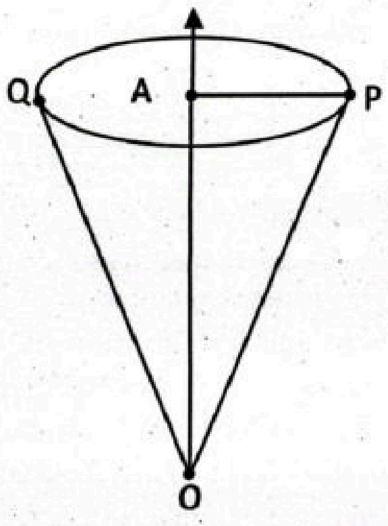
CONIC SECTION

After studying this unit, students will be able to:

- Demonstrate conics and members of its family i.e. circle, parabola, ellipse and hyperbola.
- Derive and apply equation of a circle in standard form i.e. $(x-h)^2+(y-k)^2=r^2$.
- Find the equation of circle passing through: three non collinear points, two points having its centre on a given line, two points and equation of tangent at one of these points is known, two points and touching a given line.
- Find the condition when a line intersects the circle and when it touches the circle.
- Find the equation of tangent to a circle in slope form and a normal to a circle at a point.
- Find the length of tangent to a circle from a given external point.
- Derive and apply the standard equation of a parabola.
- Sketch graphs of parabolas and find their elements.
- Find the equation of a parabola with the following elements: focus and vertex, focus and directrix, vertex and directrix.
- Find the condition when a line is tangent to a parabola at a point and hence write the equation of a tangent line in slope form.
- Find the equation of tangent and normal to a parabola at a point.
- Derive and apply the standard form of equation of an ellipse and identify its elements.
- Convert a given equation to the standard form of equation of an ellipse, find its elements and draw the graph.
- Find points of intersection of an ellipse with a line including the condition of tangency.
- Find the equation of a tangent to an ellipse in slope form.
- Find the equation of a tangent and normal to an ellipse at a point.
- Derive and apply the standard form of equation of a hyperbola and identify its elements.
- Find the equation of a hyperbola with the following given elements: transverse and conjugate axes with centre at origin, two points, eccentricity, latera recta and transverse axes, focus eccentricity and centre, focus, centre and directrix.
- Find points of intersection of hyperbola with a line including the condition of tangency.
- Find the equation of tangent to a hyperbola in slope form.
- Find the equation of tangent and a normal to a hyperbola at a point.
- Apply concepts of conics to real life problems (such as suspension and reflection problems related to parabola, satellite system, elliptic movement of electrons in the atom around the nucleus, radio system uses as hyperbolic functions, flashlights, conics in architecture).

7.1 Conics and Its Family Members



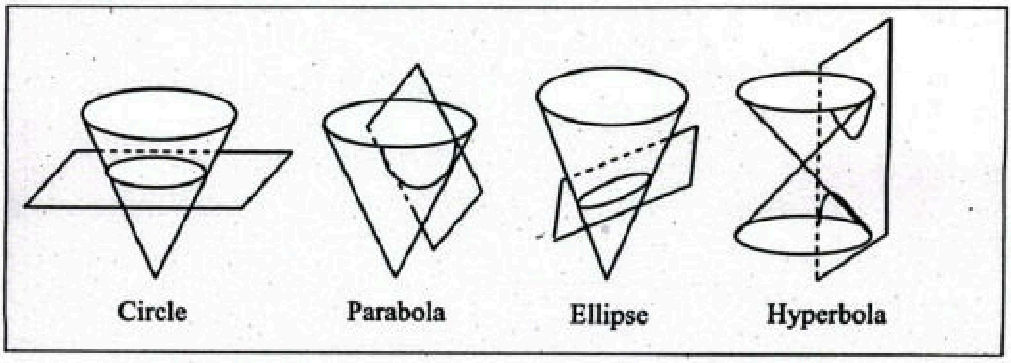
The plane shapes like circle, ellipse, parabola and hyperbola, which are formed with the intersection of a right circular cone and a plane are known as conic section. First of all, we will discuss that what is a right circular cone and a plane. When the line segment $\overline{OP}$ rotates about the circumference of a circle of any radius greater than zero with O as fixed point the shape formed is called a **right circular cone.** The fixed point O is called the **vertex** of the cone. The ray $\overrightarrow{OA}$ is called the axis of the cone, as $\overrightarrow{OA}$ is perpendicular to the radius $\overline{AP}$ of the circle that's why this cone is called **right** circular cone. The line segment $\overline{OP}$ (or any line segment) which join the point **O** with the circumference of the circle is called generator (ruling). Note that a flat or two-dimensional surface that extends indefinitely is called a plane.

Circle: When we cut a cone with a plane so that plane is perpendicular to the axis of cone and not passing through vertex then the intersection is a circle.

Parabola: When we cut a cone with the plane such that plane is parallel to any generator or ruling not passing through the vertex then the intersection is a parabola.

Ellipse: When we cut a cone with the plane such that plane is slightly tilted not passing through the vertex then the intersection is an ellipse.

Hyperbola: When a cone is inverted and is joined with an erected cone such that the vertices of the two cones lie at the same point and their axes of are also same then a shape is formed shown in the figure. When plane intersects the cone parallel to the axis of the cone such that the plane does not pass through the vertex O, then the shape formed on the plane is called hyperbola.



Note: In case when the plane passes through the vertex then the intersection of plane and the cone is a point or a pair of intersecting lines. These conic sections are known as degenerate conics.

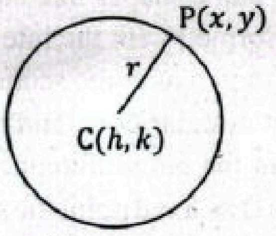
7.2 Circle

It is set of all points in the plane which are equidistant from a fixed point in the plane.

The fixed point is called the centre of the circle and distance of any point on the circle from the centre is called the radius of the circle.

7.2.1 Equation of Circle in Standard Form

Consider a circle with centre at $C(h,k)$ and radius r. Take any point $P(x,y)$ on the circle. Then by definition of the circle.



$|CP| = r$

$\Rightarrow \sqrt{(x-h)^2+(y-k)^2} = r$

Squaring both sides

$\Rightarrow (x-h)^2+(y-k)^2 = r^2$

This is the standard form of equation of circle with centre at $C(h,k)$ and radius r.

In particular if the centre of the circle is at origin, i.e. $(h,k) = (0,0)$ then equation of circle is

$$(x-0)^2+(y-0)^2 = r^2 \quad \Rightarrow \quad x^2+y^2 = r^2$$

Example 1: Find the equation of circle with centre at $(2,-5)$ and radius 3 units.

Solution: Given that the centre $(h,k) = (2,-5)$ and radius $r = 3$.

Equation of circle is $(x-h)^2+(y-k)^2 = r^2$

Putting values of h, k and r, we get required equation of circle as follows:

$$(x-2)^2+(y-(-5))^2 = 3^2 \quad \Rightarrow \quad (x-2)^2+(y+5)^2 = 9$$

7.2.2 General Form of Equation of Circle

As we know that the standard form of the equation of circle is:

$(x-h)^2+(y-k)^2 = r^2$

$\Rightarrow x^2-2hx+h^2+y^2-2ky+k^2 = r^2$

$\Rightarrow x^2+y^2-2hx-2ky+h^2+k^2-r^2 = 0 \qquad$ (i)

Letting $h^2+k^2-r^2 = c$, $-2h = 2g$ and $-2k = 2f$, the equation (i) becomes:

$$x^2+y^2+2gx+2fy+c = 0$$

Which is the general form of the equation of circle.

Theorem 7.1: Prove that $x^2+y^2+2gx+2fy+c = 0$, represents a circle in general.

Proof: Given equation is:

$x^2+y^2+2gx+2fy+c = 0$

$\Rightarrow (x^2+2gx)+(y^2+2fy)+c = 0$

$\Rightarrow (x^2+2gx+g^2-g^2)+(y^2+2fy+f^2-f^2)+c = 0$

$\Rightarrow (x^2+2gx+g^2)+(y^2+2fy+f^2)+(-g^2-f^2+c) = 0$

$\Rightarrow (x+g)^2+(y+f)^2 = g^2+f^2-c$

$\Rightarrow [x-(-g)]^2+[y-(-f)]^2 = \left(\sqrt{g^2+f^2-c}\right)^2$

Comparing it with $(x-h)^2+(y-k)^2 = r^2$, we have $h = -g$; $k = -f$; $r = \sqrt{g^2+f^2-c}$.

Thus, the given equation represents a circle with centre $= (h,k) = (-g,-f)$ and radius $= r = \sqrt{g^2+f^2-c}$

Note: As $r = \sqrt{g^2+f^2-c}$, so

i. r is a real if $g^2+f^2-c > 0$ i.e. $g^2+f^2 > c$

ii. r is imaginary if $g^2+f^2-c < 0$ i.e. $g^2+f^2 < c$

Thus, condition for a real circle is $g^2+f^2 > c$ and for an imaginary circle is $g^2+f^2 < c$

Example 2: Find the centre and radius of the circle $x^2 + y^2 - 6x - 10y + 18 = 0$.

Solution:

To find the centre and radius we convert the given equation of circle into its standard form

$(x^2 - 6x) + (y^2 - 10y) + 18 = 0$

$(x^2 - 6x + 9 - 9) + (y^2 - 10y + 25 - 25) + 18 = 0$

$(x^2 - 6x + 9) + (y^2 - 10y + 25) - 9 - 25 + 18 = 0$

$(x - 3)^2 + (y - 5)^2 - 16 = 0 \Rightarrow \quad (x - 3)^2 + (y - 5)^2 = 16$

$(x - 3)^2 + (y - 5)^2 = 4^2$

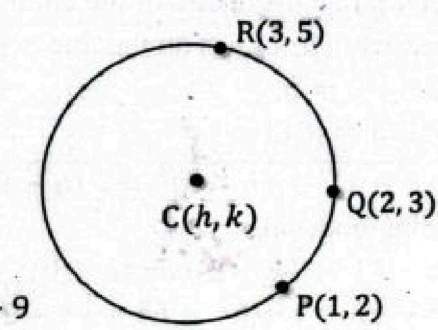
Comparing it with $(x - h)^2 + (y - k)^2 = r^2$, we have:

Centre $= (h, k) = (3, 5)$ and radius $= r = 4$

7.2.3 Equation of a Circle with Different Conditions

Example 3: Find the equation of the circle passing through the points (1, 2), (2, 3) and (3, 5).

Solution:



Consider a circle which passes through the points P(1, 2), Q(2, 3) and R(3, 5). Let $C(h, k)$ be the centre of the circle

Then, $|PC| = |QC|$

$\Rightarrow \sqrt{(h - 1)^2 + (k - 2)^2} = \sqrt{(h - 2)^2 + (k - 3)^2}$

Squaring both sides:

$(h - 1)^2 + (k - 2)^2 = (h - 2)^2 + (k - 3)^2$

$\Rightarrow h^2 - 2h + 1 + k^2 - 4k + 4 = h^2 - 4h + 4 + k^2 - 6k + 9$

$\Rightarrow 2h + 2k = 8$

$\Rightarrow h + k = 4 \qquad$ (i)

Also $\quad |PC| = |RC|$

$\Rightarrow \sqrt{(h - 1)^2 + (k - 2)^2} = \sqrt{(h - 3)^2 + (k - 5)^2}$

Squaring both sides:

$(h - 1)^2 + (k - 2)^2 = (h - 3)^2 + (k - 5)^2$

$\Rightarrow \quad h^2 - 2h + 1 + k^2 - 4k + 4 = h^2 - 6h + 9 + k^2 - 10k + 25$

$\Rightarrow \quad 4h + 6k = 29 \qquad$ (ii)

Multiplying (i) with 4 and subtracting (ii) from it, we get:

$\Rightarrow 4h + 4k = 16$

$\underline{-4h \pm 6k = -29}$

$-2k = -13 \qquad \Rightarrow k = \frac{13}{2}$

Putting in (i), we get:

$\Rightarrow h + \frac{13}{2} = 4 \Rightarrow h = 4 - \frac{13}{2} = -\frac{5}{2}$

Thus, the centre of circle $= C\left(\frac{-5}{2}, \frac{13}{2}\right)$

The radius of circle is:

$r = |CP| = \sqrt{\left(1 + \frac{5}{2}\right)^2 + \left(2 - \frac{13}{2}\right)^2} = \sqrt{\left(\frac{7}{2}\right)^2 + \left(-\frac{9}{2}\right)^2} = \sqrt{\frac{49}{4} + \frac{81}{4}} = \sqrt{\frac{130}{4}}$

Equation of the circle with centre $C\left(-\frac{5}{2},\frac{13}{2}\right)$ and radius $r=\sqrt{\frac{130}{4}}$ is:

$$\left[x-\left(-\frac{5}{2}\right)\right]^2+\left[y-\frac{13}{2}\right]^2=\left(\sqrt{\frac{130}{4}}\right)^2 \Rightarrow \left(x+\frac{5}{2}\right)^2+\left(y-\frac{13}{2}\right)^2=\frac{130}{4}$$

$$\Rightarrow x^2+5x+\frac{25}{4}+y^2-13y+\frac{169}{4}=\frac{130}{4}$$

$$\Rightarrow x^2+y^2+5x-13y+\frac{25}{4}+\frac{169}{4}-\frac{130}{4}=0$$

$$\Rightarrow x^2+y^2+5x-13y+16=0$$

Example 4: Find the equation of the circle passing through the points (1, 0), (0, 1) and having its centre on the line $x-2y+3=0$.

Solution: Consider a circle which is passing through the two given points P(1, 0) and Q(0, 1). Let $C(h,k)$ be the centre of the circle.

Given that $C(h,k)$ lies on the line $x-2y+3=0$. Thus:

$h-2k+3=0$ (i)

Also $|PC|=|QC|$

$\Rightarrow \sqrt{(h-1)^2+(k-0)^2}=\sqrt{(h-0)^2+(k-1)^2}$

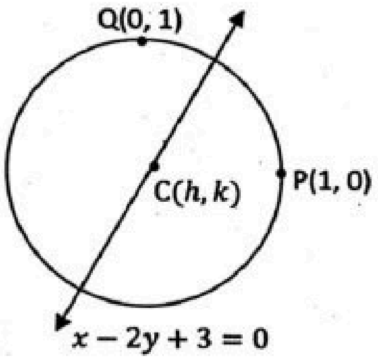
Squaring both sides

$\Rightarrow (h-1)^2+(k-0)^2=(h-0)^2+(k-1)^2$

$\Rightarrow h^2-2h+1+k^2=h^2+k^2-2k+1$

$\Rightarrow -2h+2k=0 \Rightarrow -h+k=0$

$\Rightarrow h=k$ (ii)



Using equation (ii) in equation (i), we get:

$k-2k+3=0 \Rightarrow -k=-3 \Rightarrow k=3$

Putting the value of k in (ii), we get $h=3$.

Therefore, the centre of the circle is $C(h,k)=C(3,3)$.

The radius of the circle is:

$r=|CP|=\sqrt{(3-1)^2+(3-0)^2}=\sqrt{4+9}=\sqrt{13}$

The equation of the circle with centre at (3, 3) and radius $\sqrt{13}$ is:

$$(x-3)^2+(y-3)^2=\left(\sqrt{13}\right)^2$$

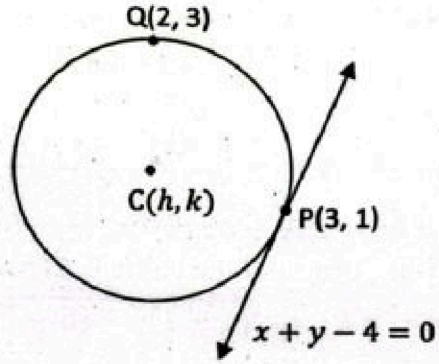
$$\Rightarrow x^2-6x+9+y^2-6y+9=13 \Rightarrow x^2+y^2-6x-6y+5=0$$

Example 5: Find the equation of the circle passing through the point (2, 3) and the line $x+y-4=0$ is the tangent to the circle at (3, 1).

Solution: Consider a circle passing through the point Q(2, 3) and having tangent line $x+y+3=0$ touching the circle at point P(3, 1). Let $C(h,k)$ be the centre of the circle. Now

$$|CP|=|CQ|$$

$$\Rightarrow \sqrt{(h-3)^2+(k-1)^2}=\sqrt{(h-2)^2+(k-3)^2}$$



Squaring both sides;

$(h-3)^2+(k-1)^2=(h-2)^2+(k-3)^2$

$\Rightarrow h^2-6h+9+k^2-2k+1=h^2-4h+4+k^2-6k+9$

$\Rightarrow -2h+4k=3$

$\Rightarrow 2h-4k=-3$ (i)

Since $\overline{CP}$ is perpendicular to the tangent line $x+y-4=0$.

Thus, (slope of $\overline{CP}$) × (slope of tangent line) = −1

$\Rightarrow \left(\frac{k-1}{h-3}\right)(-1)=-1 \Rightarrow k-1=h-3$

$\Rightarrow h-k=2$ (ii)

Solving equation (i) and (ii)), we have: $h=\frac{11}{2}$ and $k=\frac{7}{2}$

Thus, the centre of the circle is $C\left(\frac{11}{2},\frac{7}{2}\right)$. The radius of the circle is:

$$\Rightarrow r=|CP|=\sqrt{\left(\frac{11}{2}-3\right)^2+\left(\frac{7}{2}-1\right)^2}=\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{5}{2}\right)^2}=\sqrt{\frac{25}{4}+\frac{25}{4}}=\sqrt{\frac{50}{4}}=\frac{5\sqrt{2}}{2}$$

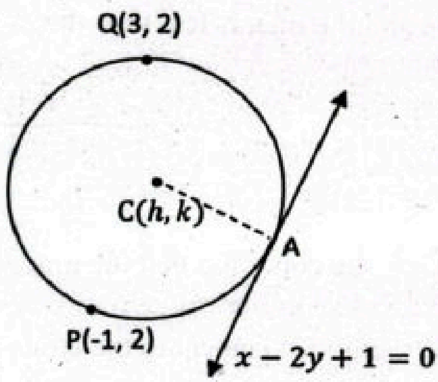
Therefore, the equation of the circle with centre at $\left(\frac{11}{2},\frac{7}{2}\right)$ and radius $\frac{5\sqrt{2}}{2}$ is:

$$\left(x-\frac{11}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\left(\frac{5\sqrt{2}}{2}\right)^2 \Rightarrow x^2-11x+\frac{121}{4}+y^2-7y+\frac{49}{4}=\frac{50}{4}$$

$$\Rightarrow x^2+y^2-11x-7y+\frac{121}{4}+\frac{49}{4}-\frac{50}{4}=0 \Rightarrow x^2+y^2-11x-7y+30=0$$

Example 6: Find the equation of the circle passing through the point (−1, 2), (3, 2) and touching the line $x-2y+1=0$.

