$ 5. $a_1 = 8, a_2 = 5, a_3 = 2, a_4 = -1$
6. $a_{87} = 347$ 7. $a_{20} = 70$ 8. $a_{56} = -\frac{105}{2}$ 9. $d = \frac{a-c}{2ac}$
10. $a_8 = 240$ feet 11. $S_{20} =$ Rs 39000 12. $a_8 = 7$

13. (i) 12 (ii) 5 (iii) $4\sqrt{5}$ (iv) $\frac{7y}{2}+4$. 14. $b=0$ 15. $x=-9, y=24$
16. $A_1=9, A_2=13$ 17. $A_1=-3, A_2=-8, A_3=-13$

Exercise 4.3

1. $S_n=116$ 2. $S_n=10100$ 3. $S_n=10500$ 4. $S_n=375$ 5. $S_n=240$, 6. -210 7. $S_n=240$
8. $S_n=2550$ 9. $S_n=2500$ 10. $S_n=34036$ 11. $S_n=-140$ 12. $S_n=1155$ 13. 162
14. 104 15. $S_n=1060$ 16. $S_n=387$ 17. $S_n=816$ 18. $S_n=162$ 19. $S_n=-220$
20. $a_1=7, a_2=19, a_3=31$ 21. $a_1=1$, $a_2=5$, $a_3=9$ 22. $a_1=6$, $a_2=36$, $a_3=66$
23. $a_{25}=62{,}950$ 24. 45 25. 12,280,000 26. 38,750

Exercise 4.4

1. The sequence is not geometric
2. The sequence is not geometric 3. The sequence is geometric $(r=\frac{3}{2})$
4. The sequence is not geometric 5. $a_1=3, a_2=-6, a_3=12, a_4=-24$
6. $a_1=27$, $a_2=-9$, $a_3=3$, $a_4=-1$ 7. $a_1=12, a_2=6, a_3=3, a_4=\frac{3}{2}$
8. $a_4=\frac{10}{3}, a_5=\frac{10}{9}$ 9. $a_4=54, a_5=162$ 10. $a_4=\frac{135}{2}, a_5=\frac{405}{4}$
11. $a_4=27, a_5=9$ 12. $a_4=1, a_5=3$ 13. $a_4=2, a_5=4$
14. $a_3=100$ 15. $a_5=32$ 16. $a_4=56$ 17. $a_5=3$
18. $a_6=-1$ 19. $a_8=\frac{1}{8}$ 20. 6, 12, 24 21. 2, 4 22. $4, 2, 1, \frac{1}{2}$
23. 15 24. 10, 20, 40 25. 14, 28, 56 26. $\frac{1}{256}$ ft 27. 151258.9(appro)
28. 3100 ft. (approximately) 29. 127 30. 81

Exercise 4.5

1. 176, 2. 93.15 3. 13,28,600 4. 947.11 5. 114681 6. 732
7. 10.66 8. 165, 9. 300 10. 189 11. 4 12. 0.51 13. 4
14. (i) $\frac{4}{9}$ (ii) 1 (iii) $\frac{5}{9}$ (iv) $\frac{2}{3}$ (v) $\frac{5}{33}$ (vi) $\frac{4}{33}$ 15. 70 16. 800

Exercise 4.6

1. $\frac{1}{27}$ 2. $-\frac{1}{7}$ 3. $-\frac{1}{77}$ 4. $\frac{1}{5n-1}$ 5. $\frac{1}{34-7n}$ 6. $\frac{1}{\frac{n+3}{2}}$
7. $\frac{1}{43}$ 8. $-\frac{1}{41}$ 9. $-\frac{1}{23}$ 10. $\frac{99}{10}$ 11. $\frac{8}{13}$ 12. $\frac{5}{23}, \frac{5}{31}, \frac{5}{39}, \frac{5}{47}$

Exercise 4.7

1. $\frac{137}{120}$ 2. $\frac{43024}{45045}$ 3. 63 4. 45π 5. $\frac{15551}{2520}$ 6. $-52{,}432$ 7. 43, 8. $\frac{10}{11}$ 9. $\sum_{k=1}^{\infty}\frac{k}{k+1}$ 10. $\sum_{k=1}^{5} 3k$
11. $\sum_{k=1}^{6}(-1)^k 2^k$ 12. $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ 14. $\frac{n^2+3n}{2}$ 15. $\frac{n}{6}(2n^2+9n+7)$ 16. $\frac{n}{2}(2n^2+5n+5)$
17. $\frac{n}{2}(2n^2+3n-1)$ 18. $\frac{2n(n+1)(2n+1)}{3}$ 19. $n(2n^3+2n^2+11n+1)$
20. $\frac{n(2n^2+3n+7)}{6}$ 21. $n(n+1)^2$
22. $n(2n^3+8n^2+7n-2)$ 23. $2^n(n-1)+1$ 24. $\frac{3ny^{n+1}-2y^{n+1}-3ny^n-y^n+2y+1}{(y-1)^2}$
25. $\frac{3.7^{n+1}-42n-21}{12.7^n}$ 26. $14-\frac{6n+7}{2^{n-1}}$ 27. 9 28. $\frac{25}{16}$ 29. $\frac{2x+1}{(1-x)^2}$ 30. $\frac{100}{27}$

Exercise 4.8

1. 780ft 2. 1065, no because the auditorium has only 1065 seats 3. 491,70044
4. $a_{20}=524288$ 5. $a_6=0.328$ or 32.8% 6. 30361.4082 7. 964615

8. 32000 9. $r = 2.95$ 10. Rs. 1420418.205 11. Rs. 100625
12. Rs. 726000 13. Rs. 69000 14. Rs. 356015.99

Miscellaneous Exercise

1. (i) a (ii) c (iii) b (iv) a (v) c (vi) d (vii) b (viii) a (ix) b (x) a (xi) c (xii) b (xiii) a (xiv) c (xv) b (xvi) b (xvii) d 2. 3, 5, 7, 9 3. 0, 1, 2, 3 4. 6, 10, 14, ⋯

5. $\frac{1}{27}[10^{n+1} - 9n - 10]$ **6.** 4, 6, 9 **7.** 6, 18, 54, 162 **8.** $n = -1$ **9.** $a_1 = \frac{6}{5}, a_2 = 1, a_3 = \frac{6}{7}, a_4 = \frac{3}{4}$

10. (i) 5 (ii) −4920 **11.** $n(n+1)^2$ **12.** $\frac{1}{6}(2n^3 - 3n^2 + 13n - 6)$

Unit 5: Polynomials

Exercise 5.1

1. (i) 5 (ii) 35 3. No 4. $y + 1$ 5. −12 6. $m = 6$ 7. Only 1 is a zero of $P(x)$
8. $2, -3, \frac{-1}{2}$ 9. $f(x) = (x-4)(x^2 + 3x - 2) + 0$ 10. $x^2 + 10x + 24$

Exercise 5.2

1. $(y+1)(y-3)(y+2)$ 2. $(x-1)(x+1)(2x-1)$ 3. $(x-2)(x+3)(2x+3)$ 4. $(x-3)(3x^2 + 4x + 12)$
5. $(t-1)(t^2 + 2t + 5)$ 6. Other two factors are $(x-6)$ and $(2x+1)$ 7. $(2x-1)(x-5)(x-2)$
8. $(2x+1)(2x^2 + x + 36)$

Exercise 5.3

1. 5cm by 12cm by 2cm 2. 650 3. 6 units by 8 units by 3 units 4. 9 units by 11 units by 25 units
5. Length of one side of square ABFG is $x+4$. Area = $(x+4)^2$. The length of rectangle ACED = $3x+7$.
6. $y+1, y-1$

REVIEW EXERCISE

1. (i) (d) (ii) (a) (iii) (b) (iv) (a) (v) (b) (vi) (c) (vii) (b) (viii) (c)
2. $16y^2 + 4y + 4$ 3. $25y^2 + 10y + 4$ 4. Yes 5. $5x^3 - 13x^2 - 34x + 24$
6. −48 7. $x + 4$ 8. $y + 5$

Unit 6: Permutation, & Combination

Exercise 6.1

1. i. 3628800 ii. 7920 iii. 11/63 iv. n − 1 v. 7/90
2. i. $\frac{14!}{10!}$ ii. $\frac{9!}{4! \times 16}$ iii. $\frac{(n+1)!}{(n-2)!}$ iv. $\frac{(n-1)!}{n(n-4)(n-4)!}$
5. i. 6 ii. 6 7. i. 31 ii. 8 iii. 3 iv. 10 v. 6 vi. 121 vii. 11 viii. 2 ix. 4 x. 5

