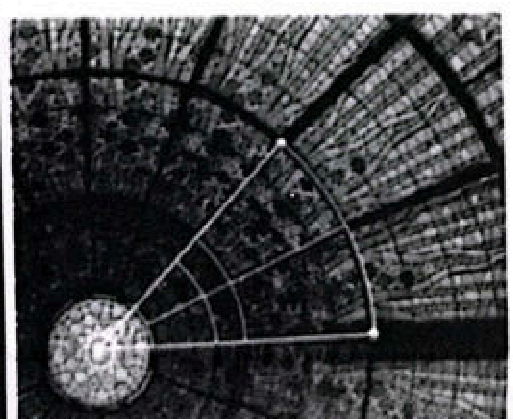
INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS

After studying this unit, students will be able to:

- Find domain and range of the principal trigonometric functions and inverse trigonometric functions.
- Draw graphs of the inverse trigonometric functions of cosine, sine, tangent, secant, cosecant and cotangent within the domain from -2π to 2π.
- State, prove and apply the addition and subtraction formulae of inverse trigonometric functions.
- Apply concepts of inverse trigonometric functions to real life word problems, such as mechanical engineering, architecture to find height of the building, angle of elevation and depression, identifying the angle of bridges to build scale models.

Inverse Trigonometric Functions play a crucial role in mathematics, particularly in calculus, geometry, and engineering, by enabling the determination of angles from known trigonometric function values. These functions are also essential in fields such as physics, engineering, and computer graphics. They are widely used to solve equations involving trigonometric functions, analyze periodic phenomena, and develop algorithms for 3D rendering and modeling. A solid understanding of these functions and their properties allows us to address complex problems in both theoretical and applied mathematics, making them a powerful tool in any mathematician's toolkit.

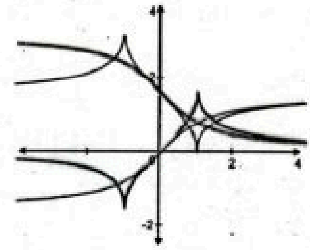


8.1 The Inverse Trigonometric Functions

Inverse trigonometric functions are the inverse operations of basic trigonometric functions cosine, sine, tangent, secant, cosecant, and cotangent.

The primary inverse trigonometric functions include:

- *Arccos* or Cos^{-1} the inverse of cosine function.
- *Arcsin* or Sin^{-1} the inverse of sine function.
- *Arctan* or Tan^{-1} the inverse of tangent function.
- *Arcsec* or Sec^{-1} the inverse of secant function.
- *Arccsc* or Csc^{-1} the inverse of cosecant function.
- *Arccot* or Cot^{-1} the inverse of cotangent function.



Overlay of all six trigonometric function on one graph

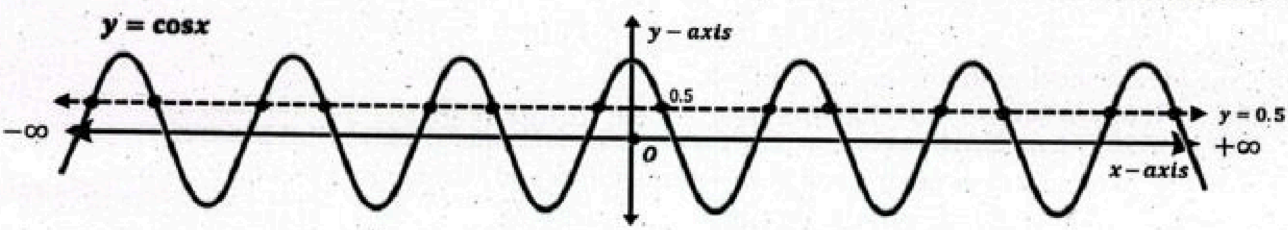
These functions are used to find the angle whose trigonometric function value is given. For example, if we know that the cosine of an angle is 0.5, we can use the arccosine function to find the angle which is 60° or $\frac{\pi}{3}$ radians.

8.1.1 The Principal Cosine Function

Consider the following graph of cosine function. You will observe that an infinite number of x values are found for which $y = \cos x = 0.5$ for each $x \in (-\infty, +\infty)$.

By this, one can record infinitely many solutions to the equation $\cos x = k$ where $k \in [-1, 1]$.

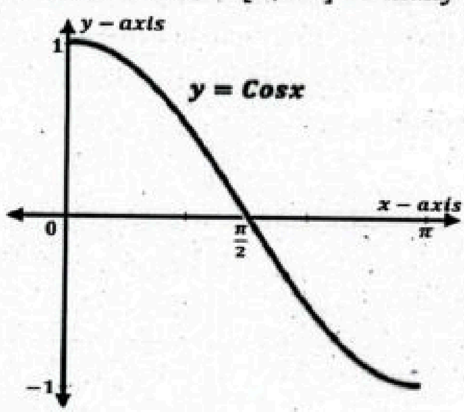
> Note that Principal and Inverse Trigonometric functions are written in capital letters; e.g., cosine function:
> $y = \cos x$ with $x\epsilon(-\infty, +\infty)$, $y\epsilon(-\infty, +\infty)$
> Principal Cosine function:
> $y = \text{Cos}\, x$ with $x\epsilon\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y\epsilon[-1, 1]$
> Cosine inverse function:
> $y = \text{Cos}^{-1} x$ with $x\epsilon[-1, 1]$, $y\epsilon\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



If we need an interval for x where there is only one solution to $\cos x = k$ for $k \in [-1, 1]$, we choose the interval from 0 to π. We have an option to consider an interval $x \in [\pi, 2\pi]$ or many others. It is a mathematical convention to choose $x \in [0, \pi]$.

In the interval $[0, \pi]$, we can find a **unique solution** to the equation $\text{Cos}\, x = k$, where $k \in [-1, 1]$.
We write this solution as $x = \text{Cos}^{-1} k$.
In other words, "x is a real number in the interval $[0, \pi]$ whose cosine is k".

The cosine function defined on $x \in [0, \pi]$ for which there is only **one solution** of the equation $\text{Cos}\, x = k$ where $k \in [-1, 1]$ is called the **Principal Cosine Function.**



Example 1: Find the principal value of $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$

Solution: Let $y = \text{Cos}^{-1}\left(-\frac{1}{2}\right)$ then $\text{Cos}\, y = -\frac{1}{2}$, where $y \in [0, \pi]$

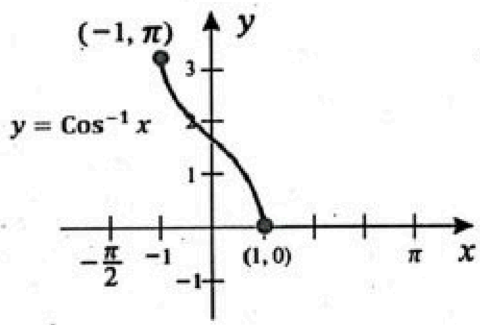
Consider $\text{Cos}\, y = -\frac{1}{2}$ [We need to find y whose cosine value is $-\frac{1}{2}$]

$\Rightarrow \text{Cos}\, y = \text{Cos}\left(\frac{2\pi}{3}\right)$ Since $\text{Cos}\, y < 0 \Rightarrow y$ lies in Quad II.

$\Rightarrow y = \frac{2\pi}{3} \Rightarrow \text{Cos}^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

8.1.2 The Inverse Cosine Function

For the cosine function $y = \text{Cos}\, x$ where $x \in [0, \pi]$ and $y \in [-1, 1]$, we define an inverse cosine function $y = \text{Cos}^{-1}(x)$ where $x \in [-1, 1]$ and $y \in [0, \pi]$.



For $x \in [-1, 1]$, $y = \text{Cos}^{-1} x$ is a real number in the interval $[0, \pi]$ whose cosine is x.

In view of above, we observe that:

$\text{Cos}^{-1}(1) = 0$ since $\text{Cos}(0) = 1$

$\text{Cos}^{-1}(0) = \frac{\pi}{2}$ since $\text{Cos}\left(\frac{\pi}{2}\right) = 0$

$\text{Cos}^{-1}(-1) = \pi$ since $\text{Cos}\, \pi = -1$

> If $y = \text{Cos}\, x$ with $x \in [0, \pi]; y \in [-1,1]$ then $y = \text{Cos}^{-1}(x)$ with $x \in [-1, 1]; y \in [0, \pi]$

8.1.3 The Domain and Range of Inverse Cosine Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the cosine function $y = \text{Cos}\, x$

$$\textit{Domain} = [0, \pi] \text{ and } \textit{Range} = [-1,1]$$

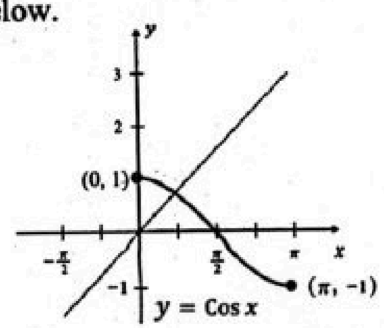
For the inverse cosine function $y = \text{Cos}^{-1} x$

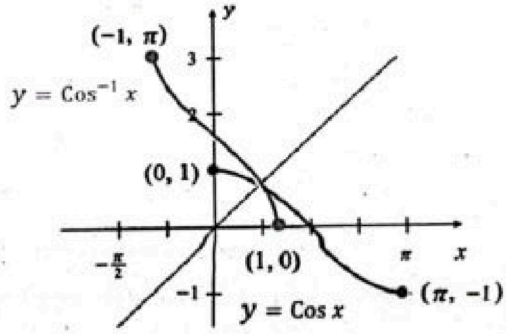
$$\textit{Domain} = [-1, 1] \text{ and } \textit{Range} = [0, \pi]$$

> The inverse trigonometric function's domain is a real number and its range is a real angle measured in degree or radian.

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown below.





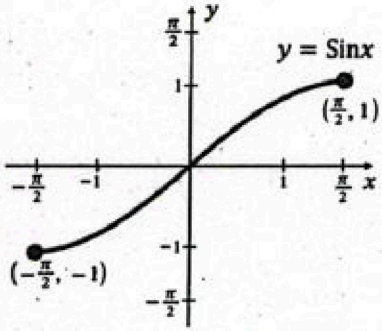
8.1.4 The Principal Sine Function

To define principal sine function, we need an interval for x where there is only one solution to $\sin x = k$ for $k \in [-1, 1]$. Such a solution is possible in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can find a **unique solution** to the equation $\text{Sin}\, x = k$, where $k \in [-1, 1]$. We write this solution as $x = \text{Sin}^{-1} k$.

In other words, "x is a real number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine value is k".

The sine function defined on $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which there is only **one solution** of the equation $\text{Sin}\, x = k$ where $k \in [-1, 1]$ is called the **Principal Sine Function.**



$\text{Sin}x = \sin x$; $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

- Sin^{-1} and **sin** are not inverses of each other (They do not cancel each other)
- **sin** does not have an inverse
- The functions that are inverses are Sin^{-1} and **Sin.**

Example 2: Find the principal value of $\text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution: Let $y = \text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ then $\text{Sin}\, y = -\frac{1}{\sqrt{2}}$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Consider $\text{Sin}\, y = -\frac{1}{\sqrt{2}}$ [We need to find y whose sine value is $-\frac{1}{\sqrt{2}}$.]

$\Rightarrow \text{Sin}\, y = \text{Sin}\left(-\frac{\pi}{4}\right)$ Since $\text{Sin}y < 0 \Rightarrow y$ lies in Quad-IV.

$\Rightarrow y = -\frac{\pi}{4}$

$\Rightarrow \text{Sin}^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

Check Point

Find the principal value of

(i) $\text{Sin}^{-1}\left(\frac{1}{2}\right)$

(ii) $\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$

8.1.5 The Inverse Sine Function

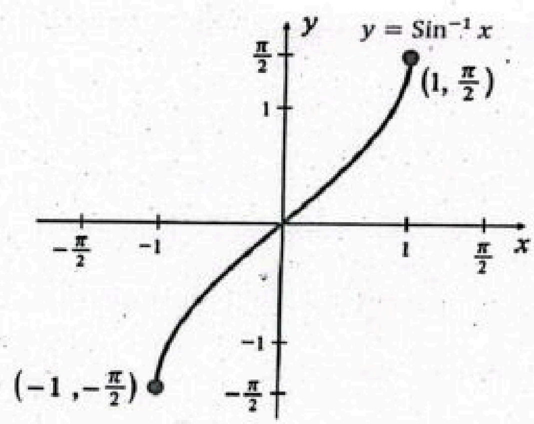
For the sine function $y = \text{Sin}\, x$ where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$ we define an inverse sine function $y = \text{Sin}^{-1}(x)$ where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

In view of above, we observe that:

$\text{Sin}^{-1}(-1) = -\frac{\pi}{2}$ since $\text{Sin}\left(-\frac{\pi}{2}\right) = -1$

$\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ since $\text{Sin}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\text{Sin}^{-1}(1) = \frac{\pi}{2}$ since $\text{Sin}\left(\frac{\pi}{2}\right) = 1$



8.1.6 The Domain and Range of Inverse Sine Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the sine function $y = \text{Sin}\, x$

$Domain = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $Range = [-1,1]$

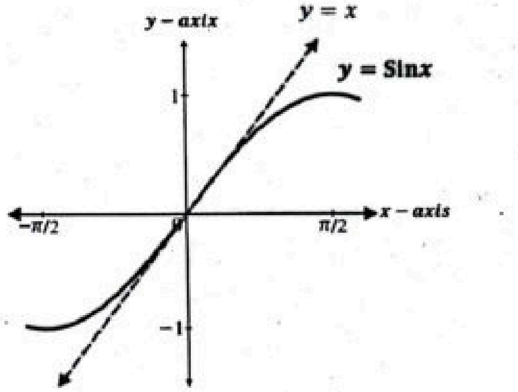
For the inverse sine function $y = \text{Sin}^{-1} x$

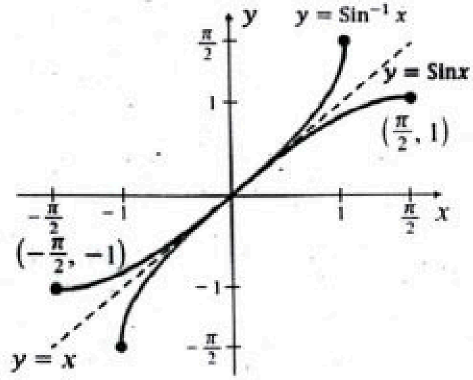
$Domain = [-1, 1]$ and $Range = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

> If $y = \text{Sin}\, x$ with $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; y \in [-1,1]$
> Then $y = \text{Sin}^{-1}(x)$ with
> $x \in [-1, 1]; y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown below.





Key Facts

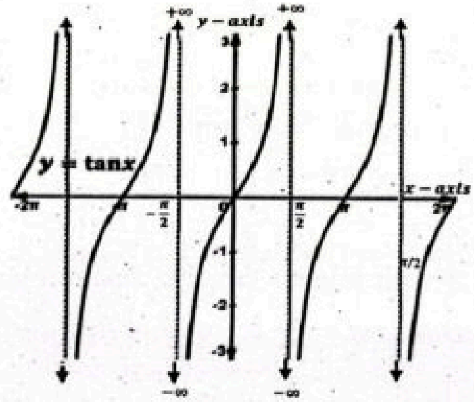
- The expression $\text{Sin}^{-1} x$ is not the same as $\frac{1}{\text{Sin}x}$. In other words, -1 is not an exponent. It simply means the inverse function.
- Remember that there will be no solution if k lies outside the interval $[-1, 1]$
- $\textbf{Cos}x = \textbf{cos}x\,;\ x \in [0, \pi]$
- $\textbf{Cos}^{-1}$ and $\textbf{cos}$ are not inverses of each other. (They do not cancel each other)
- **cos** does not have an inverse The functions that are inverses are: $\textbf{Cos}^{-1}$ and **Cos**

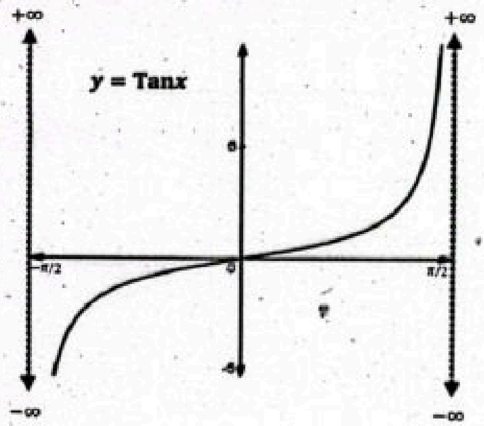
8.1.7 The Principal Tangent Function

We know that an infinite number of x values can be found from $y = \tan x = 1$ for each $x \in R - (2n + 1)\frac{\pi}{2}$ where $n \in Z$. By this, one can record infinitely many solutions to the equation $\tan x = k$ where $k \in (-\infty, +\infty)$. If we need an interval for x where there is only one solution to $\tan x = k$ for $k \in (-\infty, +\infty)$, we choose the interval between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

In the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can find a **unique solution** to the equation $\text{Tan}x = k$, where $k \in (-\infty, \infty)$.

We write this solution as $x = \text{Tan}^{-1} k$. In other words, "x is a real number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent value is k". The tangent function defined on $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which there is only **one solution** of the equation $\tan x = k$ where $k \in (-\infty, \infty)$ is called the **Principal Tangent Function.**





$\text{Tan}x = \tan x\,;\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Tan^{-1} and **tan** are not inverses of each other
(They do not cancel each other)

tan does not have an inverse

The functions that are inverses are Tan^{-1} and **Tan**

Example 3: Find the principal value of $\text{Tan}^{-1}(\sqrt{3})$.

Solution: Let $y = \text{Tan}^{-1}(\sqrt{3})$ if and only if $\text{Tan}\,y = \sqrt{3}$, where $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

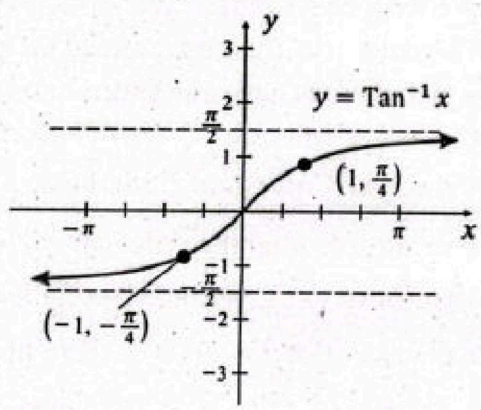
Consider $\text{Tan}\,y = \sqrt{3}$ [We need to find y whose tangent value is $\sqrt{3}$.]

$\Rightarrow \text{Tan}\,y = \text{Tan}\left(\frac{\pi}{3}\right)$ Since $\tan y > 0 \Rightarrow y$ lies in Quad I.

$\Rightarrow y = \frac{\pi}{3} \Rightarrow \text{Tan}^{-1}(\sqrt{3}) = \frac{\pi}{3}$

8.1.8 The Inverse Tangent Function

For the tangent function $y = \text{Tan}\,x$ where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $y \in (-\infty, +\infty)$, we define an inverse tangent function $y = \text{Tan}^{-1}(x)$ where $x \in (-\infty, +\infty)$ and $y(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



In view of above, we observe that:

$\text{Tan}^{-1}(-\infty) = -\frac{\pi}{2}$ since $\text{Tan}\left(-\frac{\pi}{2}\right) = \infty$

$\text{Tan}^{-1}(1) = \frac{\pi}{4}$ since $\text{Tan}\left(\frac{\pi}{4}\right) = 1$

$\text{Tan}^{-1}(0) = 0$ since $\text{Tan}(0) = 0$

$\text{Tan}^{-1}(\infty) = \frac{\pi}{2}$ since $\text{Tan}\left(\frac{\pi}{2}\right) = \infty$

Check Point

Find the principal value of $\text{Tan}^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

8.1.9 The Domain and Range of Inverse Tangent Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the tangent function $y = \text{Tan}\, x$

$Domain = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $Range = (-\infty, \infty)$

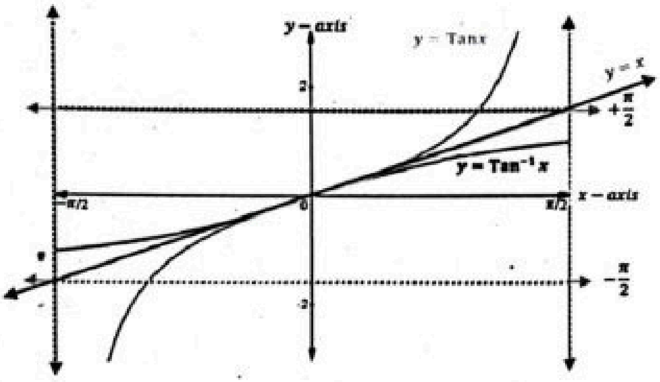
For the inverse tangent function $y = \text{Tan}^{-1} x$

$Domain = (-\infty, \infty)$ and $Range = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

If $y = \text{Tan}\, x$ with
$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); y \in (-\infty, \infty)$
then
$y = \text{Tan}^{-1}(x)$ with
$x \in (-\infty, \infty); y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in figure.