Solution:



Consider a circle passing through the given points P(−1, 2) and Q(3, 2). Also, the circle touches the line $x-2y+1=0$ at point A. Let $C(h,k)$ be the centre of the circle. From figure:

$|CP|=|CQ|$

$\Rightarrow \sqrt{(h+1)^2+(k-2)^2}=\sqrt{(h-3)^2+(k-2)^2}$

Squaring both sides;

$(h+1)^2+(k-2)^2=(h-3)^2+(k-2)^2$

$\Rightarrow h^2+2h+1+k^2-4k+4=h^2-6h+9+k^2-4k+4$

$\Rightarrow 8h=8 \quad \Rightarrow h=1$ (i)

Also, $|CP|=|CA|$, where $|CA|$ is the distance of point C from the line $x-2y+1=0$.

Therefore,

$\sqrt{(h+1)^2+(k-2)^2}=\frac{|h-2k+1|}{\sqrt{1^2+(-2)^2}}$ (ii)

Putting $h=1$ in above equation (ii).

$\Rightarrow \sqrt{(1+1)^2+(k-2)^2}=\frac{|1-2k+1|}{\sqrt{1+4}} \Rightarrow \sqrt{4+(k-2)^2}=\frac{|2-2k|}{\sqrt{5}}$

Squaring both sides:

$$\Rightarrow 4+(k-2)^2=\frac{(2-2k)^2}{5}\Rightarrow 4+k^2-4k+4=\frac{4k^2-8k+4}{5}$$

$$\Rightarrow k^2-4k+8=\frac{4k^2-8k+4}{5}\Rightarrow 5k^2-20k+40=4k^2-8k+4$$

$$\Rightarrow k^2-12k+36=0\Rightarrow (k-6)^2=0\Rightarrow k-6=0\ \Rightarrow\ k=6$$

Thus, centre of the circle is $C(h,k)=(1,6)$ and radius of the circle is:

$$r=|CP|=\sqrt{(1+1)^2+(6-2)^2}=\sqrt{4+16}=\sqrt{20}$$

Equation of the circle with centre at $(1,6)$ and radius $\sqrt{20}$ is:

$$(x-1)^2+(y-6)^2=\left(\sqrt{20}\right)^2$$

$$\Rightarrow x^2-2x+1+y^2-12y+36=20$$

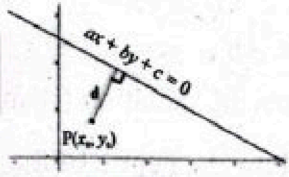
$$\Rightarrow x^2+y^2-2x-12y+17=0$$

Which is required equation of circle.

Key Facts

The distance 'd' of a point $P(x_0, y_0)$ from a line $ax+by+c=0$, is the length of the perpendicular drawn from the point to the given line as shown below:

$$d=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$



7.3 Line and a Circle

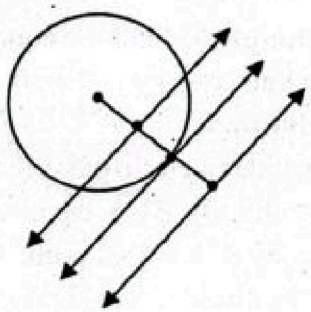
Consider a circle $x^2+y^2=r^2$ (i)

and a line $y=mx+c$ which implies:

$$mx-y+c=0 \quad \text{(ii)}$$

The centre of the circle is at $(0,0)$ and its radius is r.

i. When line intersects the circle at two distinct points.



When the line intersects the circle then the distance between the centre and the line, is less than the radius of the circle. i.e.,

$$\frac{|m(0)-0+c|}{\sqrt{m^2+(-1)^2}}<r$$

$$\Rightarrow\ \frac{|c|}{\sqrt{m^2+1}}<r\quad\text{or}\quad |c|<r\sqrt{m^2+1}$$

$$\Rightarrow\ c^2<r^2(m^2+1)$$

Which is the condition that the line will intersect the circle at two distinct points.

ii. When the line is tangent to circle.

When the line is tangent to the circle then distance between the centre and the line is equal to the radius of the circle. i.e.,

$$\frac{|m(0)-0+c|}{\sqrt{m^2+1}}=r\ \Rightarrow\ \frac{|c|}{\sqrt{m^2+1}}=r\ \Rightarrow\ |c|=r\sqrt{m^2+1}$$

$$\Rightarrow\ c=\pm r\sqrt{m^2+1}$$

Putting the value of c in $y=mx+c$, we get:

$$y=mx\pm r\sqrt{m^2+1}$$

Which are the equations of the tangent lines to the circle $x^2+y^2=r^2$.

iii. When the line neither touches nor intersects the circle

When the line neither touches nor intersects the circle then distance of the centre from the line is greater than the radius of the circle i.e.;

$\frac{|m(0)-0+c|}{\sqrt{m^2+1}} > r \Rightarrow \frac{|c|}{\sqrt{m^2+1}} > r \quad \Rightarrow |c| > r\sqrt{m^2+1}$

$\Rightarrow c^2 > r^2(m^2+1)$

Example 7: Check whether the line $x + 2y - 3 = 0$ is tangent to circle $x^2 + y^2 = r^2$ or not.

Solution: Equation of circle is $x^2 + y^2 = 16$ or $x^2 + y^2 = 4^2$

$\Rightarrow$ radius of the circle $= r = 4$

Equation the line is $x + 2y - 3 = 0$.

$\Rightarrow 2y = -x + 3 \quad \Rightarrow \quad y = -\frac{1}{2}x + \frac{3}{2} \Rightarrow m = -\frac{1}{2}$ and $c = \frac{3}{2}$

The line is tangent to the circle if $c = r\sqrt{m^2+1}$

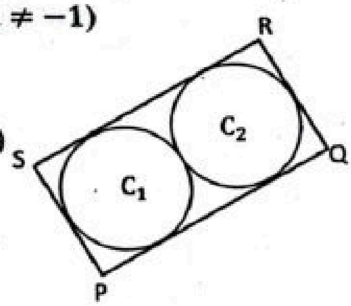
$\Rightarrow \left|\frac{3}{2}\right| = 4\sqrt{\left(-\frac{1}{2}\right)^2 + 1} = 4\sqrt{\frac{5}{4}} = 4 \times \frac{\sqrt{5}}{2} = 2\sqrt{5} \Rightarrow \left|\frac{3}{2}\right| = 2\sqrt{5}$

Which is not true. Thus, the given line is not a tangent to the circle.

Exercise 7.1

1. Find the equation of circle when its centre and radius is given:
   (i) Centre at $(3, -1)$ and radius 2 (ii) Centre at $\left(-\frac{1}{2}, -\frac{1}{3}\right)$ and radius $\frac{1}{5}$
   (iii) Centre at $\left(\frac{1}{a}, a\right)$ and radius is $a; a \neq 0$
2. Convert the following equations of circle into standard form and hence find their centre and radius:
   (i) $x^2 + y^2 - 4x + 6y - 36 = 0$ (ii) $5x^2 + 5y^2 - 2x + 4y - 27 = 0$
   (iii) $4x^2 + 4y^2 + 2ax + by - a^2 = 0$
3. Find the equation of circle passing through given three non-collinear points.
   (i) $(0,2), (2,0), (1,3)$ (ii) $(1,3), (3,6), (5,7)$
   (iii) $(0,0), (a,0), (0,b); a \neq 0, b \neq 0$
4. Find the equation of circle with centre lying on the line $x + y = 2$ and passing through the points $(2,-2)$ and $(3,4)$.
5. Find the equation of circle passing through the points $(1,-2)$ and $(4,-3)$ and whose centre lies on the line $3x + 4y = 7$.
6. Find the equation of the circle passing through the points $(2,1)$ and touching the line $x + 2y - 1 = 0$ at the point $(3,-1)$.
7. A circle touches the line $2x - 3y + 1 = 0$ at point $(1,1)$ and passes through the point of intersection of the lines $x + y + 1 = 0$ and $x - 3y + 5 = 0$. Find the equation of circle.
8. Equations of two diameters of a circle are $x - y = 3$ and $3x + y = 5$ and its radius is 5. Find the equation of circle.
9. A circle touches both the axes in the first quadrant and area of the circle is 13π square units. Find the equation of the circle.
10. The two points $A(4,3)$ and $B(2,5)$ lie on the circle and the centre of the circle lies on the perpendicular bisector of the chord $\overline{AB}$. The distance between the centre and the chord $\overline{AB}$ is $\sqrt{7}$. Find the equation of the circle.

11. Find the equation of the circle passing through the intersection of the circles $C_1: x^2+y^2-8x-2y+7=0$ and $C_2: x^2+y^2-4x+10y+8=0$ and passes through $(-1,-2)$. (Hint: equation of the required circle is $C_1+\lambda C_2=0; \lambda \neq -1$)
12. The diagram shows a rectangle PQRS and the circles C_1 and C_2. Both the circles touch each other and three sides of the rectangle. The coordinates of the points P, Q, R and S are $(0,4), (1,1), (7,3)$ and $(6,6)$. Find the equation of the circles C_1 and C_2.



13. A circle has its centre at the point $C(0,1)$ and a line touches the circle. The point $P(3,5)$ lies on the line touching the circle. The distance between P and C is five times the radius of the circle. Find the equation of the circle and the point where the line touches the circle.
14. The three lines $2x-y+1=0; 2x+y-3=0$ and $x-2y+4=0$ touch the circle. Find the centre of the circle. Also find the equation of circle.

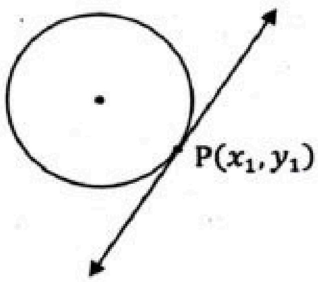
7.4 Equation of Tangent and Normal to a Circle at a Point on the Circle

7.4.1 Equation of Tangent at $P(x_1, y_1)$ on a Circle

Consider a circle $x^2+y^2+2gx+2fy+c=0$ (i)

Let $P(x_1, y_1)$ be a given point on the circle.

Differentiating equation (i) w.r.t x, we get:



$$2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$$

$$\Rightarrow (2x+2y)\frac{dy}{dx}=-(2x+2g) \Rightarrow \frac{dy}{dx}=-\frac{2x+2g}{2y+2f}=-\frac{x+g}{y+f}$$

$$\frac{dy}{dx} \text{ at } P(x_1, y_1)=m=-\frac{x_1+g}{y_1+f}$$

Which is the slope of the tangent line at the point $P(x_1, y_1)$.

By point-slope formula, equation of the tangent line at point P is:

$$y-y_1=-\left(\frac{x_1+g}{y_1+f}\right)(x-x_1)$$

$$\Rightarrow (y_1+f)(y-y_1)=-(x_1+g)(x-x_1)$$

$$\Rightarrow (y_1+f)y-(y_1+f)y_1=-(x_1+g)x+(x_1+g)x_1$$

$$\Rightarrow (y_1+f)y-(y_1+f)y_1+(x_1+g)x-(x_1+g)x_1=0$$

$$\Rightarrow (x_1+g)x+(y_1+f)y-y_1^2-fy_1-x_1^2-gx_1=0$$

$$\Rightarrow (x_1+g)x+(y_1+f)y-(x_1^2+y_1^2+gx_1+fy_1)=0$$

Since the point $P(x_1, y_1)$ lies on the circle $x^2+y^2+2gx+2fy+c=0$

So, $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

or $x_1^2+y_1^2+gx_1+fy_1+(gx_1+fy_1+c)=0$

or $x_1^2+y_1^2+gx_1+fy_1=-(gx_1+fy_1+c)=0$ (ii)

Putting in equation (ii), we get:

$$(x_1+g)x+(y_1+f)y-[-(gx_1+fy_1+c)]=0$$

$$\Rightarrow (x_1+g)x+(y_1+f)y+(gx_1+fy_1+c)=0$$

Which is the required equation of tangent line at $P(x_1, y_1)$.

Key Facts

Derivative at point of the curve is the slope of the tangent line to the curve at that point.

7.4.2 Equation of Normal at $P(x_1, y_1)$ on a Circle

Since normal line is perpendicular to the tangent line, so its slope is:

$$m_1 = -\frac{1}{m} = \frac{y_1 + f}{x_1 + g}$$

Key Facts

A line which is perpendicular to the tangent line at the point of tangency is called the normal line at that point.

By point-slope formula the equation of normal line is:

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

$$\Rightarrow (x_1 + g)(y - y_1) = (y_1 + f)(x - x_1)$$

$$\Rightarrow (x_1 + g)y - (x_1 + g)y_1 = (y_1 + f)x - (y_1 + f)x_1$$

$$\Rightarrow (x_1 + g)y - (x_1 + g)y_1 - (y_1 + f)x + (y_1 + f)x_1 = 0$$

$$\Rightarrow -(y_1 + f)x + (x_1 + g)y - x_1y_1 - gy_1 + x_1y_1 + fx_1 = 0$$

$$\Rightarrow -(y_1 + f)x + (x_1 + g)y - gy_1 + fx_1 = 0$$

$$\Rightarrow (y_1 + f)x - (x_1 + g)y + gy_1 - fx_1 = 0 \quad \text{(Multiplying both sides of equation by } -1)$$

$$\Rightarrow (y_1 + f)x - (x_1 + g)y - (fx_1 - gy_1) = 0$$

Which is the equation of the normal line at point $P(x_1, y_1)$.

Example 8: Find the equation of tangent and normal to the circle $x^2 + y^2 - 4x + 2y - 5 = 0$ at point $P(1, 2)$.

Solution:

Given equation of circle is $x^2 + y^2 - 4x + 2y - 5 = 0$

Differentiating w.r.t x, we have:

$$2x + 2y\frac{dy}{dx} - 4 + 2\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow (2y + 2)\frac{dy}{dx} = -2x + 4 \Rightarrow \frac{dy}{dx} = \frac{-2x+4}{2y+2} = \frac{-x+2}{y+1}$$

$$\frac{dy}{dx} \text{ at } P(1, 2) = m = \frac{-1+2}{2+1} = \frac{1}{3}$$

By the point-slope formula equation of the tangent line is:

$$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 2 = \frac{1}{3}(x - 1)$$

$$\Rightarrow \quad 3y - 6 = x - 1 \qquad \Rightarrow \quad x - 3y + 5 = 0$$

Since normal is perpendicular to the tangent line. Thus, slope of the normal line is -3 and by the point-slope formula, equation of the normal line is:

$$y - 2 = -3(x - 1) \Rightarrow y - 2 = -3x + 3 \quad \Rightarrow 3x + y - 5 = 0$$

Alternatively

Equation of circle is $x^2 + y^2 - 4x + 2y - 5 = 0$

Comparing it with $x^2 + y^2 + 2gx + 2fy + c = 0$, we have:

$$2g = -4 \Rightarrow g = -2$$

$$2f = 2 \quad \Rightarrow f = 1 \quad \text{and} \quad c = -5$$

Given point is $P(1, 2)$. i.e., $x_1 = 1, y_1 = 2$

Equation of tangent line is:

$$(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0 \qquad \text{(i)}$$

Substituting the values of x_1, y_1, g, f and c in equation (i), we have:

$(1-2)x+(2+1)y+((-2)1+1(2)-5)=0 \quad \Rightarrow -x+3y+5=0$

Equation of normal is:

$(y_1+f)x-(x_1+g)y-(fx_1-gy_1)=0 \qquad$ (ii)

Substituting the values of x_1, y_1, g and f in equation (ii), we have:

$(2+1)x-(1-2)y-(1\times 1-(-2\times 2)=0 \quad \Rightarrow 3x+y-5$

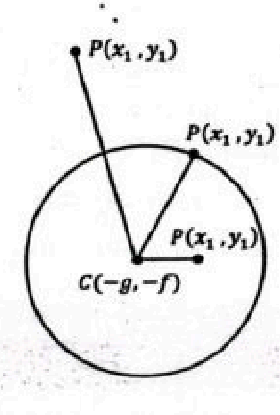
Exercise 7.2

1. Find the points of intersection of the given line and the circle.
   (i) $x-y+1=0;\ x^2+y^2-3x-8=0$
   (ii) $2x+y+4\sqrt{5}=0;\ x^2+y^2-2x+4y-11=0$
   (iii) $3x+2y+1=0;\ x^2+y^2-x+y+2=0$
2. The equation of a circle is $x^2+y^2-4x+y-7=0$ and the equation of line is $2x-y+c=0$. Find the value(s) of "c" such that the line:
   (i) intersects the circle at two distinct points.
   (ii) is tangent to circle.
   (iii) has no common point with the circle.
3. The tangents to a circle $(x-1)^2+(y-2)^2=3^2$ are perpendicular to the line $x+3y-6=0$. Find the equations of the tangent lines.
4. Find the equations of tangent and normal to the circle $x^2+y^2+3x+2y+3=0$ at the point $(2,-1)$.
5. Find the equation of the tangent to the circle $36x^2+36y^2-72y+11=0$ at the point on the circle with abscissa $\frac{1}{2}$.
6. Find the equation of the normal to the circle $x^2+y^2-2x+2y-11=0$ at the point on the circle with ordinate -4.
7. Circles with equations $x^2+y^2-2y-3=0$ and $x^2+y^2-8x+4y+11=0$ touch each other externally. Find the equation of the common tangent to the given circles.
8. Show that the tangent line at any point P on the circle is always perpendicular to the radial line through point P.
9. $A(-5,-1)$ and $B(1,5)$ are two points on the circle $x^2+y^2=26$. Find the point of intersection of the tangents to the circle at A and B. Show that the point of intersection of the tangent lines, the midpoint of chord $\overline{AB}$ and the centre of the circle are collinear.
10. A normal line cuts the circle with centre at $(1,2)$ at the point $(3,5)$. Find the other point of intersection of the normal and the circle. Also find the equation of the circle.

7.5 Position of a Point with respect to a Circle

Let $x^2+y^2+2gx+2fy+c=0$ be the equation of a circle and $P(x_1, y_1)$ be any point in the plane. We want to check that the given point lies outside, on or inside the circle.