Exercise 6.2

2. i. 7 ii. 13 iii. 15 iv. 8 v. 29 vi. 6 vii. 9 viii. 8 ix. 10
3. i. 4 ii. 5 iii. 2 iv. 8 v. 2 vi. 3 vii. 41
4. 60 5. 60480 6. 1296 7. 18 8. 576 9. 1260 10. 720; 120
11. 45360 12. 108 13. 4320 14. 210 15. 8640 16. 360
17. 72 18. 94 19. 30,240 20. 86400 21. 6, 6, 6, 24 22. HOELRA 22. MULTAN

Exercise 6.3

2. i. 13 ii. 22 iii. 51 iv. 6 v. 11 vi. 5 3. i. 3 ii. 3 iii. 6 iv. 5
4. i. 9; 3 ii. 62; 27 iii. 10; 5 iv. 14; 4
5. i. 4368 ii. 3003 6. 55 7. i. 120 ii. 186 iii. 186 8. i. 45 ii. 120
9. ${}^nC_2 - n$ 10. 120 11. 10 12. 300500200 13. 63 14. 175616

REVIEW EXERCISE

1. i. b ii. c iii. a iv. a v. d vi. d vii. b viii. d ix. c x. c
2. 24 3. 90 4. 600 5. 360 6. 32,659,200

Unit 7: Mathematical Induction and Binomial Theorem

Exercise 7.2

1. i. $32x^{\frac{5}{2}} + 80x^{\frac{3}{2}} + 80\sqrt{x} + \frac{40}{\sqrt{x}} + \frac{10}{x^{\frac{3}{2}}} + \frac{1}{x^{\frac{5}{2}}}$

   ii. $\frac{729}{x^6} + \frac{729y}{x^5} + \frac{1215y^2}{4x^4} + \frac{135y^3}{2x^3} + \frac{135y^4}{16x^2} + \frac{9y^5}{16x} + \frac{y^5}{64}$

   iii. $128 - 448x^{\frac{3}{2}} + 672x^3 - 560x^{\frac{9}{2}} + 280x^6 - 84x^{\frac{15}{2}} + 14x^9 - x^{\frac{21}{2}}$

   iv. $\frac{x^{10}}{y^{10}} - 5\frac{x^{\frac{15}{2}}}{y^{\frac{15}{2}}} + 10\frac{x^5}{y^5} - 10\frac{x^{\frac{5}{2}}}{y^{\frac{5}{2}}} + 5\frac{y^{\frac{5}{2}}}{x^{\frac{5}{2}}}$

2. i. $\frac{32x^5}{243} - \frac{40x^3}{27} + \frac{20x}{3} - \frac{15}{x} + \frac{45}{x^2} - \frac{243}{32x^5}$

   ii. $x^6 - 6\frac{x^5}{y} + \frac{15x^4}{y^2} - \frac{20x^3}{y^3} + \frac{15x^2}{y^4} - \frac{6x}{y^5} + \frac{1}{y^6}$

   iii. $2187u^7 - 5103u^6 + 5103u^5 - 2835u^4 + 945u^3 - 189u^2 + 21u - 1$

   iv. Iv. $a^5 2^{\frac{5}{2}} + 20ab\sqrt{3} + 30ab^2\sqrt{2} + 20ab^3\sqrt{3} + 45ab^4\sqrt{2} + b^5 3^{\frac{5}{2}}$

   v. V. $1 + 8x - 4y + 24x^2 - 24xy + 6y^2 + 32x^3 - 12x^2y + 6xy^3 - y^3 + 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

   vi. Vi. $\frac{1}{x^4} + \frac{8}{x^3y} + \frac{24}{x^2y^2} + \frac{32}{xy^3} + \frac{16}{y^4} + \frac{12}{zx^3} + \frac{72}{x^2yz} + \frac{144}{xy^2z} + \frac{96}{y^3z} + \frac{54}{x^2z^2} + \frac{216}{xyz^2} + \frac{216}{y^2z^2} + \frac{108}{xz^3} + \frac{216}{yz^3} + \frac{81}{x^4}$

3. i. $2 + 1200x^2 + 20000x^4$ ii. $8x^2 - 12x + 2 - \frac{3}{x}$

   iii. $1 - 3x + 2x^2 + 10x^3 + 5x^4 - 11x^5 + 8x^6 - 2x^7$

   iv. 2.01090301

   v. $\frac{32}{x^4} + 2 + \frac{x^4}{128}$ vi. $8a^6\sqrt{a^2-1} + 8a^2\left(\sqrt{a^2-1}\right)^3$

4. $\frac{15}{16}x^6y^7$

5. i. $-\frac{15309}{8}x^5$ ii. $0.946176x^4$ iii. $70a^4$ iv. $\frac{673596a^6}{x^{12}}$

6. i. $\frac{15}{2}a^{14}b^6$ ii. $\frac{40095}{16}p^{16}q^8$ iii. $-12x^4y^3$ iv. $-270y^8x^3$

7. i. 1980 ii. 5

8. $T_{r+2} = \binom{n}{r+1}a^{n-r-1}b^{r+1}$

10. -5940 11. 14 13. 12, 2 18. 1, 6, 15, 20, 15, 6, 1

19. 21 times 20. 56 times

Exercise 7.3

1. i. $1 + 3x^{\frac{1}{2}} + 6x + 10x^{\frac{3}{2}}$ $0 < x < 1$

   ii. $\frac{1}{3^{\frac{1}{3}}} - \frac{2}{3^{\frac{7}{3}}x} + \frac{24}{3^{\frac{13}{3}}x^2} - \frac{80}{3^{\frac{22}{3}}x^3}$ $-\frac{2}{3} > x > \frac{2}{3}$

   iii. $\left(\frac{5}{2}\right)^{2} - \frac{3}{\sqrt{2}\sqrt{5}x^2} - \frac{3^2}{2^{\frac{3}{2}}5^{\frac{3}{2}}x^4} + \frac{3^3}{2^{\frac{3}{2}}5^{\frac{7}{2}}x^6}$ $|x^2| > \frac{6}{5}$

   iv. $1 + \frac{2x}{3} + \frac{2x^2}{3^2} + \frac{2x^3}{3^3} + \frac{x^4}{3^4}$ $-3 < x < 3$

v. $\frac{1}{3}-\left(\frac{1}{4.3^{\frac{3}{2}}}+\frac{2}{\sqrt{3}}\right)x+\left(\frac{1}{32.3^{\frac{5}{2}}}+\frac{1}{128.3^{\frac{3}{2}}}\right)x^2-\left(\frac{5}{128.3^{\frac{7}{2}}}+\frac{1}{16.3^{\frac{3}{2}}}\right)x^4$ $\quad -6 < x < 6$ $\quad$ vi. 1

2. i. 2.0052 ii. 1.2963 iii. 1.0099 iv. 0.9859 3. $\frac{315\sqrt{2}}{4096}$

7. $\frac{(-1)^n}{2}(n^2+7n+8)x^n$

8. i. $\frac{1}{\sqrt[3]{4}}$ ii. $\left(\frac{5}{6}\right)^{\frac{1}{3}}$ iii. $(-3)^{\frac{6}{5}}$ iv $2\sqrt{2}$

Exercise 7.4

1. 1, 8, 4, 7 2. a. 5 b. 2 3. 1

4. a. 5, 25 b. 3, 33 c. 1, 01 8. 8 9. 8

REVIEW EXERCISE

1. i. d ii. c iii. d iv. d v. b vi. c vii. b viii. a ix. b x. b

3. 232 6. 78

Unit 8: Fundamentals of Trigonometry

EXERCISE 8.1

1. (i) $\cos(180^\circ+60^\circ)=-\cos 60^\circ$, $\cos(180^\circ-60^\circ)=-\cos 60^\circ$, $\sin(180^\circ+60^\circ)=-\sin 60^\circ$, $\sin(180^\circ-60^\circ)=\sin 60^\circ$, $\tan(180^\circ+60^\circ)=\tan 60^\circ$, $\tan(180^\circ-60^\circ)=-\tan 60^\circ$

(ii) $\cos(90^\circ+60^\circ)=-\sin 60^\circ$, $\cos(90^\circ-60^\circ)=\sin 60^\circ$, $\sin(90^\circ+60^\circ)=\cos 60^\circ$, $\sin(90^\circ-60^\circ)=\cos 60^\circ$, $\tan(90^\circ+60^\circ)=-\cot 60^\circ$, $\tan(90^\circ-60^\circ)=\cot 60^\circ$

(iii) $\cos(180^\circ+30^\circ)=-\cos 30^\circ$, $\cos(180^\circ-30^\circ)=-\cos 30^\circ$, $\sin(180^\circ+30^\circ)=-\sin 30^\circ$, $\sin(180^\circ-30^\circ)=\sin 30^\circ$, $\tan(180^\circ+30^\circ)=\tan 30^\circ$, $\tan(180^\circ-30^\circ)=-\tan 30^\circ$