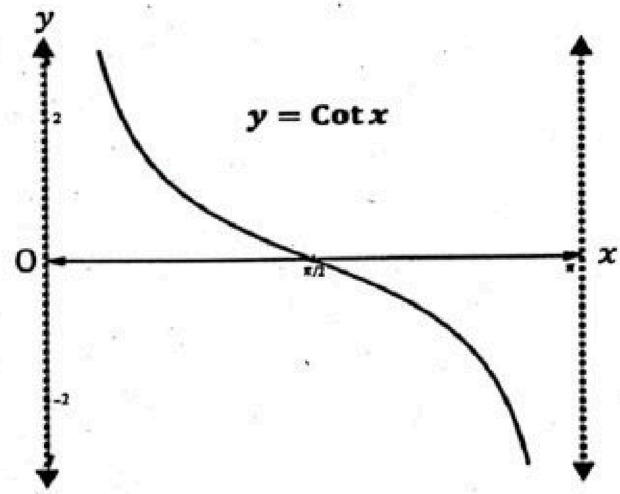


8.1.10 The Principal Cotangent Function

To define principal cotangent function, we need an interval for x where there is only one solution to $\cot x = k$ for $k \in (-\infty, +\infty)$. Such a solution is possible in the interval between 0 and π. In the interval $(0, \pi)$, we can find a **unique solution** to the equation $\text{Cot}\, x = k$, where $k \in (-\infty, \infty)$.

We write this solution as $x = \text{Cot}^{-1} k$.

The cotangent function defined on $x \in (0, \pi)$ for which there is only **one solution** of the equation $\text{Cot}\, x = k$ where $k \in (-\infty, \infty)$ is called the **Principal Cotangent Function.**



Example 4: Find the principal value of $\text{Cot}^{-1}(-\sqrt{3})$

Solution: Let $y = \text{Cot}^{-1}(-\sqrt{3})$ if and only if $\text{Cot}\, y = -\sqrt{3}$, where $y \in (0, \pi)$.

Consider $\text{Cot}\, y = -\sqrt{3}$ [We need to find y whose cotangent value is $-\sqrt{3}$.]

$\text{Tan}\, y = -\frac{1}{\sqrt{3}}$ Since, $\text{Tan}\, y < 0 \Rightarrow \text{Cot} y < 0 \Rightarrow y$ lies in Quad II.

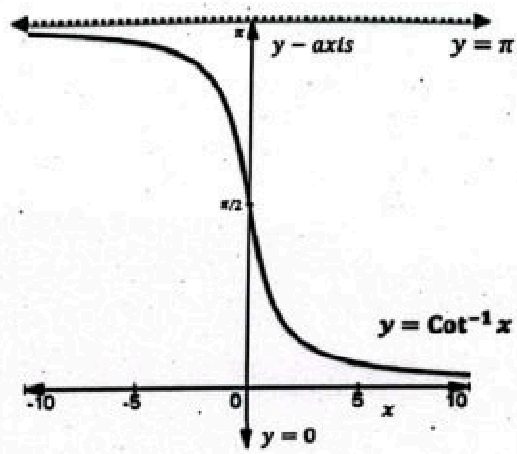
$\Rightarrow \text{Tan}\, y = \text{Tan}\left(-\frac{\pi}{3}\right)$

$\Rightarrow y = -\frac{\pi}{3} \quad \Rightarrow \text{Cot}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Check Point

Find the principal value of $\text{Cot}^{-1}(1)$.

8.1.11 The Inverse Cotangent Function



For the cotangent function $y = \text{Cot}\,x$ where $x \in (0, \pi)$ and $f(x) \in (-\infty, +\infty)$, we define an inverse cotangent function $y = \text{Cot}^{-1}(x)$ where $x \in (-\infty, +\infty)$ and $y \in (0, \pi)$.

In view of above, we observe that:

$\text{Cot}^{-1}(\infty) = 0$ since $\text{Cot}(0) = \infty$

$\text{Cot}^{-1}(0) = \frac{\pi}{2}$ since $\text{Cot}\left(\frac{\pi}{2}\right) = 0$

$\text{Cot}^{-1}(-\infty) = \pi$ since $\text{Cot}(\pi) = -\infty$

8.1.12 The Domain and Range of Inverse Cotangent Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the cotangent function $y = \text{Cot}\,x$

$Domain = (0, \pi)$ and $Range = (-\infty, \infty)$

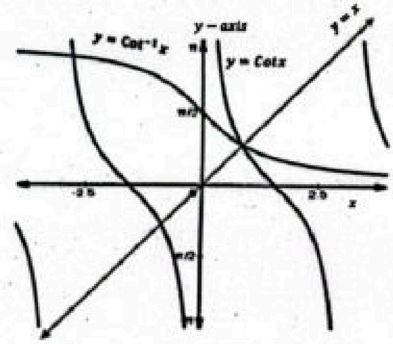
For the inverse tangent function $y = \text{Cot}^{-1}x$

$Domain = (-\infty, \infty)$ and $Range = (0, \pi)$

> If $y = \text{Cot}\,x$ with $x \in (0, \pi); y \in (-\infty, \infty)$ then $y = \text{Cot}^{-1}(x)$ with $x \in (-\infty, \infty); y \in (0, \pi)$

Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in figure.

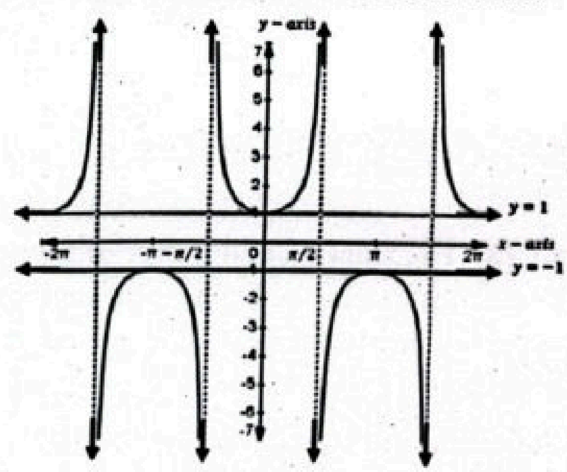


8.1.13 The Principal Secant Function

Consider the following graph of secant function and observe that an infinite number of x values found for which $y = \sec x = 1$ for each $x \in R - (2n + 1)\frac{\pi}{2}$ where $n \in Z$.

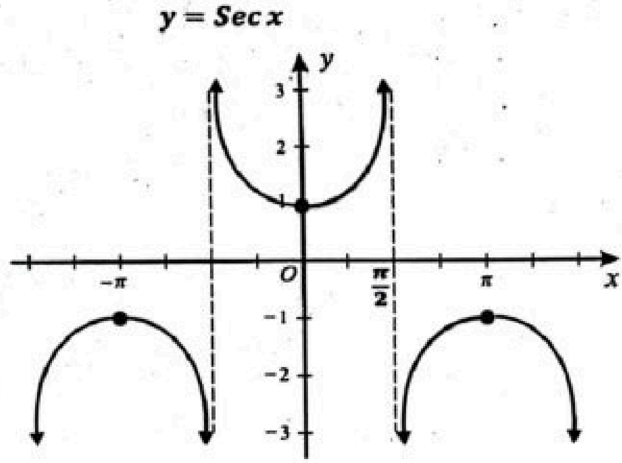
By this, one can record infinitely many solutions to the equation $\sec x = k$ where $k \in R - (-1, 1)$.

If we need an interval for x where there is only one solution to $\sec x = k$ for $k \in R - (-1, 1)$, we choose the interval from 0 to π except $\frac{\pi}{2}$. We have an option to consider an interval $x \in (\pi, 2\pi), x \neq \frac{3\pi}{2}$ or many others. It is a mathematical convention to choose $x \in [0, \pi], x \neq \frac{\pi}{2}$.



$y = \sec x$

In the interval $x \in [0, \pi], x \neq \frac{\pi}{2}$, we can find a **unique solution** to the equation $\text{Sec}\, x = k$, where $k \in R - (-1, 1)$. We write this solution as $x = \text{Sec}^{-1} k$. In other words, "x is a real number in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ whose secant value is k". The secant function defined on $x \in [0, \pi], x \neq \frac{\pi}{2}$ for which there is only **one** solution of the equation $\sec x = k$ where, $k \in R - (-1, 1)$ is called the **Principal Secant Function.**



Example 5: Find the principal value of $\text{Sec}^{-1}(-2)$.

Solution: Let $y = \text{Sec}^{-1}(-2)$ if and only if $\text{Sec}\, y = (-2)$, where $y \in [0, \pi], y \notin \frac{\pi}{2}$.

Consider $\text{Sec}\, y = (-2)$ [We need to find y whose secant value is (-2).]

$\Rightarrow \text{Cos}\, y = -\frac{1}{2}$ Since $\text{Cos}\, y < 0 \Rightarrow \text{Sec} y < 0 \Rightarrow y$ lies in Quad II.

$\Rightarrow \text{Cos}\, y = \text{Cos}\left(\frac{2\pi}{3}\right)$

$\Rightarrow y = \frac{2\pi}{3}$

$\Rightarrow \text{Sec}^{-1}(-2) = \frac{2\pi}{3}$

Check Point

Find the principal value of $\text{Sec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

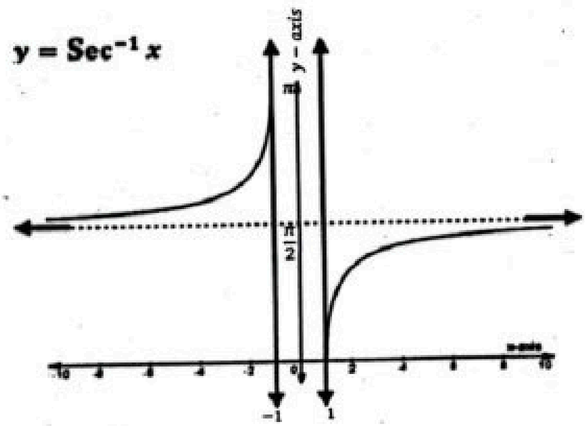
8.1.14 The Inverse Secant Function

For the secant function $y = \text{Sec}\, x$

where $x \in [0, \pi], x \neq \frac{\pi}{2}$ and $y \in R - (-1, 1)$,

we define an inverse secant function $y = \text{Sec}^{-1}(x)$

where $x \in R - (-1, 1)$ and $y \in [0, \pi], x \neq \frac{\pi}{2}$.



In view of above, we observe that:

$\text{Sec}^{-1}(1) = 0$ since $\text{Sec}(0) = 1$

$\text{Sec}^{-1}(-1) = \pi$ since $\text{Sec}(\pi) = -1$

8.1.15 The Domain and Range of Inverse Secant Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function. For the secant function $y = \text{Sec}\, x$

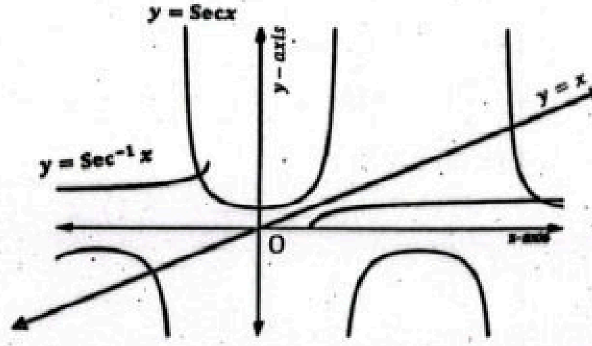
$$Domain = x \in [0, \pi], x \neq \frac{\pi}{2} \text{ and } Range = R - (-1, 1)$$

For the inverse secant function $y = \text{Sec}^{-1} x$

$$Domain = R - (-1, 1) \text{ and } Range = x \in [0, \pi], x \neq \frac{\pi}{2}$$

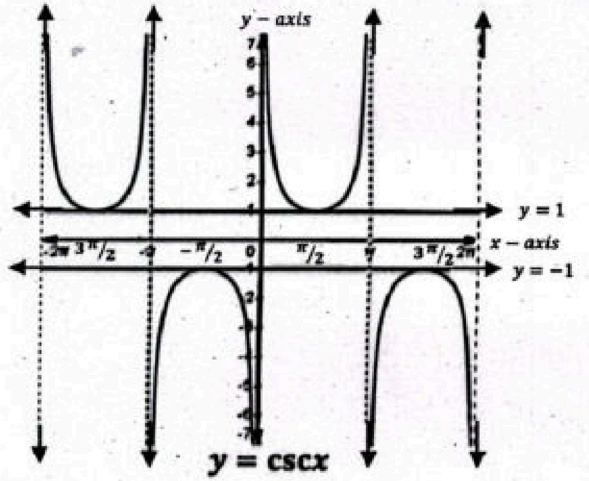
Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in the adjoining figure.



8.1.16 The Principal Cosecant Function

The graph of cosecant function from -2π to $+2\pi$ is shown in the adjoining figure.

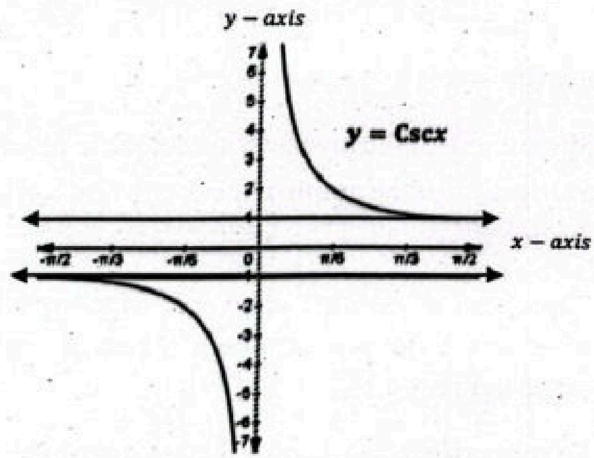


From the graph it is clear that $\csc x = k$ for $k \in R - (-1, 1)$, has many solutions.

To define principal cosecant function, we need an interval for x where there is only one solution to $\csc x = k$ for $k \in R - (-1, 1)$.

Such a solution is possible in the interval from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ except 0. In the interval $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $x \neq 0$, we can find a **unique solution** to the equation $\text{Csc}\, x = k$, where $k \in R - (-1, 1)$.

We write this solution as $x = \text{Csc}^{-1} k$. In other words, "x is a real number in the interval $\left[-\frac{\pi}{2}\ 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant value is k". The cosecant function defined on $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$ for which there is only **one solution** of the equation $\text{Csc}\, x = k$ where $k \in R - (-1, 1)$ is called the **Principal Cosecant Function.**



Example 6: Find the principal value of $\text{Csc}^{-1}(2)$.

Solution: Let $y = \text{Csc}^{-1}(2)$ if and only if $\text{Csc}\, y = (2)$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \notin 0$.

Consider $\text{Csc}\, y = (2)$ [We need to find y whose cosecant value is (2).]

$\Rightarrow \text{Sin}\, y = \frac{1}{2}$ Since, $\text{Sin} y > 0 \Rightarrow \text{Csc}\, y > 0 \Rightarrow y$ lies in Quad I.

$\Rightarrow \text{Sin}\, y = \text{Sin}\left(\frac{\pi}{6}\right)$

$\Rightarrow y = \frac{\pi}{6}$

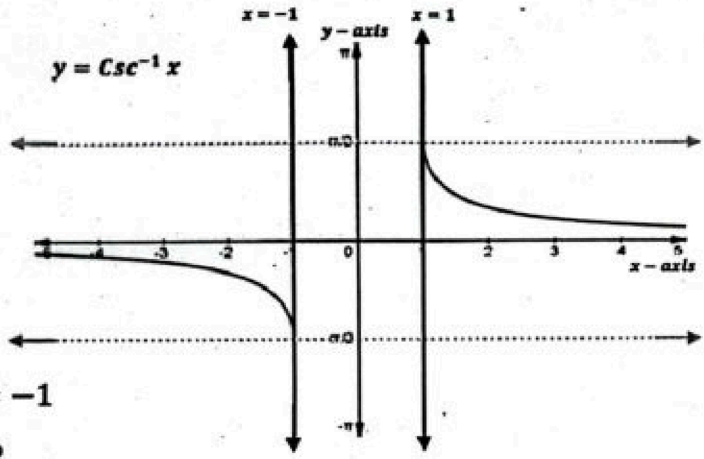
$\Rightarrow \text{Csc}^{-1}(2) = \frac{\pi}{6}$

Check Point

Find the principal value of $\text{Csc}^{-1}(-\sqrt{2})$.

8.1.17 The Inverse Cosecant Function

For the cosecant function $y = \text{Csc}\, x$ where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$ and $f(x) \in R - (-1, 1)$, we define an inverse cosecant function $y = \text{Csc}^{-1}(x)$ where $x \in R - (-1, 1), x \neq 0$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



In view of above, we observe that:

$\text{Csc}^{-1}(-1) = -\frac{\pi}{2}$ since $\text{Csc}\left(-\frac{\pi}{2}\right) = -1$

$\text{Csc}^{-1}(\infty) = 0$ since $\text{Csc}(0) = \infty$

$\text{Csc}^{-1}(\sqrt{2}) = \frac{\pi}{4}$ since $\text{Csc}\left(\frac{\pi}{4}\right) = \sqrt{2}$

8.1.18 The Domain and Range of Inverse Cosecant Function

To find the domain and range of inverse trigonometric function, switch the domain and range of the original function.

For the cosecant function $y = \text{Csc}\, x$

$Domain = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$ and

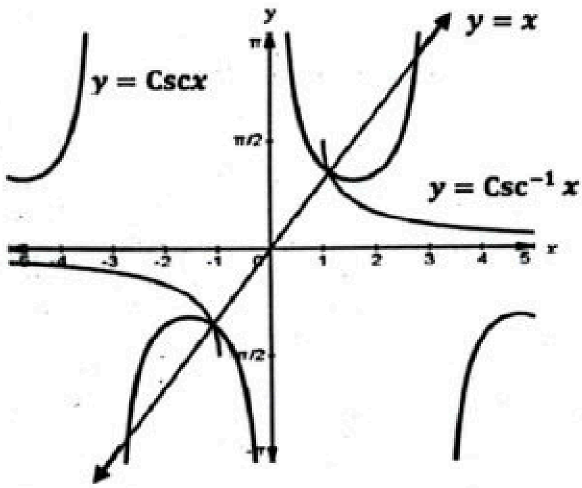
$Range = R - (-1, 1)$

For the inverse cosecant function

$y = \text{Csc}^{-1} x$

$Domain = R - (-1, 1)$ and

$Range = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \neq 0$



Since graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

Therefore, to graph the inverse trigonometric function, we use the graph of the trigonometric function restricted to the domain specified earlier and reflect the graph about the line $y = x$ as shown in the figure.