The centre of the circle is $C(-g,-f)$ and the radius of the circle is $r=\sqrt{g^2+f^2-c}$. Observe that P lies outside the circle if $|CP|>r$, P lies on the circle if $|CP|=r$ and P lies inside the circle if $|CP|<r$. Combining all these; we may write $|CP| \gtreqless r$



$$\Rightarrow \sqrt{(x_1+g)^2+(y_1+f)^2} \lesseqqgtr \sqrt{g^2+f^2-c}$$

Squaring both sides

$$(x_1+g)^2+(y_1+f)^2 \lesseqqgtr g^2+f^2-c$$

$$\Rightarrow x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2 \lesseqqgtr g^2+f^2-c$$

$$\Rightarrow x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-g^2-f^2+c \lesseqqgtr 0$$

$$\Rightarrow x_1^2+2gx_1+g^2+y_1^2+2fy_1+c \lesseqqgtr 0$$

Which is the condition that a point lies inside, on or outside the circle.

Example 9: Check whether the point $P(2,4)$ lies outside, on or inside the circle:

$$2x^2+2y^2-6x+8x+1=0$$

Solution: Given equation of circle is $2x^2+2y^2-6x+8x+1=0$.

First make the coefficients of x^2 and y^2 one. Dividing both sides by 2, we have:

$$x^2+y^2-3x+4x+\frac{1}{2}=0$$

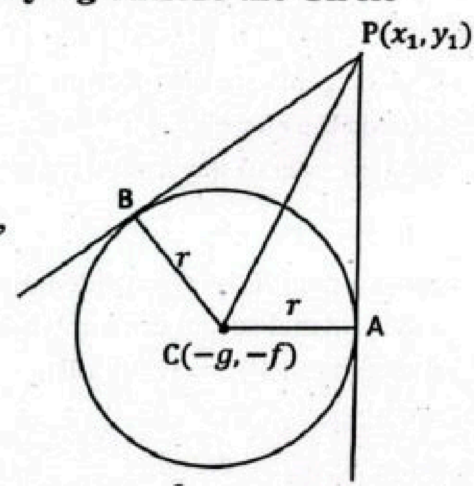
Given point is $(2,4)$. So:

$$x_1^2+y_1^2-3x_1+4x_1+\frac{1}{2}=(2)^2+(4)^2-3(2)+4(4)+\frac{1}{2}$$

$$=4+16-6+16+\frac{1}{2}=\frac{61}{2}>0$$

Thus, the point lies outside the circle.

7.6 Length of a Tangent Drawn from a Point Lying outside the Circle



Let $x^2+y^2+2gx+2fy+c=0$ be the equation of a circle and $P(x_1, y_1)$ lies outside the circle.

The centre of the circle is $C(-g,-f)$ and the radius of the circle is $r=\sqrt{g^2+f^2-c}$.

Two tangents can be drawn from the point P to circle. Thus, both tangents have the same length. i.e., $|AP|=|BP|$.

Since PCA is a right-angled triangle, so by Pythagoras theorem:

$$|CA|^2+|AP|^2=|CP|^2$$

$$\Rightarrow r^2+|AP|^2=\left(\sqrt{(x_1+g)^2+(y_1+f)^2}\right)$$

$$\Rightarrow \left(\sqrt{g^2+f^2-c}\right)^2+|AP|^2=\left(\sqrt{x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2}\right)^2$$

$$\Rightarrow g^2+f^2-c+|AP|^2=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2$$

$$\Rightarrow |AP|^2=x_1^2+y_1^2+2gx_1+2fy_1+c$$

$$\Rightarrow |AP|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Which is the length of tangent line.

Example 10:

Find the length of the tangent line to the circle $x^2 + y^2 + 3x - 4y + 15 = 0$ from the point $(1, 2)$.

Solution:

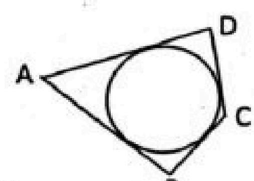
Equation of circle is $x^2 + y^2 + 3x - 4y + 15 = 0$ and the point $(1, 2)$ lies outside the circle.

Length of tangent line $= \sqrt{x_1^2 + y_1^2 + 3x_1 - 4y_1 + 15} = \sqrt{1^2 + 2^2 + 3(1) - 4(2) + 15}$

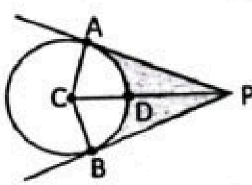
$= \sqrt{1 + 4 + 3 - 8 + 15} \qquad = \sqrt{15}$ units

Exercise 7.3

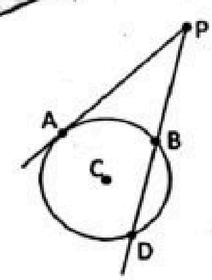
1. Check whether the given point lies inside, outside or on the given circle
   (i) $(3,5)$; $x^2 + y^2 - 6x + 2y - 18 = 0$
   (ii) $(1,6)$; $2x^2 + 2y^2 - 4x - 16y + 5 = 0$
   (iii) $(-1,-2)$; $x^2 + y^2 + 6x + 4y + 9 = 0$
2. Find the length of the tangent to the circle $x^2 + y^2 - 18x + 16y + 10 = 0$ from the point $(-1,-1)$ lying outside the circle.
3. Find the length of tangent to the circle $3x^2 + 3y^2 + 18x - 24y + 50 = 0$ drawn from a point $(-3, 1)$ lying outside the circle.
4. The point $P(-11,-10)$ lies outside the circle $x^2 + y^2 + 6x + 8y + 5 = 0$. Find the equations of the tangents to the circle drawn from point P. Also find the points of contact.
5. A quadrilateral ABCD is circumscribing a circle. Prove that $\overline{AB} + \overline{CD} = \overline{AD} + \overline{BC}$



6. Two tangents are drawn from a point $P(6, 1)$ lying outside the circle $x^2 + y^2 - 8x - 2y + 14 = 0$. Find the area of the shaded region.



7. From a point P lying outside the circle a tangent and secant lines are drawn. Prove that $|\overline{AP}|^2 = |\overline{PB}| \cdot |\overline{PD}|$



7.7 Parabola

It is the set of all the points in the plane which are equidistant from a fixed point and a fixed line in the plane (fixed point not lying on the fixed line).

The fixed point is called focus of the parabola and the fixed line is called its directrix. The ratio of distance at any point on the parabola from its focus to its directrix is called eccentricity of the parabola and is denoted by "e". Since by the definition, points on the parabola are equidistant from the focus and the directrix thus eccentricity of the parabola is 1.

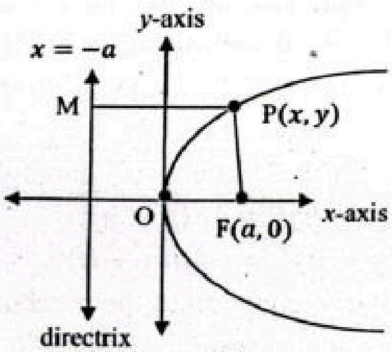
7.7.1 Standard Equation of Parabola

Consider a parabola with focus $F(a, 0)$ where $a > 0$ and directrix $x = -a$ or $x + a = 0$.

Take a point $P(x, y)$ on the parabola then by definition of the parabola:

$|PF| = |PM|$

Where $|PM|$ is the distance of the point P from the directrix.



(a)

$\Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = \dfrac{|x+a|}{\sqrt{1^2+0^2}}$

$\Rightarrow \sqrt{x^2 - 2ax + a^2 + y^2} = |x + a|$

Squaring both sides, we have:

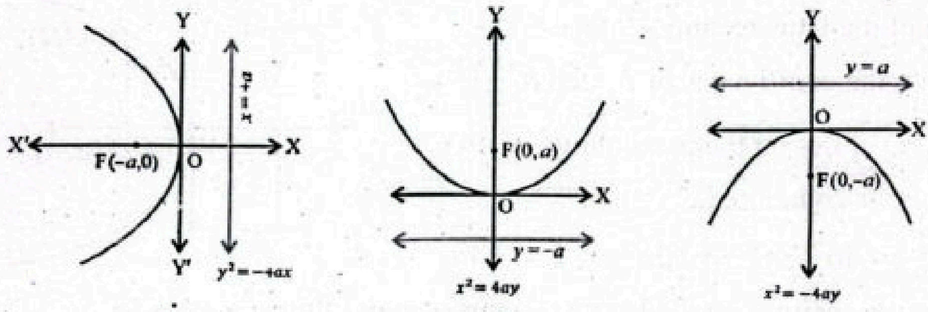
$\Rightarrow x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$\Rightarrow -2ax + y^2 = 2ax \quad \Rightarrow y^2 = 4ax$

Which is the equation of the parabola.

Similarly,

- If we take focus at $F(-a, 0)$ the equation of parabola is $y^2 = -4ax$. (b)
- If we take focus at $F(0, a)$ the equation of parabola is $x^2 = 4ay$ (c)
- If we take focus at $F(0, -a)$ the equation of parabola is $x^2 = -4ay$ (d)



(b) (c) (d)

7.7.2 Elements of Parabola

Directrix: The fixed line is called the directrix of parabola.

Focus: The fixed point (not lying on the directrix) is called the focus of the parabola.

Axis of Parabola (Axis of symmetry)

The line which passes through the focus and is perpendicular to the directrix is called axis of parabola and its equation is $y = 0$. Axis of parabola is also known as axis of symmetry.

Vertex of Parabola:

The point where the parabola cuts its axis is called vertex of the parabola. In this case $O(0, 0)$ is the vertex of the parabola which is closest to the focus.

Chord of a Parabola:

A line segment with its end points on the parabola is called chord of parabola.

Focal Chord:

A chord of the parabola which passes through the focus of parabola is called focal chord.

Focal Distance:

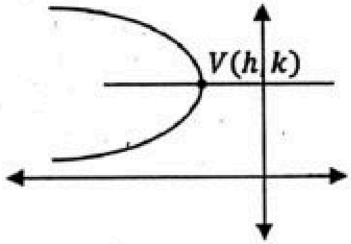
The distance of any point of the parabola from the focus is called focal distance.

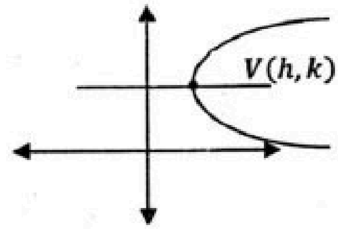
Latus rectum of Parabola:

A focal chord of the parabola which is parallel to the directrix of a parabola is called latus rectum of the parabola and its length is $4a$.

7.7.3 *Standard* Equation of Parabola

When the vertex is at any arbitrary point $V(h, k)$ and the axis of symmetry is parallel to x-axis then equation of parabola then:

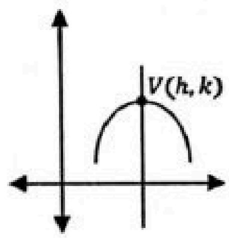


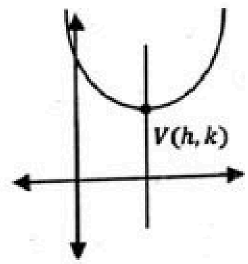


$$(y-k)^2 = 4a(x-h)$$

or $(y-k)^2 = -4a(x-h)$

Similarly, equation of the parabola with vertex at any arbitrary point $V(h, k)$ and the axis of symmetry parallel to y-axis is



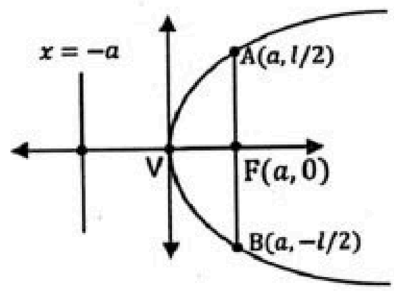


$$(x-h)^2 = 4a(y-k)$$

or $(x-h)^2 = -4a(y-k)$

7.7.4 Length of Latus Rectum of Parabola

Consider the parabola $y^2 = 4ax$. Its focus is at $F(a, 0)$ and vertex is at $V(0, 0)$.



Let l be the length of the latus rectum $\overline{AB}$ then $|\overline{FA}| = \frac{l}{2}$. Therefore, the coordinates of A are $\left(a, \frac{l}{2}\right)$.

Since the point $A\left(a, \frac{l}{2}\right)$ lies on the parabola $y^2 = 4ax$

Thus, it must satisfy its equation. i.e.;

$$\left(\frac{l}{2}\right)^2 = 4a(a) \Rightarrow \frac{l^2}{4} = 4a^2 \Rightarrow l^2 = 16a^2 \Rightarrow l = \pm 4a$$

Since length is always positive, so the length of latus rectum of the parabola is:

$$l = 4a$$

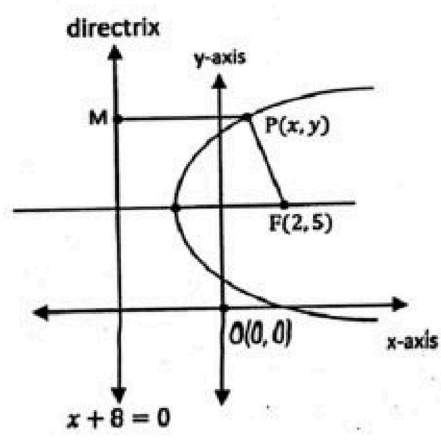
Example 11:

Find the equation of parabola with focus at $(2, 5)$ and the equation of directrix is $x + 8 = 0$.

Solution:

Consider any point $P(x, y)$ on the parabola with Focus at $F(2, 5)$ and equation of directrix $x + 8 = 0$.

By the definition of the parabola



$$|PF| = |PM|$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-5)^2} = \frac{|x+8|}{\sqrt{1^2 + 0^2}}$$

Squaring both sides:

$\Rightarrow (x-2)^2+(y-5)^2=(x+8)^2 \quad \Rightarrow \quad (y-5)^2=(x+8)^2-(x-2)^2$

$\Rightarrow (y-5)^2=x^2+16x+64-x^2+4x-4$

$\Rightarrow (y-5)^2=18x+60 \quad \Rightarrow (y-5)^2=18\left(x+\frac{10}{3}\right)$

Which is the required equation of parabola.

7.7.5 Elements of Parabola and its Graph

To find the elements of parabola when its equation is given; first we convert the given equation into the standard form and then compare it with one of the four standard equations and then find the elements of parabola.

Example 12: Find elements of parabola with equation $y=x^2-3x+7$ and draw its graph.

Solution: Given equation of parabola is:

$$y=x^2-3x+7 \quad \Rightarrow x^2-3x=y-7$$

$$\Rightarrow x^2-3x+\frac{9}{4}=y-7+\frac{9}{4} \quad \Rightarrow \left(x-\frac{3}{2}\right)^2=y-\frac{19}{4}$$

Let $x-\frac{3}{2}=X$ and $y-\frac{19}{4}=Y$ then $X^2=Y$ or $X^2=4\left(\frac{1}{4}\right)Y$

Which is of the form $X^2=4aY$ where $a=\frac{1}{4}$.

Now we write the elements of the parabola.

Vertex: We know that vertex is at $(0,0)$ i.e. $(X,Y)=(0,0)$

$\Rightarrow X=0$ and $Y=0$

$\Rightarrow x-\frac{3}{2}=0$ and $y-\frac{19}{4}=0 \quad \Rightarrow x=\frac{3}{2}$ and $y=\frac{19}{4}$

Thus, the vertex of the given parabola is at $\left(\frac{3}{2},\frac{19}{4}\right)$.

Focus: The focus of the parabola is at $(0,a)$. i.e., $(X,Y)=(0,a)$

$$\Rightarrow X=0 \text{ and } Y=a \quad \Rightarrow x-\frac{3}{2}=0 \text{ and } y-\frac{19}{4}=\frac{1}{4} \quad \Rightarrow x=\frac{3}{2} \text{ and } y=5$$

Thus, the focus of the parabola is at $\left(\frac{3}{2},5\right)$.

Axis of Parabola: Equation of axis of parabola is $X=0$.

$\Rightarrow x-\frac{3}{2}=0$ or $x=\frac{3}{2}$

Which is the equation of axis of parabola.

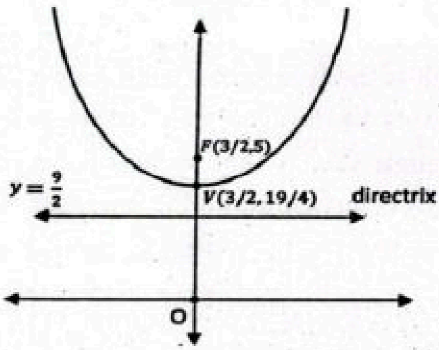
Directrix: Equation of directrix of parabola is $Y=-a$ or $Y+a=0$

$\Rightarrow y-\frac{19}{4}+\frac{1}{4}=0 \quad \Rightarrow y-\frac{9}{2}=0.$

Which is the equation of directrix of parabola.

Length of Latus Rectum:

Length of latus rectum is $4a=4\left(\frac{1}{4}\right)=1$ unit



Graph: To draw the graph of parabola, we find its x-intercept and y-intercept. For x-intercept put $y=0$ in the equation $y=x^2-3x+7$. We have:

$x^2-2x+7=0$

$\Rightarrow x = \frac{-(-3)\pm\sqrt{(-3)^2-4(1)(7)}}{2(1)} = \frac{3\pm\sqrt{9-28}}{2} = \frac{3\pm i\sqrt{19}}{2}$

Which are complex numbers thus parabola has no x-intercept. For y-intercept put $x = 0$ in equation $y = x^2 - 3x + 7$. We have:

$y = 0 - 0 + 7 \Rightarrow y = 7$

Thus $(0, 7)$ is the y-intercept of parabola.

7.7.6 General equation of Parabola

Prove that the equation $y = ax^2 + bx + c$ where a, b, c are real numbers with $a \neq 0$ represents a parabola.

Proof: Given equation is

$$y = ax^2 + bx + c \quad \Rightarrow ax^2 + bx = y - c$$

Since $a \neq 0$; dividing both sides by a.

$$x^2 + \frac{b}{a}x = \frac{1}{a}(y - c)$$

Adding $\frac{b^2}{4a^2}$ to both sides, we have:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{1}{a}(y - c) + \frac{b^2}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left[y - c + \frac{b^2}{4a}\right] = \frac{1}{a}\left[y + \frac{b^2-4ac}{4a}\right] = 4\left(\frac{1}{4a}\right)\left[y + \frac{b^2-4ac}{4a}\right]$$

$$\Rightarrow \left(x - \frac{-b}{2a}\right)^2 = 4\left(\frac{1}{4a}\right)\left[y - \frac{-b^2+4ac}{4a}\right] \qquad \text{(i)}$$

Which is of the form $(x - h)^2 = 4p(y - k)$ where $h = -\frac{b}{2a}$; $k = -\frac{b^2-4ac}{4a}$ and $p = \frac{1}{4a}$.