(iv) $\cos(\pi+\frac{\pi}{3})=-\cos\frac{\pi}{3}$, $\cos(\pi-\frac{\pi}{3})=-\cos\frac{\pi}{3}$, $\sin(\pi+\frac{\pi}{3})=-\sin\frac{\pi}{3}$, $\sin(\pi-\frac{\pi}{3})=\sin\frac{\pi}{3}$, $\tan(\pi+\frac{\pi}{3})=\tan\frac{\pi}{3}$, $\tan(\pi-\frac{\pi}{3})=-\tan\frac{\pi}{3}$

(v) $\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)$, $\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{6}\right)$, $\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)$, $\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)$, $\tan\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=-\cot\left(\frac{\pi}{6}\right)$, $\tan\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\cot\left(\frac{\pi}{6}\right)$

(vi) $\cos\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)$, $\cos\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$, $\tan\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)=-\cot\left(\frac{\pi}{4}\right)$, $\tan\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)=\cot\left(\frac{\pi}{4}\right)$

2. (a) $\cos 15^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$ (b) $\cos 165^\circ=-\frac{1+\sqrt{3}}{2\sqrt{2}}$ (c) $\cos 345^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$ (d) $\sin 75^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$

3. (a) $\cos 120^\circ = -\frac{1}{2}$ (b) $\sin 120^\circ = \frac{\sqrt{3}}{2}, \tan 120^\circ = -\sqrt{3}$ (c) $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(d) $\cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}}$ (e) $\cos 285^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (f) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

4. (i) $\cos 9\theta$ (ii) $\cos 5\theta$ (iii) $\sin\frac{\theta}{2}$ (iv) $\sin 92^\circ$ (v) $\tan 30^\circ$ (vi) $\tan 2\pi$

5. $\cos(\alpha+\beta) = \frac{16}{65}$, $\cos(\alpha-\beta) = \frac{56}{65}$

6. (i) $\sin(\alpha-\beta) = \frac{416}{425}$ (ii) $\cos(\alpha-\beta) = \frac{87}{425}$ (iii) $\tan(\alpha-\beta) = \frac{416}{87}$

7. (i) $\sin(\alpha+\beta) = \frac{56}{65}$ (ii) $\cos(\alpha+\beta) = -\frac{33}{65}$ (iii) $\tan(\alpha+\beta) = -\frac{56}{33}$

8. (i) $\csc(\alpha+\beta) = \frac{65}{16}$ (ii) $\sec(\alpha+\beta) = \frac{65}{63}$ (iii) $\cot(\alpha+\beta) = \frac{63}{16}$

9. (i) $\sin(\alpha+\beta) = -\frac{7}{5\sqrt{2}}, \sin(\alpha-\beta) = \frac{1}{5\sqrt{2}}$ (ii) $\cos(\alpha+\beta) = -\frac{1}{5\sqrt{2}}, \cos(\alpha-\beta) = \frac{7}{5\sqrt{2}}$

(iii) $\tan(\alpha+\beta) = 7, \tan(\alpha-\beta) = \frac{1}{7}$

13. (i) $12\sin\theta - 5\cos\theta = r\sin(\theta+\varphi)$ where $r = 13$ and $\varphi = \tan^{-1}\left(-\frac{5}{12}\right)$

(ii) $3\sin\theta + 4\cos\theta = r\sin(\theta+\varphi)$ where $r = 5$ and $\varphi = \tan^{-1}\left(\frac{4}{3}\right)$ (iii) Do yourself.

14. (a) $\alpha = 45^\circ$ (b) $\sin\theta = \frac{7}{\sqrt{58}}, \cos\theta = \frac{3}{\sqrt{58}}$ (c) 0.9285 (d) 22°

EXERCISE 8.2

1. $\cos 2\theta = -\frac{7}{25}$, $\sin 2\theta = -\frac{24}{25}$, III quadrant

2. $\sin 2\alpha = -2y\sqrt{1-y^2}$, $\cos 2\alpha = 1-2y^2$, $\tan 2\alpha = -\frac{2y\sqrt{1-y^2}}{1-2y2}$ 3. $\cos 15^\circ = \sqrt{\frac{1+\cos 30^\circ}{2}} = 0.966$

4. (i) $\sin 2\theta = \frac{24}{25}, \cos 2\theta = -\frac{7}{25}, \tan 2\theta = -\frac{24}{7}, \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}}, \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}}, \tan\frac{\theta}{2} = \frac{1}{2}$

(ii) $\sin 2\theta = \frac{120}{169}, \cos 2\theta = -\frac{119}{169}, \tan 2\theta = \frac{120}{119}, \sin\frac{\theta}{2} = \frac{3}{\sqrt{13}}, \cos\frac{\theta}{2} = \frac{-2}{\sqrt{13}}, \tan\frac{\theta}{2} = -\frac{3}{2}$

(iii) $\sin 2\theta = -\frac{336}{625}, \cos 2\theta = \frac{527}{625}, \tan 2\theta = -\frac{336}{527}, \sin\frac{\theta}{2} = \frac{1}{5\sqrt{2}}, \cos\frac{\theta}{2} = \frac{-7}{5\sqrt{2}}, \tan\frac{\theta}{2} = -\frac{1}{7}$

(iv) $\sin 2\theta = -\frac{4}{5}, \cos 2\theta = -\frac{3}{5}, \tan 2\theta = \frac{4}{3}, \sin\frac{\theta}{2} = \sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}, \cos\frac{\theta}{2} = -\sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}, \tan\frac{\theta}{2} = -\sqrt{\frac{\sqrt{5}-1}{\sqrt{5}+1}}$

(v) $\sin 2\theta = -1, \cos 2\theta = 0, \tan 2\theta =$ undefined, $\sin\frac{\theta}{2} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}, \cos\frac{\theta}{2} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}, \tan\frac{\theta}{2} = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}$

(vi) $\sin 2\theta = -\frac{\sqrt{3}}{2}, \cos 2\theta = -\frac{1}{2}, \tan 2\theta = \sqrt{3}, \sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}, \cos\frac{\theta}{2} = \frac{1}{2}, \tan\frac{\theta}{2} = \sqrt{3}$

5. (i) $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$ (ii) $\sin\theta = \frac{4}{5}, \cos\theta = -\frac{3}{5}, \tan\theta = -\frac{4}{3}$

(iii) $\sin\theta = \frac{15}{17}, \cos\theta = -\frac{8}{17}, \tan\theta = -\frac{15}{8}$ (iv) $\sin\theta = \frac{7}{13\sqrt{2}}, \cos\theta = -\frac{17}{13\sqrt{2}}, \tan\theta = -\frac{7}{17}$

6. (i) $\frac{1}{4}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $\frac{\sqrt{3}}{2}$ (v) $\frac{1}{\sqrt{3}}$

7. (i) $\frac{1-\cos 4\alpha}{8}$ (ii) $\frac{1}{16}[1 - 7\cos 2\alpha - \cos 4\alpha + \cos 2\alpha\cos 4\alpha]$ (iii) $\frac{1}{128}[3 - 4\cos 4\alpha + \cos 8\alpha]$

EXERCISE 8.3

1. (i) $2[\sin 26x + \sin 6x]$ (ii) $5[\cos 16y + \cos 4y]$ (iii) $\sin 8t - \sin 2t$
   (iv) $3[\sin 15x + \sin 5x]$ (v) $\frac{1}{2}[\cos 6u - \cos 4u]$ (vi) $\cos 120^\circ - \cos 80^\circ$
   (vii) $\frac{1}{2}[\sin 40^\circ - \sin 6^\circ]$ (viii) $\sin 104^\circ - \sin 8^\circ$ (ix) $\cos 60^\circ - \cos 90^\circ$
   (x) $2[\sin u + \sin v]$ (xi) $\sin 2u - \sin 2v$
2. (i) $2 \sin 50^\circ \cos 20^\circ$ (ii) $2 \cos 45^\circ \sin 31^\circ$ (iii) $2 \cos 35^\circ \cos 23^\circ$
   (iv) $2 \cos \frac{p}{2} \cos \frac{q}{2}$ (v) $-2 \sin 15^\circ \cos 5^\circ$

REVIEW EXERCISE

1. (i) a (ii) b (iii) c (iv) d (v) b (vi) a
   (vii) c (viii) d (ix) b (x) c (xi) d (xii) a
2. (i) $\frac{56}{65}$ (ii) $-\frac{56}{33}$ (iii) $-\frac{16}{33}$
3. (i) $\sin(\beta + 45^\circ)$ or $\cos(\beta - 45^\circ)$ (ii) $\sin 120^\circ$ or $\cos 30^\circ$
4. (i) $\tan 60^\circ = 1.732$ (ii) $\cos 90^\circ = 0$ 5. $\tan\theta = 2$ 9. 1