Properties of Inverse Trigonometric Functions

Property I

- $\text{Cos}^{-1}(\text{Cos}\, x) = x$ for $x \in [0, \pi]$ and $\text{Sin}^{-1}(\text{Sin}\, x) = x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\text{Tan}^{-1}(\text{Tan}\, x) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\text{Cot}^{-1}(\text{Cot}\, x) = x$ for $x \in (0, \pi)$
- $\text{Sec}^{-1}(\text{Sec}\, x) = x$ for $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
- $\text{Csc}^{-1}(\text{Csc}\, x) = x$ for $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Property II

- $\text{Cos}(\text{Cos}^{-1}x) = x$ for $x \in [-1,1]$ and $\text{Sin}(\text{Sin}^{-1}x) = x$ for $x \in [-1,1]$
- $\text{Tan}(\text{Tan}^{-1}x) = x$ for $x \in (-\infty,+\infty)$ and $\text{Cot}(\text{Cot}^{-1}x) = x$ for $x \in (-\infty,+\infty)$
- $\text{Sec}(\text{Sec}^{-1}x) = x$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Csc}(\text{Csc}^{-1}x) = x$ for $x \in (-\infty,-1] \cup [1,\infty)$

Property III

- $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x$ for $x \in [-1,1]$
- $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$ for $x \in [-1,1]$
- $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x)$ for $x \in (-\infty,+\infty)$
- $\text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}(x)$ for $x \in (-\infty,+\infty)$
- $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Csc}^{-1}(-x) = -\text{Csc}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$

Property IV

- $\text{Sin}^{-1}\left(\frac{1}{x}\right) = \text{Csc}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Sec}^{-1}(x)$ for $x \in (-\infty,-1] \cup [1,\infty)$
- $\text{Tan}^{-1}\left(\frac{1}{x}\right) = \begin{cases} \text{Cot}^{-1}(x) & \text{if } x > 0 \\ -\pi + \text{Cot}^{-1}(x) & \text{if } x < 0 \end{cases}$

Property V

- $\text{Sin}^{-1}(x) + \text{Cos}^{-1}(x) = \frac{\pi}{2}$ for $x \in [-1,1]$
- $\text{Tan}^{-1}(x) + \text{Cot}^{-1}(x) = \frac{\pi}{2}$ for $x \in (-\infty,+\infty)$
- $\text{Sec}^{-1}(-x) + \text{Csc}^{-1}(-x) = \frac{\pi}{2}$ for $x \in (-\infty,-1] \cup [1,\infty)$

Property VI

- $\text{Sin}^{-1}(x) = \text{Cos}^{-1}(\sqrt{1-x^2}) = \text{Tan}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \text{Cot}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{Sec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{Csc}^{-1}\left(\frac{1}{x}\right)$
- $\text{Cos}^{-1}(x) = \text{Sin}^{-1}(\sqrt{1-x^2}) = \text{Tan}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \text{Cot}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \text{Sec}^{-1}\left(\frac{1}{x}\right) = \text{Csc}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$
- $\text{Tan}^{-1}(x) = \text{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \text{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \text{Cot}^{-1}\left(\frac{1}{x}\right) = \text{Sec}^{-1}(\sqrt{1+x^2}) = \text{Csc}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

Exercise 8.1

1. Find the principal values of each of the following without using a calculator.

i.	$\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$	ii.	$\text{Sin}^{-1}(1)$	iii.	$\text{Tan}^{-1}(\sqrt{3})$
iv.	$\text{Cot}^{-1}\left(\frac{\sqrt{3}}{3}\right)$	v.	$\text{Sec}^{-1}\left(\frac{2\sqrt{3}}{3}\right)$	vi.	$\text{Csc}^{-1}(-\sqrt{2})$
vii.	$\text{Cos}^{-1}\left(-\frac{1}{2}\right)$	viii.	$\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$	ix.	$\text{Csc}^{-1}(-2)$

2. Find sum of principal values of the following inverse trigonometric expressions without using calculator.

   i. $\text{Tan}^{-1}(1) + \text{Cos}^{-1}\left(-\frac{1}{2}\right) + \text{Sin}^{-1}\left(-\frac{1}{2}\right)$ ii. $\text{Cos}^{-1}\left(\frac{1}{2}\right) + 2\,\text{Sin}^{-1}\left(\frac{1}{2}\right)$

   iii. $\text{Tan}^{-1}(\sqrt{3}) - \text{Sec}^{-1}(-2)$ iv. $\text{Cot}^{-1}(-\sqrt{3}) + \text{Csc}^{-1}(-2) - \text{Cos}^{-1}\left(\frac{\sqrt{2}}{2}\right)$

3. Find the exact real number value of each without using a calculator.

   i. $\text{Cos}^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right]$ ii. $\text{Sin}^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$ iii. $\text{Cos}^{-1}\left(\sin\frac{11\pi}{6}\right)$

   iv. $\text{Cos}^{-1}\left(\cos\frac{\pi}{6}\right)$ v. $\text{Sin}\left(\text{Tan}^{-1}\frac{3}{4}\right)$ vi. $\text{Cos}\left(2\,\text{Sin}^{-1}\frac{\sqrt{2}}{2}\right)$

   vii. $\text{Sin}\left[2\,\text{Sin}^{-1}\left(\frac{4}{5}\right)\right]$ viii. $\text{Cos}\left(\text{Sin}^{-1}\frac{5}{13}\right)$ ix. $\text{Sin}\left[\text{Cos}^{-1}\left(-\frac{3}{5}\right)\right]$

   x. $\text{Sin}\left[\text{Sin}^{-1}\frac{2}{3} + \text{Cos}^{-1}\frac{1}{2}\right]$ xi. $\text{Cos}\left(\text{Sin}^{-1}\frac{3}{4} + \text{Cos}^{-1}\frac{5}{13}\right)$

   xii. $\text{Cos}[\text{Sec}^{-1}(3) + \text{Tan}^{-1}(2)]$

4. Find the unknown angles and use a calculator to evaluate the following as real numbers to three decimal places:

   i. $\text{Cos}^{-1}\left(\frac{3}{5}\right) = \text{Sin}^{-1}(\ )$ ii. $\text{Sin}^{-1}\left(\frac{2}{3}\right) = \text{Cos}^{-1}(\ )$ iii. $\text{Sin}^{-1}\left(-\frac{1}{\sqrt{5}}\right) = -\text{Cos}^{-1}(\ )$

   iv. $\text{Tan}^{-1}\left(\frac{1}{4}\right) = \text{Cos}^{-1}(\ )$ v. $\text{Tan}^{-1}(-1.2) = -\text{Cos}^{-1}(\ )$

   vi. $\text{Cot}^{-1}\left(-\frac{3}{4}\right) = -\text{Sin}^{-1}(\ )$ vii. $\text{Sec}^{-1}(2.041) = \text{Tan}^{-1}(\ )$

   viii. $\text{Sec}^{-1}(-\sqrt{5}) = \text{Cot}^{-1}(\ )$ ix. $\text{Csc}^{-1}(1.172) = \text{Sin}^{-1}(\ )$

   x. $\text{Csc}^{-1}\left(-\frac{5}{3}\right) = \text{Tan}^{-1}(\ )$

8.2 Graphs of Inverse Trigonometric Functions

Drawing graphs of trigonometric functions is a fundamental skill. It allows the problem solver to interpret and understand the relationship visually between two variable quantities. Graphs provide an intuitive way to analyze the behavior, identify the trend and solve problems.

Graphing becomes valuable in case of inverse trigonometric functions, where the restricted domain and range were not enough to understand the problem.

Plotting these graphs, we gain insights the characteristics, such as symmetry, continuity, and asymptotic behavior. Whether it is engineering or navigation, the ability to draw accurately and interpret graphs of inverse trigonometric functions is essential for solving real-world problems involving angles and distances.

Following table helps in drawing graphs of the inverse trigonometric functions.

Inverse cosine function	Inverse sine function	Inverse tangent function	Inverse secant function	Inverse cosecant function	Inverse cotangent function
Domain $[-1,1]$	Domain $[-1,1]$	Domain R	Domain $(-\infty,-1] \cup [1,\infty)$	Domain $(-\infty,-1] \cup [1,\infty)$	Domain R
Range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	Range $[0,\pi]$	Range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	Range $[0,\pi]-\left\{\frac{\pi}{2}\right\}$	Range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	Range $(0,\pi)$
non periodic function	non periodic function	non periodic function	non periodic function	non periodic function	non periodic function
odd function	neither even nor odd function	odd function	odd function	neither even nor odd function	neither even nor odd function
strictly increasing function	strictly decreasing function	strictly increasing function	strictly decreasing function with respect to its domain	strictly decreasing function with respect to its domain	strictly decreasing function
one to one function	one to one function	one to one function	one to one function	one to one function	one to one function
graphically $\text{Cos}^{-1}(x)$ is a reflection of $\text{Cos}(x)$ across the line $y=x$	graphically $\text{Sin}^{-1}(x)$ is a reflection of $\text{Sin}(x)$ across the line $y=x$	graphically $\text{Tan}^{-1}(x)$ is a reflection of $\text{Tan}(x)$ across the line $y=x$	graphically $\text{Sec}^{-1}(x)$ is a reflection of $\text{Sec}(x)$ across the line $y=x$	graphically $\text{Csc}^{-1}(x)$ is a reflection of $\text{Csc}(x)$ across the line $y=x$	graphically $\text{Cot}^{-1}(x)$ is a reflection of $\text{Cot}(x)$ across the line $y=x$
		S-shaped curve approaches but never reaches the horizontal asymptotes $y=\pm\frac{\pi}{2}$	Smooth curve approaches but never reaches the vertical asymptote $y=\frac{\pi}{2}$	Smooth curve approaches but never reaches the vertical asymptote $y=0$	S-shaped curve approaches but never reaches the horizontal asymptotes $y=0$ and $y=\pi$

key points include $(-1,\pi)$, $\left(0,\frac{\pi}{2}\right)$, $(1,0)$.	key points include $\left(-1,\frac{\pi}{2}\right)$, $(0,0)$, $\left(1,\frac{\pi}{2}\right)$	key points include $\left(-1,-\frac{\pi}{4}\right)$, $(0,0)$, $\left(1,\frac{\pi}{4}\right)$			
Adding or subtracting constants affects the graphs horizontally or vertically (horizontal and vertical shifts).					
Multiplying by constants to stretch or compress the graph (stretching and compressing).					
Multiplying by negative values shows reflection of the graphs across axes (Reflections).					

8.2.1 Graph of $y = \text{Cos}^{-1} x$

The graph of the function $y = \text{Cos}^{-1} x$ represents the angle y whose cosine is x. The domain of this function is $x \in [-1,1]$, and the range is $y \in [0,\pi]$. This means the output values y will always be within this interval, corresponding to angles in radians from 0 to π.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from -1 to 1 and y-axis with values from 0 to π.
- Choose an appropriate scale.
- Mark the points $(1,0)$, $\left(0.9,\frac{\pi}{6}\right)$, $\left(0.5,\frac{\pi}{3}\right)$, $\left(0,\frac{\pi}{2}\right)$, $\left(-0.5,\frac{2\pi}{3}\right)$, $\left(-0.9,\frac{5\pi}{6}\right)$, $(-1,\pi)$ on the graph.
- Connect these points with a smooth curve.

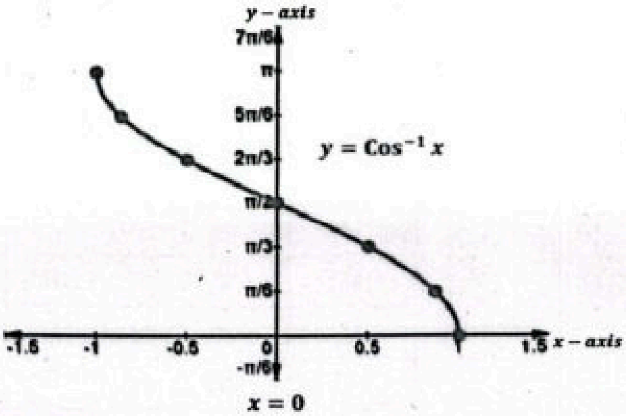
Properties:

- **Domain and Range:** The domain is $[-1,1]$ and the range is $[0,\pi]$.
- **Key Points:** $(1,0)$, $\left(0,\frac{\pi}{2}\right)$, $(-1,\pi)$
- **Appearance:** The graph appears as a smooth curve starting from $(1,0)$, decreasing through the key points, passes through the point $(1,0)$ and ending at $(-1,\pi)$.
- **Monotonicity:** The graph is monotonically decreasing smoothly (it always decreases as x increases).
- **Neither even nor odd function**
- **Reflection:** The graph of $y = \text{Cos}^{-1}x$, is a reflection of $y = \text{Cos}\, x$ across the line $y = x$.
- This means that the x and y coordinates of the points on the graph of $y = \text{Cos}\, x$ are switched to obtain the points on the graph of $y = \text{Cos}^{-1}x$
- **Reflection Symmetry:** $\begin{bmatrix} \text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x \text{ for } x \in [-1,1] \\ \left(\text{Symmetric about the line } y = \frac{\pi}{2}\right) \end{bmatrix}$

  Symmetry along the range $[0,\pi]$ reflects across the vertical axis at $x = 0$.
- **Inverse Function:** $\text{Cos}(\text{Cos}^{-1}x) = x$ for $x \in [-1,1]$ and

  $\text{Cos}^{-1}(\text{Cos}\, x) = x$ for $x \in [0,\pi]$

- **Identities:** $\text{Cos}^{-1}x = \text{Sin}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [0,1]$ and
  $\text{Cos}^{-1}x = \pi - \text{Sin}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [-1,0)$



x	$y = \text{Cos}^{-1}x$
1	0 or 0°
0.9	$\pi/6$ or 30°
0.5	$\pi/3$ or 60°
0	$\pi/2$ or 90°
−0.5	$2\pi/3$ or 120°
−0.9	$5\pi/6$ or 150°
−1	π or 180°

8.2.2 Graph of $y = \text{Sin}^{-1}x$

The graph of the function $y = \text{Sin}^{-1}x$ represents the angle y whose sine is x. The domain of this function is $x \in [-1, 1]$, and the range is $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This means the output values y will always be within this interval, corresponding to angles in radians from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Construction Steps:

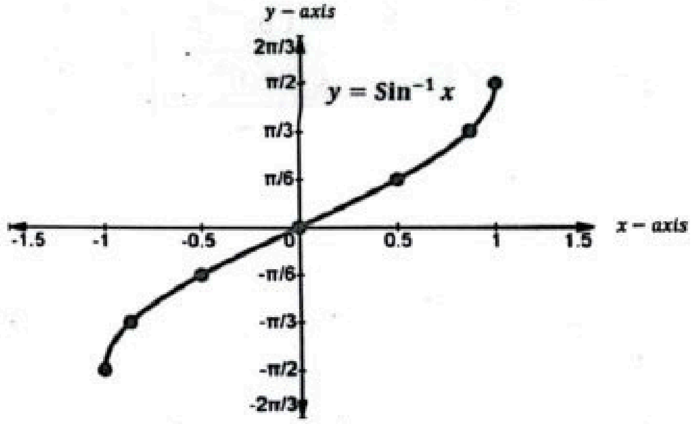
- Draw the coordinate axes. Mark x-axis with values from -1 to 1 and y-axis with values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and choose an appropriate scale.
- Mark the points $\left(1,\frac{\pi}{2}\right)$, $\left(0.9,\frac{\pi}{3}\right)$, $\left(0.5,\frac{\pi}{6}\right)$, $(0,0)$, $\left(-0.5,-\frac{\pi}{6}\right)$, $\left(-0.9,-\frac{\pi}{3}\right)$, $\left(-1,-\frac{\pi}{2}\right)$ on the graph and connect these points with a smooth curve.

Properties:

- **Domain and Range:** The domain is $[-1,1]$ and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
- **Key Points:** $\left(1,\frac{\pi}{2}\right)$, $(0,0)$, $\left(-1,-\frac{\pi}{2}\right)$
- **Appearance:** The graph appears as a smooth curve starting from $\left(-1,-\frac{\pi}{2}\right)$, increasing through the key points, passes through the point $(0,0)$ and ending at $\left(1,\frac{\pi}{2}\right)$.
- **Monotonicity:** The graph is monotonically increasing smoothly.
- **Odd Inverse Function:** $\left[\begin{array}{c}\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x \text{ for } x \in [-1,1] \\ \text{(Symmetric about origin)}\end{array}\right]$

  The symmetry along range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ shows that graph is reflected across origin.
- **Reflection:** The graph of $y = \text{Sin}^{-1}x$, is a reflection of $y = \text{Sin}\,x$ across the line $y = x$.
- **Inverse Function:** $\text{Sin}(\text{Sin}^{-1}x) = x$ for $x \in [-1,1]$ and
  $\text{Sin}^{-1}(\text{Sin}\,x) = x$ for $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

- **Identities:** $\text{Sin}^{-1}x = \text{Cos}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [0,1]$ and
  $\text{Sin}^{-1}x = -\text{Cos}^{-1}\left(\sqrt{1-x^2}\right)$ for $x \in [-1,0)$



x	$y = \text{Sin}^{-1} x$
1	$\pi/2$ or $90°$
0.9	$\pi/3$ or $60°$
0.5	$\pi/6$ or $30°$
0	0 or $0°$
-0.5	$-\pi/6$ or $-30°$
-0.9	$-\pi/3$ or $-60°$
-1	$-\pi/2$ or $-90°$

8.2.3 Graph of $y = \text{Tan}^{-1} x$

The graph of the function $y = \text{Tan}^{-1} x$ represents the angle y whose tangent is x. The domain of this function is $x \in (-\infty, +\infty)$, and the range is $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means the output values y will always be within this interval, corresponding to angles in radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and choose an appropriate scale.
- Mark the points $\left(-\infty, -\frac{\pi}{2}\right)$, $\left(-1.7, -\frac{\pi}{3}\right)$, $\left(-1, -\frac{\pi}{4}\right)$, $\left(-0.6, -\frac{\pi}{6}\right)$, $(0,0)$, $\left(1, \frac{\pi}{4}\right)$, $\left(1.7, \frac{\pi}{3}\right)$, $\left(0.6, \frac{\pi}{6}\right)$, $\left(\infty, \frac{\pi}{2}\right)$ on the graph and connect these points with a smooth curve.

Properties:

- **Domain and Range:** The domain is $(-\infty, +\infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- **Key Points:** $\left(-\infty, -\frac{\pi}{2}\right)$, $\left(-1.7, -\frac{\pi}{3}\right)$, $(0,0)$, $\left(1.7, \frac{\pi}{3}\right)$ $\left(\infty, \frac{\pi}{2}\right)$
- **Asymptotes:** Draw two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ approaching the tangent curve at $x = \pm\infty$.
- **Appearance:** The graph appears as a smooth curve starting from $\left(-\infty, -\frac{\pi}{2}\right)$, increasing through the key points, passes through the point $(0,0)$ and ending at $\left(+\infty, \frac{\pi}{2}\right)$.
- **Monotonicity:** The graph is monotonically increasing smoothly between two horizontal asymptotes.
- **Odd Inverse Function:** $\left[\begin{array}{c}\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x) \text{ for } x \in (-\infty, +\infty) \\ \text{(Symmetric about origin)}\end{array}\right]$

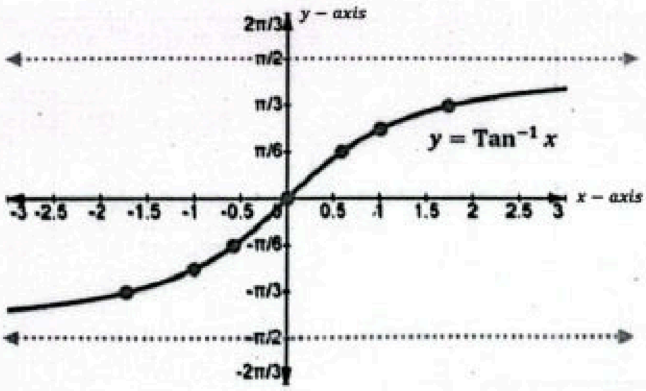
The symmetry along range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ shows that graph is reflected across origin.

- **Reflection:** The graph of $y = \mathrm{Tan}^{-1}x$, is a reflection of $y = \mathrm{Tan}\,x$ across the line $y = x$.
- **Inverse Function:** $\mathrm{Tan}(\mathrm{Tan}^{-1}x) = x$ for $x \in (-\infty, +\infty)$ and

$$\mathrm{Tan}^{-1}(\mathrm{Tan}\,x) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- **Identities:** $\mathrm{Tan}^{-1}(x) + \mathrm{Tan}^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$ for $x > 0$ and

$$\mathrm{Tan}^{-1}(x) + \mathrm{Tan}^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2} \text{ for } x < 0$$



x	$y = \mathrm{Tan}^{-1}x$
$-\infty$	$-\pi/2$ or $-90°$
-1.7	$-\pi/3$ or $-60°$
-1	$-\pi/4$ or $-45°$
-0.6	$-\pi/6$ or $-30°$
0	0 or $0°$
0.6	$\pi/6$ or $30°$
1	$\pi/4$ or $45°$
1.7	$\pi/3$ or $60°$
∞	$\pi/2$ or $90°$

8.2.4 Graph of $y = \mathrm{Cot}^{-1}x$

The graph of the function $y = \mathrm{Cot}^{-1}x$ represents the angle y whose cotangent is x. The domain of this function is $x \in (-\infty, +\infty)$, and the range is $y \in (0, \pi)$. This means the output values y will always be within this interval, corresponding to angles in radians between 0 and π.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from 0 to π and choose an appropriate scale.
- Mark the points $(-\infty, \pi)$, $\left(-1.7, \frac{5\pi}{6}\right)$, $\left(-1, \frac{3\pi}{4}\right)$, $\left(-0.6, \frac{2\pi}{3}\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(0.6, \frac{\pi}{3}\right)$ $\left(1, \frac{\pi}{4}\right)$, $\left(1.7, \frac{\pi}{6}\right)$, $(+\infty, 0)$ on the graph.
- Connect these points with a smooth curve.