(i) is the equation of parabola with vertex $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$ and its branches open upwards or downwards according as $p > 0$ or $p < 0$.

Example 13: Find elements of parabola with vertex at $(2, 3)$ and focus at $(7, 3)$.

Solution: Given that vertex is $V(2, 3)$ and focus is $F(7, 3)$. Therefore, $h = 2$ and $k = 3$. Observe that y-coordinate of both V and F is same, thus $y = 3$ is the axis of parabola. Since x-coordinate of V is less than x-coordinate of F. Thus branches of parabola open on the right side.

The distance between F and V is "a". i.e.,

$$a = |FV| = \sqrt{(7-2)^2 + (3-3)^2} = \sqrt{25 + 0} = 5$$

So, the equation of parabola is:

$$(y - k)^2 = 4a(x - h)$$

$$\Rightarrow (y - 3)^2 = 4(5)(x - 2) \Rightarrow (y - 3)^2 = 20(x - 2)$$

Which is the required equation of parabola.

Example 14: Find elements of parabola with focus at $(3, -1)$ and its directrix is $3x - 4y + 1 = 0$.

Solution: Given that focus of the parabola is at $F(3, -1)$ and directrix is $3x - 4y + 1 = 0$. If $P(x, y)$ is any point on the parabola then by definition of parabola.

$|PF| = |PM|$; where $|PM|$ is the distance of P from directrix. Therefore:

$$\sqrt{(x-3)^2 + (y+1)^2} = \frac{3x-4y+1}{\sqrt{(3)^2+(-4)^2}}$$

Squaring both sides, we have:

$$(x-3)^2+(y+1)^2=\frac{(3x-4y+1)^2}{25}$$
$$\Rightarrow 25[x^2-6x+9+y^2+2y+1]=9x^2+16y^2+1-24xy+6x-8y$$
$$\Rightarrow 25x^2-150x+25y^2+50y+250=9x^2+16y^2+1-24xy+6x-8y$$
$$\Rightarrow 16x^2+9y^2+24xy-156x+58y+249=0$$

Which is the equation of parabola.

Example 15: Find elements of parabola with vertex at $(2,3)$ and equation of directrix is $y=-4$.

Solution: As we know that the focus of the parabola is on the opposite side of vertex as that of directrix, so in this case it opens upwards. Therefore its equation is:

$$(x-h)^2=4a(y-k) \qquad \text{(i)}$$

Given that the vertex is $(2,3)$, so $h=2$ and $k=3$. Also, the distance between vertex and directrix is $y=-4$ or $y+4=0$. Therefore, $a=\frac{|3+4|}{\sqrt{0^2+1^2}}=7$

Putting this value in equation (i), we get:

$$(x-2)^2=4(7)(y-3) \quad \text{or} \quad (x-2)^2=28(y-3)$$

Which is the required equation of parabola.

Exercise 7.4

1. Find focus, vertex, axis of symmetry, directrix, length of latus rectum, end points of the latus rectum of parabola with the given equations. Also draw the graph of the parabola.
   (i) $y^2=6x$ (ii) $y^2=-\frac{3}{2}x$ (iii) $x^2=24y$ (iv) $x^2=-5y$
   (v) $y^2-2y-12x-71=0$ (vi) $3x^2+42x+y+149=0$
   (vii) $4y^2+4y=15-32x$ (viii) $9x^2-6x=108y+26$
2. Find the equation of the parabola in each of the following.
   (i) Vertex at origin and focus $(0,-\frac{1}{32})$ (ii) Vertex $(-8,-9)$ and focus $(-\frac{31}{4},-9)$
   (iii) Vertex $(-6,-9)$ and directrix: $x=-\frac{47}{8}$ (iv) Vertex $(5,-1)$ and y-intercept: $-\frac{27}{2}$
   (v) Focus $(-\frac{3}{4},-1)$ and directrix $y=\frac{2}{5}$
   (vi) Opens left or right with vertex $(7,6)$ and passes through $(-11,9)$.
   (vii) Opens up or down and passes through the points $(11,15),(7,7)$ and $(4,22)$.
   (viii) Vertex $(10,0)$; axis of symmetry: $y=0$; length of latus rectum $=1$; $a<0$
   (ix) Vertex $(4,2)$; axis of symmetry: $x=4$; length of latus rectum $=\frac{1}{3}$; $a>0$
   (x) Vertex at origin; opens left; distance between focus and vertex is $\frac{1}{8}$ units.
3. Find the equation of the parabola with the focus at $(p\sin\theta, p\cos\theta)$ whose directrix is $x\cos\theta+y\sin\theta=p$.
4. Find the coordinates of the vertex of each parabola by differentiating its equation both sides and then solving for horizontal (or vertical) tangent.
   (i) $y=x^2-4x+10$ (ii) $4x^2+24x+39-3y=0$
   (iii) $y^2-10y+4x+28=0$ (iv) $y^2-14y=-3x-45y$

7.8 Equation of Tangent and Normal of Parabola

7.8.1 Condition for a Line to be Tangent to a Parabola

Consider a parabola $(y-k)^2 = 4a(x-h)$ (1)

and a line $y = mx + c$ (2)

On solving these equations, we will get the points of intersections of the line and the parabola. Using equation (2) in equation (1), we have:

$$[(mx+c)-k]^2 = 4a(x-h)$$
$$\Rightarrow [mx+(c-k)]^2 = 4a(x-h)$$
$$\Rightarrow m^2x^2 + 2m(c-k)x + (c-k)^2 = 4ax - 4ah$$
$$\Rightarrow m^2x^2 + \{2m(c-k)x - 4ax\} + (c-k)^2 + 4ah = 0$$
$$\Rightarrow m^2x^2 + \{2m(c-k) - 4a\}x + (c-k)^2 + 4ah = 0$$

On solving this equation, we will get at most two values of x. But for the line to be tangent to the parabola it must intersect only at a point. i.e., Both values of x should be same, thus discriminant of the above quadratic equation must be zero.

$$\{2m(c-k)-4a\}^2 - 4m^2\{(c-k)^2+4ah\} = 0$$
$$\Rightarrow 4m^2(c-k)^2 - 16am(c-k) + 16a^2 - 4m^2(c-k)^2 - 16ahm^2 = 0$$
$$\Rightarrow -16am(c-k) + 16a^2 - 16ahm^2 = 0$$
$$\Rightarrow -16a[m(c-k) - a + hm^2] = 0 \quad \Rightarrow \quad m(c-k) - a + hm^2 = 0$$
$$\Rightarrow m(c-k) = a - m^2h \quad \Rightarrow \quad c - k = \frac{a-m^2h}{m}$$
$$\Rightarrow c = \frac{a-m^2h}{m} + k = \frac{a-m^2h+mk}{m}$$

Putting this value of c in equation (2), we get:

$$y = mx + \frac{a-m^2h+mk}{m} = mx - \frac{m^2h-mk-a}{m}$$

Which is the equation of tangent to the parabola. Here m is the slope of the tangent line.

Particular Case:

When the vertex of the parabola is at $(0,0)$ i.e.; $h=0$ and $k=0$ then equation of the tangent line is:

$$y = mx - \frac{m^2(0)-m(0)-a}{m} \quad \Rightarrow \quad y = mx - \frac{-a}{m} \quad \Rightarrow \quad y = mx + \frac{a}{m}$$

Example 16: Find the equation of the tangent to the parabola $y^2 - 6y - 16x + 25 = 0$ with the slope $1/2$.

Solution: Equation of parabola is:

$$y^2 - 6y - 16x + 25 = 0$$
$$\Rightarrow y^2 - 6y = 16x - 25 \Rightarrow y^2 - 6y + 9 = 16x - 25 + 9$$
$$\Rightarrow (y-3)^2 = 16x - 16 = 16(x-1)$$
$$\Rightarrow (y-3)^2 = 4(4)(x-1)$$

Which is of the form $(y-k)^2 = 4a(x-h)$

Here $h = 4$; $k = 3$ and $a = 4$ and given that slope is $m = \frac{1}{2}$, therefore equation of tangent to the parabola is:

$$y = mx - \frac{m^2h-mk-a}{m}$$

Putting values, we get:

$$y = \frac{1}{2}x - \frac{\left(\frac{1}{2}\right)^2 4 - \frac{1}{2}(3) - 4}{\frac{1}{2}} \qquad \Rightarrow \quad y = \frac{1}{2}x - \frac{1 - \frac{3}{2} - 4}{\frac{1}{2}}$$

$$\Rightarrow \quad y = \frac{1}{2}x - (2 - 3 - 8) = \frac{1}{2}x + 9 \qquad \Rightarrow \quad 2y = x + 18$$

$$\Rightarrow \quad x - 2y + 18 = 0$$

7.8.2 Equation of Tangent Line to the Parabola at a Given Point

Consider a parabola $(y - k)^2 = 4a(x - h)$ (1)

and let $P(x_1, y_1)$ be a given point on the parabola. Differentiating equation (1) w. r. t. x

$$2(y - k)\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y - k}$$

Slope of tangent at $P(x_1, y_1) = m = \frac{dy}{dx}$ at $P(x_1, y_1) = \frac{2a}{y_1 - k}$

Thus, equation of the tangent line at $P(x_1, y_1)$: is

$$y - y_1 = m(x - x_1) \qquad \Rightarrow \quad y - y_1 = \frac{2a}{y_1 - k}(x - x_1)$$

$$\Rightarrow \quad (y_1 - k)(y - y_1) = 2a(x - x_1) \quad \Rightarrow \quad (y_1 - k)y - (y_1 - k)y_1 = 2ax - 2ax_1$$

$$\Rightarrow \quad 2ax - (y_1 - k)y - 2ax_1 + (y_1 - k)y_1 = 0$$

Which is the equation of the tangent line at $P(x_1, y_1)$.

In particular if vertex is at $(0, 0)$ then $h = 0$, $k = 0$. Then, the equation of tangent line is:

$$2ax - (y_1 - 0)y - 2ax_1 + (y_1 - 0)y_1 = 0 \quad \Rightarrow \quad 2ax - y_1y - 2ax_1 + y_1^2 = 0$$

Since (x_1, y_1) lies on the parabola, so $y_1^2 = 4ax_1$ and we have:

$$2ax - y_1y - 2ax_1 + 4ax_1 = 0 \qquad \Rightarrow \quad 2ax - y_1y + 2ax_1 = 0$$

7.8.3 Equation of Normal Line to the Parabola at a Given Point

As, we know that normal line is perpendicular to the tangent line.

Thus, slope of the normal line is

$$m = \frac{-1}{\text{slope of tangent line}} = -\frac{(y_1 - k)}{2a}$$

Equation of the normal line at point $P(x_1, y_1)$ is:

$$y - y_1 = -\frac{(y_1 - k)}{2a}(x - x_1)$$

$$\Rightarrow \quad 2a(y - y_1) = -(y_1 - k)(x - x_1) \quad \Rightarrow \quad (y_1 - k)(x - x_1) + 2a(y - y_1) = 0$$

Which is the equation of normal line at point P.

In particular if vertex of the parabola is at $(0, 0)$, the equation of normal becomes:

$$(y_1 - 0)(x - x_1) + 2a(y - y_1) = 0 \quad \text{or} \quad y_1(x - x_1) + 2a(y - y_1) = 0$$

Example 17: Find the equations of the tangent and normal to the parabola $y^2 - 6y + 8x - 9 = 0$ at point $P(\frac{1}{4}, -1)$.

Solution: Equation of parabola is $y^2 - 6y + 8x - 9 = 0$

Diff. w. r. t. x, we have:

$$2y\frac{dy}{dx} - 6\frac{dy}{dx} + 8 = 0 \quad \Rightarrow \quad (2y - 6)\frac{dy}{dx} = -8 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-8}{2y - 8} = \frac{-4}{y - 3}$$

Slope of tangent line $= m = \frac{dy}{dx}$ at $P = -\frac{-4}{-1 - 3} = 1$

Thus, equation of tangent line at P is:

$$y-(-1)=1\left(x-\frac{1}{4}\right) \quad \Rightarrow \quad y+1=x-\frac{1}{4}$$

$$\Rightarrow \quad 4y+4=4x-1 \quad \Rightarrow \quad 4x-4y-5=0$$

Since normal line is perpendicular to the tangent line, so slope of the normal line is $\frac{-1}{1}=-1$.

Equation of normal line is:

$$y-(-1)=-1\left(x-\frac{1}{4}\right) \quad \Rightarrow \quad y+1=-x+\frac{1}{4}$$

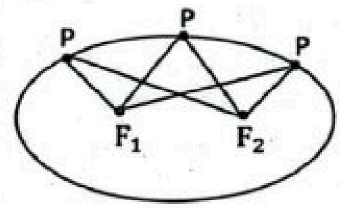
$$\Rightarrow \quad 4y+4=-4x+1 \quad \Rightarrow \quad 4x+4y+3=0$$

Which is the equation of normal line.

Exercise 7.5

1. Find the points of intersections of the line $x-2y+3=0$ and the parabola $y^2=8x+1$. Also find the chord intercepted. Is the chord a focal chord?
2. For what value(s) of c the line $2x+2y-c=0$ never touches or intersects at different points of the parabola $x^2-x+2y+3=0$.
3. Find the value of a so that the line $ax-2y+3=0$ is tangent to the parabola:
$$y^2-2y+3x+7=0\,;\ a\neq 0$$
4. Prove that the line $3x+y-5=0$ is tangent to the parabola $y^2-2y+6x-6=0$. Also find the point of contact.
5. Find the equation of tangent and normal to the parabola $2y^2-3y+11x-16=0$ at the point $(1,-1)$.
6. Find the equation of tangent and normal to the parabola $x^2-5x+2y+6=0$ at the point where abscissa is 1.
7. Find the equation of tangent and normal to the parabola $y^2=18x$ at the end points of its latus rectum.
8. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2=4ax$. Prove that the chord joining P and Q is a focal chord if $t_1t_2=-1$.
9. A tangent line is drawn to the parabola at any point P. Prove that the line segment of the tangent cut off between P and directrix subtends a right angle at the focus.
10. Prove that tangents at the end points of any focal chord intersect at right angles on the directrix.
11. Prove that tangents to the parabola at any point P on the parabola make equal angle with line joining P and its focus and the line through P parallel to the axis of parabola. (Reflecting property of parabola).
12. Prove that the semi latus rectum is a harmonic mean between the segments of any focal chord.

7.9 Ellipse



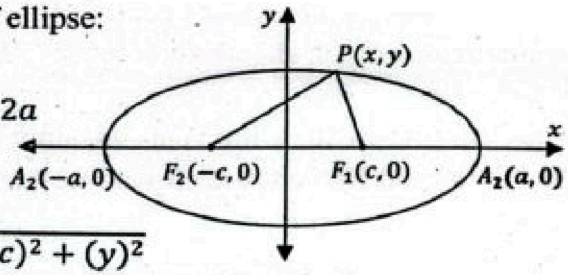
It is the set of all the points in the plane such that the sum of the distances of each point from two fixed points in the plane remains same. The two fixed points are known as foci (plural of focus) of the ellipse.

The midpoint of the foci is called the centre of ellipse.

7.9.1 Standard Equation of an Ellipse

Consider an ellipse with centre at origin and the foci on x-axis. Let the foci be $F_1(-c,0)$ and $F_2(c,0)$. Also suppose that sum of the distance of each pint of ellipse from foci is $2a$ which is constant.

Take any point $P(x,y)$ on ellipse then by definition of ellipse:



$$|PF_1| + |PF_2| = 2a$$

$$\Rightarrow \sqrt{(x+c)^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2+(y)^2} = 2a - \sqrt{(x-c)^2+(y)^2}$$

Squaring both sides, we get:

$$(x+c)^2 + y^2 = 4a^2 + ((x-c)^2+y^2) - 4a\sqrt{(x-c)^2+(y)^2}$$

$$\Rightarrow x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow 4a\sqrt{(x-c)^2+y^2} = 4a^2 - 4cx \quad \Rightarrow \quad a\sqrt{(x-c)^2+y^2} = a^2 - cx$$

Again, squaring both sides, we have:

$$a^2[(x-c)^2+y^2] = a^4 + c^2x^2 - 2a^2cx$$

$$\Rightarrow a^2[x^2+c^2-2cx+y^2] = a^4 + c^2x^2 - 2a^2cx$$

$$\Rightarrow a^2x^2 + a^2c^2 - 2a^2cx + a^2y^2 = a^4 + c^2x^2 - 2a^2cx$$

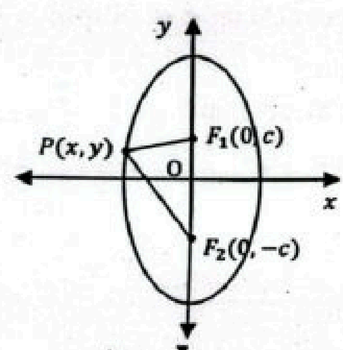
$$\Rightarrow (a^2x^2 - c^2x^2) + a^2y^2 = a^4 - a^2c^2 \quad \Rightarrow \quad (a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2) \qquad \text{(i)}$$

Since $a > c \;\Rightarrow\; a^2 > c^2 \;\Rightarrow\; a^2 - c^2 > 0$

Let $a^2 - c^2 = b$ (say), thus, equation (i) becomes:

$$b^2x^2 + a^2y^2 = a^2b^2$$

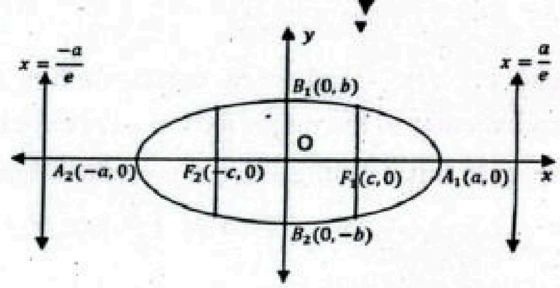
Dividing both sides by a^2b^2, we have: $\dfrac{b^2x^2}{a^2b^2} + \dfrac{a^2y^2}{a^2b^2} = \dfrac{a^2b^2}{a^2b^2}$



$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of ellipse in standard form.