Unit 9: Trigonometric Functions

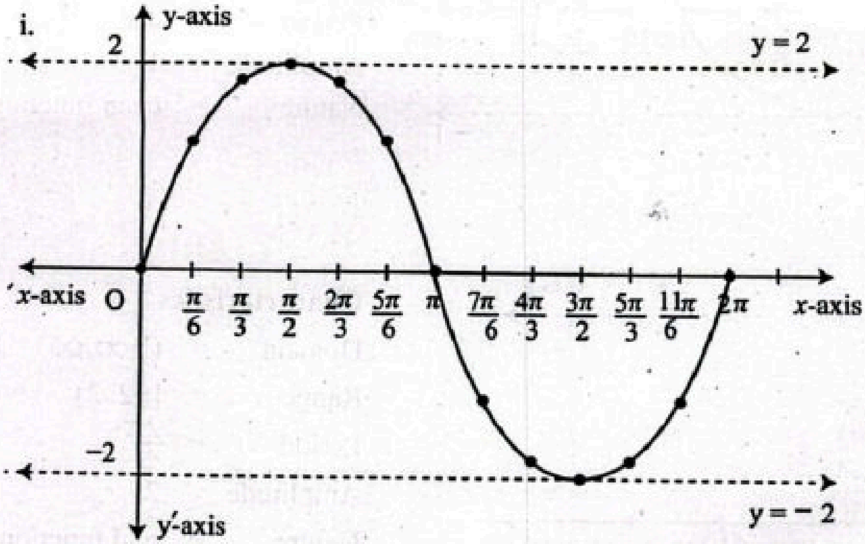
EXERCISE 9.1

1. i. Maximum value (M) $= 4$; Minimum value (m) $= 0$
   ii. Maximum value (M) $= \frac{7}{6}$; Minimum value (m) $= \frac{1}{6}$
   iii. Maximum value (M) $= \frac{11}{5}$; Minimum value (m) $= -\frac{9}{5}$
   iv. Maximum value (M) $= \frac{38}{5}$; Minimum value (m) $= \frac{32}{5}$
2. i. Maximum value (M) $= 1$; Minimum value (m) $= \frac{1}{7}$
   ii. Maximum value (M) $= \frac{2}{11}$; Minimum value (m) $= -\frac{2}{9}$
   iii. Maximum value (M) $= \frac{3}{13}$; Minimum value (m) $= -\frac{3}{11}$
   iv. Maximum value (M) $= \frac{5}{13}$; Minimum value (m) $= \frac{5}{17}$
3. i. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-7, 7]$
   ii. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-1, 1]$
   iii. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-1, 1]$
   iv. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = 0
   v. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = 0
   vi. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-6, 6]$
4. i. π ii. $\frac{2\pi}{5}$ iii. 4π iv. $\frac{2\pi}{3}$ v. π vi. 2π

EXERCISE 9.2

1. i. odd ii. even iii. even iv. even
   v. even vi. even vii. odd viii. odd

2. i.



Characteristics

Domain = $(-\infty, \infty) = R$

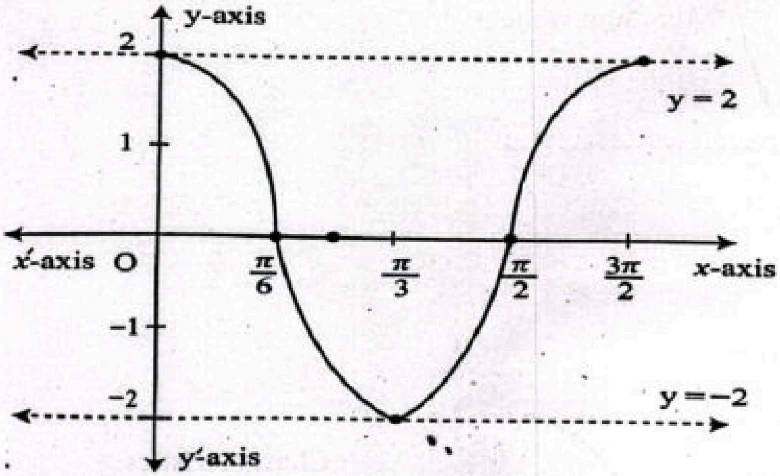
Range = $[-2, 2]$

Period = 2π

Amplitude = 2

Nature = odd function

ii.



Characteristics

Domain = $(-\infty, \infty) = R$

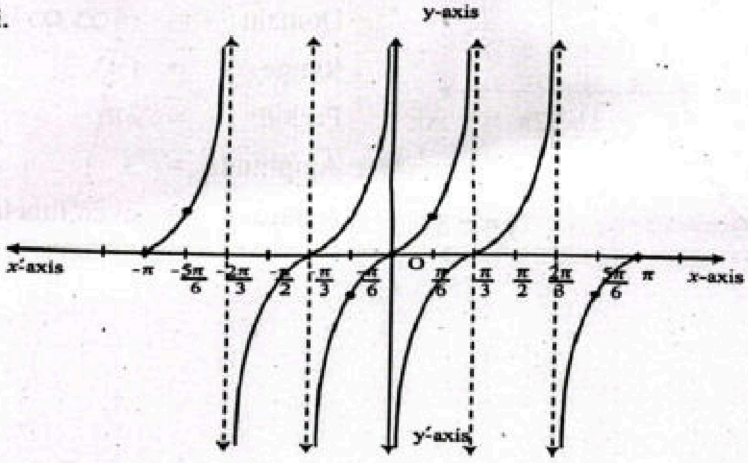
Range = $[-2, 2]$

Period = $\frac{2\pi}{3}$

Amplitude = 2

Nature = even function

iii.



Characteristics

Domain = $\frac{\pi n}{2} < x < \frac{\pi}{4} + \frac{\pi n}{2}$

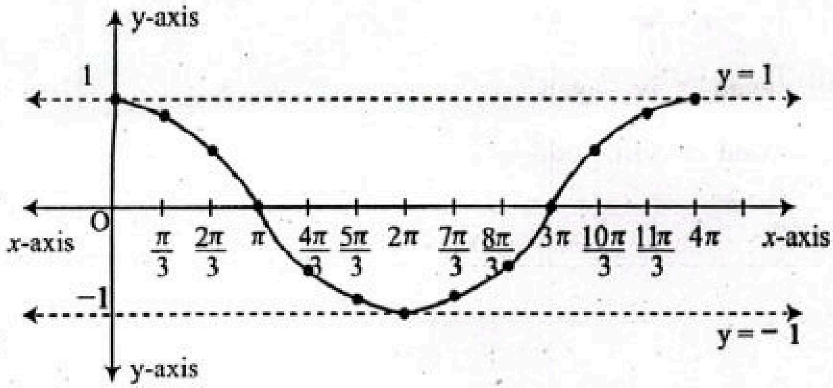
Range = $]-\infty, \infty[$

Period = $\frac{\pi}{2}$

Amplitude = Nil

Nature = odd function

iv.



Characteristics

Domain = $(-\infty,\infty) = R$

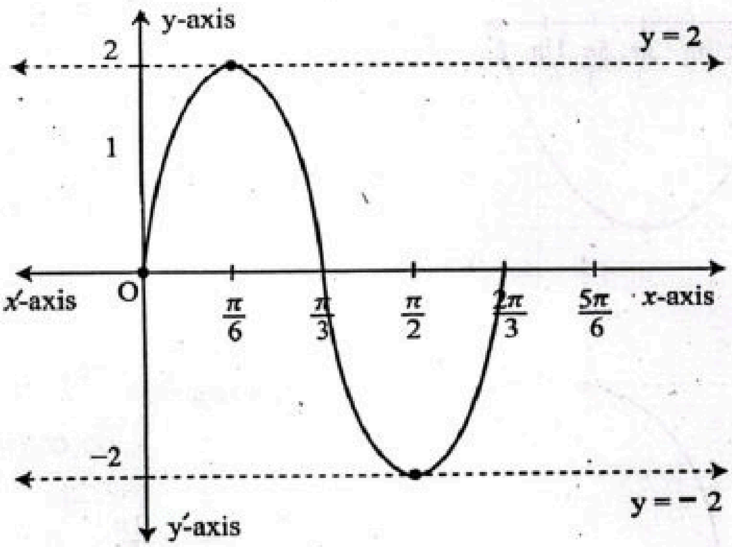
Range = $[-1, 1]$

Period = 4π

Amplitude = 1

Nature = even function

v.