Properties:

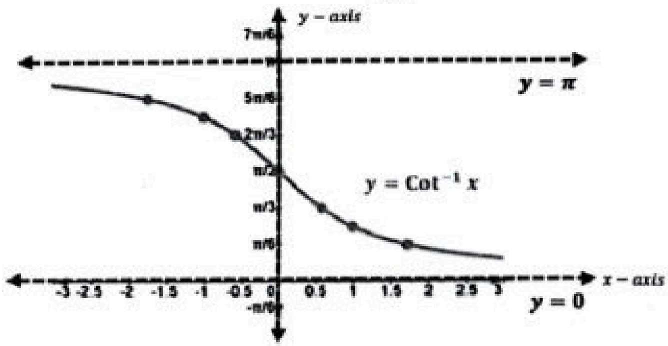
- **Domain and Range:** The domain is $(-\infty, +\infty)$ and the range is $(0, \pi)$.
- **Key Points:** $(-\infty, \pi)$, $\left(-1.7, \frac{5\pi}{6}\right)$, $(0, 0)$, $\left(1.7, \frac{\pi}{6}\right)$ $(+\infty, 0)$
- **Asymptotes:** Draw two horizontal asymptotes $y = 0$ and $y = \pi$ approaching the cotangent curve at $x = \pm\infty$.
- **Appearance:** The graph appears as a smooth curve starting from $(-\infty, \pi)$, increasing through the key points, passes through the point $(0, \frac{\pi}{2})$ and ending at $(+\infty, 0)$.
- **Monotonicity:** The graph is monotonically decreasing smoothly between two horizontal asymptotes.
- **Neither even nor odd function**
- **Reflection:** The graph of $y = \mathrm{Cot}^{-1}x$, is a reflection of $y = \mathrm{Cot}\,x$ across the line $y = x$.

This means that the x and y coordinates of the points on the graph of $y = \text{Cot}\,x$ are switched to obtain points on the graph of $y = \text{Cot}^{-1}x$.

- **Reflection Symmetry:** $\begin{bmatrix} \text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}(x) \text{ for } x \in (-\infty, +\infty) \\ \left(\text{about the line } y = \frac{\pi}{2}\right) \end{bmatrix}$

  The symmetry along range $(0, \pi)$reflects across the vertical axis at $x = 0$.
- **Inverse Function:** $\text{Cot}(\text{Cot}^{-1}x) = x$ for $x \in (-\infty, +\infty)$ and

  $\text{Cot}^{-1}(\text{Cot}\,x) = x$ for $x \in (0, \pi)$
- **Identities:** $\text{Cot}^{-1}(x) = \text{Tan}^{-1}\left(\frac{1}{x}\right)$ for $x \neq 0$



x	$y = \text{Cot}^{-1}x$
$+\infty$	0 or 0°
1.7	$\pi/6$ or 30°
1	$\pi/4$ or 45°
0.6	$\pi/3$ or 60°
0	$\pi/2$ or 90°
−0.6	$2\pi/3$ or 120°
−1	$3\pi/4$ or 135°
−1.7	$5\pi/6$ or 150°
$-\infty$	π or 180°

8.2.5 Graph of $y = \text{Sec}^{-1}x$

The graph of the function $y = \text{Sec}^{-1}x$ represents the angle y whose secant is x. The domain of this function is $x \in (-\infty, -1] \cup [1, \infty)$, and the range is $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. This means the output values y will always be within this interval, corresponding to angles in radians between 0 and π.

Construction Steps:

- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from 0 to π.
- Choose an appropriate scale.
- Mark the points $\left(+\infty, \frac{\pi}{2}\right)$, $\left(2, \frac{\pi}{3}\right)$, $\left(1.4, \frac{\pi}{4}\right)$, $\left(1.2, \frac{\pi}{6}\right)$, $(1, 0)$, $(-1, \pi)$, $\left(-1.2, \frac{5\pi}{6}\right)$, $\left(-1.4, \frac{3\pi}{4}\right)$, $\left(-2, \frac{2\pi}{3}\right)$, $\left(-\infty, \frac{\pi}{2}\right)$ on the graph.
- Connect these points with a smooth curve.

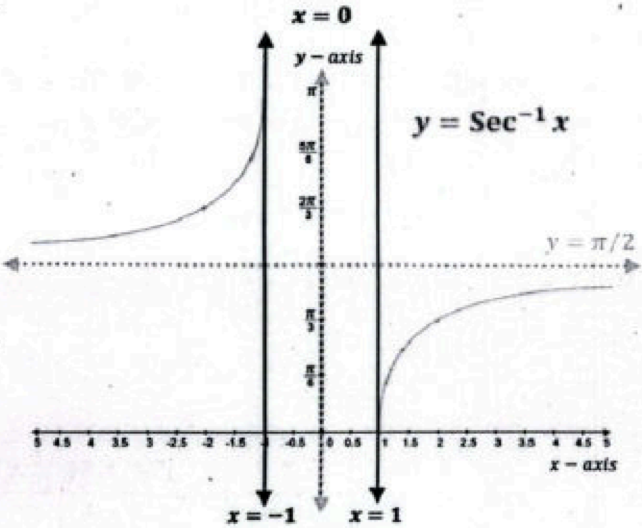
Properties:

- **Domain and Range:** The domain is $(-\infty, -1] \cup [1, \infty)$ and the range is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.
- **Key Points:** $\left(-\infty, \frac{\pi}{2}\right)$, $\left(-2, \frac{2\pi}{3}\right)$, $(-1, \pi)$, $(1, 0)$, $\left(2, \frac{\pi}{3}\right)$, $\left(+\infty, \frac{\pi}{2}\right)$
- **Asymptotes:** Draw a horizontal asymptote $y = \frac{\pi}{2}$ approaching the secant curve at $x = \pm\infty$.
- **Appearance:** The secant graph appears increasing from $(1, 0)$, moves along the horizontal asymptote $y = \frac{\pi}{2}$ and ends at $\left(+\infty, \frac{\pi}{2}\right)$. The secant graph appears decreasing from $(-1, \pi)$, moves along the horizontal asymptote $y = \frac{\pi}{2}$ and ends at $\left(-\infty, \frac{\pi}{2}\right)$.

- **Monotonicity:** The graph is monotonically increasing and decreasing smoothly (it always increases/decreases as x increases/increases) along the horizontal asymptote $y = \frac{\pi}{2}$.
- **Neither even nor odd function**
- **Reflection:** The graph of $y = \text{Sec}^{-1}x$, is a reflection of $y = \text{Sec}\,x$ across the line $y = x$. This means that the x and y coordinates of the points on the graph of $y = \text{Sec}\,x$ are switched to obtain the points on the graph of $y = \text{Sec}^{-1}x$.
- **Reflection Symmetry:** $\begin{bmatrix} \text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}(x) \text{ for } x \in (-\infty,-1] \cup [1,\infty) \\ \left(\text{symmetric about the line } y = \frac{\pi}{2}\right) \end{bmatrix}$

  The symmetry along range $[0,\pi]/\left\{\frac{\pi}{2}\right\}$ reflects across the vertical axis at $x = 0$.
- **Inverse Function:** $\text{Sec}(\text{Sec}^{-1}x) = x$ for $x \in (-\infty,-1] \cup [1,\infty)$ and

  $$\text{Sec}^{-1}(\text{Sec}\,x) = x \text{ for } x \in \left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$$
- **Identities:** $\text{Sec}^{-1}(x) = \text{Cos}^{-1}\left(\frac{1}{x}\right)$ for $x \in (-\infty,-1] \cup [1,\infty)$



x	$y = Sec^{-1}x$
∞	$\pi/2$ or $90°$
2	$\pi/3$ or $60°$
1.4	$\pi/4$ or $45°$
1.2	$\pi/6$ or $30°$
1	0 or $0°$
-1	π or $180°$
-1.2	$5\pi/6$ or $150°$
-1.4	$3\pi/4$ or $135°$
-2	$2\pi/3$ or $120°$
$-\infty$	$\pi/2$ or $90°$

8.2.6 Graph of $y = \text{Csc}^{-1}x$

The graph of the function $y = \text{Csc}^{-1}x$ represents the angle y whose cosecant is x. The domain of this function is $x \in (-\infty,-1] \cup [1,\infty)$, and the range is $y \in \left[-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right]$. This means the output values y will always be within this interval, corresponding to angles in radians between 0 and π.

Construction Steps:

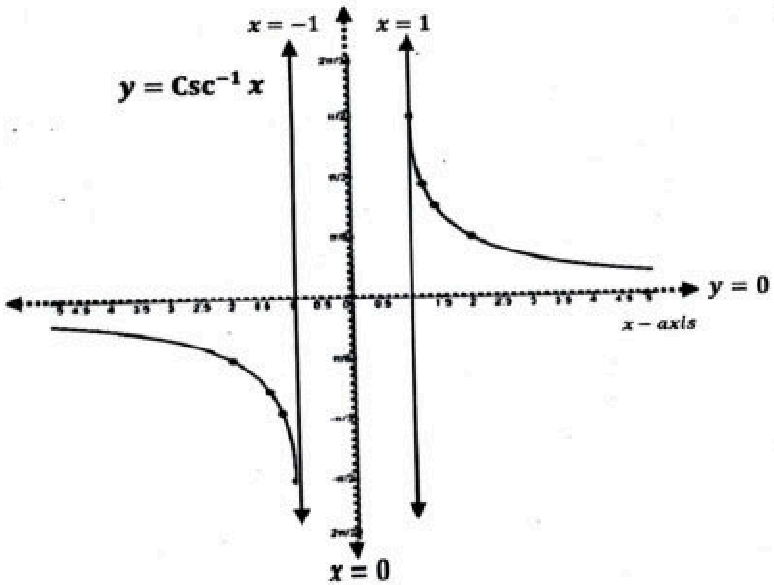
- Draw the coordinate axes. Mark x-axis with values from $-\infty$ to $+\infty$ and y-axis with values from 0 to π and choose an appropriate scale.
- Mark the points $\left(1,\frac{\pi}{2}\right)$, $\left(1.2,\frac{\pi}{3}\right)$, $\left(1.4,\frac{\pi}{4}\right)$, $\left(2,\frac{\pi}{6}\right)$, $(+\infty,+0)$, $\left(-1,-\frac{\pi}{2}\right)$, $\left(-1.2,-\frac{\pi}{3}\right)$, $\left(-1.4,-\frac{\pi}{4}\right)$, $\left(-2,-\frac{\pi}{6}\right)$, $(-\infty,-0)$ on the graph and connect them smoothly.

Properties:

- **Domain and Range:** The domain is $(-\infty,-1]\cup[1,\infty)$ and the range is $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$.
- **Key Points:** $\left(-\infty,\frac{\pi}{2}\right)$, $\left(-2,\frac{2\pi}{3}\right)$, $(-1,\pi)$, $(1,0)$, $\left(2,\frac{\pi}{3}\right)$, $\left(+\infty,\frac{\pi}{2}\right)$
- **Asymptotes:** Draw a horizontal asymptote $y=0$ approaching the cosecant curve at $x=\pm\infty$.
- **Appearance:** The cosecant graph appears decreasing from $\left(1,\frac{\pi}{2}\right)$, moves along the horizontal asymptote $y=0$ and ends at $(+\infty,0)$. The cosecant graph appears increasing from $\left(-1,-\frac{\pi}{2}\right)$, moves along the horizontal asymptote $y=0$ and ends at $(-\infty,-0)$.
- **Monotonicity:** The graph is monotonically increasing and decreasing smoothly (it always increases/decreases as x increases/increases) along the horizontal asymptote $y=0$.
- **Odd Inverse Function:** $\left[\begin{array}{c}\text{Csc}^{-1}(-x)=-\text{Csc}^{-1}(x)\ \text{for } x\in(-\infty,-1]\cup[1,\infty)\\ \text{(Symmetric about origin)}\end{array}\right]$

  The symmetry along range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]/\{0\}$ shows that graph is reflected across origin.
- **Reflection:** The graph of $y=\text{Csc}^{-1}x$, is a reflection of $y=\text{Csc}\,x$ across the line $y=x$. This means that the x and y coordinates of the points on the graph of $y=\text{Csc}\,x$ are switched to obtain the points on the graph of $y=\text{Csc}^{-1}x$.
- **Inverse Function:** $\text{Csc}(\text{Csc}^{-1}x)=x$ for $x\in(-\infty,-1]\cup[1,\infty)$ and

  $$\text{Csc}^{-1}(\text{Csc}\,x)=x \text{ for } x\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$$
- **Identities:** $\text{Csc}^{-1}(x)=\text{Sin}^{-1}\left(\frac{1}{x}\right)$ for $x\in(-\infty,-1]\cup[1,\infty)$



x	$y=\text{Csc}^{-1}x$
$+\infty$	$+0$ or $+0^\circ$
2	$\pi/6$ or 30°
1.4	$\pi/4$ or 45°
1.2	$\pi/3$ or 60°
1	$\pi/2$ or 90°
-1	$-\pi/2$ or -90°
-1.2	$-\pi/3$ or -60°
-1.4	$-\pi/4$ or -45°
-2	$-\pi/6$ or -30°
$-\infty$	-0 or -0°

4. i. $1-i$ ii. $\frac{-5}{2}-\frac{5\sqrt{3}}{2}i$ iii. $-2i$ iv. $-2\sqrt{3}+2i$ v. $\sqrt{3}+i$

vi. $\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ vii. $6.43+7.66i$ viii. $-1+i$

5. i. . $x+y=1$ ii. $x^2+y^2=4$ iii. $-\sqrt{3}\le\frac{y}{x-4}\le\sqrt{3}$

iv. $(-\sqrt{3}-1)x+(\sqrt{3}-1)y+4(\sqrt{3}+1)\le 0\le -x+y+4$

v. $x^2+y^2=1$ vi. $\sqrt{3}(x^2-y^2+1)-2xy=0$

6. i. $\frac{\sqrt{2}}{500}(1+i)$ ii. $\frac{1}{500}(1+\sqrt{3}i)$ iii. $\frac{1}{500}(\sqrt{3}+i)$ 7. i. $1+2\sqrt{3}i$ ii. 0.4756

8. i. $\frac{5}{2}-\frac{35}{2}i$ ii. $-\frac{10}{73}(77+38i)$ 9. 0.3 cost 10. 0.8 cost

11. $7.936\cos(t+39.39°)$ 12. $\sqrt{61}\cos(t+86.33°)$

REVIEW EXERCISE

1. i. c ii. b iii. a iv. d v b vi d vii c viii d ix b x b
2. i. 0 ii. $\sqrt{2}$ iii. $\sqrt{221}$ iv. $-\frac{9}{34}-\frac{19}{34}i$ 3. i. $3(x-6i)(x+6i)$ ii. $4(x-\sqrt{10}i)(x+\sqrt{10}i)$
4. $z=x$ 5. $z=\frac{-14}{29}+\frac{64}{29}i$ 6. $2+11i$ 7. $z=\frac{11\pm i\sqrt{71}}{4}$ 8. 2

Unit 2: Matrices and Determinants

Exercise 2.1

1. i. 2 x 3 ii. 3x2 iii. 3x1 iv. 1x4 v. 1x1 vi. 2x2
2. i. rectangular ii. Square iii. column iv. square v. row vi. square
3. i. lower triangular ii. scalar iii. diagonal iv. identity

   v. diagonal vii. upper triangular viii. diagonal ix. scaler

4. i. $\begin{bmatrix}2 & \sqrt{5} & 1\\ 0 & 6 & 9\end{bmatrix}$ neither symmetric nor skew symmetric ii. $\begin{bmatrix}1\\6\\2\\0\end{bmatrix}$ neither symmetric nor skew symmetric iii. $\begin{bmatrix}2 & 6\\ 6 & 2\end{bmatrix}$ symmetric

iv. $\begin{bmatrix}0 & -1 & -9\\ 1 & 0 & 5\\ 9 & 5 & 0\end{bmatrix}$ skew symmetric v. $\begin{bmatrix}3 & -6 & 9\\ -6 & 2 & 0\\ 9 & 0 & 0\end{bmatrix}$ symmetric vi. $\begin{bmatrix}9 & 0 & 0\\ 0 & 6 & 0\\ 1 & 3 & 1\end{bmatrix}$ neither symmetric nor skew symmetric

Exercise 2.2

1. i. $\begin{bmatrix}2 & 7/2\\ 5/2 & 4\end{bmatrix}$ ii. $\begin{bmatrix}1/2 & 1\\ 1 & 2\end{bmatrix}$ iii. $\begin{bmatrix}1 & 1/2\\ 2 & 1\end{bmatrix}$ iv. $\begin{bmatrix}-1/3 & -4/3\\ 1/3 & -2/3\end{bmatrix}$

2. i. $\begin{bmatrix}0 & -1/3 & -2/3\\ 1 & 2/3 & 1/3\\ 8/3 & 7/3 & 2\end{bmatrix}$ ii. $\begin{bmatrix}0 & -3/2 & -4\\ 3/4 & 0 & -5/4\\ 8/9 & 5/9 & 0\end{bmatrix}$ iii. $\begin{bmatrix}2/3 & 1/2 & 2/5\\ 2/5 & 1/3 & 2/7\\ 2/7 & 1/4 & 2/9\end{bmatrix}$

iv. $\begin{bmatrix}1 & 5/3 & 5/2\\ 5/3 & 2 & 13/5\\ 5/2 & 13/5 & 3\end{bmatrix}$ 3. $C=\begin{bmatrix}-5 & 0 & -9\\ 0 & -8 & 0\\ 4 & -4 & 1\end{bmatrix}$

4. i. $A=\begin{bmatrix}-5 & 7/2\\ 8 & -11/2\end{bmatrix}$ ii. $\begin{bmatrix}1/2 & 1 & 0\\ 2 & 4 & 1\end{bmatrix}$ iii. $\begin{bmatrix}7x\\ x\end{bmatrix}$ *where* $x \in \mathbb{R}$

iv. $z=4, t=0,\ x^2+y^2=20$ v. $\alpha=-10; \beta=9$ vi. $-4, 3$

6. $\alpha=-9;\quad \beta=-1$ 10. skew symmetric

12. $X=\begin{bmatrix}1 & 3 & -1\\ 1 & 0 & 3\end{bmatrix}$, $Y=\begin{bmatrix}1 & 0 & 1\\ 0 & -1 & -1\end{bmatrix}$ 13. $X=\begin{bmatrix}\frac{3}{5} & \frac{4}{5} & \frac{3i}{5}\\ \frac{-3+12i}{5} & 2-i & \frac{7}{5}\\ \frac{1}{5} & \frac{1}{5} & \frac{-6+6i}{5}\end{bmatrix}$, $Y=\begin{bmatrix}\frac{1}{5} & \frac{13}{5} & \frac{6i}{5}\\ \frac{19-i}{5} & -1+3i & \frac{14}{5}\\ \frac{7}{5} & \frac{7}{5} & \frac{18-3i}{5}\end{bmatrix}$

Exercise 2.3

1. i. 15 ii. 1 iii. – 6 iv. 16 + 8i
2. i. – 17 ii. 27 iii. 1 – 16i iv. – 17 + 11i
3. singular ii. Non-singular iii. Non-singular iv. singular
4. i. 16/23 ii. – 4 iii. – 1 + 5i iv. $-\frac{7}{100}-\frac{i}{100}$
5. i. $\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0\\ \frac{-1}{9} & \frac{2}{9} & \frac{-1}{3}\\ \frac{5}{9} & \frac{-1}{9} & \frac{-1}{3}\end{bmatrix}$ ii. $\begin{bmatrix}\frac{-1}{3} & \frac{-4}{9} & \frac{26}{9}\\ \frac{-1}{3} & \frac{-1}{9} & \frac{11}{9}\\ \frac{1}{3} & \frac{4}{9} & \frac{-17}{9}\end{bmatrix}$ iii. $\begin{bmatrix}\frac{-4i}{5} & 0 & \frac{1}{5}\\ \frac{8-i}{5} & -1 & \frac{-1+2i}{5}\\ \frac{1}{5} & 0 & \frac{-i}{5}\end{bmatrix}$ iv. $\begin{bmatrix}\frac{3}{11} & \frac{2+2i}{11} & \frac{-2+i}{22}\\ 0 & \frac{1-i}{2} & \frac{1+i}{4}\\ \frac{-2i}{11} & \frac{-1+i}{22} & \frac{5-i}{44}\end{bmatrix}$ 6. $\begin{bmatrix}\frac{1}{3} & \frac{-1}{2} & \frac{1}{6}\\ 0 & 1 & 0\\ \frac{-1}{9} & 0 & \frac{1}{9}\end{bmatrix}$