Note: If we take foci on y-axis i.e. $F_1(0,-c)$ and $F_2(0,c)$ then equation of ellipse will be: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

Elements of Ellipse



Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Following are its elements.

Foci

The two fixed points $F_1(-c,0)$ and $F_2(c,0)$ are known as foci of ellipse.

Major Axis

The line which passes through both the foci of ellipse is called major axis of the ellipse and its length is $2a$. In this case major axis is x-axis.

Vertices

The points where the ellipse cuts its major axis are known as the vertices of ellipse. In this case vertices are $A_1(-a, 0)$ and $A_2(a, 0)$.

Centre

The mid-point of both the foci (or both the vertices) is called the centre of the ellipse. In this case $(0, 0)$ is the centre of ellipse.

Minor Axis

The line passing through the centre of the ellipse and perpendicular to the major axis is known as minor axis of the ellipse. Its length is $2b$. In this case y-axis is the minor axis.

Co-Vertices

The points where the ellipse cuts its minor axis are known as co-vertices of the ellipse. In this case $B_1(0, -b)$ and $B_2(0, b)$ are co-vertices of the ellipse. Note that the mid point of the co-vertices is also the centre of ellipse.

Chord

A line segment with its end points on the ellipse is called chord of the ellipse.

Focal Chord

A chord which passes through any of the foci is called focal chord. e.g., B_1B_2 is focal chord.

Latus Rectum

A focal chord which is perpendicular to the major axis is called latus rectum of the ellipse. There are two latus rectums (latera recta) of an ellipse through each of the foci. The length of both the latus rectums is same and is $\frac{2b^2}{a}$.

Eccentricity

The eccentricity e of ellipse is $\frac{c}{a}$. i.e., $e = \frac{c}{a}$ since $c < a$ thus eccentricity of the ellipse is always less than 1.

As we know that:

$$b^2 = a^2 - c^2 \quad \Rightarrow \quad \frac{b^2}{a^2} = \frac{a^2}{a^2} - \frac{c^2}{a^2} \quad \Rightarrow \quad \left(\frac{b}{a}\right)^2 = 1 - \left(\frac{c}{a}\right)^2$$

$$\Rightarrow \quad \left(\frac{b}{a}\right)^2 = 1 - e^2 \quad \Rightarrow \quad e^2 = 1 - \left(\frac{b}{a}\right)^2 = \frac{a^2 - b^2}{a^2}$$

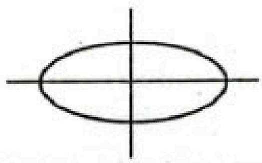
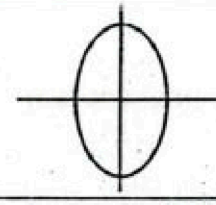
or $\quad e = \frac{\sqrt{a^2 - b^2}}{a}$

Directrices

There are two fixed lines corresponding to each foci lying outside of the ellipse and are perpendicular to the major axis at a specific distance $\frac{a}{e}$. These lines are known as directrices of the ellipse. In this case these are the vertical lines and their equations are:

$$x = \pm\frac{a}{e} \text{ or } x = \pm\frac{a}{e^2} \text{ or } x = \pm\frac{a^2}{c}$$

The table shows summary of the elements of ellipse.

	 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	 $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
Centre	$O(0,0)$	$O(0,0)$
Foci	$(-c,0)$ & $(c,0)$	$(0,-c)$ & $(0,c)$
Major Axis	x-axis with equation $y=0$	y-axis with equation $x=0$
Minor Axis	y-axis with equation $x=0$	x-axis with equation $y=0$
Vertices	$(-a,0)$ & $(a,0)$	$(0,-a)$ & $(0,a)$
Co-Vertices	$(0,-b)$ & $(0,b)$	$(-b,0)$ & $(b,0)$
Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$

Note: Equation of the ellipse with centre at arbitrary point (h,k) is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{or} \quad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

Example 18: Find the equation of an ellipse with foci $(-3,0)$ and $(3,0)$ and the sum of the distance of any point from the foci is 10.

Solution: Given that foci of the ellipse are $F_1(-3,0)$ and $F_2(3,0)$. Take any point $P(x,y)$ on the ellipse, then by definition of ellipse:

$$|PF_1|+|PF_2|=10$$

$$\Rightarrow \sqrt{(x+3)^2+(y-0)^2}+\sqrt{(x-3)^2+(y-0)^2}=10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2}=10-\sqrt{(x-3)^2+y^2}$$

Squaring both sides:

$$(x+3)^2+y^2=100+((x-3)^2+y^2)-20\sqrt{(x-3)^2+y^2}$$

$$\Rightarrow \cancel{y^2}+6x+\cancel{9}+\cancel{y^2}=100+\cancel{x^2}-6x+\cancel{9}+\cancel{y^2}-20\sqrt{(x-3)^2+y^2}$$

$$\Rightarrow 20\sqrt{(x-3)^2+y^2}=100-12x \quad \text{or} \quad 5\sqrt{(x-3)^2+y^2}=25-3x$$

Again, squaring both sides:

$$25((x-3)^2+y^2)=625+9x^2-150x$$

$$\Rightarrow 25(x^2-6x+9+y^2)=625+9x^2-150x$$

$$\Rightarrow 25x^2-150x+225+25y^2)=625+9x^2-150x$$

$$\Rightarrow 16x^2+25y^2=400$$

Divide both sides by 400, we get the required equation of ellipse as:

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

Example 19: Convert the equation of ellipse $4x^2+9y^2+8x-36y+4=0$ in standard form. Also find its elements and draw the graph.

Solution: Given equation of ellipse is:

$$4x^2+9y^2+8x-36y+4=0 \qquad \Rightarrow \quad (4x^2+8x)+(9y^2-36y)=-4$$

$$\Rightarrow 4(x^2+2x)+9(y^2-4y)=-4$$

$$\Rightarrow 4[x^2+2x+1-1]+9[y^2-4y+4-4]=-4$$

$$\Rightarrow 4(x+1)^2-4+9(y-2)^2-36=-4 \quad \text{or} \quad 4(x+1)^2+9(y-2)^2=36$$

Dividing both sides by 36, we get:

$$\frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1 \qquad \text{or} \qquad \frac{(x+1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$$

Which is the standard form of the equation of ellipse.

Letting $x+1=X$ and $y-2=Y$, the equation becomes:

$$\frac{X^2}{3^2}+\frac{Y^2}{2^2}=1$$

Its major axis is along x-axis. Hence $a=3$ and $b=2$.

Now we find its elements.

Centre: As the centre of ellipse is $(0,0)$, therefore $(X,Y)=(0,0)$

$\Rightarrow X=0$	$Y=0$
$\Rightarrow x+1=0$	$y-2=0$
$\Rightarrow x=-1$	$y=2$

Thus, the centre of ellipse is at $(-1,2)$.

Foci: As we know that $c=\pm\sqrt{a^2-b^2}$ hence for the given equation $c=\pm\sqrt{9-4}=\pm\sqrt{5}$ therefore foci are $F_1(-c,0)=(-\sqrt{5},0)$ and $F_2(-c,0)=(-\sqrt{5},0)$.

Major Axis: As major axis of ellipse is along x-*axis* hence its equation is $Y=0$ $\Rightarrow y-2=0 \qquad \Rightarrow y=2$ is equation of major axis.

Minor Axis: Equation of minor axis is $X=0 \qquad \Rightarrow x+1=0 \qquad \Rightarrow x=-1$

Vertices: The vertices of given ellipse are $(-a,0)=(-3,0)$ and $(a,0)=(3,0)$.

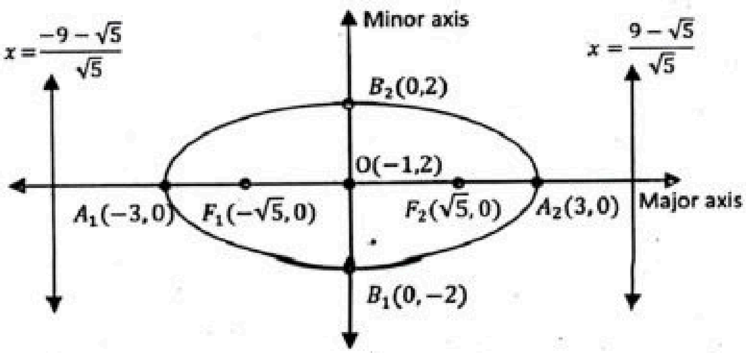
Co-Vertices: Co-vertices of given ellipse are $(0,-b)=(0,-2)$ and $(0,b)=(0,2)$.

Directrices: The equation of directrices is $X=\pm\frac{a^2}{c}$ i.e. $x+1=\pm\frac{9}{\sqrt{5}}$

$$\text{or } x=\pm\frac{9}{\sqrt{5}}-1=\frac{9-\sqrt{5}}{\sqrt{5}}=\frac{-9-\sqrt{5}}{\sqrt{5}}$$

$x=\frac{-9-\sqrt{5}}{\sqrt{5}}$

The graph is shown in the adjoining figure.



Exercise 7.6

1. Find the centre, vertices, co-vertices, foci, eccentricity, length and equation of major axis, length and equation of minor axis, directrices, and length of latus rectums for the following equations of ellipse. Also draw the ellipse in each case
   (i) $\frac{x^2}{9}+\frac{y^2}{16}=1$ (ii) $2x^2+3y^2=30$
   (iii) $\frac{x^2}{49}+\frac{(y-3)^2}{64}=1$ (iv) $x^2+9y^2+6x-90y+225=0$
   (v) $16x^2+9y^2-32x+36y-92=0$
2. From the given information, find the equation of ellipse in each of the following.
   (i) Vertices $(-4,6),(-16,6)$ and co-vertices $(-10,2),(-10,10)$
   (ii) Centre at $(-8,5)$; vertex $(-8,15)$; Focus $(-8,5+\sqrt{5})$
   (iii) Eccentricity is $\frac{\sqrt{15}}{4}$; centre $(-5,5)$; co-vertex $(-8,5)$
   (iv) Eccentricity is $\frac{3}{4}$ and passes through the point $(3,1)$ with centre at $(0,0)$; major axis along y-axis.
   (v) Focus at $(3,4)$; directrix $x=5$ and eccentricity $\frac{2}{3}$.
   (vi) Centre at $(1,-1)$; horizontal tangents are $y=7$ and $y=-9$ and vertical tangents are $x=6$ and $x=-4$.
   (vii) Length of latus rectum is 4; centre at $(0,0)$ and eccentricity is $\frac{1}{3}$; major axis along y-axis.
   (viii) Centre at $(1,1)$; eccentricity is $\frac{2}{5}$ and one of the directrices is $x=5$.
3. Find the coordinates of the vertices and co-vertices of the following ellipse by differentiating both sides of equations and solving for horizontal and vertical tangent lines
   (i) $2x^2+3y^2+2x-3y-5=0$ (ii) $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{16}=1$
   (iii) $\frac{1}{2}x^2+\frac{3}{4}y^2+x-y-5=0$ (iv) $x^2+2y^2-6x-4=0$

7.10 Equation of Tangent and Normal to Ellipse

7.10.1 Condition of Tangency

Consider the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ ______ (1)

and the line $y=mx+c$ ______ (2)

on solving (1) and (2) simultaneously we will get the common points, from (2) put the value of y in equation (1)

$$\frac{(x-h)^2}{a^2}+\frac{(mx+c-k)^2}{b^2}=1$$

$$b^2(x-h)^2+a^2(mx+c-k)^2=a^2b^2$$

$$b^2(x^2-2hx+h^2)+a^2\{m^2x^2+2m(c-k)+(c-k)^2\}=a^2b^2$$

$$(b^2+a^2m^2)x^2+\left(-2b^2h+2a^2m(c-k)\right)x+b^2h^2+a^2(c-k)^2-a^2b^2=0$$

which is quadratic equation in x.

If Disc > 0 then equation will have two distinct real roots so, line will cut the ellipse at two different points.

If Disc < 0 then equation will have no real roots, i.e. there is no common point.

If Disc = 0 then equation will have one real root i.e. there is only one common point. Thus line will be tangent to the ellipse if Disc = 0.

$$\Rightarrow \{-2b^2h + 2a^2m(c-k)\}^2 - 4(b^2 + a^2m^2)[b^2h^2 + a^2(c-k)^2 - a^2b^2] = 0$$

$$\Rightarrow 4\{-b^2h + a^2m(c-k)\}^2 - 4(b^2 + a^2m^2)[b^2h^2 + a^2(c-k)^2 - a^2b^2] = 0$$

$$\Rightarrow b^4h^2 + a^4m^2(c-k)^2 - 2a^2b^2hm(c-k) - b^4h^2 - a^2b^2(c-k)^2 + a^2b^4 - a^2b^2m^2h^2$$
$$- a^4b^2m^2h^2 - a^4m^2(c-k)^2 - a^4b^2m^2 = 0$$

Taking common $-a^2b^2$

$$-a^2b^2[2hm(c-k) + (c-k)^2 - b^2 + m^2h^2 + a^2m^2] = 0$$

$$\Rightarrow 2hm(c-k) + (c-k)^2 - b^2 + m^2(a^2 + h^2) = 0$$

$$\Rightarrow b^2 = 2hm(c-k) + (c-k)^2 + m^2(a^2 + h^2)$$

is the condition for the line to be tangent to the ellipse. In particular when centre of the ellipse is at (0,0) then $h = 0$ and $k = 0$. In this case the condition reduces to

$$b^2 = 2(0)m(c-0) + (c-0)^2 + m^2(a^2 + 0)$$

$$\Rightarrow b^2 = c^2 + m^2a^2$$

$$\Rightarrow c^2 = b^2 - m^2a^2$$

$$\Rightarrow c = \sqrt{b^2 - m^2a^2} \qquad \text{provided that } b^2 - m^2a^2 \geq 0$$

Put value in equation (2)

$$y = mx \pm \sqrt{b^2 - m^2a^2}$$

Are the equations of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the slope form.

Example 20: Find the equations of the tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{16} = 1$ with slope $\frac{1}{2}$.

Solution: Equation of the ellipse is $\frac{x^2}{2} + \frac{y^2}{16} = 1$.

Here $a^2 = 2$; $b^2 = 16$ and given that $m = \frac{1}{2}$.

Equations of the tangent to ellipse are $y = mx \pm \sqrt{b^2 - m^2a^2}$

Putting the values; we have:

$$y = \frac{1}{2}x \pm \sqrt{16 - \frac{1}{4}(2)} \Rightarrow y = \frac{1}{2}x \pm \frac{\sqrt{62}}{2} \Rightarrow 2y = x \pm \sqrt{62}$$

are the required equations of the tangent lines.

Example 21: prove that the line $\sqrt{3}x - 2y + 8 = 0$ is tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Solution:

Equation of line is $\sqrt{3}x - 2y + 8 = 0$ (1)

and equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$$x^2 + 4y^2 = 16 \qquad (2)$$

From equation (1) $\quad 2y = \sqrt{3}x + 8 \Rightarrow y = \frac{\sqrt{3}x+8}{2}$

Put in equation (2)

$$x^2 + 4\left(\frac{\sqrt{3}x + 8}{2}\right)^2 = 16$$

$$\Rightarrow x^2 + 4\left(\frac{3x^2 + 16\sqrt{3}x + 64}{4}\right) = 16$$

$$\Rightarrow x^2 + 3x^2 + 16\sqrt{3}x + 64 = 16$$

$$\Rightarrow 4x^2 + 16\sqrt{3}x + 48 = 0$$

Dividing both sides by 4

$$\Rightarrow x^2 + 4\sqrt{3}x + 12 = 0$$

Taking its discriminant

$$\Rightarrow Disc. = \left(4\sqrt{3}\right)^2 - 4(1)(12) = 48 - 48$$

$\Rightarrow Disc. = 0$ this shows that line is tangent to the ellipse.

7.11 Equations of Tangent and Normal to an Ellipse at a Given Point

Consider the equation of ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow b^2(x-h)^2 + a^2(y-k)^2 = a^2b^2 \qquad (1)$$

Let $P(x_1, y_1)$ be any point on this ellipse. Differentiating equation (1) w. r. t. x

$$\Rightarrow 2b^2(x-h) + 2a^2(y-k)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{(x-h)}{(y-k)}$$

At point $P(x_1, y_1)$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{(x_1-h)}{(y_1-k)}$$

is the slope of the tangent line at point $P(x_1, y_1)$. Thus, equation of tangent line at this point P is

$$y - y_1 = -\frac{b^2}{a^2}\frac{(x_1-h)}{(y_1-k)}(x - x_1)$$

$$\Rightarrow a^2(y_1-k)(y-y_1) = -b^2(x_1-h)(x-x_1)$$

$$\Rightarrow b^2(x_1-h)(x-h+h-x_1) + a^2(y_1-k)(y-k+k-y_1) = 0$$

$$\Rightarrow b^2(x_1-h)[(x-h)-(x_1-h)] + a^2(y_1-k)[(y-k)-(y_1-k)] = 0$$

$$\Rightarrow b^2(x_1-h)(x-h) - b^2(x_1-h)^2 + a^2(y_1-k)(y-k) - a^2(y_1-k)^2 = 0$$

$$\Rightarrow b^2(x_1-h)(x-h) + a^2(y_1-k)(y-k) = b^2(x_1-h)^2 + a^2(y_1-k)^2$$

Since, the point P lies on the ellipse, thus from equation (1) we have

$$b^2(x_1-h)^2+a^2(y_1-k)^2=a^2b^2$$

$$\Rightarrow b^2(x_1-h)(x-h)+a^2(y_1-k)(y-k)=a^2b^2$$

Dividing both sides by a^2b^2

$$\Rightarrow \frac{(x_1-h)(x-h)}{a^2}+\frac{(y_1-k)(y-k)}{b^2}=1$$

is the required equation of the tangent line.

In particular case if the centre of the ellipse lies at origin, then $h=0, k=0$. In this case equation of tangent is

$$\frac{(x_1-0)(x-0)}{a^2}+\frac{(y_1-0)(y-0)}{b^2}=1$$

$$\Rightarrow \frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

7.11.1 Equation of Normal

As the normal line is perpendicular to the tangent line thus,

$$\text{slope of the normal line}=\frac{-1}{\text{slope of tangent line}}$$

$$=\frac{-1}{-\frac{b^2}{a^2}\frac{(x_1-h)}{(y_1-k)}}=\frac{a^2(y_1-k)}{b^2(x_1-h)}$$

Thus, equation of normal line at point P is

$$y-y_1=\frac{a^2(y_1-k)}{b^2(x_1-h)}(x-x_1)$$

$$\Rightarrow b^2(x_1-h)(y-y_1)=a^2(y_1-k)(x-x_1)$$

$$\Rightarrow a^2(y_1-k)(x-x_1)-b^2(x_1-h)(y-y_1)=0$$

is the equation of the normal line.