Characteristics

Domain = $(-\infty,\infty) = R$

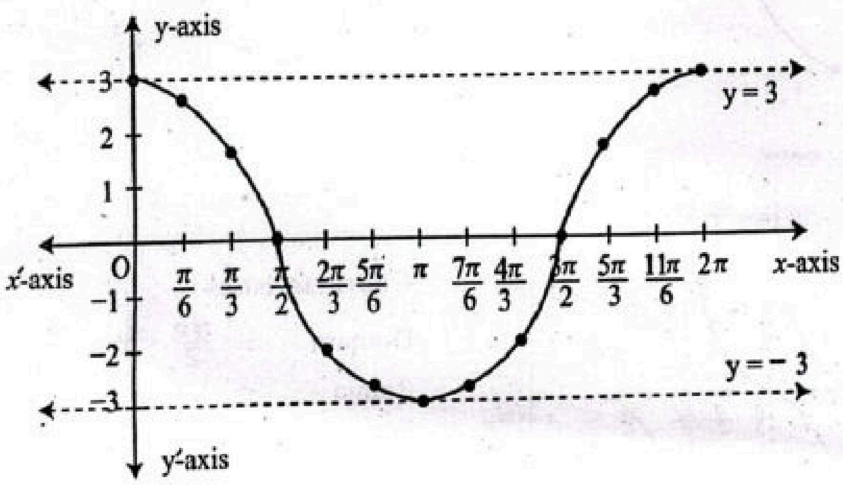
Range = $[-2, 2]$

Period = $\frac{2\pi}{3}$

Amplitude = 2

Nature = odd function

vi.



Characteristics

Domain = $(-\infty,\infty) = R$

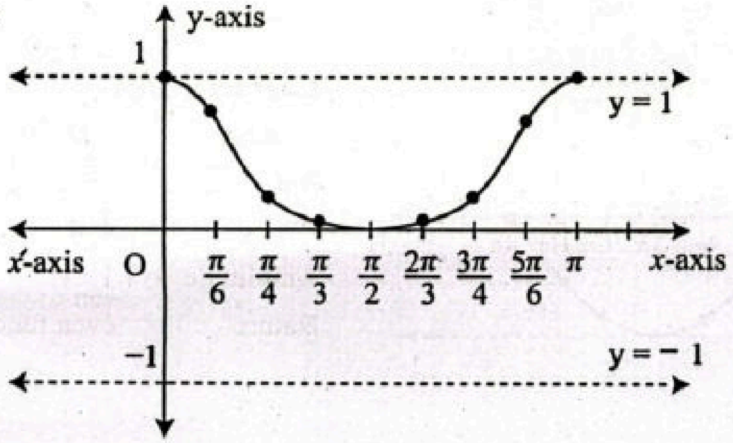
Range = $[-3, 3]$

Period = 2π

Amplitude = 3

Nature = even function

vii.



Characteristics

Domain = $(-\infty,\infty) = R$

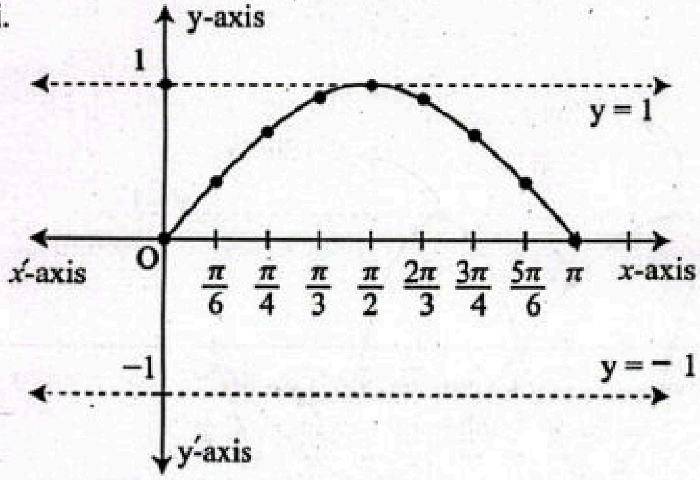
Range = $[0, 1]$

Period = π

Amplitude = 1

Nature = even function

viii.



Characteristics

Domain = $(-\infty,\infty) = R$

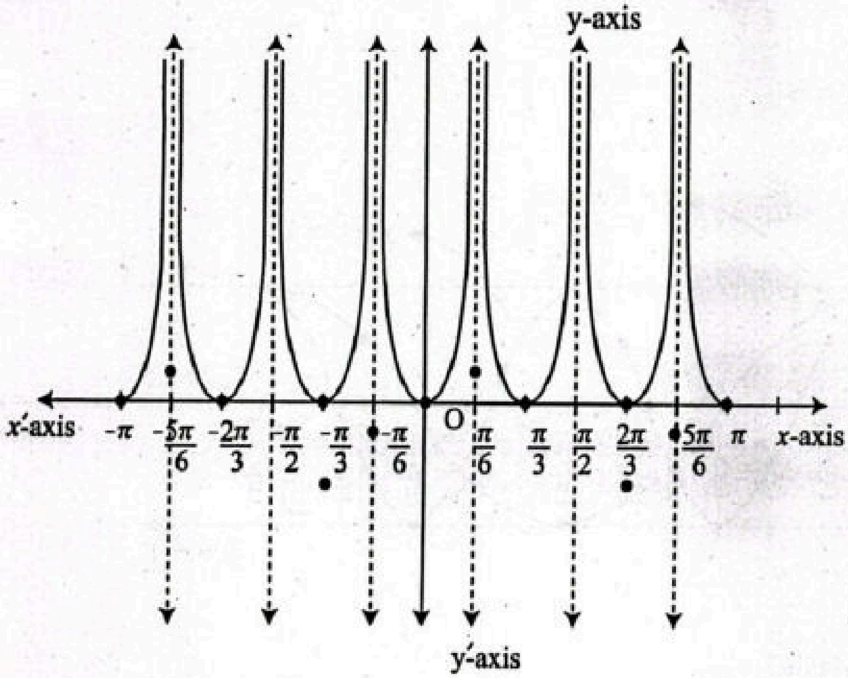
Range = $[0, 1]$

Period = π

Amplitude = 1

Nature = even function

ix.



Characteristics

Domain = $\pi n \le x < \frac{\pi}{2} + \pi n$

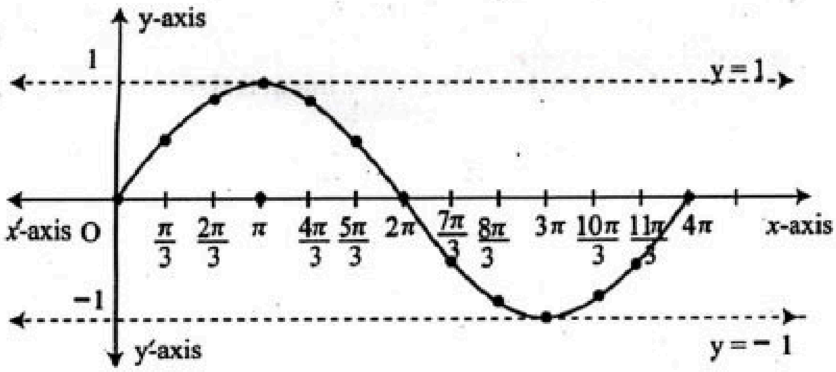
Range = $f(x) \geq 0$

Period = π

Amplitude = Nil

Nature = even function

x.



Characteristics

Domain = $(-\infty, \infty) = R$

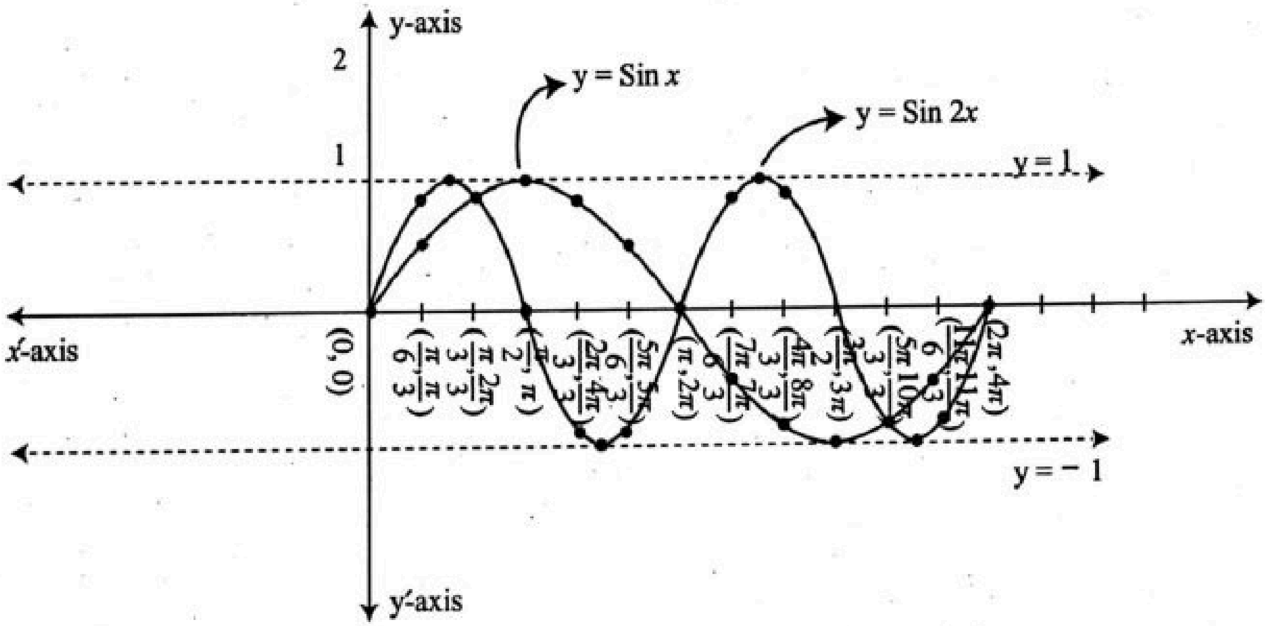
Range = $[-1, 1]$

Period = 4π

Amplitude = 1

Nature = even function

3.