Exercise 2.5

1. i. $\begin{bmatrix}1 & 3 & 5\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ii. $\begin{bmatrix}1 & -18\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$ iii. $\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 27\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

iv. $\begin{bmatrix}1 & -2 & 3/2\\ 0 & 1 & -8/9\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ v. $\begin{bmatrix}1 & -8 & -6\\ 0 & 1 & 4/5\end{bmatrix}\begin{bmatrix}1 & 0 & 2/5\\ 0 & 1 & 4/5\end{bmatrix}$ vi. $\begin{bmatrix}0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

2. i. 3 ii. 2 iii. 3 iv. 2
3. $\frac{1}{2}\begin{bmatrix}-12 & -5 & -3\\ -4 & -1 & -1\\ 2 & 1 & 1\end{bmatrix}$ ii. $\frac{1}{6}\begin{bmatrix}-2 & 0 & 2\\ 19 & -15 & -16\\ -6 & 6 & 6\end{bmatrix}$ iii. $\frac{1}{12}\begin{bmatrix}0 & -6 & 6\\ 3 & -9 & 6\\ 2 & -4 & 6\end{bmatrix}$ iv. $\begin{bmatrix}-8 & 5 & 2\\ -18 & 18 & -9\\ 15 & -6 & 3\end{bmatrix}$

Exercise 2.6

1. i. $\begin{bmatrix}x_3\\ 2x_3\\ x_3\end{bmatrix}$ ii. $\begin{bmatrix}\frac{-7}{5}x_3\\ \frac{2}{5}x_3\\ x_3\end{bmatrix}$ iii. $\begin{bmatrix}x_3\\ 2x_3\\ x_3\end{bmatrix}$ iv. does not exist.
2. i. $\lambda=\frac{-7}{11}; \begin{bmatrix}\frac{-10}{13}x_3\\ \frac{11}{13}x_3\\ x_3\end{bmatrix}$ ii. $\lambda=2; \begin{bmatrix}-x_3\\ \frac{1}{2}x_3\\ x_3\end{bmatrix}$ *or* $\lambda=-7; \begin{bmatrix}17x_3\\ 5x_3\\ x_3\end{bmatrix}$
3. i. $;\frac{66}{19};-\frac{63}{19}$ ii. No solution iii. $\frac{5}{4};\frac{5}{4};\frac{-1}{2}$ iv. $-7;-7;5$
4. i. 3;1;2 ii. $-\frac{1}{7};\frac{1}{7};0$ iii. solution not possible as A is singular iv. $\frac{6}{11};-\frac{7}{11};\frac{2}{11}$
5. $\frac{1}{11};-\frac{3}{11};\frac{70}{11}$ ii. $\frac{37}{12};\frac{7}{3};\frac{11}{12}$ iii. $\frac{7}{4};-\frac{23}{2};-\frac{29}{4}$ iv. 2; 3; 5
6. $\begin{bmatrix}-3/62 & 9/62 & 5/62\\ 13/31 & -8/31 & -1/31\\ 19/62 & 5/62 & -11/62\end{bmatrix}$ 1; 1; 1
7. $\lambda=\pm 4$, no solution; $\lambda \neq \pm 4$ unique solution

10. $\begin{bmatrix}1 & 0 & 0\\0 & 1 & -5\\0 & 0 & 1\end{bmatrix}$; $x'^2 + y'^2 + 10y' + 16 = 0$

11. $\begin{bmatrix}1 & 0 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $x'^2 + 8x' - 3y' + 4 = 0$

12. $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $2x'^2 - 5y'^2 - 4x' - 8 = 0$

13. $\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$; $2x' - 7y' + 11 = 0$

REVIEW EXERCISE

1. i. b ii. d iii. d iv. b v. c vi. b vii. d viii. c ix. c x. d
2. − 11, 3, 10; 87 4. 1/3

Unit 3: Vectors

Exercise 3.1

1. i. $-7\hat{\imath} - 5\hat{\jmath}$ ii. $-22\hat{\imath} - 16\hat{\jmath}$ iii. $-7\hat{\imath} + 28\hat{\jmath}$ iv. $-\frac{35}{2}\hat{\imath} - \frac{13}{2}\hat{\jmath}$ v. $-18\hat{\imath} + 155\hat{\jmath}$

3. i. $\frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath}$ ii. $\vec{u} = -5\hat{\imath} + 8\hat{\jmath}$; $\vec{v} = 7\hat{\imath} - 11\hat{\jmath}$ iii. $8\hat{\imath} - \hat{\jmath} - \hat{k}$

4. i. $m = 4/3$ ii. $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $-2\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$, $\lambda\hat{\imath} - \lambda\hat{k}$

5. i. $\frac{1}{\sqrt{38}}\hat{\imath} - \frac{1}{\sqrt{38}}\hat{\jmath} + \frac{6}{\sqrt{38}}\hat{k}$ ii. $\sqrt{5}$; 3 7. i. $\lambda = \pm 2\sqrt{11}$ ii. $|\vec{a}| = |\vec{b}|$; $\vec{a} \neq \vec{b}$

8. i. $\frac{5}{\sqrt{10}}\hat{\jmath} - \frac{15}{\sqrt{10}}\hat{k}$ ii. $-\frac{3}{\sqrt{257}}\hat{\imath} + \frac{36}{7\sqrt{257}}\hat{\jmath} - \frac{24}{7\sqrt{257}}\hat{k}$ 9. i. $\frac{7}{5}\hat{\imath} + \frac{1}{5}\hat{k}$ ii. $\hat{\imath} - 12\hat{\jmath} + 5\hat{k}$

10. i. D(− 2, 1) ii. x = 6 and y = 3 14. i. $PS = \vec{s} - \vec{r}$

16. $\overrightarrow{AC} = \vec{a} + \vec{b}$, $\overrightarrow{CD} = \vec{b} - \vec{a}$, $\overrightarrow{EF} = -\vec{b}$, $\overrightarrow{DA} = -2\vec{a}$, $\overrightarrow{EB} = 2(\vec{a} - \vec{b})$, $\overrightarrow{FA} = \vec{a} - \vec{b}$, $\overrightarrow{FC} = 2\vec{a}$

Exercise 3.2

1. i. 15 ii. 90 iii. − 16 iv. 147 v. 4

2. i. $\theta = \cos^{-1}\left(\frac{-5}{2\sqrt{13}}\right)$ ii. $\theta = \cos^{-1}\left(\frac{57}{\sqrt{6342}}\right)$ iii. $\theta = \cos^{-1}\left(\frac{-30}{\sqrt{1870}}\right)$
   iv. $\theta = \cos^{-1}\left(\frac{-15}{\sqrt{357}}\right)$ v. $\theta = \cos^{-1}\left(\frac{18}{\sqrt{438}}\right)$

3. i. $\cos^{-1}\left(\frac{1}{4}\right)$ ii. 90°

4. i. $\lambda = \frac{29}{44}$
   i. $\cos\alpha = \frac{2}{\sqrt{29}}$; $\cos\beta = \frac{-3}{\sqrt{29}}$; $\cos\gamma = \frac{4}{\sqrt{29}}$ ii. $\frac{2}{\sqrt{114}}$; $\frac{21}{\sqrt{62}}$

6. i. 45°, 45° ii. $\pm\frac{5}{\sqrt{3}}\hat{\imath} \pm \frac{5}{\sqrt{3}}\hat{\jmath} \pm \frac{5}{\sqrt{3}}\hat{k}$

7. i. − 45/2

8. $\vec{r} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$
   14. $350/\sqrt{11}\,joules$ 15. 28 units 16. $150\sqrt{3}$

Exercise 3.3

1. i. (4, − 15, − 7) ii. (30, 11, −27) iii. (4, − 6, 2)
2. i. (− 18, - 8, 3) ii. (3, 15, 6)
3. i. $\frac{\sqrt{78}}{\sqrt{29}\sqrt{26}}$ ii. $\frac{3\sqrt{62}}{\sqrt{29}\sqrt{83}}$ 4. i. $\left(\frac{40}{\sqrt{1533}}, \frac{185}{\sqrt{1533}}, \frac{50}{\sqrt{1533}}\right)$
   ii. parallel $\frac{-1}{5}\hat{\imath} + \frac{1}{10}\hat{\jmath} - \frac{3}{10}\hat{k}$; perpendicular $\frac{26}{5}\hat{\imath} + \frac{19}{10}\hat{\jmath} - \frac{27}{10}\hat{k}$ 5. ii. $\vec{d} = -\frac{2}{5}\hat{\imath} - \frac{1}{5}\hat{\jmath} + \frac{4}{5}\hat{k}$
6. i. $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ 9. i. $\frac{15\sqrt{15}}{4}$ ii. 6

10. $i.\ 3\sqrt{59}$ ii. $5/\sqrt{2}$ $\alpha = 74.21°, \beta = 60.50°, \gamma = 45.29°$

11. $\sqrt{75}$ 12. Either $\vec{a} = 0$ or $\vec{b} = 0$ or Both are zero

14. i. $\sqrt{181}$ ii. $11\sqrt{6}$ 15. i. $\frac{1}{4}(7-3t)\hat{\imath} + \frac{1}{2}(1-t)\hat{\jmath} + t\hat{k}$ ii. $\left(-\frac{1}{6}(4t+21), -\frac{1}{3}(2t+3), t\right)$

16. $11\hat{\imath} + \hat{\jmath} - 5\hat{k}$ 17. $15\hat{\imath} - 20\hat{\jmath} + 7\hat{k};\ -9\hat{\imath} - 26\hat{\jmath} + 19\hat{k};\ 6\hat{\imath} - 46\hat{\jmath} + 26\hat{k}$

Exercise 3.4

1. i. -14 ii. -20
2. i. 2 ii. -8
3. ii. $\lambda = -1/2$
4. i. $\lambda = 2$ 6. i. zero ii. 68 7. i. 27/6 ii. 3

REVIEW EXERCISE

1. i. d ii. c iii. b iv. b v. c vi. d vii. b viii. b ix. c x. a
2. i. 2/3 ii. - 3/20
3. $\sqrt{2}$ 4. $\sqrt{19}$ 5. $-11/2$

8. Ground speed $\approx 235.492\ km/h$ true course$\approx 64.872°$

9. Speed $\approx 237.816\ km/h$ direction $\approx 107.980°$

Unit 4: Sequences and Series

Exercise 4.1

1. (i) $a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_{10} = 31, a_{15} = 46$
   (ii) $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11, a_{10} = 29, a_{15} = 44$
   (iii) $a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, a_3 = \frac{3}{4}, a_4 = \frac{4}{5}, a_{10} = \frac{10}{11}, a_{15} = \frac{15}{16}$
   (iv) $a_1 = 2, a_2 = 5, a_3 = 10, a_4 = 17, a_{10} = 101, a_{15} = 226$
   (v) $a_1 = -1, a_2 = 0, a_3 = 3, a_4 = 8, a_{10} = 80, a_{15} = 95$
   (vi) $a_1 = 0, a_2 = \frac{3}{5}, a_3 = \frac{4}{5}, a_4 = \frac{15}{17}, a_{10} = \frac{99}{101}, a_{15} = \frac{112}{113}$
   (vii) $a_1 = 1, a_2 = -\frac{1}{2}, a_3 = \frac{1}{4}, a_4 = -\frac{1}{8}, a_{10} = -\frac{1}{512}, a_{15} = \frac{1}{16384}$
   (viii) $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_{10} = 100, a_{15} = 225$
   (ix) $a_1 = -4, a_2 = 5, a_3 = -6, a_4 = 7, a_{10} = 13, a_{15} = -18$
   (x) $a_1 = -2, a_2 = -1, a_3 = 4, a_4 = -7, a_{10} = -25, a_{15} = 40$
2. (i) $a_8 = 29$ (ii) $a_9 = 56$ (iii) $a_7 = 225,$ (iv) $a_{12} = -23.5$
   (v) $a_{22} = 528{,}528,$ (vi) $a_{20} = \frac{441}{400}$ (vii) $a_{43} = 43$ (viii) $a_{67} = 67$
3. (i) $a_n = 2n - 1$ (ii) $a_n = 3^n$ (iii) $a_n = \sqrt{2n}$ (iv) $a_n = n(n+1)$

Exercise 4.2

1. **(i)** $a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13$ **(ii)** $a_1 = 7, a_2 = 12, a_3 = 17, a_4 = 22$
   (iii) $a_1 = 16, a_2 = 14, a_3 = 12, a_4 = 10$ **(iv)** $a_1 = 38, a_2 = 34, a_3 = 30, a_4 = 26$
   (v) $a_1 = \frac{3}{4}, a_2 = 1, a_3 = \frac{5}{4}, a_4 = \frac{3}{2}$ **(vi)** $a_1 = \frac{3}{8}, a_2 = 1, a_3 = \frac{13}{8}, a_4 = \frac{9}{4}$
2. **(i)** The next three terms of the sequence are $17, 21, 25$
   (ii) The next three terms of the sequence are $20, 23, 26$
   (iii) The next three terms of the sequence are $\frac{7}{2}, \frac{9}{2}, \frac{11}{2}$
   (iv) The next three terms of the sequence are 0.22, 0.27, 0.32 3. $a_{11} = 0.57$
4. $a_1 = 19, a_2 = \frac{33}{2}, a_3 = 14, a_4 = \frac{23}{2}$ 5. $a_1 = 8, a_2 = 5, a_3 = 2, a_4 = -1$
6. $a_{87} = 347$ 7. $a_{20} = 70$ 8. $a_{56} = -\frac{105}{2}$ 9. $d = \frac{a-c}{2ac}$
10. $a_8 = 240$ feet 11. $S_{20} =$ Rs 39000 12. $a_8 = 7$

13. (i) 12 (ii) 5 (iii) $4\sqrt{5}$ (iv) $\frac{7y}{2}+4$. 14. $b=0$ 15. $x=-9, y=24$
16. $A_1=9, A_2=13$ 17. $A_1=-3, A_2=-8, A_3=-13$

Exercise 4.3

1. $S_n=116$ 2. $S_n=10100$ 3. $S_n=10500$ 4. $S_n=375$ 5. $S_n=240$, 6. -210 7. $S_n=240$
8. $S_n=2550$ 9. $S_n=2500$ 10. $S_n=34036$ 11. $S_n=-140$ 12. $S_n=1155$ 13. 162
14. 104 15. $S_n=1060$ 16. $S_n=387$ 17. $S_n=816$ 18. $S_n=162$ 19. $S_n=-220$
20. $a_1=7, a_2=19, a_3=31$ 21. $a_1=1$, $a_2=5$, $a_3=9$ 22. $a_1=6$, $a_2=36$, $a_3=66$
23. $a_{25}=62{,}950$ 24. 45 25. 12,280,000 26. 38,750

Exercise 4.4

1. The sequence is not geometric
2. The sequence is not geometric 3. The sequence is geometric $(r=\frac{3}{2})$
4. The sequence is not geometric 5. $a_1=3, a_2=-6, a_3=12, a_4=-24$
6. $a_1=27$, $a_2=-9$, $a_3=3$, $a_4=-1$ 7. $a_1=12, a_2=6, a_3=3, a_4=\frac{3}{2}$
8. $a_4=\frac{10}{3}, a_5=\frac{10}{9}$ 9. $a_4=54, a_5=162$ 10. $a_4=\frac{135}{2}, a_5=\frac{405}{4}$
11. $a_4=27, a_5=9$ 12. $a_4=1, a_5=3$ 13. $a_4=2, a_5=4$
14. $a_3=100$ 15. $a_5=32$ 16. $a_4=56$ 17. $a_5=3$
18. $a_6=-1$ 19. $a_8=\frac{1}{8}$ 20. 6, 12, 24 21. 2, 4 22. 4, 2, 1, $\frac{1}{2}$
23. 15 24. 10, 20, 40 25. 14, 28, 56 26. $\frac{1}{256}$ ft 27. 151258.9(appro)
28. 3100 ft. (approximately) 29. 127 30. 81

Exercise 4.5

1. 176, 2. 93.15 3. 13,28,600 4. 947.11 5. 114681 6. 732
7. 10.66 8. 165, 9. 300 10. 189 11. 4 12. 0.51 13. 4
14. (i) $\frac{4}{9}$ (ii) 1 (iii) $\frac{5}{9}$ (iv) $\frac{2}{3}$ (v) $\frac{5}{33}$ (vi) $\frac{4}{33}$ 15. 70 16. 800

Exercise 4.6

1. $\frac{1}{27}$ 2. $-\frac{1}{7}$ 3. $-\frac{1}{77}$ 4. $\frac{1}{5n-1}$ 5. $\frac{1}{34-7n}$ 6. $\frac{1}{\frac{n+3}{2}}$
7. $\frac{1}{43}$ 8. $-\frac{1}{41}$ 9. $-\frac{1}{23}$ 10. $\frac{99}{10}$ 11. $\frac{8}{13}$ 12. $\frac{5}{23}, \frac{5}{31}, \frac{5}{39}, \frac{5}{47}$

Exercise 4.7

1. $\frac{137}{120}$ 2. $\frac{43024}{45045}$ 3. 63 4. 45π 5. $\frac{15551}{2520}$ 6. $-52{,}432$ 7. 43, 8. $\frac{10}{11}$ 9. $\sum_{k=1}^{\infty}\frac{k}{k+1}$ 10. $\sum_{k=1}^{5} 3k$
11. $\sum_{k=1}^{6}(-1)^k 2^k$ 12. $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ 14. $\frac{n^2+3n}{2}$ 15. $\frac{n}{6}(2n^2+9n+7)$ 16. $\frac{n}{2}(2n^2+5n+5)$
17. $\frac{n}{2}(2n^2+3n-1)$ 18. $\frac{2n(n+1)(2n+1)}{3}$ 19. $n(2n^3+2n^2+11n+1)$
20. $\frac{n(2n^2+3n+7)}{6}$ 21. $n(n+1)^2$
22. $n(2n^3+8n^2+7n-2)$ 23. $2^n(n-1)+1$ 24. $\frac{3ny^{n+1}-2y^{n+1}-3ny^n-y^n+2y+1}{(y-1)^2}$
25. $\frac{3.7^{n+1}-42n-21}{12.7^n}$ 26. $14-\frac{6n+7}{2^{n-1}}$ 27. 9 28. $\frac{25}{16}$ 29. $\frac{2x+1}{(1-x)^2}$ 30. $\frac{100}{27}$

Exercise 4.8

1. 780ft 2. 1065, no because the auditorium has only 1065 seats 3. 491,70044
4. $a_{20}=524288$ 5. $a_6=0.328$ or 32.8% 6. 30361.4082 7. 964615

8. 32000 9. $r = 2.95$ 10. Rs. 1420418.205 11. Rs. 100625
12. Rs. 726000 13. Rs. 69000 14. Rs. 356015.99

Miscellaneous Exercise

1. **(i)** a **(ii)** c **(iii)** b **(iv)** a **(v)** c **(vi)** d **(vii)** b **(viii)** a **(ix)** b **(x)** a **(xi)** c **(xii)** b **(xiii)** a **(xiv)** c **(xv)** b **(xvi)** b **(xvii)** d **2.** 3, 5, 7, 9 **3.** 0, 1, 2, 3 **4.** 6, 10, 14, ⋯

5. $\frac{1}{27}[10^{n+1} - 9n - 10]$ **6.** 4, 6, 9 **7.** 6, 18, 54, 162 **8.** $n = -1$ **9.** $a_1 = \frac{6}{5}, a_2 = 1, a_3 = \frac{6}{7}, a_4 = \frac{3}{4}$

10. (i) 5 (ii) -4920 **11.** $n(n+1)^2$ **12.** $\frac{1}{6}(2n^3 - 3n^2 + 13n - 6)$

Unit 5: Polynomials

Exercise 5.1

1. (i) 5 (ii) 35 3. No 4. $y + 1$ 5. -12 6. $m = 6$ 7. Only 1 is a zero of $P(x)$
8. $2, -3, \frac{-1}{2}$ 9. $f(x) = (x-4)(x^2 + 3x - 2) + 0$ 10. $x^2 + 10x + 24$