In particular if the centre of the ellipse is at origin, then $h=0$ and $k=0$. So, equation of normal becomes

$$a^2(y_1-0)(x-x_1)-b^2(x_1-0)(y-y_1)=0$$

$$\Rightarrow a^2y_1(x-x_1)-b^2x_1(y-y_1)=0$$

Example 22: Find the equations of tangent and normal to the ellipse $x^2+2y^2-2x+4y=0$ at point $P(0,0)$.

Solution:

Equation of ellipse is $x^2+2y^2-2x+4y=0$

Differentiating w. r. t. x

$$2x+4y\frac{dy}{dx}-2+4\frac{dy}{dx}=0$$

$$\Rightarrow (4y+4)\frac{dy}{dx}=-(2x-2)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2x-2}{4y+4}=-\frac{x-1}{2y+2}$$

At point (0,0)

$$\Rightarrow \frac{dy}{dx} = -\frac{0-1}{2(0)+2} = \frac{1}{2}$$

is the slope of the tangent line at $P(0,0)$. Thus, equation of tangent line is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 0 = \frac{1}{2}(x - 0) \Rightarrow 2y = x \quad \Rightarrow x - 2y = 0$$

Now slope of the normal line is $-\frac{1}{1/2}$. Thus, equation of normal line at point $P(0,0)$ is

$$y - 0 = -2(x - 0) \Rightarrow y = -2x$$

$$2x + y = 0$$

Alternatively

Equation of ellipse is $x^2 + 2y^2 - 2x + 4y = 0$, convert it into standard form, i.e.;

$$(x^2 - 2x) + 2(y^2 + 2y) = 0$$

$$\Rightarrow (x^2 - 2x + 1 - 1) + 2(y^2 + 2y + 1 - 1) = 0$$

$$\Rightarrow [(x-1)^2 - 1] + 2[(y+1)^2 - 1] = 0$$

$$\Rightarrow (x-1)^2 - 1 + 2(y+1)^2 - 2 = 0$$

$$\Rightarrow (x-1)^2 + 2(y+1)^2 = 3$$

Dividing both sides by 3

$$\Rightarrow \frac{(x-1)^2}{3} + \frac{2(y+1)^2}{3} = 1$$

$$\Rightarrow \frac{(x-1)^2}{3} + \frac{(y-(-1))^2}{3/2} = 1$$

Compare it with standard equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; h = 1; k = -1; a^2 = 3; b^2 = 3/2$

Equation of tangent line is

$$\frac{(x_1 - h)(x - h)}{a^2} + \frac{(y_1 - k)(y - k)}{b^2} = 1$$

Putting the values

$$\frac{(0-1)(x-1)}{3} + \frac{(0-(-1))(y-(-1))}{3/2} = 1$$

$$\Rightarrow \frac{-x+1}{3} + \frac{2(y+1)}{3} = 1 \Rightarrow -x + 1 + 2y + 2 = 3 \Rightarrow -x + 2y = 0$$

$$\Rightarrow x - 2y = 0$$

And the equation of normal line is

$$a^2(y_1 - k)(x - x_1) - b^2(x_1 - h)(y - y_1) = 0$$

Putting the values

$$\Rightarrow 3(0-(-1))(x-0) - 3/2(0-1)(y-0) = 0$$

$$\Rightarrow 3x + \frac{3}{2}y = 0$$

$\Rightarrow 2x + y = 0$, is the equation of normal line.

Exercise 7.7

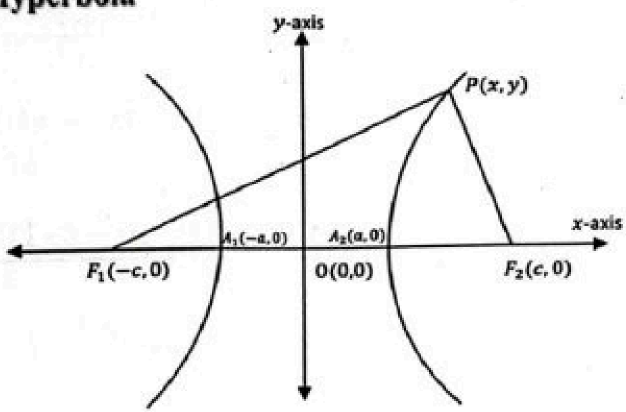
1. Find the points of intersection of the line and ellipse and also find the length of chord intercepted $x - y + 1 = 0;\ x^2 + 2y^2 + 3x - 7y - 11 = 0$.
2. Find the value of m so that the line $5x + y - 20 = 0, m \neq 0$ touches the ellipse $\frac{x^2}{4} + \frac{y^2}{5} = 1$. Also find the point where it touches the ellipse.
3. Find the equation of tangent to the ellipse $x^2 + 2y^2 - 3x + 5y - 3 = 0$ with slope 1.
4. Find the equations of tangent to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.
5. Find the equations of the tangent and normal to the ellipse $2x^2 + 3y^2 - 5x - 10y + 5 = 0$ at the point $(3,2)$.
6. Find the equations of the tangents and normal to the ellipse $15x^2 + 6y^2 - 8x - 5y + 2 = 0$ at the point with ordinate $\frac{1}{3}$.
7. Prove that tangent at any point to the ellipse make equal angles with the line joining the point with the foci (reflecting property of ellipse).

7.12 Hyperbola

It is the set of all points in a plane such that the difference of the distances of each point from two fixed points in the plane is same. The two fixed points are known as foci of the hyperbola. The midpoint of foci is known centre of the hyperbola.

7.12.1 Standard Form of the Equation of Hyperbola

Consider a hyperbola with centre at origin and foci on $x - axis$. Let the foci be $F_1(-c, 0)$ and $F_2(c, 0)$ and the difference of the distances of each point from the foci is $2a$.



Take any points $P(x, y)$ on hyperbola then by the definition of the hyperbola

$|PF_1| - |PF_2| = 2a$

$$\Rightarrow \sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

squaring both sides

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + (y)^2} + ((x-c)^2 + y^2)$$

$$\Rightarrow x^2 + c^2 + 2cx + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + x^2 + c^2 - 2cx + y^2$$

$$\Rightarrow 4a\sqrt{(x-c)^2+y^2} = 4cx - 4a^2$$

$$\Rightarrow a\sqrt{(x-c)^2+y^2} = cx - a^2$$

Again, squaring both sides

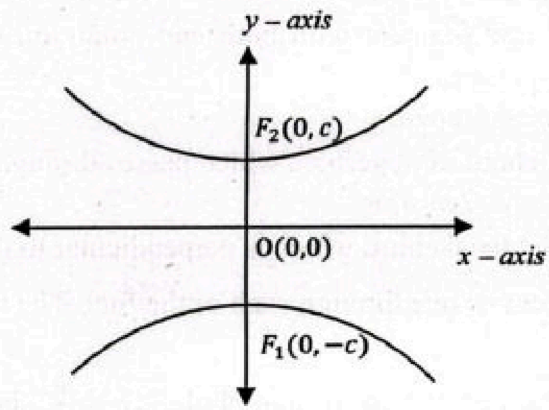
$$a^2[(x-c)^2+y^2] = c^2x^2 + a^4 - 2a^2cx$$

$$\Rightarrow a^2[x^2+c^2-2cx+y^2] = c^2x^2+a^4-2a^2cx$$

$$\Rightarrow a^2x^2 + a^2c^2 - \cancel{2a^2cx} + a^2y^2 = c^2x^2 + a^4 - \cancel{2a^2cx}$$

$$\Rightarrow (c^2x^2 - a^2x^2) - a^2y^2 = a^2c^2 - a^4$$

$$\Rightarrow (c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2)$$



since $c > a \Rightarrow \ c^2 > a^2$

$$\Rightarrow \ c^2 - a^2 > 0$$

Let $c^2 - a^2 = b^2$

Therefore $b^2x^2 - a^2y^2 = a^2b^2$

Dividing both sides by a^2b^2, we have:

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is the standard form of the equation of hyperbola. Similarly, if we take the foci on $y - axis$ the equation of hyperbola will be

$$\Rightarrow \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Elements of Hyperbola

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci

The two fixed points $F_1(-c, 0)$ and $F_1(c, 0)$ are the foci of the hyperbola.

Centre

The midpoint of both the foci is called the centre of the hyperbola.

Focal Axis or Transverse Axis or Real Axis

The line which passes through both the foci of hyperbola is called the focal axis. In this case x-axis is the focal axis with equation $y = 0$ and its length is $2a$.

Conjugate Axis or Imaginary Axis

The line passing through the centre of the hyperbola and perpendicular to its focal axis is known as conjugate axis or imaginary axis. In this case y-axis is conjugate axis and its equation is $x = 0$. The length of conjugate axis is $2b$.

Vertices

The points where the hyperbola cuts its focal axis are known as the vertices of the hyperbola. In this case $A_1(-a,0)$ and $A_2(a,0)$ are vertices of the hyperbola. Note that the mid point of the vertices is the centre of the hyperbola.

Co-Vertices

The points $(0,-b)$ and $(0,b)$ lying on the conjugate axis and equidistant from the centre are known as co-vertices of the hyperbola.

Focal Length

The distance between the two foci is called focal length and its value is $2c$.

Chord

A line segment with both end points on the same branch of hyperbola is called chord of the hyperbola.

Focal Chord

A chord of hyperbola which passes through the focus of the hyperbola is called focal chord.

Latus Rectum

The focal chord which is perpendicular to the focal axis is called latus rectum. There are two latus rectums one through each of the foci. The length of each latus rectum is $\frac{2b^2}{a}$.

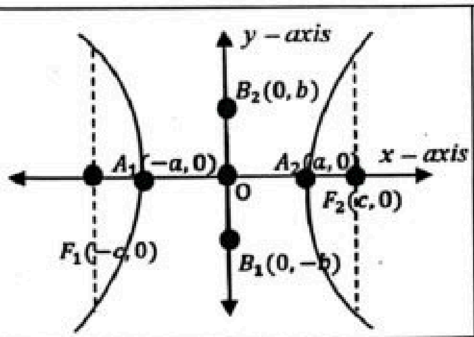
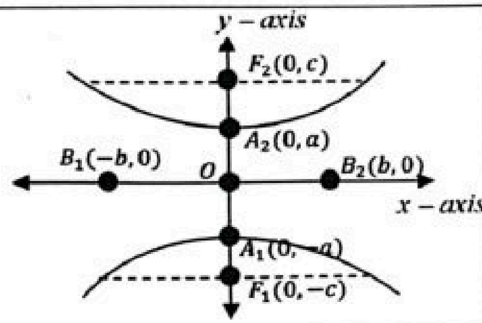
Eccentricity

The eccentricity of hyperbola is $e=\frac{c}{a}$ since $c>a$ thus eccentricity of the hyperbola is always greater than 1.

The equation of hyperbola with centre at (h,k) when the axis of symmetry remains parallel to the coordinate axis is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad \text{or} \quad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

The table shows the summary of elements of hyperbola.

Sketch and Equation		
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$
Centre	$O(0,0)$	$O(0,0)$
Foci	$(-c,0)$ & $(c,0)$	$(0,-c)$ & $(0,c)$
Vertices	$A_1(-a,0)$ & $A_2(a,0)$	$A_1(0,-a)$ & $A_2(0,a)$
Co-Vertices	$B_1(0,-b)$ & $B_2(0,b)$	$B_1(-b,0)$ & $B_2(b,0)$

Focal Axis	x-axis with equation $y = 0$	y-axis with equation $x = 0$
Conjugate Axis	y-axis with equation $x = 0$	x-axis with equation $y = 0$
Eccentricity	$e = \frac{c}{a} > 1$	$e = \frac{c}{a} > 1$
Directrices	$x = \pm\frac{c}{a}$ or $x = \pm\frac{a}{e}$	$y = \pm\frac{c}{a}$ or $y = \pm\frac{a}{e}$
Latus Rectum	length of latus rectum $2\frac{b^2}{a}$	length of latus rectum $2\frac{b^2}{a}$

Example 23: **Convert the equation of hyperbola $4x^2 - 9y^2 - 16x - 18y - 29 = 0$ into standard form and find its elements.**

Solution: Given equation is:

$4x^2 - 9y^2 - 16x - 18y - 29 = 0$ $\quad\Rightarrow (4x^2 - 16x) - (9y^2 + 18y) = 29$

$\Rightarrow 4(x^2 - 4x) - 9(y^2 + 2y) = 29$

$\Rightarrow 4(x^2 - 4x + 4 - 4) - 9(y^2 + 2y + 1 - 1) = 29$

$\Rightarrow 4[(x-2)^2 - 4] - 9[(y+1)^2 - 1] = 29$ $\quad\Rightarrow 4(x-2)^2 - 16 - 9(y+1)^2 + 9 = 29$

$\Rightarrow 4(x-2)^2 - 9(y+1)^2 = 36$

Dividing both sides by 36:

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{4} = 1 \quad \text{or} \quad \frac{(x-2)^2}{3^2} - \frac{(y+1)^2}{2^2} = 1$$

is the standard form of the equation of the hyperbola.

Let $x - 21 = X$ and $y + 1 = Y$

the equation reduces to $\frac{X^2}{3^2} - \frac{Y^2}{2^2} = 1.$

Centre: Centre of hyperbola is at $(0, 0)$. i.e., $(X, Y) = (0, 0)$

$\Rightarrow X = 0$	$Y = 0$
$\Rightarrow x - 2 = 0$	$y + 1 = 0$
$\Rightarrow x = 2$	$y = -1$

Therefore, $(2, -1)$ is the centre of hyperbola.

Vertices: The vertices of given hyperbola are $(\pm a, 0)$ i.e. $(X, Y) = (\pm a, 0)$

$\Rightarrow X = \pm a$	$Y = 0$
$\Rightarrow x - 2 = \pm 3$	$y + 1 = 0$
$\Rightarrow x = \pm 3 + 2$	$y = -1$
$\Rightarrow x = 5, x = -1$	

Hence vertices are $(-1, -1)$ and $(5, -1)$.

Covertices: Covertices hyperbola are $(0, \pm b)$ i.e. $(X, Y) = (0, \pm b)$

$\Rightarrow X = 0$	$\Rightarrow Y = \pm b$
$\Rightarrow x - 2 = 0$	$\Rightarrow y + 1 = \pm 2$
$\Rightarrow x = 2$	$\Rightarrow y = -1 \pm 2$
	$\Rightarrow y = -3, x = 1$

Thus, covertices are $(2, -3)$ and $(2,1)$.

Foci: Foci of the hyperbola are $(\pm c, 0)$ i.e. $(X, Y) = (\pm c, 0)$

$\Rightarrow X = \pm c$	$\Rightarrow Y = 0$
$\Rightarrow x - 2 = \pm c$	$\Rightarrow y + 1 = 0$
$\Rightarrow x = 2 \pm c$	$\Rightarrow y = -1$

As we know that $b^2 = c^2 - a^2$

$\Rightarrow 2^2 = c^2 - 3^2 \quad \Rightarrow c^2 = 13 \quad \Rightarrow c = \sqrt{13}$

Thus, $x = 2 \pm \sqrt{13}$ and $y = 1$

$\Rightarrow (2 \pm \sqrt{13}, -1)$ are foci of the hyperbola.

Eccentricity: Eccentricity of the hyperbola is $e = \frac{c}{a} = \frac{\sqrt{13}}{3}$

Focal Axis: Equation of focal axis is $Y = 0 \Rightarrow y + 1 = 0$ and the length of focal axis is $2a = 2(3) = 6$ units

Conjugate Axis: Equation of conjugate axis is $X = 0 \Rightarrow x - 2 = 0$ and the length of conjugate axis is $2b = 2(2) = 4$ units.

Directrices: The equation of directrices of hyperbola are $X = \pm\frac{c}{a}$

$$\Rightarrow x - 2 = \pm\frac{\sqrt{13}}{3} = 2 \pm \frac{\sqrt{13}}{3}$$

Latus Rectum: The length of each latus rectum of hyperbola is $2\frac{b^2}{a} = 2\frac{2^2}{3} = \frac{8}{3}$ units

Example 24: Find the equation of hyperbola with centre at (2,4) and the length of semi transverse and conjugate axis are 3 and 5 units respectively. Also, the transverse axis is parallel to $x - axis$

Solution: Given that centre of the hyperbola is at (2,4) and length of semi transverse and conjugate axis are 3 and 5 respectively. So $a = 3$ and $b = 5$

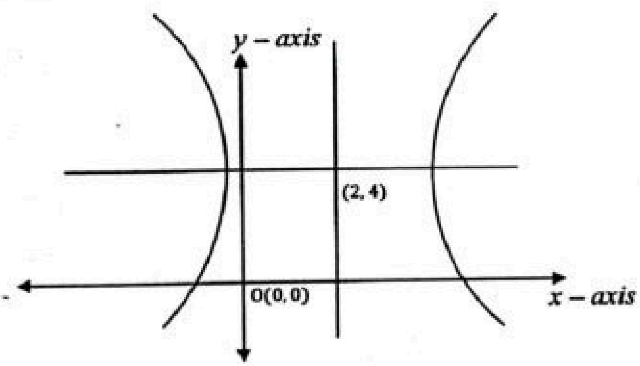
Since transverse axis is parallel to x-axis, so equation of hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Putting the values we have

$$\frac{(x-2)^2}{3^2} - \frac{(y-4)^2}{5^2} = 1$$

$$\Rightarrow \quad \frac{(x-2)^2}{9} - \frac{(y-4)^2}{25} = 1$$



Example 25: Find the equation of hyperbola with centre at $(0, 2)$ and it passes through the points $(1, 1)$ and $(4, 7)$.