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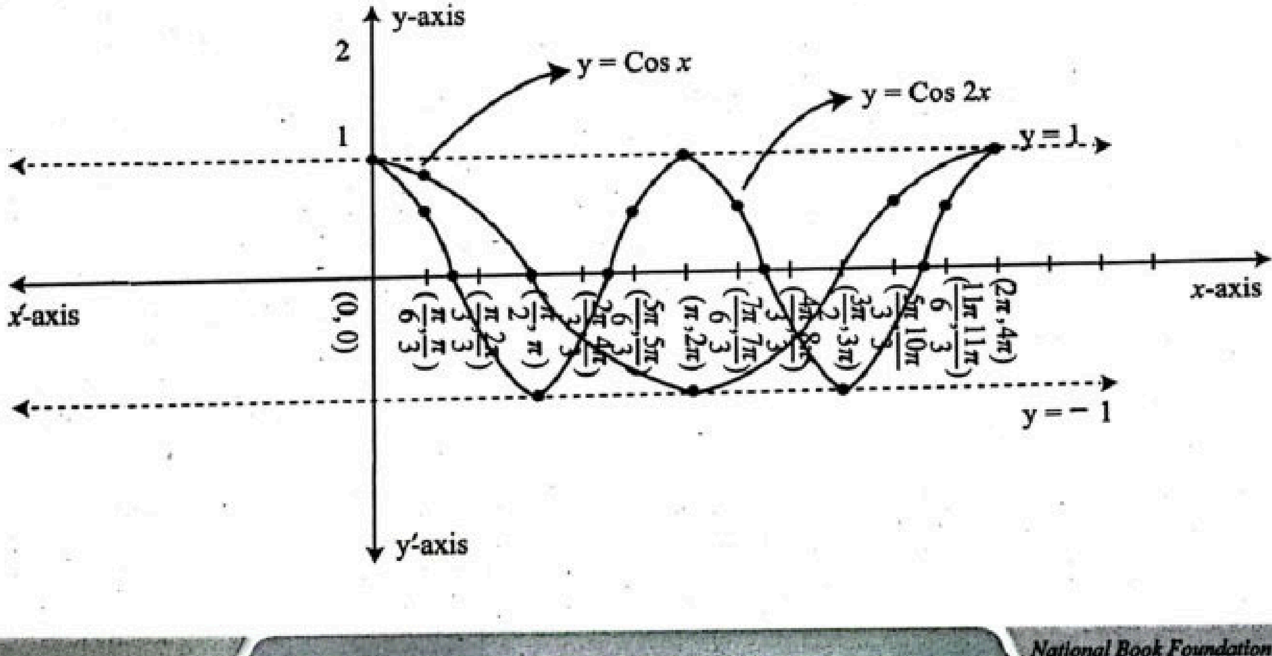
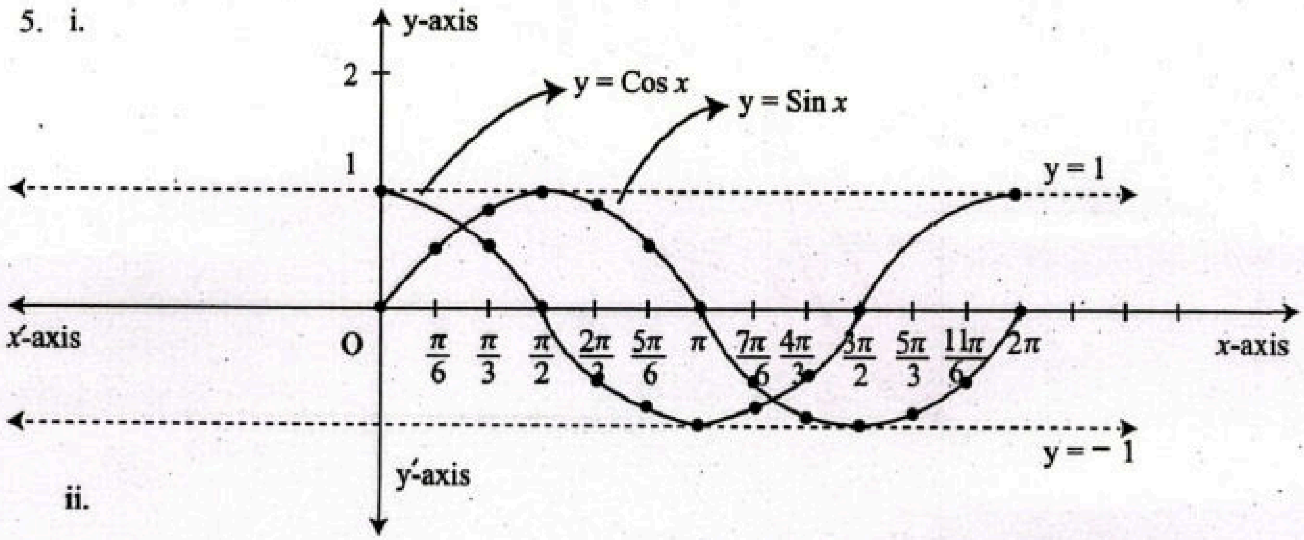
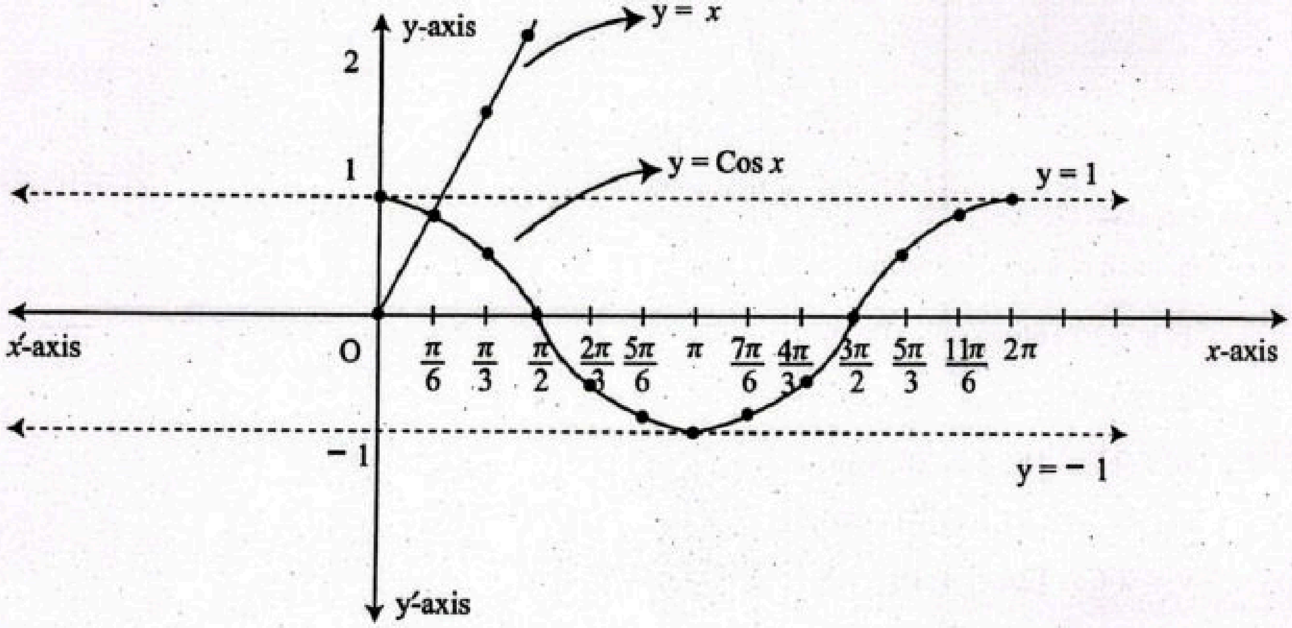


5. i.