Exercise 5.2

1. $(y+1)(y-3)(y+2)$ 2. $(x-1)(x+1)(2x-1)$ 3. $(x-2)(x+3)(2x+3)$ 4. $(x-3)(3x^2 + 4x + 12)$
5. $(t-1)(t^2 + 2t + 5)$ 6. Other two factors are $(x-6)$ and $(2x+1)$ 7. $(2x-1)(x-5)(x-2)$
8. $(2x+1)(2x^2 + x + 36)$

Exercise 5.3

1. 5cm by 12cm by 2cm 2. 650 3. 6 units by 8 units by 3 units 4. 9 units by 11 units by 25 units
5. Length of one side of square ABFG is $x+4$. Area = $(x+4)^2$. The length of rectangle ACED = $3x+7$.
6. $y+1, y-1$

REVIEW EXERCISE

1. (i) (d) (ii) (a) (iii) (b) (iv) (a) (v) (b) (vi) (c) (vii) (b) (viii) (c)
2. $16y^2 + 4y + 4$ 3. $25y^2 + 10y + 4$ 4. Yes 5. $5x^3 - 13x^2 - 34x + 24$
6. -48 7. $x + 4$ 8. $y + 5$

Unit 6: Permutation, & Combination

Exercise 6.1

1. i. 3628800 ii. 7920 iii. 11/63 iv. n − 1 v. 7/90
2. i. $\frac{14!}{10!}$ ii. $\frac{9!}{4! \times 16}$ iii. $\frac{(n+1)!}{(n-2)!}$ iv. $\frac{(n-1)!}{n(n-4)(n-4)!}$
5. i. 6 ii. 6 7. i. 31 ii. 8 iii. 3 iv. 10 v. 6 vi. 121 vii. 11 viii. 2 ix. 4 x. 5

Exercise 6.2

2. i. 7 ii. 13 iii. 15 iv. 8 v. 29 vi. 6 vii. 9 viii. 8 ix. 10
3. i. 4 ii. 5 iii. 2 iv. 8 v. 2 vi. 3 vii. 41
4. 60 5. 60480 6. 1296 7. 18 8. 576 9. 1260 10. 720; 120
11. 45360 12. 108 13. 4320 14. 210 15. 8640 16. 360
17. 72 18. 94 19. 30,240 20. 86400 21. 6, 6, 6, 24 22. HOELRA 22. MULTAN

Exercise 6.3

2. i. 13 ii. 22 iii. 51 iv. 6 v. 11 vi. 5 3. i. 3 ii. 3 iii. 6 iv. 5
4. i. 9; 3 ii. 62; 27 iii. 10; 5 iv. 14; 4
5. i. 4368 ii. 3003 6. 55 7. i. 120 ii. 186 iii. 186 8. i. 45 ii. 120
9. ${}^nC_2 - n$ 10. 120 11. 10 12. 300500200 13. 63 14. 175616

REVIEW EXERCISE

1. i. b ii. c iii. a iv. a v. d vi. d vii. b viii. d ix. c x. c

2. 24 3. 90 4. 600 5. 360 6. 32,659,200

Exercise 7.2

1. i. $32x^{\frac{5}{2}} + 80x^{\frac{3}{2}} + 80\sqrt{x} + \frac{40}{\sqrt{x}} + \frac{10}{x^{\frac{3}{2}}} + \frac{1}{x^{\frac{5}{2}}}$

   ii. $\frac{729}{x^6} + \frac{729y}{x^5} + \frac{1215y^2}{4x^4} + \frac{135y^3}{2x^3} + \frac{135y^4}{16x^2} + \frac{9y^5}{16x} + \frac{y^5}{64}$

   iii. $128 - 448x^{\frac{3}{2}} + 672x^3 - 560x^{\frac{9}{2}} + 280x^6 - 84x^{\frac{15}{2}} + 14x^9 - x^{\frac{21}{2}}$

   iv. $\frac{x^{10}}{y^{10}} - 5\frac{x^{\frac{15}{2}}}{y^{\frac{15}{2}}} + 10\frac{x^5}{y^5} - 10\frac{x^{\frac{5}{2}}}{y^{\frac{5}{2}}} + 5\frac{y^{\frac{5}{2}}}{x^{\frac{5}{2}}}$

2. i. $\frac{32x^5}{243} - \frac{40x^3}{27} + \frac{20x}{3} - \frac{15}{x} + \frac{45}{x^2} - \frac{243}{32x^5}$

   ii. $x^6 - 6\frac{x^5}{y} + \frac{15x^4}{y^2} - \frac{20x^3}{y^3} + \frac{15x^2}{y^4} - \frac{6x}{y^5} + \frac{1}{y^6}$

   iii. $2187u^7 - 5103u^6 + 5103u^5 - 2835u^4 + 945u^3 - 189u^2 + 21u - 1$

   iv. Iv. $a^5 2^{\frac{5}{2}} + 20ab\sqrt{3} + 30ab^2\sqrt{2} + 20ab^3\sqrt{3} + 45ab^4\sqrt{2} + b^5 3^{\frac{5}{2}}$

   v. V. $1 + 8x - 4y + 24x^2 - 24xy + 6y^2 + 32x^3 - 12x^2y + 6xy^3 - y^3 + 16x^4 - 32x^3y +$
   $24x^2y^2 - 8xy^3 + y^4$

   vi. Vi. $\frac{1}{x^4} + \frac{8}{x^3y} + \frac{24}{x^2y^2} + \frac{32}{xy^3} + \frac{16}{y^4} + \frac{12}{zx^3} + \frac{72}{x^2yz} + \frac{144}{xy^2z} + \frac{96}{y^3z} + \frac{54}{x^2z^2} + \frac{216}{xyz^2} + \frac{216}{y^2z^2} + \frac{108}{xz^3} + \frac{216}{yz^3} + \frac{81}{x^4}$

3. i. $2 + 1200x^2 + 20000x^4$ ii. $8x^2 - 12x + 2 - \frac{3}{x}$

   iii. $1 - 3x + 2x^2 + 10x^3 + 5x^4 - 11x^5 + 8x^6 - 2x^7$

   iv. 2.01090301

   v. $\frac{32}{x^4} + 2 + \frac{x^4}{128}$ vi. $8a^6\sqrt{a^2 - 1} + 8a^2\left(\sqrt{a^2 - 1}\right)^3$

4. $\frac{15}{16}x^6y^7$

5. i. $-\frac{15309}{8}x^5$ ii. $0.946176x^4$ iii. $70a^4$ iv. $\frac{673596a^6}{x^{12}}$

6. i. $\frac{15}{2}a^{14}b^6$ ii. $\frac{40095}{16}p^{16}q^8$ iii. $-12x^4y^3$ iv. $-270y^8x^3$

7. i. 1980 ii. 5

8. $T_{r+2} = \binom{n}{r+1} a^{n-r-1} b^{r+1}$

10. -5940 11. 14 13. 12, 2 18. 1, 6, 15, 20, 15, 6, 1

19. 21 times 20. 56 times

Exercise 7.3

1. i. $1 + 3x^{\frac{1}{2}} + 6x + 10x^{\frac{3}{2}}$ $0 < x < 1$

   ii. $\frac{1}{3^{\frac{1}{3}}} - \frac{2}{3^{\frac{7}{3}}x} + \frac{24}{3^{\frac{13}{3}}x^2} - \frac{80}{3^{\frac{22}{3}}x^3}$ $-\frac{2}{3} > x > \frac{2}{3}$

   iii. $\left(\frac{5}{2}\right)^{\frac{1}{2}} - \frac{3}{\sqrt{2}\sqrt{5}x^2} - \frac{3^2}{2^{\frac{3}{2}}5^{\frac{3}{2}}x^4} + \frac{3^3}{2^{\frac{3}{2}}5^{\frac{7}{2}}x^6}$ $|x^2| > \frac{6}{5}$

   iv. $1 + \frac{2x}{3} + \frac{2x^2}{3^2} + \frac{2x^3}{3^3} + \frac{x^4}{3^4}$ $-3 < x < 3$

v. $\frac{1}{3}-\left(\frac{1}{4.3^{\frac{3}{2}}}+\frac{2}{\sqrt{3}}\right)x+\left(\frac{1}{32.3^{\frac{5}{2}}}+\frac{1}{128.3^{\frac{3}{2}}}\right)x^2-\left(\frac{5}{128.3^{\frac{7}{2}}}+\frac{1}{16.3^{\frac{3}{2}}}\right)x^4$ $\quad -6 < x < 6$ $\quad$ vi. 1

2. i. 2.0052 $\quad$ ii. 1.2963 $\quad$ iii. 1.0099 $\quad$ iv. 0.9859 $\quad$ 3. $\frac{315\sqrt{2}}{4096}$

7. $\frac{(-1)^n}{2}(n^2+7n+8)x^n$

8. i. $\frac{1}{\sqrt[3]{4}}$ $\quad$ ii. $\left(\frac{5}{6}\right)^{\frac{1}{3}}$ $\quad$ iii. $(-3)^{\frac{6}{5}}$ $\quad$ iv $2\sqrt{2}$

Exercise 7.4

1. 1, 8, 4, 7 $\quad$ 2. a. 5 $\quad$ b. 2 $\quad$ 3. 1

4. a. 5, 25 $\quad$ b. 3, 33 $\quad$ c. 1, 01 $\quad$ 8. 8 $\quad$ 9. 8

REVIEW EXERCISE

1. i. d $\quad$ ii. c $\quad$ iii. d $\quad$ iv. d $\quad$ v. b $\quad$ vi. c $\quad$ vii. b $\quad$ viii. a $\quad$ ix. b $\quad$ x. b

3. 232 $\quad$ 6. 78

Unit 8: Fundamentals of Trigonometry

EXERCISE 8.1

1. (i) $\cos(180^\circ+60^\circ)=-\cos 60^\circ$, $\cos(180^\circ-60^\circ)=-\cos 60^\circ$, $\sin(180^\circ+60^\circ)=-\sin 60^\circ$, $\sin(180^\circ-60^\circ)=\sin 60^\circ$, $\tan(180^\circ+60^\circ)=\tan 60^\circ$, $\tan(180^\circ-60^\circ)=-\tan 60^\circ$

(ii) $\cos(90^\circ+60^\circ)=-\sin 60^\circ$, $\cos(90^\circ-60^\circ)=\sin 60^\circ$, $\sin(90^\circ+60^\circ)=\cos 60^\circ$, $\sin(90^\circ-60^\circ)=\cos 60^\circ$, $\tan(90^\circ+60^\circ)=-\cot 60^\circ$, $\tan(90^\circ-60^\circ)=\cot 60^\circ$

(iii) $\cos(180^\circ+30^\circ)=-\cos 30^\circ$, $\cos(180^\circ-30^\circ)=-\cos 30^\circ$, $\sin(180^\circ+30^\circ)=-\sin 30^\circ$, $\sin(180^\circ-30^\circ)=\sin 30^\circ$, $\tan(180^\circ+30^\circ)=\tan 30^\circ$, $\tan(180^\circ-30^\circ)=-\tan 30^\circ$

(iv) $\cos(\pi+\frac{\pi}{3})=-\cos\frac{\pi}{3}$, $\cos(\pi-\frac{\pi}{3})=-\cos\frac{\pi}{3}$, $\sin(\pi+\frac{\pi}{3})=-\sin\frac{\pi}{3}$, $\sin(\pi-\frac{\pi}{3})=\sin\frac{\pi}{3}$, $\tan(\pi+\frac{\pi}{3})=\tan\frac{\pi}{3}$, $\tan(\pi-\frac{\pi}{3})=-\tan\frac{\pi}{3}$

(v) $\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)$, $\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{6}\right)$, $\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)$, $\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)$, $\tan\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=-\cot\left(\frac{\pi}{6}\right)$, $\tan\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\cot\left(\frac{\pi}{6}\right)$

(vi) $\cos\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)$, $\cos\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$, $\tan\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)=-\cot\left(\frac{\pi}{4}\right)$, $\tan\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)=\cot\left(\frac{\pi}{4}\right)$

2. (a) $\cos 15^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$ $\quad$ (b) $\cos 165^\circ=-\frac{1+\sqrt{3}}{2\sqrt{2}}$ $\quad$ (c) $\cos 345^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$ $\quad$ (d) $\sin 75^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$

3. (a) $\cos 120^\circ = -\frac{1}{2}$ (b) $\sin 120^\circ = \frac{\sqrt{3}}{2}, \tan 120^\circ = -\sqrt{3}$ (c) $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(d) $\cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}}$ (e) $\cos 285^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (f) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

4. (i) $\cos 9\theta$ (ii) $\cos 5\theta$ (iii) $\sin \frac{\theta}{2}$ (iv) $\sin 92^\circ$ (v) $\tan 30^\circ$ (vi) $\tan 2\pi$

5. $\cos(\alpha+\beta) = \frac{16}{65}$, $\cos(\alpha-\beta) = \frac{56}{65}$

6. (i) $\sin(\alpha-\beta) = \frac{416}{425}$ (ii) $\cos(\alpha-\beta) = \frac{87}{425}$ (iii) $\tan(\alpha-\beta) = \frac{416}{87}$

7. (i) $\sin(\alpha+\beta) = \frac{56}{65}$ (ii) $\cos(\alpha+\beta) = -\frac{33}{65}$ (iii) $\tan(\alpha+\beta) = -\frac{56}{33}$

8. (i) $\csc(\alpha+\beta) = \frac{65}{16}$ (ii) $\sec(\alpha+\beta) = \frac{65}{63}$ (iii) $\cot(\alpha+\beta) = \frac{63}{16}$

9. (i) $\sin(\alpha+\beta) = -\frac{7}{5\sqrt{2}}, \sin(\alpha-\beta) = \frac{1}{5\sqrt{2}}$ (ii) $\cos(\alpha+\beta) = -\frac{1}{5\sqrt{2}}, \cos(\alpha-\beta) = \frac{7}{5\sqrt{2}}$

(iii) $\tan(\alpha+\beta) = 7, \tan(\alpha-\beta) = \frac{1}{7}$

13. (i) $12\sin\theta - 5\cos\theta = r\sin(\theta+\varphi)$ where $r = 13$ and $\varphi = \tan^{-1}\left(-\frac{5}{12}\right)$

(ii) $3\sin\theta + 4\cos\theta = r\sin(\theta+\varphi)$ where $r = 5$ and $\varphi = \tan^{-1}\left(\frac{4}{3}\right)$ (iii) Do yourself.

14. (a) $\alpha = 45^\circ$ (b) $\sin\theta = \frac{7}{\sqrt{58}}, \cos\theta = \frac{3}{\sqrt{58}}$ (c) 0.9285 (d) 22°

EXERCISE 8.2

1. $\cos 2\theta = -\frac{7}{25}$, $\sin 2\theta = -\frac{24}{25}$, III quadrant

2. $\sin 2\alpha = -2y\sqrt{1-y^2}$, $\cos 2\alpha = 1-2y^2$, $\tan 2\alpha = -\frac{2y\sqrt{1-y^2}}{1-2y2}$ 3. $\cos 15^\circ = \sqrt{\frac{1+\cos 30^\circ}{2}} = 0.966$

4. (i) $\sin 2\theta = \frac{24}{25}, \cos 2\theta = -\frac{7}{25}, \tan 2\theta = -\frac{24}{7}, \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}}, \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}}, \tan\frac{\theta}{2} = \frac{1}{2}$

(ii) $\sin 2\theta = \frac{120}{169}, \cos 2\theta = -\frac{119}{169}, \tan 2\theta = \frac{120}{119}, \sin\frac{\theta}{2} = \frac{3}{\sqrt{13}}, \cos\frac{\theta}{2} = \frac{-2}{\sqrt{13}}, \tan\frac{\theta}{2} = -\frac{3}{2}$

(iii) $\sin 2\theta = -\frac{336}{625}, \cos 2\theta = \frac{527}{625}, \tan 2\theta = -\frac{336}{527}, \sin\frac{\theta}{2} = \frac{1}{5\sqrt{2}}, \cos\frac{\theta}{2} = \frac{-7}{5\sqrt{2}}, \tan\frac{\theta}{2} = -\frac{1}{7}$

(iv) $\sin 2\theta = -\frac{4}{5}, \cos 2\theta = -\frac{3}{5}, \tan 2\theta = \frac{4}{3}, \sin\frac{\theta}{2} = \sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}, \cos\frac{\theta}{2} = -\sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}, \tan\frac{\theta}{2} = -\sqrt{\frac{\sqrt{5}-1}{\sqrt{5}+1}}$

(v) $\sin 2\theta = -1, \cos 2\theta = 0, \tan 2\theta =$ undefined, $\sin\frac{\theta}{2} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}, \cos\frac{\theta}{2} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}, \tan\frac{\theta}{2} = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}$

(vi) $\sin 2\theta = -\frac{\sqrt{3}}{2}, \cos 2\theta = -\frac{1}{2}, \tan 2\theta = \sqrt{3}, \sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}, \cos\frac{\theta}{2} = \frac{1}{2}, \tan\frac{\theta}{2} = \sqrt{3}$

5. (i) $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$ (ii) $\sin\theta = \frac{4}{5}, \cos\theta = -\frac{3}{5}, \tan\theta = -\frac{4}{3}$

(iii) $\sin\theta = \frac{15}{17}, \cos\theta = -\frac{8}{17}, \tan\theta = -\frac{15}{8}$ (iv) $\sin\theta = \frac{7}{13\sqrt{2}}, \cos\theta = -\frac{17}{13\sqrt{2}}, \tan\theta = -\frac{7}{17}$

6. (i) $\frac{1}{4}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $\frac{\sqrt{3}}{2}$ (v) $\frac{1}{\sqrt{3}}$

7. (i) $\frac{1-\cos 4\alpha}{8}$ (ii) $\frac{1}{16}[1 - 7\cos 2\alpha - \cos 4\alpha + \cos 2\alpha\cos 4\alpha]$ (iii) $\frac{1}{128}[3 - 4\cos 4\alpha + \cos 8\alpha]$

EXERCISE 8.3

1. (i) $2[\sin 26x + \sin 6x]$ (ii) $5[\cos 16y + \cos 4y]$ (iii) $\sin 8t - \sin 2t$

   (iv) $3[\sin 15x + \sin 5x]$ (v) $\frac{1}{2}[\cos 6u - \cos 4u]$ (vi) $\cos 120^\circ - \cos 80^\circ$

   (vii) $\frac{1}{2}[\sin 40^\circ - \sin 6^\circ]$ (viii) $\sin 104^\circ - \sin 8^\circ$ (ix) $\cos 60^\circ - \cos 90^\circ$

   (x) $2[\sin u + \sin v]$ (xi) $\sin 2u - \sin 2v$

2. (i) $2 \sin 50^\circ \cos 20^\circ$ (ii) $2 \cos 45^\circ \sin 31^\circ$ (iii) $2 \cos 35^\circ \cos 23^\circ$

   (iv) $2 \cos\frac{p}{2} \cos\frac{q}{2}$ (v) $-2 \sin 15^\circ \cos 5^\circ$

REVIEW EXERCISE

1. (i) a (ii) b (iii) c (iv) d (v) b (vi) a

   (vii) c (viii) d (ix) b (x) c (xi) d (xii) a

2. (i) $\frac{56}{65}$ (ii) $-\frac{56}{33}$ (iii) $-\frac{16}{33}$

3. (i) $\sin(\beta + 45^\circ)$ or $\cos(\beta - 45^\circ)$ (ii) $\sin 120^\circ$ or $\cos 30^\circ$

4. (i) $\tan 60^\circ = 1.732$ (ii) $\cos 90^\circ = 0$ 5. $\tan\theta = 2$ 9. 1

Unit 9: Trigonometric Functions

EXERCISE 9.1

1. i. Maximum value (M) $= 4$; Minimum value (m) $= 0$

   ii. Maximum value (M) $= \frac{7}{6}$; Minimum value (m) $= \frac{1}{6}$

   iii. Maximum value (M) $= \frac{11}{5}$; Minimum value (m) $= -\frac{9}{5}$

   iv. Maximum value (M) $= \frac{38}{5}$; Minimum value (m) $= \frac{32}{5}$

2. i. Maximum value (M) $= 1$; Minimum value (m) $= \frac{1}{7}$

   ii. Maximum value (M) $= \frac{2}{11}$; Minimum value (m) $= -\frac{2}{9}$

   iii. Maximum value (M) $= \frac{3}{13}$; Minimum value (m) $= -\frac{3}{11}$

   iv. Maximum value (M) $= \frac{5}{13}$; Minimum value (m) $= \frac{5}{17}$

3. i. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-7, 7]$

   ii. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-1, 1]$

   iii. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-1, 1]$

   iv. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = 0

   v. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = 0

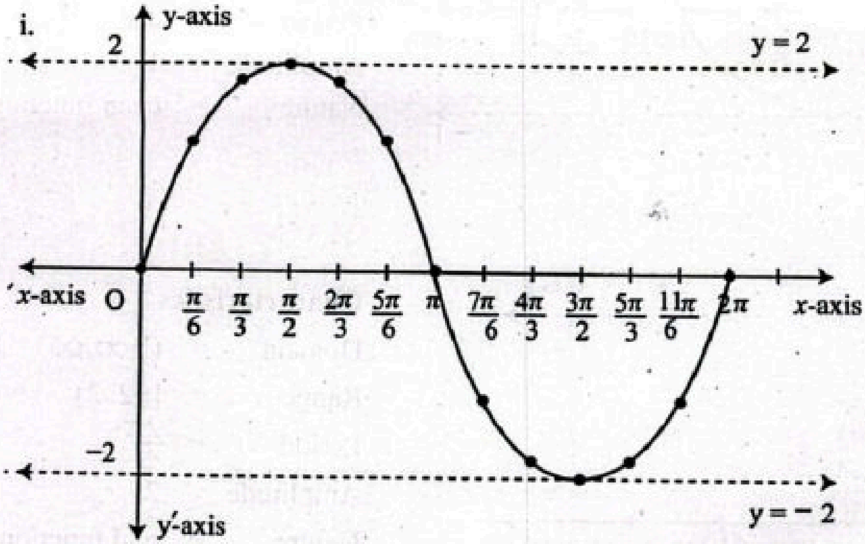
   vi. Domain = Dy = $]-\infty, \infty[$ = R ; Range = Ry = $[-6, 6]$

4. i. π ii. $\frac{2\pi}{5}$ iii. 4π iv. $\frac{2\pi}{3}$ v. π vi. 2π

EXERCISE 9.2

1. i. odd ii. even iii. even iv. even
   v. even vi. even vii. odd viii. odd

2. i.