Solution: Let the equation of hyperbola be $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$

Given that centre is at $(0, 2)$ thus equation is

$$\frac{(x-0)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{(y-2)^2}{b^2} = 1 \qquad (1)$$

Given that it passes through the point (1,1); so from Eq (1)

$$\frac{1^2}{a^2} - \frac{(1-2)^2}{b^2} = 1 \qquad \Rightarrow \frac{1}{a^2} - \frac{1}{b^2} = 1 \qquad ....(2)$$

Also, it passes through the point (4,7), so from Eq (1)

$$\frac{4^2}{a^2}-\frac{(7-2)^2}{b^2}=1$$

$$\frac{16}{a^2}-\frac{25}{b^2}=1 \qquad \text{........(3)}$$

Multiply Eq (2) by 16 and then subtract Eq (3) from it

$$\frac{16}{a^2}-\frac{16}{b^2}=16$$

$$\pm\frac{16}{a^2}\mp\frac{25}{b^2}=\mp1$$

$$\frac{9}{b^2}=15$$

$b^2=\frac{9}{15}$ put in Eq (2)

$$\frac{1}{a^2}-\frac{1}{\left(\frac{9}{15}\right)}=1$$

$$\Rightarrow\frac{1}{a^2}-\frac{15}{9}=1 \quad \Rightarrow\frac{1}{a^2}=1+\frac{15}{9}$$

$$\Rightarrow\frac{1}{a^2}=\frac{24}{9} \qquad \Rightarrow a^2=\frac{9}{24}$$

putting the values in Eq (1)

$$\Rightarrow\frac{x^2}{\left(\frac{9}{24}\right)}-\frac{y^2}{\left(\frac{9}{15}\right)}=1$$

or

$$\Rightarrow\frac{24x^2}{9}-\frac{15y^2}{9}=1$$

is the required Equation of hyperbola.

Example 26: Find the equation of hyperbola with centre at origin with eccentricity $\frac{3}{2}$ and length of its latus rectum is 3.

Solution:

Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{........(1)}$$

Given that $e=\frac{3}{2}$ and $2\frac{b^2}{a}=3$ ....(2)

Since $c=ae \Rightarrow c=a\left(\frac{3}{2}\right)=\frac{3a}{2}$

also $b^2=c^2-a^2=\left(\frac{3a}{2}\right)^2-a^2$

$$b^2=\frac{9a^2}{4}-a^2=\frac{5a^2}{4} \qquad \text{........(3)}$$

put in Eq (2)

$$\frac{2\left(\frac{5a^2}{4}\right)}{a}=3 \quad \Rightarrow a=\frac{6}{5} \qquad \Rightarrow a^2=\frac{36}{25}$$

put in eq (3)

$$b^2=\frac{5}{4}\left(\frac{36}{25}\right)=\frac{9}{5}$$

put the values of a^2 and b^2 in Eq (1)

$$\Rightarrow\frac{x^2}{\left(\frac{36}{25}\right)}-\frac{y^2}{\left(\frac{9}{5}\right)}=1$$

or

$$\Rightarrow\frac{25x^2}{36}-\frac{5y^2}{9}=1$$

is the equation of hyperbola

Example 27: Find the equation of hyperbola in the standard form with centre at origin and eccentricity $\frac{4}{3}$; also, length of its semi conjugate axis is 3, which is along $y-axis$.

Solution:

Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{.......(1)}$$

Given that $e=\frac{4}{3}$ and $b=3$

$$\therefore b^2=c^2-a^2=(ae)^2-a^2=a^2(e^2-1)$$

$$=a^2\left(\frac{16}{9}-1\right)=\frac{7}{9}a^2$$

put the value of b

$$3^2=\frac{7}{9}a^2 \quad \Rightarrow a^2=\frac{81}{7}$$

put the values of a^2 and b^2 in Eq (1)

$$\Rightarrow \frac{x^2}{\left(\frac{81}{7}\right)}-\frac{y^2}{3^2}=1 \Rightarrow \frac{7x^2}{81}-\frac{y^2}{9}=1$$

is the required equation of hyperbola

Example 28: Find the equation of hyperbola with centre at origin and eccentricity $\sqrt{3}$. Its one focus is $(3,0)$ and the directrix is $x=1$.

Solution: As the y coordinate of focus is zero; so its transverse axis is along $x-axis$. Equation of directrix is $x=1$ or $x-1=0$.

Take any point $P(x,y)$ on the hyperbola then by definition

$$e=\frac{\text{distance of point from focus}}{\text{distance of point from line}} \qquad \text{........(1)}$$

distance of point P from focus $=\sqrt{(x-3)^2+(y-0)^2}$

$$=\sqrt{x^2-6x+9+y^2}=\sqrt{x^2+y^2-6x+9}$$

distance of point from directrix$=\frac{|x-1|}{\sqrt{1^2+0^2}}=|x-1|$

putting values in Eq (1)

$$\sqrt{3}=\frac{\sqrt{x^2+y^2-6x+9}}{|x-1|}$$

$\Rightarrow \sqrt{3}|x-1|=\sqrt{x^2+y^2-6x+9}$

squaring both sides

$3(x-1)^2=x^2+y^2-6x+9$

$3x^2-6x+3=x^2+y^2-6x+9$

$\Rightarrow 2x^2-y^2=6$

Dividing both sides by 6

$$\frac{x^2}{3}-\frac{y^2}{6}=1$$

is the required equation of hyperbola

Alternate Method

Given that $e=\sqrt{3}$ and focus is $(3,0) \Rightarrow c=3$

$\Rightarrow ae=3 \Rightarrow a\sqrt{3}=3 \Rightarrow a=\sqrt{3} \Rightarrow a^2=3$

also $b^2=c^2-a^2$

$\Rightarrow b^2=3^2-\left(\sqrt{3}\right)^2=9-3=6$

putting values in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\frac{x^2}{3}-\frac{y^2}{6}=1$ is the required equation of hyperbola.

Exercise 7.8

1. Find the centre, vertices, co-vertices, foci, eccentricity, length and equation of transverse axis, length and equation of conjugate axis, directrices and length of latus rectums for the given equations of hyperbola. Also sketch the hyperbola in each case.

   (i) $\frac{(x-5)^2}{36}-\frac{(y-4)^2}{81}=1$

   (ii) $\frac{(y-10)^2}{10}-\frac{(x-1)^2}{15}=1$

   (iii) $-16x^2+9y^2+32x+144y-16=0$

   (iv) $x^2-4y^2+2x-40y-135=0$

2. From the given information find the equation of hyperbola in each of the following.

   (i) Vertices: $(8,14),(8,-10)$; conjugate axis of length 6 units.

   (ii) Vertices $\left(-2,\frac{5}{2}\right),\left(-16,\frac{5}{2}\right)$ and end points of conjugate axis are $\left(-9,\frac{15}{2}\right);\left(-9,-\frac{5}{2}\right)$.

   (iii) Vertices are $(-5,1),(-5,-7)$ and foci are $\left(-5,-3+\sqrt{97}\right),\left(-5,-3-\sqrt{97}\right)$.

   (iv) Foci are $\left(8,-5+\sqrt{53}\right),\left(8,-5-\sqrt{53}\right)$ and end points of conjugate axis are $(15,5);(1,-5)$.

   (v) Vertices are $(0,9),(0,-9)$ and passes through the point $(8,15)$.

   (vi) Eccentricity is $\frac{5}{4}$; centre at $(1,1)$ and passes through $(-7,2)$, focal axis is horizontal.

   (vii) Focus $(5,3)$; directrix $y=-2$ and eccentricity $\frac{5}{3}$.

   (viii) Centre at $(0,0)$; length of latus rectum is 5, eccentricity $\frac{5}{4}$; conjugate axis along x-$axis$.

3. Find the vertices of the following hyperbolas by differentiating the given equation and solving for horizontal/vertical tangent lines.

   (i) $\frac{(x-\frac{1}{2})^2}{3}-\frac{(y+3)^2}{5}=1$

   (ii) $\frac{(y-1)^2}{2^2}-\frac{(x+1)^2}{19}=1$

   (iii) $5x^2-5y^2+25x-5y+20=0$

   (iv) $-x^2+y^2-10x-4y-28=0$

7.13 Equation of Tangent and Normal to a Hyperbola

Consider the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ ....(1)

and the line $y = mx + c$ ....(2)

on solving Eqs (1) and (2) simultaneously we wil get the points of intersections of the line and hyperbola; for this use Eq (2) in (1)

$$\frac{(x-h)^2}{a^2} - \frac{(mx+c-k)^2}{b^2} = 1$$

$$\Rightarrow b^2(x-h)^2 - a^2(mx+(c-k))^2 = a^2b^2$$

$$\Rightarrow b^2x^2 - 2b^2hx + b^2h^2 - a^2m^2x^2 - 2a^2m(c-k)x - a^2(c-k)^2 - a^2b^2 = 0$$

$$\Rightarrow (b^2 - a^2m^2)x^2 - 2(b^2h + a^2m(c-k))x + (b^2h^2 - a^2(c-k)^2 - a^2b^2) = 0$$

which is quadratic equation in 'x'.

If discriminant of this equation is greater than zero or positive the equation will have two distinct values of x so line will cut the hyperbola at two distinct points. If discriminant of this equation is negative then the roots of the equation will be imaginary i.e. there is no point of intersection and if value of discriminant is zero then equation has repeated roots i.e. there is only one point of intersection. In this case the line will be tangent to the hyperbola, so condition for the line to be tangent to hyperbola is

$$Disc = 0$$

$$\Rightarrow \left(2(b^2h + a^2m(c-k))\right)^2 - 4(b^2 - a^2m^2)(b^2h^2 - a^2b^2 - a^2(c-k)^2) = 0$$

$$\Rightarrow 4(b^2h + a^2m(c-k))^2 - 4(b^2 - a^2m^2)(b^2h^2 - a^2b^2 - a^2(c-k)^2) = 0$$

Dividing both sides by '4'

$$(b^2h + a^2m(c-k))^2 - (b^2 - a^2m^2)(b^2h^2 - a^2b^2 - a^2(c-k)^2) = 0$$

$$\Rightarrow b^4h^2 + a^4m^4(c-k)^2 + 2a^2b^2mh(c-k)^2 - b^4h^2 + a^2b^4 + a^2b^2(c-k)^2 + a^2b^2m^2h^2 - a^4b^2m^2 - a^4m^2(c-k)^2 = 0$$

$$\Rightarrow a^2b^2[2mh(c-k)^2 + b^2 + (c-k)^2 + m^2h^2 - a^2m^2] = 0$$

Dividing both sides by a^2b^2

$$\Rightarrow (2mh+1)(c-k)^2 + b^2 + m^2h^2 - a^2m^2 = 0$$

is condition for tangency.

In particular when centre of the hyperbola is at (0,0); then $h = 0, k = 0$;

In this case condition of tangency is

$$(2m(0)+1)(c-0)^2 + b^2 + m^2(0)^2 - a^2m^2 = 0$$

$$c^2 + b^2 - a^2m^2 = 0$$

$$c^2 = a^2m^2 - b^2$$

from here we have $c = \pm\sqrt{a^2m^2 - b^2}$ provided that $a^2m^2 - b^2 \geq 0$.

Put in $y = mx + c$

$$\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$$

are the equations of the tangents to the hyperbola with slope m.

Example 29: Find the equations of tangents to the hyperbola $\frac{x^2}{3}-\frac{y^2}{2}=1$ with slope $\frac{7}{3}$.

Solution:

Equation of hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$

here $a^2=3$; $b^2=2$ and the slope is given to be $\frac{7}{3}$ i.e. $m=\frac{7}{3}$

Equation of tangents to the hyperbola are

$y=mx\pm\sqrt{a^2m^2-b^2}$

put the values

$$y=\frac{7}{3}x\pm\sqrt{3\left(\frac{7}{3}\right)^2-2}$$

$y=\frac{7}{3}x\pm\frac{1}{3}\sqrt{129}$ or $3y=7x\pm\sqrt{129}$ are the required equations of the tangent lines.

Example 30: Check whether the line $x-y-1=0$ is the tangent to the hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$

Solution:

The equation of hyperbola is $\frac{x^2}{4}-\frac{y^2}{3}=1$ or $3x^2-4y^2=12$ ....(1)

Equation of line is $x-y-1=0$

$\Rightarrow y=x-1$ put in Eq (1)

$\Rightarrow 3x^2-4(x-1)^2=12$ $\Rightarrow 3x^2-4x^2+8x-4=12$

or $\Rightarrow x^2-8x+16=0$

compute its discriminant

$Disc=(-8)^2-4(1)(16)=64-64=0$

this shows that the line is tangent to the hyperbola.

7.14 Equation of Tangent and Normal to the Hyperbola at a Given Point

Consider the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ and point $P(x_1,y_1)$ be any point on the hyperbola.

Equation of hyperbola may be written as

$b^2(x-h)^2-a^2(y-k)^2=a^2b^2$ ....(1)

Differentiate w.r.t. 'x'

$2b^2(x-h)-2a^2(y-k)\frac{dy}{dx}=0$ $\Rightarrow\frac{dy}{dx}=\frac{b^2}{a^2}\left(\frac{x-h}{y-k}\right)$

at point $P(x_1,y_1)$

$\Rightarrow\frac{dy}{dx}=\frac{b^2}{a^2}\left(\frac{x_1-h}{y_1-k}\right)$ is the slope of the tangent line. So, equation of the tangent line is

$$y-y_1=\frac{b^2}{a^2}\left(\frac{x_1-h}{y_1-k}\right)(x-x_1)$$

$\Rightarrow a^2(y_1-k)(y-y_1)=b^2(x_1-h)(x-x_1)$

$\Rightarrow b^2(x_1-h)x - a^2(y_1-k)y - b^2(x_1-h)x_1 + a^2(y_1-k)y_1 = 0$ ....(2)

Since the point $P(x_1, y_1)$ lies on the hyperbola so from Eq (1)

$b^2(x_1-h)^2 - a^2(y_1-k)^2 = a^2b^2$

$\Rightarrow b^2(x_1-h)(x_1-h) - a^2(y_1-k)(y_1-k) = a^2b^2$

$\Rightarrow b^2x_1(x_1-h) - b^2h(x_1-h) - a^2y_1(y_1-k) + a^2k(y_1-k) = a^2b^2$

$\Rightarrow -b^2x_1(x_1-h) - a^2y_1(y_1-k) = a^2k(y_1-k) - b^2h(x_1-h) - a^2b^2$

put in Eq (2)

$b^2(x_1-h)x - a^2(y_1-k)y + a^2k(y_1-k) - b^2h(x_1-h) - a^2b^2 = 0$

is equation of tangent line at point $P(x_1, y_1)$.

In particular if centre of the hyperbola is at (0,0) then $h = 0, k = 0$ so equation of tangent line reduces to

$b^2(x_1-0)x - a^2(y_1-0)y + a^2(0)(y_1-0) - b^2(0)(x_1-0) - a^2b^2 = 0$

$\Rightarrow b^2x_1x - a^2y_1y - a^2b^2 = 0$

or

$b^2x_1x - a^2y_1y = a^2b^2$ dividing both sides by a^2b^2

$\frac{x_1x}{a^2} - \frac{yy_1}{b^2} = 1$

Now we find the equation of the normal line.

Since normal line is perpendicular to tangent line, thus

$$\textit{slope of normal line} = -\frac{1}{\textit{slope of tangent line}}$$

$$= -\frac{1}{\frac{b^2}{a^2}\left(\frac{x_1-h}{y_1-k}\right)} = -\frac{a^2(y_1-k)}{b^2(x_1-h)}$$

the equation of the normal line at point P is

$y - y_1 = -\frac{a^2(y_1-k)}{b^2(x_1-h)}(x - x_1)$

$\Rightarrow b^2(x_1-h)(y-y_1) = -a^2(y_1-k)(x-x_1)$

or

$a^2(y_1-k)(x-x_1) + b^2(x_1-h)(y-y_1) = 0$

is equation of normal line.

In particular if centre of the hyperbola is at (0,0) then $h = 0, k = 0$ so equation of normal line in this case is

$a^2(y_1-0)(x-x_1) + b^2(x_1-0)(y-y_1) = 0$

$\Rightarrow a^2y_1(x-x_1) + b^2x_1(y-y_1) = 0$

$\Rightarrow a^2y_1x - a^2x_1y_1 + b^2x_1y - b^2x_1y_1 = 0$

$\Rightarrow a^2y_1x + b^2x_1y - (a^2+b^2)x_1y_1 = 0$

Example 31: Find the equation of tangent and normal to the hyperbola $4x^2 - 9y^2 - 16x - 18y - 29 = 0$ at the point $(5, -1)$.

Solution: Equation of the hyperbola is $4x^2 - 9y^2 - 16x - 18y - 29 = 0$

Differentiating w.r.t. 'x'

$$8x - 18y\frac{dy}{dx} - 16 - 18\frac{dy}{dx} = 0$$

$$\Rightarrow -18(y+1)\frac{dy}{dx} + 8(x-2) = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{8(x-2)}{18(y+1)} = \frac{4(x-2)}{9(y+1)}$$

at point $(5, -1)$

$$\Rightarrow \frac{dy}{dx} = \frac{4(5-2)}{9(-1+1)} \quad = \infty \text{ undefined}$$

it means tangent line is vertical; so its equation is $x = constant$

$\Rightarrow x = 5$ or $x - 5 = 0$

$$Slope\ of\ normal\ line = -\frac{1}{Slope\ of\ tangent\ line} \quad = -\frac{1}{\infty} = 0$$

it means normal line is horizontal; so its equation is $y = constant$

$\Rightarrow y = -1$ or $y + 1 = 0$

Exercise 7.9

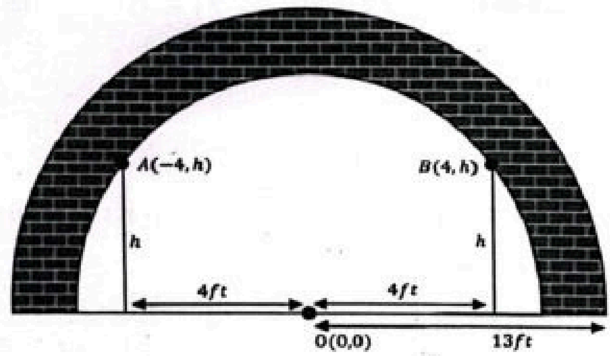
1. Find the length of chord intercepted by the line $2x + y - 1 = 0$ and the hyperbola $2x^2 - 3y^2 + 7x - 4y + 13 = 0$.
2. Find the value of m so that the line $3x + 4y + m = 0$ is tangent to the hyperbola $x^2 - y^2 - 7x - 2y + 13 = 0$.
3. Find the equation of tangent to the hyperbola $x^2 - 2y^2 + 4x - 6y + 11 = 0$ which is parallel to the line $4x - 8y + 7 = 0$.
4. Find the equations of tangent to the hyperbola $2x^2 - 4y^2 + 6x - 8y - 7 = 0$ which is perpendicular to the line $x + 2y + 5 = 0$. Also find the point of tangency.
5. Find the equations of the tangent and normal to the hyperbola $6x^2 - 6y^2 - 14x + 21y - 5 = 0$ at the point $(2, 3)$.
6. Find the equations of the tangents and normal to the hyperbola $\frac{\left(y-\frac{7}{4}\right)^2}{\frac{25}{48}} - \frac{\left(x-\frac{7}{6}\right)^2}{\frac{25}{72}} = 1$ at the point whose ordinate is 3 and abscissa is an integer.
7. Prove that the tangent at any point on the hyperbola makes equal angles with the lines joining the point with the foci of the hyperbola.
8. The line $12x - 5\sqrt{11} - 50 = 0$ is tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$. Find the point where the tangent line intersects the focal axis of hyperbola. Also find the ratio in which this point of intersection divides the distance between the foci.