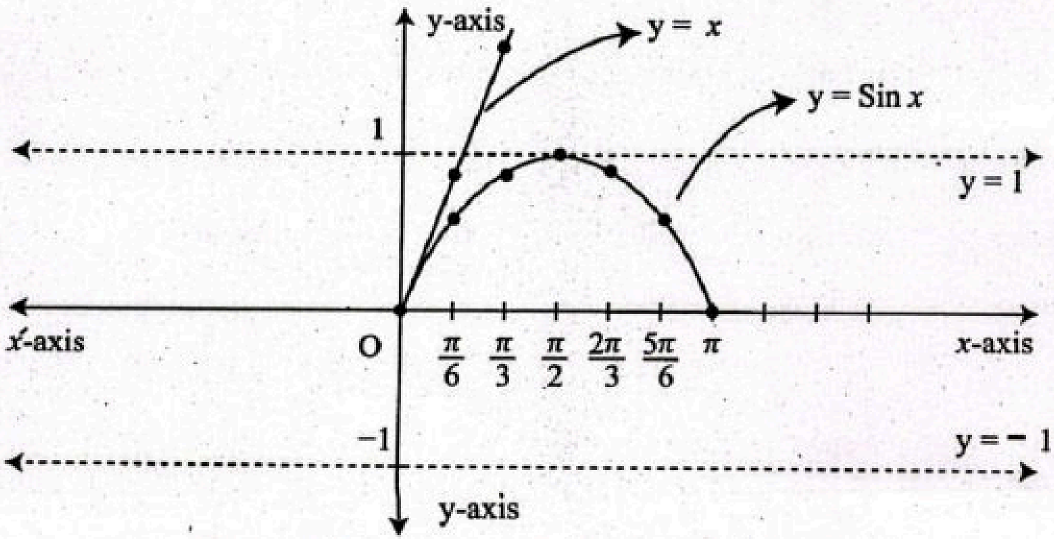
y-axis
2
y = Cos x
y = Sin x
1
y = 1
x'-axis
O
$\frac{\pi}{6}$ $\frac{\pi}{3}$ $\frac{\pi}{2}$ $\frac{2\pi}{3}$ $\frac{5\pi}{6}$ π $\frac{7\pi}{6}$ $\frac{4\pi}{3}$ $\frac{3\pi}{2}$ $\frac{5\pi}{3}$ $\frac{11\pi}{6}$ 2π
x-axis
y = − 1
y'-axis

ii.



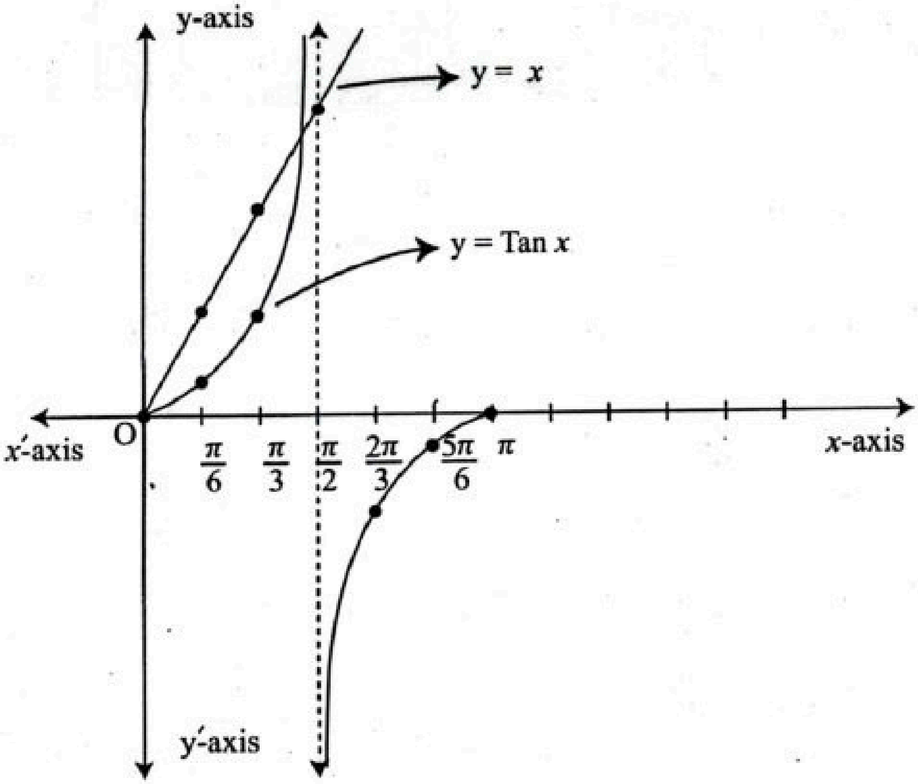
y-axis
y = x
2
y = Cos x
1
y = 1
x'-axis
O
$\frac{\pi}{6}$ $\frac{\pi}{3}$ $\frac{\pi}{2}$ $\frac{2\pi}{3}$ $\frac{5\pi}{6}$ π $\frac{7\pi}{6}$ $\frac{4\pi}{3}$ $\frac{3\pi}{2}$ $\frac{5\pi}{3}$ $\frac{11\pi}{6}$ 2π
x-axis
− 1
y = − 1
y'-axis

iii.



y-axis
y = x
y = Sin x
1
y = 1
x'-axis
O
$\frac{\pi}{6}$ $\frac{\pi}{3}$ $\frac{\pi}{2}$ $\frac{2\pi}{3}$ $\frac{5\pi}{6}$ π
x-axis
−1
y = − 1
y-axis

iv.



6. a. each cycle is $\frac{1}{56}$ second

b. k = 20,160

c. 180

d. $V(t) = 180\ Sin\ 20{,}160t$

7. $h(t) = -8.5\ Cos\left(\frac{\pi t}{25}\right) + 10.5$

8. $y = 2\ Cos\ [2\pi x] + 18$

REVIEW EXERCISE

1.

i. b	ii. a	iii. c	iv. a	v. c
vi. a	vii. a	viii. b	ix. c	x. a
xi. a	xii. b	xiii. c	xiv. b	xv. a
xvi. c	xvii. d	xviii. a		

5. a. Period $= \frac{\pi}{6}$; Amplitude $= -20$; Vertical shift $= 24$

   b. $h(t) = -20 \, Cos\,(\frac{\pi t}{8}) + 24$

   c. The height is 32 m after 5minutes.

6. a. Maximum height = 22 m, Minimum height = 2 m.
   b. The height is 12 m after 30 seconds.
   c. One complete revolution takes place in 120 seconds.

7. a. $y = 10\,Sin\,1440\,t^{\circ}, y = 10\,Sin\,1440\,(t - \frac{1}{16})^{\circ}$

   b. Domain $= \{t \,/\, t \geq 0, t \in R\}$, Range $= \{y / -10 \leq y \leq 10, y \in R\}$

8.

	Domain	Range	Period
i.	Domain $=]-\infty, \infty[= R$	Range $= [-2, 2]$	6π
ii.	Domain $=]-\infty, \infty[= R$	Range $= [-5, 5]$	$\frac{2\pi}{3}$
iii.	Domain $=]-\infty, \infty[= R$	Range $= [-\frac{1}{2}, \frac{1}{2}]$	3π
iv.	Domain $=]-\infty, \infty[= R$	Range $= [-\frac{5}{3}, \frac{5}{3}]$	$\frac{3\pi}{2}$
v.	Domain $=]-\infty, \infty[= R$	Range $= [-3, 3]$	2
vi.	Domain $=]-\infty, \infty[= R$	Range $= [-7, 7]$	$\frac{2\pi}{5}$
vii.	Domain $R - \frac{3n}{2}, n \in Z$	Range $=]-\infty, \infty[$	$\frac{3}{2}$
viii.	Domain $=]-\infty, \infty[= R$	Range $= [-9, 9]$	$\frac{2\pi}{3}$
ix.	Domain $=]-\infty, \infty[= R$	Range $= [7, 9]$	$\frac{\pi}{2}$
x.	Domain $=]-\infty, \infty[= R$	Range $= [2, 12]$	π
xi.	Domain $=]-\infty, \infty[= R$	Range $= [2, 10]$	π

G L O S S A R Y

Adjoint of a matrix: A matrix of order 2, obtained by interchanging diagonal elements and changing the signs of non-diagonal elements.

Algebraic expression: A statement in which variables or constants or both are connected by arithmetic operations (i.e. $+, -, \times, \div$).