Characteristics

Domain = $(-\infty, \infty) = R$

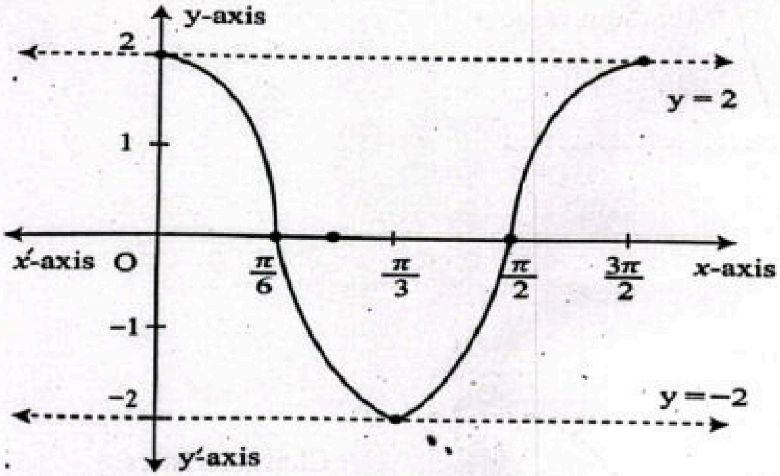
Range = $[-2, 2]$

Period = 2π

Amplitude = 2

Nature = odd function

ii.



Characteristics

Domain = $(-\infty, \infty) = R$

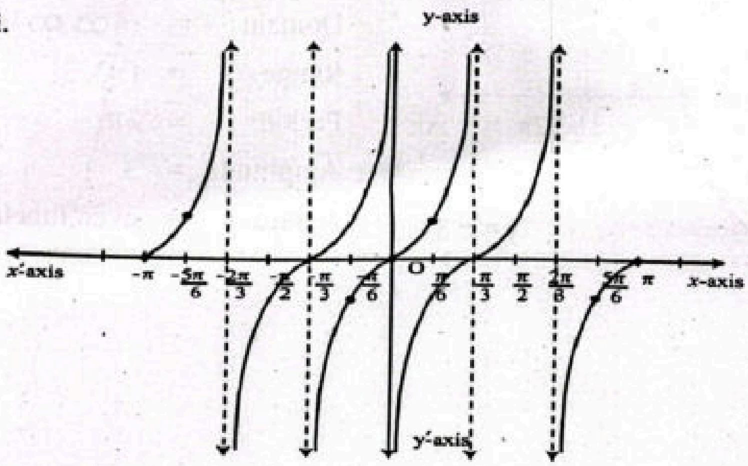
Range = $[-2, 2]$

Period = $\frac{2\pi}{3}$

Amplitude = 2

Nature = even function

iii.



Characteristics

Domain = $\frac{\pi n}{2} < x < \frac{\pi}{4} + \frac{\pi n}{2}$

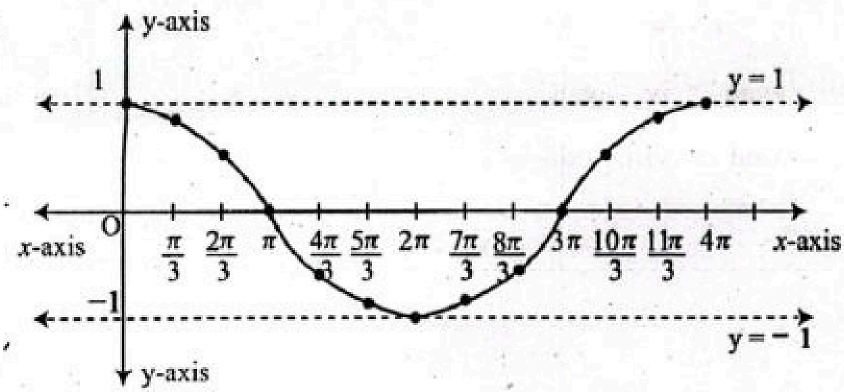
Range = $]-\infty, \infty[$

Period = $\frac{\pi}{2}$

Amplitude = Nil

Nature = odd function

iv.



Characteristics

Domain = $(-\infty,\infty) = R$

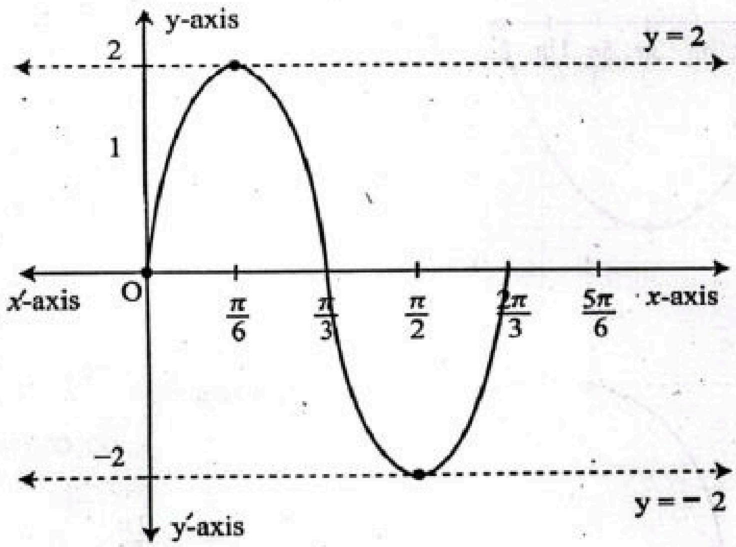
Range = $[-1, 1]$

Period = 4π

Amplitude = 1

Nature = even function

v.



Characteristics

Domain = $(-\infty,\infty) = R$

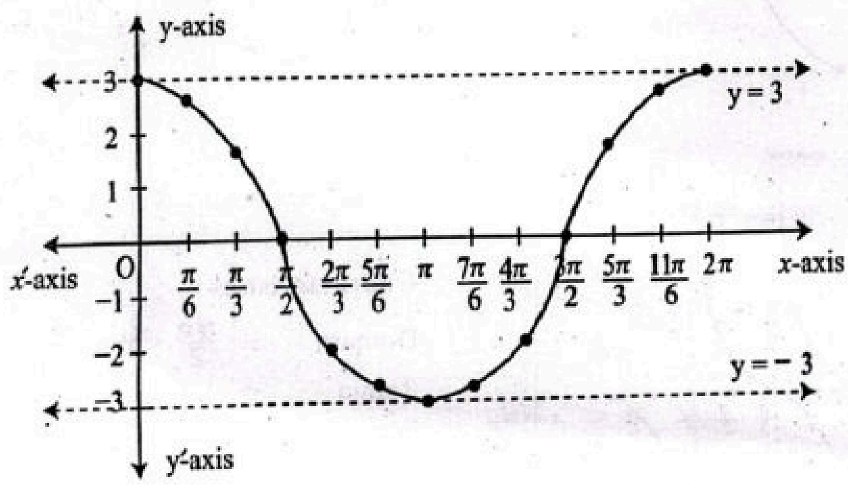
Range = $[-2, 2]$

Period = $\frac{2\pi}{3}$

Amplitude = 2

Nature = odd function

vi.



Characteristics

Domain = $(-\infty,\infty) = R$

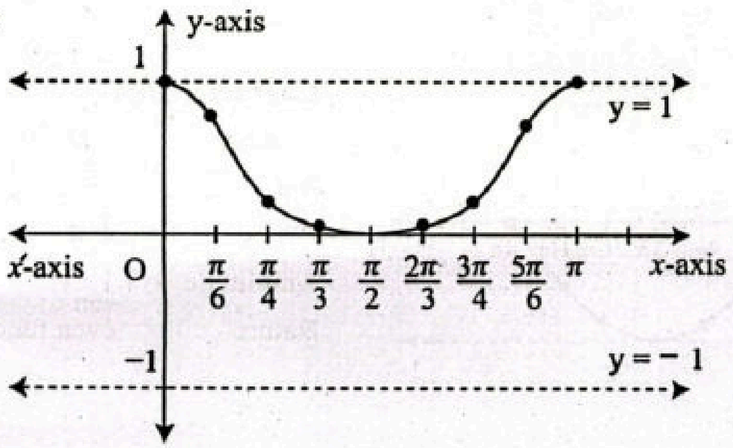
Range = $[-3, 3]$

Period = 2π

Amplitude = 3

Nature = even function

vii.



Characteristics

Domain = $(-\infty, \infty) = R$

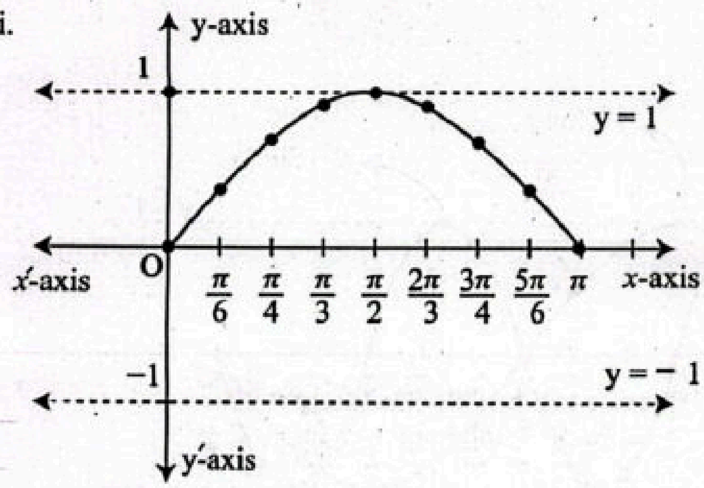
Range = $[0, 1]$

Period = π

Amplitude = 1

Nature = even function

viii.



Characteristics

Domain = $(-\infty, \infty) = R$

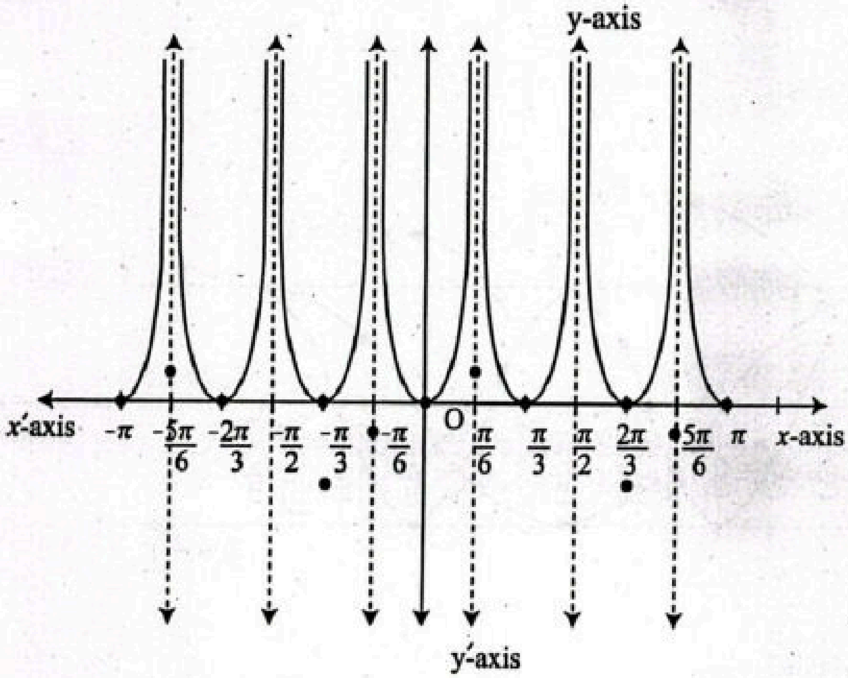
Range = $[0, 1]$

Period = π

Amplitude = 1

Nature = even function

ix.



Characteristics

Domain = $\pi n \leq x < \frac{\pi}{2} + \pi n$

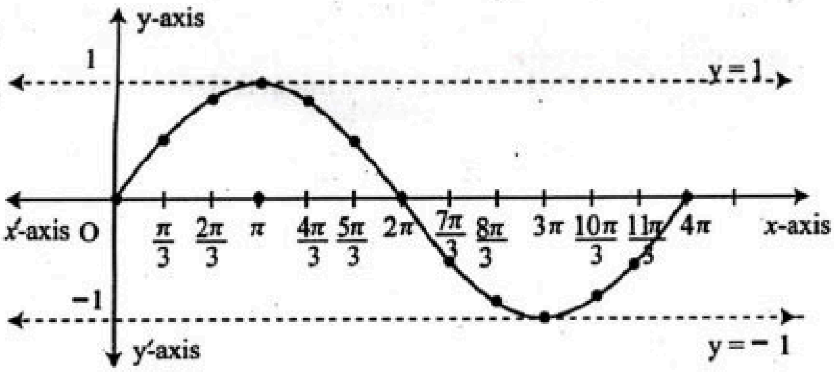
Range = $f(x) \geqslant 0$

Period = π

Amplitude = Nil

Nature = even function

x.



Characteristics

Domain = $(-\infty, \infty) = R$

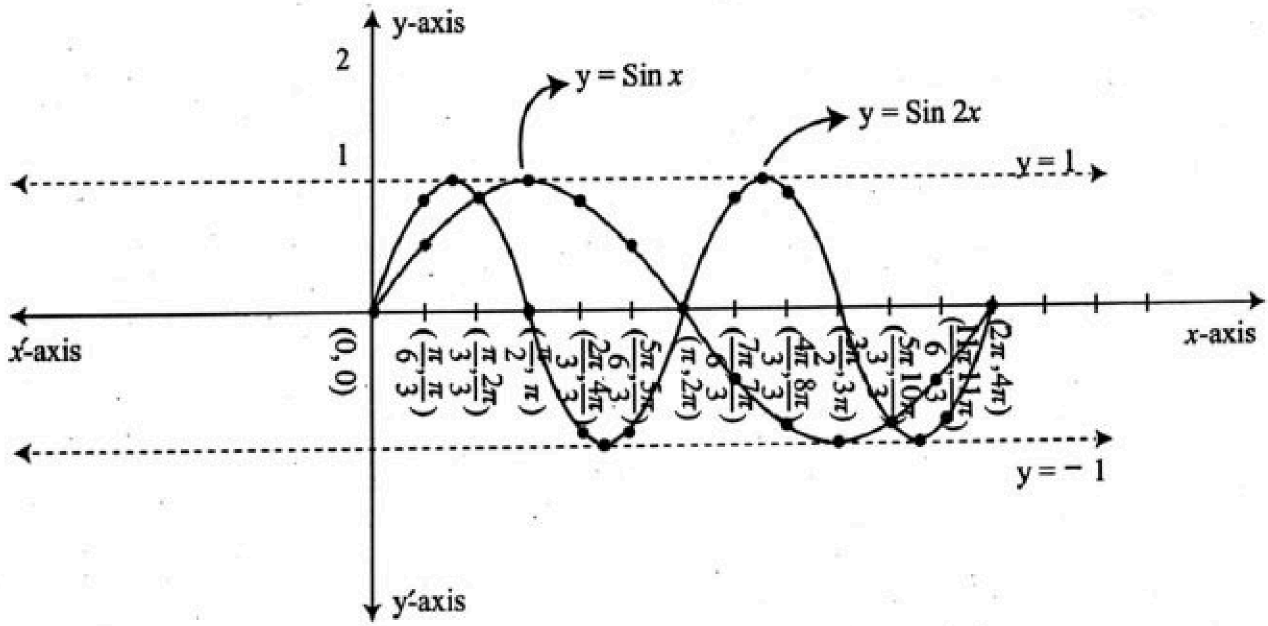
Range = $[-1, 1]$

Period = 4π

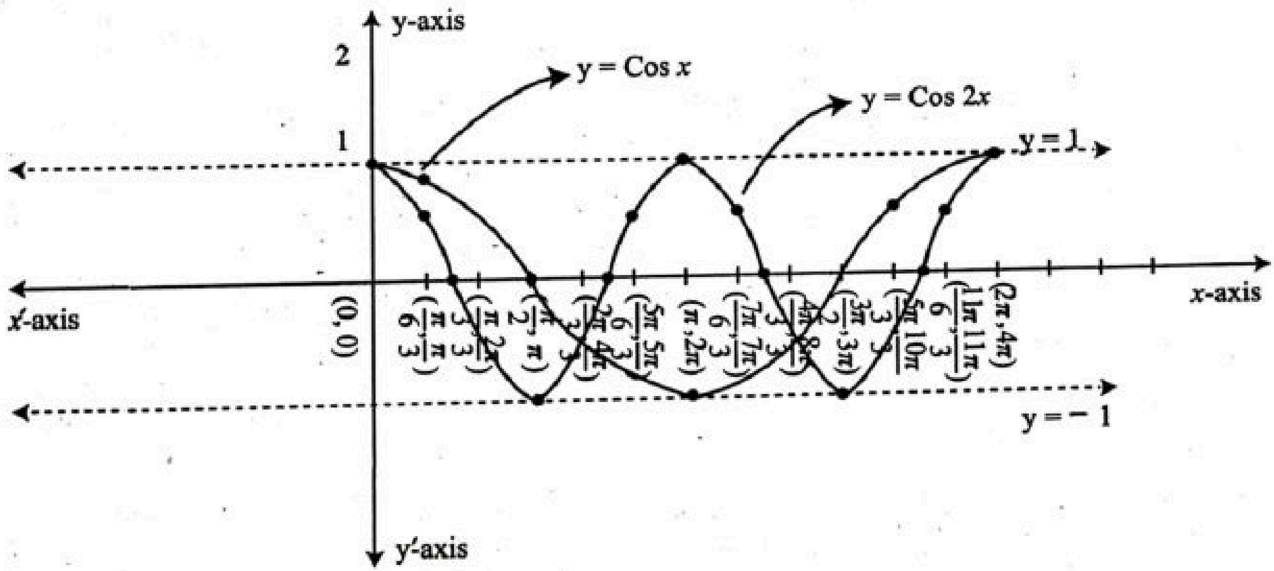
Amplitude = 1

Nature = even function

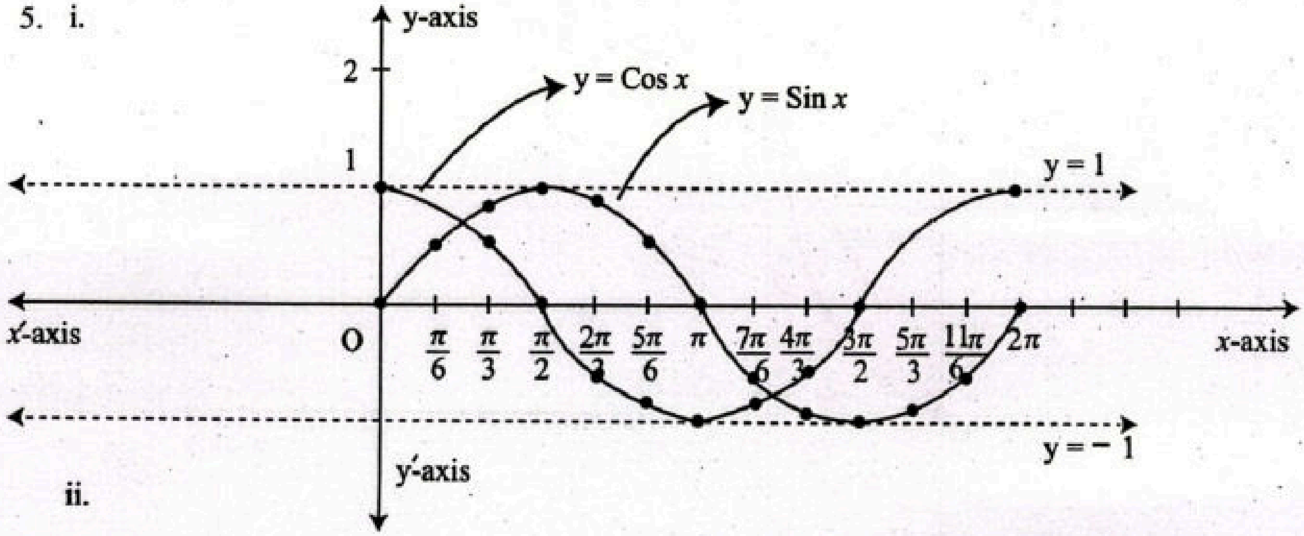
3.