7.15 Application of Conic Sections

Conic sections are counted as one of the prominent topics in Geometry and possess numerous applications in science and technology, including astronomy, optics, and even architecture. Here we are discussing some of them.

Example 32: A train track is $8ft$ wide and goes through a semi-circular tunnel of radius $13ft$. How high is the tunnel at edge of the track?

Solution:



Let the centre of the tunnel be at origin.

Given that radius of tunnel is $13ft$. So equation of the circle is $(x-0)^2+(y-0)^2=13^2$

$\Rightarrow x^2+y^2=169$ ....(1)

The track is $8ft$ wide. Let $h\ ft$ be the height of the tunnel at the edges of the track so coordinates of the A are $(-4,h)$ and B are $(4,h)$.

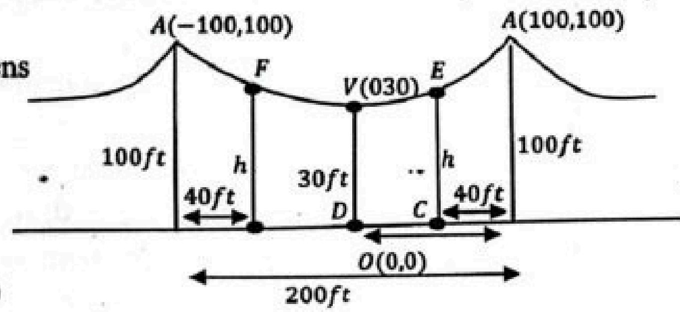
Since the point $B(4,0)$ (also the point $A(-4,0)$) lies on the circle, so from Eq (1)

$4^2+h^2=169$

$\Rightarrow h^2=169-16 \quad \Rightarrow h^2=153 \quad$ or $\quad h=\sqrt{153}$

Example 33: The cable of a suspension bridge hangs in a shape of parabola. The tower supporting the cable are $200ft$ apart and $100ft$ in height. If the lowest point of the cable is $20ft$ above the bridge at its mid point. How high is the cable from either tower at a distance of $40ft$ horizontally?

Solution:



The vertex of the parabola is $(0,30)$ and it opens upwards, so equation of parabola is:

$$(x-h)^2=4a(y-k)$$

or

$$(x-0)^2=4a(y-30)$$

$\Rightarrow x^2=4a(y-30)$ ... (1)

The coordinates of the point A are $(-100,100)$ and B are $(100,100)$. As the point $B(100,100)$ (also, the point $A(-100,100)$) lies on parabola

So, from Eq (1)

$(100)^2=4a(100-30)$

$\Rightarrow 10000=280a \Rightarrow a=\frac{10000}{280}=\frac{1000}{28}$

Put in Eq (1).

$\Rightarrow x^2=4\left(\frac{1000}{28}\right)(y-30) \quad \Rightarrow x^2=\frac{1000}{7}(y-30)$ ... (2)

The point which is at a distance of $40ft$ from tower is at a distance of $100-40=60ft$ from origin. Let h be the height of the cable at this point C.
The coordinates of E are $(60, h)$ and the point lies on the parabola, so from Eq (2):

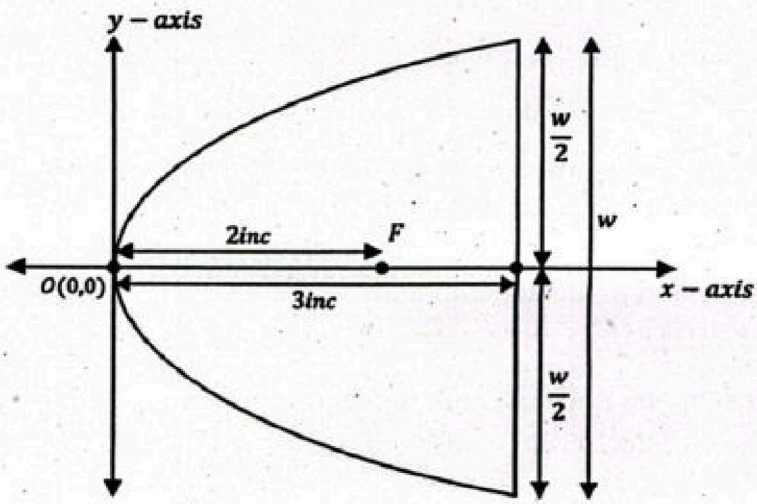
$$(60)^2 = \frac{1000}{7}(h-30)$$

$$\Rightarrow (h-30) = 3600 \times \frac{1000}{7} = 25.2 \qquad \Rightarrow h = 30 + 25.2 = 55.2ft$$

7.16 Use of Conic Section

Example 34: The head light of an automobile is in the shape of parabola. The bulb is placed at the focus which is 2 *inch* from its vertex. The depth of the head light is 3 *inches*. What is the width of the head light at its opening?

Solution:



Consider the vertex of the parabola is at origin and it opens on the right side. Given that focus is at 2*inches* from the vertex; so

$(a, 0) = (2,0) \quad \Rightarrow a = 2$

As we know equation of this parabola is $y^2 = 4ax$ so in this case

$y^2 = 8x \qquad \ldots (1)$

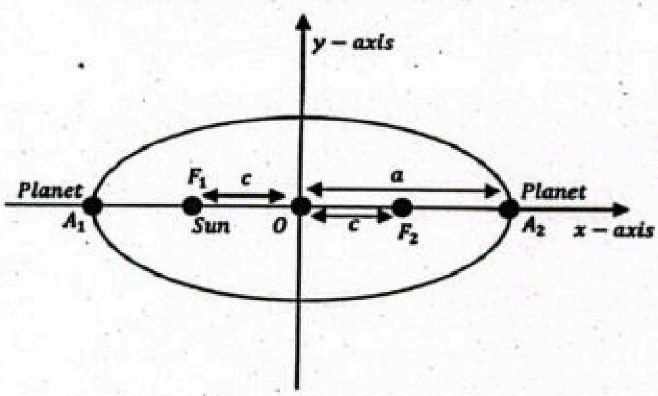
Let 'w' be the width of the parabola, so coordinates of the points A and B are $A(3, -\frac{w}{2})$ and $B(3, \frac{w}{2})$.

Since the point $B(3, \frac{w}{2})$ (also point $A(3, -\frac{w}{2})$) lies on the parabola, so from Eq (1)

$\frac{w}{2} = 8(3) \quad \Rightarrow w = 48 \text{ inches}$

Example 35: According to Kepler's law planets have elliptical orbits with Sun at one of the foci. If the longest distance of the planet from the sun is 4.4 billion *km* then write the equation of the orbit of the planet taking its centre at origin.

Solution:



Consider the ellipse with centre at origin and its major axis is along $x-axis$. Thus, its equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (1)$

Let the Sun be at its focus F_1. The planet will be at maximum distance from Sun when it is at A_2 and its distance from Sun is minimum when it is at A_1.

Given that $|F_1A_2| = 7.4 \Rightarrow a + c = 7.4$ ... (2)
and $|F_1A_1| = 4.4 \Rightarrow a - c = 4.4$ ... (3)
Adding Eq (2) and Eq (3)
$2a = 11.8 \quad \Rightarrow a = 5.9$
Subtract Eq (3) from Eq (2)
$2c = 3 \quad \Rightarrow c = 1.5$
since $b^2 = a^2 - c^2 = (5.9)^2 - (1.5)^2 \quad \Rightarrow b^2 = 32.56 \Rightarrow b = 5.7$
put the value in Eq (1)

$$\frac{x^2}{(5.9)^2} + \frac{y^2}{(5.7)^2} = 1$$

Which is the required equation of the orbit.

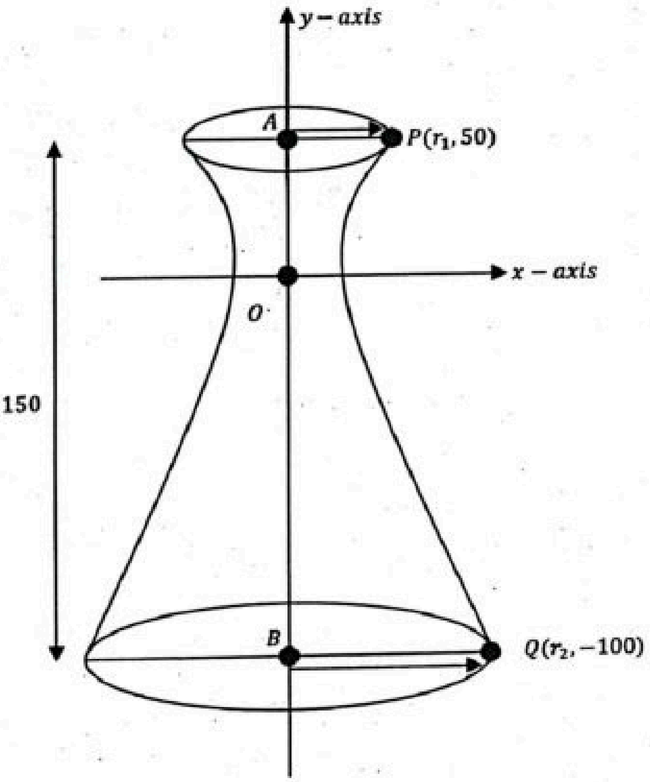
Example 36: Cross section of a nuclear cooling tower is in the shape of hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is $150m$ tall and the distance from the top of the tower to the centre of hyperbola is half the distance from the base of the tower to the centre of hyperbola. Find the diameter of the top and the base of the tower.

Solution:

Equation of the hyperbola is

$$\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1 \qquad ... (1)$$

Let r_1 and r_2 be the radii of the top and the base of the tower.



Given that

$$|OA| = \frac{1}{2}|OB| \quad \Rightarrow 2|OA| = |OB|$$

also $\Rightarrow |OA| + |OB| = |AB|$

$\Rightarrow |OA| + 2|OA| = 150 \Rightarrow |OA| = 50$

$$|OB| = 2|OA| = 2(50) = 100$$

$\Rightarrow |OB| = 100$

Thus, coordinates of the points P and Q are $(r_1, 50)$ and $(r_2, -100)$. Since the point $P(r_1, 50)$ lies on the hyperbola so from Eq (1)

$$\frac{{r_1}^2}{30^2} - \frac{(50)^2}{44^2} = 1$$

$$\Rightarrow \frac{{r_1}^2}{900} - \frac{(50)^2}{44^2} = 1 \Rightarrow \frac{{r_1}^2}{900} = 1 + \left(\frac{25}{22}\right)^2$$

$$\Rightarrow \frac{{r_1}^2}{900} = \frac{22^2 + 25^2}{22^2}$$

Taking square root

$$\Rightarrow \frac{r_1}{30} = \frac{\sqrt{22^2 + 25^2}}{22} = \frac{33.30}{22} \quad \Rightarrow r_1 = 30\left(\frac{33.30}{22}\right) = 45.41$$

The diameter of top of tower $= 2r_1 = 2(45.41)$
$= 90.82m$

Also, the point $Q(r_2, -100)$ lies on the hyperbola,
so, from Eq (1)

$$\frac{r_2^2}{30^2} - \frac{(-100)^2}{44^2} = 1 \quad \Rightarrow \frac{r_2^2}{900} - \frac{(-100)^2}{44^2} = 1 \quad \Rightarrow \frac{r_2^2}{900} = 1 + \left(\frac{25}{11}\right)^2$$

$$\Rightarrow \frac{r_2^2}{900} = \frac{11^2 + 25^2}{11^2}$$

Taking square root

$$\Rightarrow \frac{r_2}{30} = \frac{\sqrt{11^2 + 25^2}}{11} = \frac{27.31}{11} \quad \Rightarrow r_2 = 30\left(\frac{27.31}{11}\right) = 74.49$$

The diameter of base of tower $= 2r_2 = 2(74.49) = 148.98m$

Exercise 7.10

1. A cell phone company has installed three signal towers A, B and C at different locations to provide service to their customers. Tower A is located at the position $(-2, 10)$ and covers an area up to $8km$, tower B is located at $(-3, 5)$ and covers an area up to $6km$ and tower C is located at $(10, -1)$ and covers an area up to $5km$. A man is standing at a position of $(12, 3)$, which tower will provide service to the man?
2. The tyre of a car is of radius 15 inches and it revolves 1000 times in a minute. What is the speed of the car in $\frac{km}{h}$?
3. A batsman hits the ball. The ball attains the maximum height of $25m$ and drops on the ground at a distance of $80m$ from the batsman. Assuming the origin at the position of batsman write the equation of path of the ball. Is it possible for a man of height $1.6m$ to catch the ball, standing $70m$ away from the batsman?
4. Playing team coaches use parabolic shaped antennas to listen the conversation between the players in the ground. If the antenna has the cross section of 18 inches and depth of 4 inches then where the microphone should be placed to hear the conversation?
5. Man A is standing at one focus of the whispering gallery which is 16 feet from the nearest wall. The man B is standing at the other focus $100ft$ away. What is the length of the whispering gallery? How high is the elliptical ceiling at the centre?
6. The orbit of a planet is in elliptical shape with Sun setting at one of its two foci. The eccentricity of the ellipse is 0.085 and the minimum distance of the planet from the Sun is 105 million miles. What is the maximum distance of the planet from Sun?

7. The control towers are located at points Q(−500, 0) and R(500, 0) on a straight shore where the x-axis runs through. At same moment both towers send a radio signal to a ship out at sea, each travelling at $300m/\mu s$. The ship received the signal from Q, $3\mu s$ earlier the message from R. Find the equation of hyperbola containing the possible location of the ship.
8. An architect's design for a building includes some large pillars with cross sections in the shape of hyperbolas. The curve can be modelled by the equation $\frac{x^2}{16}-\frac{y^2}{9}=1$ where the units are in meters. If the pillars are $4.2m$ tall. Find the width of the top of each pillar. Also find the width at the middle of the pillar.

Review Exercise

1. Choose the correct option.

(i) The eccentricity is the ratio of distance of a point on the conic section from:
(a) Focus to directrix (b) Directrix to focus (c) Vertex to directrix (d) Directrix to vertex

(ii) Eccentricity of circle is:
(a) $e>1$ (b) $e<1$ (c) $e=1$ (d) $e=0$

(iii) The focus of the parabola $x^2=-16y$ is:
(a) $(4,0)$ (b) $(-4,0)$ (c) $(0,4)$ (d) $(0,-4)$

(iv) Length of the latus rectum of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is:
(a) $\frac{32}{9}$ (b) $\frac{9}{32}$ (c) $\frac{9}{2}$ (d) $\frac{8}{9}$

(v) Equation of the directrices of ellipse $\frac{x^2}{16}+\frac{y^2}{36}=1$ is:
(a) $y=\pm\frac{18}{\sqrt{5}}$ (b) $y=\pm\frac{\sqrt{5}}{18}$ (c) $x=\frac{18}{\sqrt{5}}$ (d) $x=\frac{\sqrt{5}}{18}$

(vi) Eccentricity of the hyperbola $\frac{x^2}{25}-\frac{y^2}{81}=1$ is:
(a) $\frac{5}{\sqrt{106}}$ (b) $\frac{\sqrt{106}}{5}$ (c) $\frac{\sqrt{106}}{9}$ (d) $\frac{9}{\sqrt{106}}$

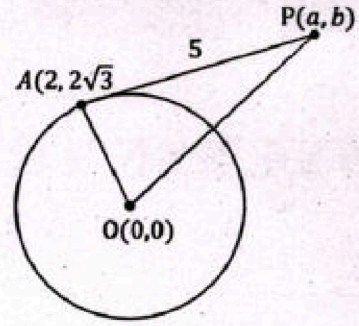
(vii) Equation of conjugate axis of the hyperbola $\frac{(x-1)^2}{4}-\frac{(y+3)^2}{12}=1$ is:
(a) $x=1$ (b) $x=-1$ (c) $y=3$ (d) $y=-3$

(viii) Length of the tangent drawn from the point (1, 2) to the circle $2x^2+2y^2+3x+2y-6=0$ is
(a) 11 (b) $\sqrt{11}$ (c) $\frac{11}{2}$ (d) $\sqrt{\frac{11}{2}}$

(ix) The chord joining the two points $(at_1{}^2, 2at_1)$ and $(at_2{}^2, 2at_2)$ on the parabola $y^2=4ax$ is focal chord if:
(a) $t_1+t_2=1$ (b) $t_1+t_2=-1$ (c) $t_1t_2=1$ (d) $t_1t_2=-1$

(x) Exactly one tangent can be drawn to a circle if point lies:
(a) outside the circle (b) on the circle (c) inside circle (d) centre of circle

2. A point P(a, b) lies outside a circle with centre at origin. From P a tangent is drawn to the circle at point A$(2, 2\sqrt{3})$. If the area of the triangle POA is 10 sq units then find equation of the circle and the coordinates of the point P.



3. On the Siri Nagar highway overhead steel bridges are constructed to cross the road. They are supported by the parabolic arc over them. The highway is $300ft$ wide and walkway is $15ft$ above the highway. The centre of the parabolic arc is $10ft$ high from the walkway. Find the equation of the parabolic arc take origin at the mid of the road.

4. If r_1 and r_2 are the minimum and maximum distance from a focus of an ellipse then prove that semi-major axis is the arithmetic mean between r_1 and r_2 and semi-minor axis is the geometric mean between r_1 and r_2.

5. A hyperbolic mirror has the property that the light directed at the focus will be reflected to the other focus. If one of the foci has coordinates $(24, 0)$ and the top mount point of mirror has coordinates $(24, 24)$. Find the vertex of the mirror.