Allied angles: The angles connected with basic angles of measure θ by a right angle or its multiple, are called allied angles.

Arithmetic mean: A number M is said to be arithmetic mean between two numbers a and b if a, M, b are in A.P.

Arithmetic sequence: An arithmetic sequence is a sequence in which each term, after the first, is found by adding a constant.

Arithmetic series: The sum of the terms of an arithmetic sequence is called an arithmetic series.

Arithmetico–geometrico sequence: This sequence is the result of term-by-term multiplication of a geometric progression with the corresponding terms of arithmetic progression.

Column: The vertical arrangement of objects.

Column matrix: A matrix having only one column.

Combination: If in the arrangements of objects their order is not important then this arrangement of objects is called combination.

Complex number: The number of the form $a + ib$, where a and b are real number and $i = \sqrt{-1}$.

Complex polynomial: If z is a complex variable, then the expression $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is called complex polynomial of degree n if $a_n \neq 0$ and n is a non-negative integer.

Conformable for matrix addition: Matrices of same order so that they may be added.

Conformable for matrix multiplication: If number of columns of first matrix is equal to the number of rows of second matrix so that they may be multiplied in that order.

Conformable for matrix subtraction: Matrices of same order, so that they may be subtracted.

Conjugate: Two complex numbers differing only in the sign of their imaginary parts.

Constant polynomial: A polynomial having degree zero is called a constant polynomial.

Consistency criteria: A system of homogeneous linear equations is consistent if $Rank\ A = Rank\ A_b$.

Consistent system: A system of equations is consistent if it has at least one solution.

Cross product of vectors: The product of vectors resulting in a vector quantity.

Cubic polynomial: A polynomial having degree three is called a cubic

Deductive reasoning: Deductive reasoning is a logical approach where someone moves from general ideas to specific conclusions.

Determinant of a matrix: A number obtained by subtracting the product of non-diagonal elements from the product of diagonal elements, in a square matrix of order two.

Diagonal: A line joining any two vertices of a polygon that are not joined by any of its edges; elements running from the upper left corner to the lower right corner of a square matrix.

Diagonal matrix: A matrix in which all the non-diagonal elements are zero but at least one element of the diagonal is non-zero.

Direction angles: The angles that a non-zero vector $\vec{r}$ makes with the coordinate axes in the positive direction are known as direction angles of $\vec{r}$.

Direction cosines: Coines of direction angles are called direction cosines.

Domain of trigonometric functions: The domain of a function f(x) is the set of all possible values of 'x' such that function f(x) is defined.

Dot product of vectors: The product of vectors resulting in a scalar quantity.

Equal vectors: Two vectors $\vec{a}$ and $\vec{b}$are equal if both have the same magnitude and direction.

Equality of complex numbers: Two complex numbers are said to be equal if both have the same real and imaginary parts.

Equality of matrices: Two matrices are equal if both have the same order and the same corresponding elements.

Even function: A function is even if and only if $f(-x) = f(x)$.

Factor theorem: A polynomial $p(x)$ has a factor $x - c$, if and only if $p(c) = 0$.

Factorial: Factorial of an integer n is denoted by $n! = 1 \times 2 \times 3 \ldots (n-1)n$.

Fundamental law of trigonometry: This law is stated as: $cos(\alpha - \beta) = cos\,\alpha\, cos\,\beta + sin\,\alpha\, sin\,\beta$

Geometric mean: If a, G, b is in a geometric sequence, then G is called the geometric mean of a and b.

Geometric sequence: A geometric sequence is one in which each term after the first is found by multiplying the previous term by a constant called the common ratio.

Geometric series: The sum of the terms of a geometric sequence is called a geometric series.

Graphic solution: Method of solving two simultaneous equations by plotting the graph of each equation.

Harmonic mean: A number H is said to be the harmonic mean between two numbers a and b if a, H, b are in H.P.

Harmonic sequence: A sequence is called a harmonic sequence if the reciprocals of its terms are in an arithmetic sequence.

Imaginary part: The coefficient i in any complex number.

Inconsistent system: A system of equations that has no solution is called inconsistent.

Inductive reasoning: It is a method of reasoning in which general principle is derived from observations.

Inequality: The relation between two comparable quantities, which are not equal.

Irrational expression: An algebraic expression that is not rational is called an irrational expression.

Linear polynomial: A polynomial having degree one is called a linear polynomial.

Lower triangular matrix: A square matrix in which all the elements lie above the main diagonal are zero.

Matrix: A rectangular arrangement of numbers enclosed within square brackets.

Modulus of a complex number: It is the distance of a complex number from its origin.

Negative of a vector: A vector having the same magnitude but the opposite direction is called the negative of the given vector.

Non-singular matrix: A matrix with non-zero determinant.

Null matrix: A matrix with all entries to be zero.

Odd function: A function is odd if and only if $f(-x) = -f(x)$.

Order of a matrix: If a matrix has m number of rows and n number of columns then the order of the matrix is m-by- n.

Ordered pair: A pair set in which x is designated the first element and y the second, denoted by (x, y).

Parallel vectors: Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if $\vec{a} = \lambda\vec{b}$.

Periodic function: A periodic function is a function where values repeat after a specific time interval.

Periodicity: The periodic behavior of trigonometric functions is called periodicity.

Permutation: The arrangement of numbers or things in a definite order is called permutation.
Polynomial: Algebraic expressions consisting of one or more terms in which exponents of the variables involved are whole numbers.
Position vector: The vector used to specify the position of a point P with respect to the origin O is called the position vector of P.
Quadratic polynomial: A polynomial having degree two is called a quadratic polynomial.
Range of trigonometric functions: The range of a function $f(x)$ is the set of all possible values of the function $f(x)$ can take, when 'x' is any number from the domain of the function.
Rational expression: An algebraic expression of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.
Rectangular matrix: A matrix having an unequal number of rows and columns.
Remainder theorem: If a polynomial $p(x)$ is divided by $x - c$, then the remainder is $p(c)$.
Row: Horizontal arrangement of elements.
Row matrix: A matrix having only one row of elements.
Rule of product: If event A can happen in m ways and event B can happen in n ways then pair (A, B) can happen in $m \times n$ or mn ways.
Sequence: A sequence is an arrangement of objects or numbers in a particular order followed by some rule.
Scalar matrix: A diagonal matrix with equal diagonal elements.
Scalar quantity: A physical quantity that can be completely specified by its magnitude only.
Simultaneous equations: Set of equations satisfied by the same solution.
Singular matrix: A matrix with zero determinant.
Skew symmetric matrix: A matrix whose transpose is not equal to the matrix itself.
Solution of equations: The solution of an equation is the process of finding the values of the unknown involved in the equation.
Square matrix: A matrix having an equal number of rows and columns.
Symmetric matrix: A matrix whose transpose is equal to the matrix itself.
Terminating decimal fraction: A decimal fraction whose decimal part is finite.
Transpose of a matrix: A matrix obtained by interchanging rows and columns of a given matrix.
Triangular matrix: A square matrix that is either upper triangular or lower triangular is called a triangular matrix.
Triangular numbers: A triangular number counts objects arranged in an equilateral triangle.
Unit matrix: A diagonal matrix having all diagonal elements equal to one.
Unit vector: A vector that has magnitude 1 is called a unit vector.
Upper triangular matrix: A square matrix in which all the elements lying below the main diagonal are zero.
Vector quantity: A physical quantity that is completely specified by its magnitude and direction.
Zero matrix: A matrix having all elements equal to zero.
Zeros of a polynomial: A value of the variable for which the value of the polynomial is zero.
Zero polynomial: A polynomial having "0" as the only term.
Zero vector: A vector in which the initial and terminal points coincide.

SYMBOLS AND ABBREVIATIONS USED IN MATH

=	→	is equal to
≠	→	is not equal to
∈	→	is member of
∉	→	is not member of
ϕ	→	empty set
∪	→	union of sets
∩	→	intersection of sets
⇔	→	if and only if
$\overline{AB}$	→	line Segment AB
AB	→	measurement of side AB
∠A	→	measurement of angle A
≅	→	is congruent to
⊥	→	is perpendicular to
Δ	→	triangle
⇒	→	implies that
∧, &	→	and
∨	→	or
<	→	is less than
>	→	is greater than
≤	→	is less than or equal to
≥	→	is greater than or equal to
@	→	at the rate of
%	→	percent
π	→	Pie
:	→	ratio
::	→	proportion
∴	→	therefore, hence
∵	→	because, since
i.e.	→	that is
≈	→	approximately equal to
√	→	square root / radical
e.g.	→	for example
/	→	such that
↔	→	corresponding to
//	→	is parallel to
!	→	factorial
^nP_r	→	permutation
^nC_r	→	combination

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