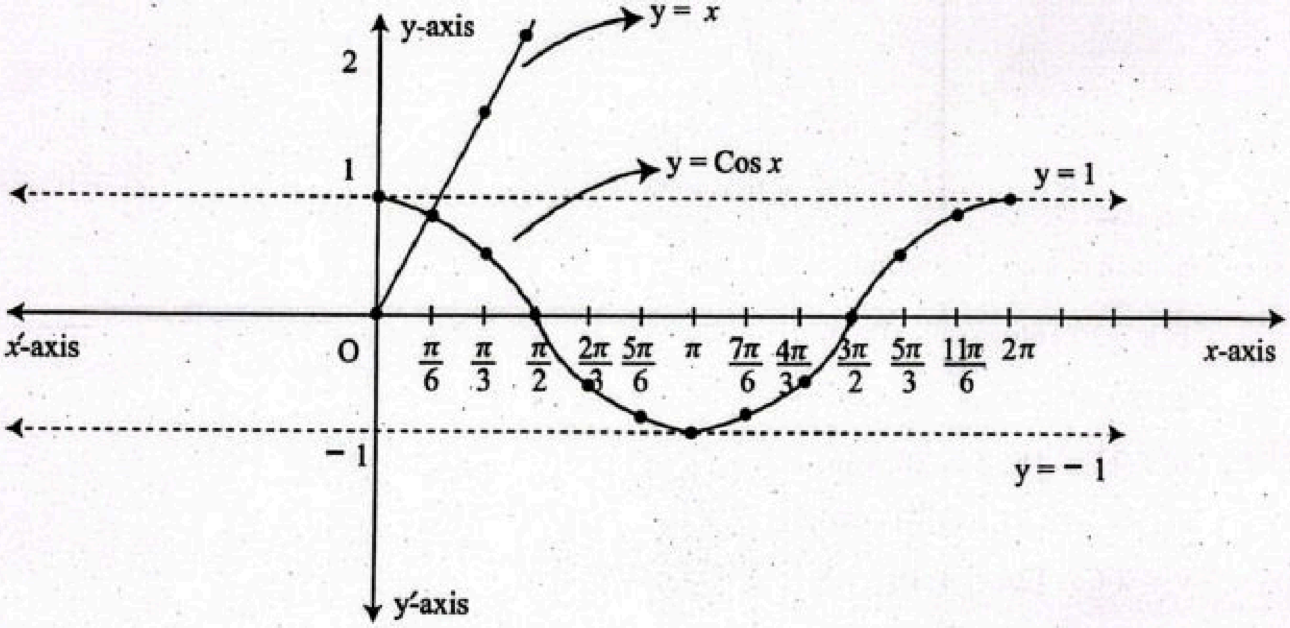
4.



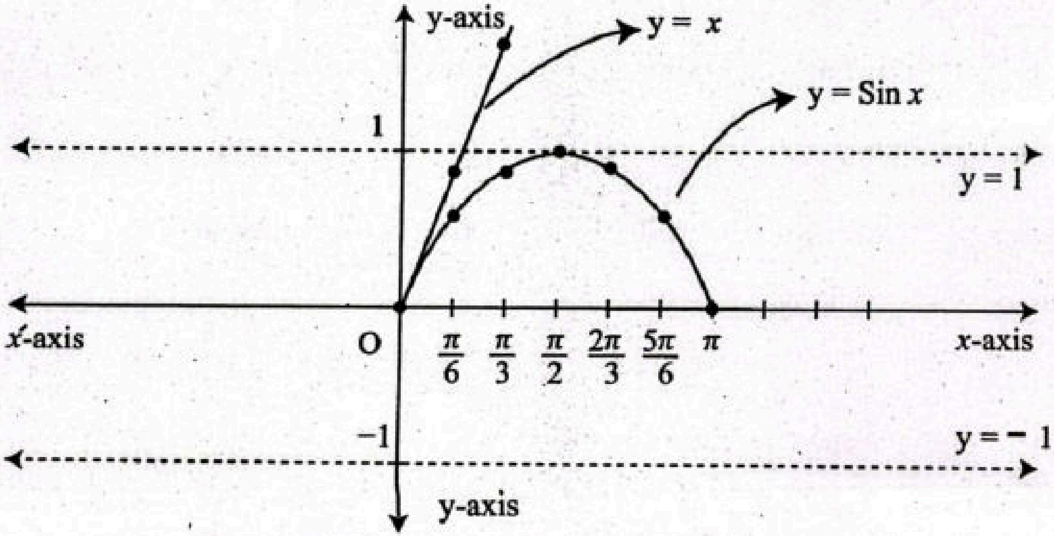
5. i.


y-axis
2
1
y = Cos x
y = Sin x
y = 1
x'-axis
O
π/6 π/3 π/2 2π/3 5π/6 π 7π/6 4π/3 3π/2 5π/3 11π/6 2π
x-axis
y = − 1
y'-axis

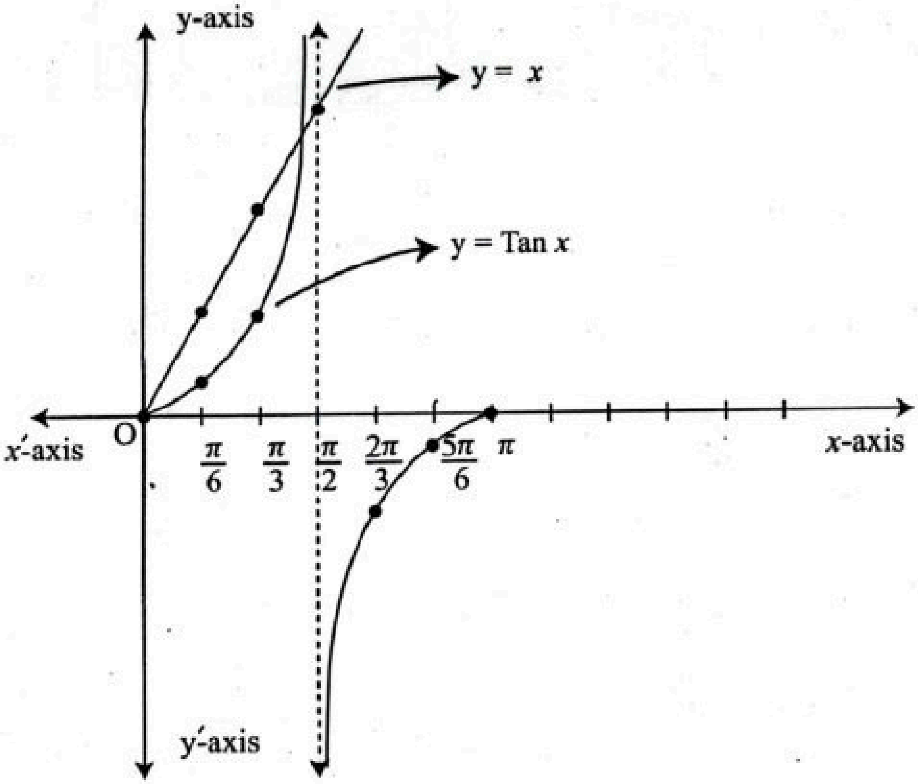
ii.


y-axis
y = x
2
1
y = Cos x
y = 1
x'-axis
O
π/6 π/3 π/2 2π/3 5π/6 π 7π/6 4π/3 3π/2 5π/3 11π/6 2π
x-axis
− 1
y = − 1
y'-axis

iii.


y-axis
y = x
y = Sin x
1
y = 1
x'-axis
O
π/6 π/3 π/2 2π/3 5π/6 π
x-axis
−1
y = − 1
y-axis

iv.



6. a. each cycle is $\frac{1}{56}$ second

b. k = 20,160

c. 180

d. $V(t) = 180\ Sin\ 20,160t$

7. $h(t) = -8.5\ Cos\ (\frac{\pi t}{25}) + 10.5$

8. $y = 2\ Cos\ [2\pi x] + 18$

REVIEW EXERCISE

1. i. b	ii. a	iii. c	iv. a	v. c
vi. a	vii. a	viii. b	ix. c	x. a
xi. a	xii. b	xiii. c	xiv. b	xv. a
xvi. c	xvii. d	xviii. a		

5. a. Period $= \frac{\pi}{6}$; Amplitude $= -20$; Vertical shift $= 24$

   b. $h(t) = -20 \cos\left(\frac{\pi t}{8}\right) + 24$

   c. The height is 32 m after 5minutes.

6. a. Maximum height = 22 m, Minimum height = 2 m.
   b. The height is 12 m after 30 seconds.
   c. One complete revolution takes place in 120 seconds.

7. a. $y = 10\, Sin\, 1440\, t^{\circ}, y = 10\, Sin\, 1440\, (t - \frac{1}{16})^{\circ}$

   b. Domain $= \{t \,/\, t \geq 0, t \in R\}$, Range $= \{y / -10 \leq y \leq 10, y \in R\}$

8.

	Domain	Range	Period
i.	Domain $=]-\infty, \infty[= R$	Range $= [-2, 2]$	6π
ii.	Domain $=]-\infty, \infty[= R$	Range $= [-5, 5]$	$\frac{2\pi}{3}$
iii.	Domain $=]-\infty, \infty[= R$	Range $= [-\frac{1}{2}, \frac{1}{2}]$	3π
iv.	Domain $=]-\infty, \infty[= R$	Range $= [-\frac{5}{3}, \frac{5}{3}]$	$\frac{3\pi}{2}$
v.	Domain $=]-\infty, \infty[= R$	Range $= [-3, 3]$	2
vi.	Domain $=]-\infty, \infty[= R$	Range $= [-7, 7]$	$\frac{2\pi}{5}$
vii.	Domain $R - \frac{3n}{2}, n \in Z$	Range $=]-\infty, \infty[$	$\frac{3}{2}$
viii.	Domain $=]-\infty, \infty[= R$	Range $= [-9, 9]$	$\frac{2\pi}{3}$
ix.	Domain $=]-\infty, \infty[= R$	Range $= [7, 9]$	$\frac{\pi}{2}$
x.	Domain $=]-\infty, \infty[= R$	Range $= [2, 12]$	π
xi.	Domain $=]-\infty, \infty[= R$	Range $= [2, 10]$	π

G L O S S A R Y

Adjoint of a matrix: A matrix of order 2, obtained by interchanging diagonal elements and changing the signs of non-diagonal elements.

Algebraic expression: A statement in which variables or constants or both are connected by arithmetic operations (i.e. +, −, ×, ÷).

Allied angles: The angles connected with basic angles of measure θ by a right angle or its multiple, are called allied angles.

Arithmetic mean: A number M is said to be arithmetic mean between two numbers a and b if a, M, b are in A.P.

Arithmetic sequence: An arithmetic sequence is a sequence in which each term, after the first, is found by adding a constant.

Arithmetic series: The sum of the terms of an arithmetic sequence is called an arithmetic series.

Arithmetico–geometrico sequence: This sequence is the result of term-by-term multiplication of a geometric progression with the corresponding terms of arithmetic progression.

Column: The vertical arrangement of objects.

Column matrix: A matrix having only one column.

Combination: If in the arrangements of objects their order is not important then this arrangement of objects is called combination.

Complex number: The number of the form $a + ib$, where a and b are real number and $i = \sqrt{-1}$.

Complex polynomial: If z is a complex variable, then the expression $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ is called complex polynomial of degree n if $a_n \neq 0$ and n is a non-negative integer.

Conformable for matrix addition: Matrices of same order so that they may be added.

Conformable for matrix multiplication: If number of columns of first matrix is equal to the number of rows of second matrix so that they may be multiplied in that order.

Conformable for matrix subtraction: Matrices of same order, so that they may be subtracted.

Conjugate: Two complex numbers differing only in the sign of their imaginary parts.

Constant polynomial: A polynomial having degree zero is called a constant polynomial.

Consistency criteria: A system of homogeneous linear equations is consistent if $Rank\ A = Rank\ A_b$.

Consistent system: A system of equations is consistent if it has at least one solution.

Cross product of vectors: The product of vectors resulting in a vector quantity.

Cubic polynomial: A polynomial having degree three is called a cubic

Deductive reasoning: Deductive reasoning is a logical approach where someone moves from general ideas to specific conclusions.

Determinant of a matrix: A number obtained by subtracting the product of non-diagonal elements from the product of diagonal elements, in a square matrix of order two.

Diagonal: A line joining any two vertices of a polygon that are not joined by any of its edges; elements running from the upper left corner to the lower right corner of a square matrix.

Diagonal matrix: A matrix in which all the non-diagonal elements are zero but at least one element of the diagonal is non-zero.

Direction angles: The angles that a non-zero vector $\vec{r}$ makes with the coordinate axes in the positive direction are known as direction angles of $\vec{r}$.

Direction cosines: Coines of direction angles are called direction cosines.

Domain of trigonometric functions: The domain of a function f(x) is the set of all possible values of 'x' such that function f(x) is defined.

Dot product of vectors: The product of vectors resulting in a scalar quantity.

Equal vectors: Two vectors $\vec{a}$ and $\vec{b}$are equal if both have the same magnitude and direction.

Equality of complex numbers: Two complex numbers are said to be equal if both have the same real and imaginary parts.

Equality of matrices: Two matrices are equal if both have the same order and the same corresponding elements.

Even function: A function is even if and only if $f(-x) = f(x)$.

Factor theorem: A polynomial p(x) has a factor $x - c$, if and only if $p(c) = 0$.

Factorial: Factorial of an integer n is denoted by $n! = 1 \times 2 \times 3 \ldots (n-1)n$.

Fundamental law of trigonometry: This law is stated as: $cos(\alpha - \beta) = cos\,\alpha\, cos\,\beta + sin\,\alpha\, sin\,\beta$

Geometric mean: If a, G, b is in a geometric sequence, then G is called the geometric mean of a and b.

Geometric sequence: A geometric sequence is one in which each term after the first is found by multiplying the previous term by a constant called the common ratio.

Geometric series: The sum of the terms of a geometric sequence is called a geometric series.

Graphic solution: Method of solving two simultaneous equations by plotting the graph of each equation.

Harmonic mean: A number H is said to be the harmonic mean between two numbers a and b if a, H, b are in H.P.

Harmonic sequence: A sequence is called a harmonic sequence if the reciprocals of its terms are in an arithmetic sequence.

Imaginary part: The coefficient i in any complex number.

Inconsistent system: A system of equations that has no solution is called inconsistent.

Inductive reasoning: It is a method of reasoning in which general principle is derived from observations.

Inequality: The relation between two comparable quantities, which are not equal.

Irrational expression: An algebraic expression that is not rational is called an irrational expression.

Linear polynomial: A polynomial having degree one is called a linear polynomial.

Lower triangular matrix: A square matrix in which all the elements lie above the main diagonal are zero.

Matrix: A rectangular arrangement of numbers enclosed within square brackets.

Modulus of a complex number: It is the distance of a complex number from its origin.

Negative of a vector: A vector having the same magnitude but the opposite direction is called the negative of the given vector.

Non-singular matrix: A matrix with non-zero determinant.

Null matrix: A matrix with all entries to be zero.

Odd function: A function is odd if and only if $f(-x) = -f(x)$.

Order of a matrix: If a matrix has m number of rows and n number of columns then the order of the matrix is m-by- n.

Ordered pair: A pair set in which x is designated the first element and y the second, denoted by (x, y).

Parallel vectors: Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if $\vec{a} = \lambda\vec{b}$.

Periodic function: A periodic function is a function where values repeat after a specific time interval.

Periodicity: The periodic behavior of trigonometric functions is called periodicity.

Permutation: The arrangement of numbers or things in a definite order is called permutation.
Polynomial: Algebraic expressions consisting of one or more terms in which exponents of the variables involved are whole numbers.
Position vector: The vector used to specify the position of a point P with respect to the origin O is called the position vector of P.
Quadratic polynomial: A polynomial having degree two is called a quadratic polynomial.
Range of trigonometric functions: The range of a function $f(x)$ is the set of all possible values of the function $f(x)$ can take, when 'x' is any number from the domain of the function.
Rational expression: An algebraic expression of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.
Rectangular matrix: A matrix having an unequal number of rows and columns.
Remainder theorem: If a polynomial $p(x)$ is divided by $x - c$, then the remainder is $p(c)$.
Row: Horizontal arrangement of elements.
Row matrix: A matrix having only one row of elements.
Rule of product: If event A can happen in m ways and event B can happen in n ways then pair (A, B) can happen in $m \times n$ or mn ways.
Sequence: A sequence is an arrangement of objects or numbers in a particular order followed by some rule.
Scalar matrix: A diagonal matrix with equal diagonal elements.
Scalar quantity: A physical quantity that can be completely specified by its magnitude only.
Simultaneous equations: Set of equations satisfied by the same solution.
Singular matrix: A matrix with zero determinant.
Skew symmetric matrix: A matrix whose transpose is not equal to the matrix itself.
Solution of equations: The solution of an equation is the process of finding the values of the unknown involved in the equation.
Square matrix: A matrix having an equal number of rows and columns.
Symmetric matrix: A matrix whose transpose is equal to the matrix itself.
Terminating decimal fraction: A decimal fraction whose decimal part is finite.
Transpose of a matrix: A matrix obtained by interchanging rows and columns of a given matrix.
Triangular matrix: A square matrix that is either upper triangular or lower triangular is called a triangular matrix.
Triangular numbers: A triangular number counts objects arranged in an equilateral triangle.
Unit matrix: A diagonal matrix having all diagonal elements equal to one.
Unit vector: A vector that has magnitude 1 is called a unit vector.
Upper triangular matrix: A square matrix in which all the elements lying below the main diagonal are zero.
Vector quantity: A physical quantity that is completely specified by its magnitude and direction.
Zero matrix: A matrix having all elements equal to zero.
Zeros of a polynomial: A value of the variable for which the value of the polynomial is zero.
Zero polynomial: A polynomial having "0" as the only term.
Zero vector: A vector in which the initial and terminal points coincide.

SYMBOLS AND ABBREVIATIONS USED IN MATH

=	→	is equal to
≠	→	is not equal to
∈	→	is member of
∉	→	is not member of
∅	→	empty set
∪	→	union of sets
∩	→	intersection of sets
⇔	→	if and only if
$\overline{AB}$	→	line Segment AB
AB	→	measurement of side AB
∠A	→	measurement of angle A
≅	→	is congruent to
⊥	→	is perpendicular to
Δ	→	triangle
⇒	→	implies that
∧, &	→	and
∨	→	or
<	→	is less than
>	→	is greater than
≤	→	is less than or equal to
≥	→	is greater than or equal to
@	→	at the rate of
%	→	percent
π	→	Pie
:	→	ratio
::	→	proportion
∴	→	therefore, hence
∵	→	because, since
i.e.	→	that is
≈	→	approximately equal to
√	→	square root / radical
e.g.	→	for example
/	→	such that
↔	→	corresponding to
//	→	is parallel to
!	→	factorial
^nP_r	→	permutation
^nC_r	→	combination

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