UNIT
06

ANALYTICAL GEOMETRY

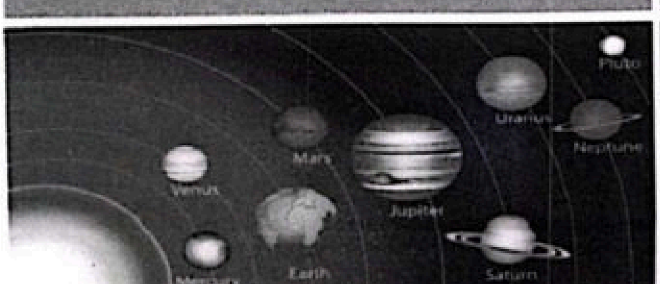
After studying this unit, students will be able to:

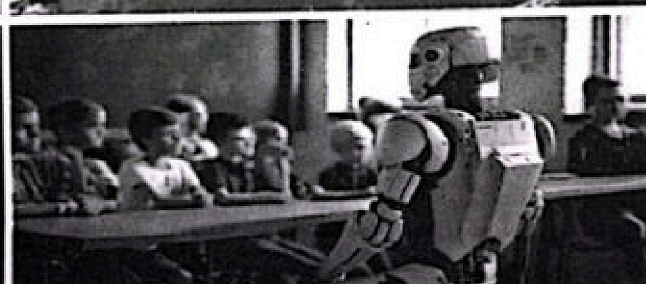
- Find the condition of concurrency of three straight lines.
- Find the equations of altitudes, right bisectors and medians of a triangle.
- Show that altitudes, right bisectors and medians of a triangle are concurrent.
- Find the area of triangular region whose vertices are given.
- Recognize homogeneous linear and quadratic equations in two variables.
- Investigate that second-degree homogeneous equation in two variables represents a pair of straight lines through the origin and find acute angle between them.
- Apply concepts of analytical geometry to real life world problems such as aviation, to track stars, distance between planets and satellites, space science and engineering.

Analytic geometry known as coordinate geometry is a branch of mathematics that combines algebra and geometry. It involves the study of the geometric shapes using the coordinates and equations.

Analytic geometry is used in physics and engineering, as well as in fields like aviation, rocketry, space science, spaceflight, computer graphics, astronomy, cartography and robotics etc.

It serves as the cornerstone for many contemporary geometric disciplines, including algebraic, differential, discrete, and computational geometry.





6.1 Point of Intersection of two Straight Lines

We know that two non-parallel lines intersect each other at one and only one point.

Let $l_1: a_1x + b_1y + c_1 = 0$ .... (1)

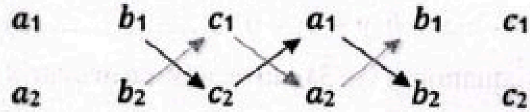
and $l_2: a_2x + b_2y + c_2 = 0$ .... (2)

be two non-parallel lines and $P(x_1, y_1)$ be the point of intersection of l_1 and l_2. Then:

$a_1x_1 + b_1y_1 + c_1 = 0$ .... (3)

$a_2x_1 + b_2y_1 + c_2 = 0$ .... (4)

For the solution of (3) and (4), we proceed as follow:



$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad x_1 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y_1 = \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}$$

$\therefore$ $P(x_1, y_1) = \left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right)$ is the required point of intersection where $a_1b_2 - a_2b_1 \neq 0$ otherwise $l_1 \parallel l_2$.

Key Facts

- If $l_1 \parallel l_2$, then $a_1b_2 - a_2b_1 = 0 \Rightarrow a_1b_2 = a_2b_1 \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$
- If $l_1 \perp l_2$, then $m_1m_2 = -1 \Rightarrow \left(-\frac{a_1}{a_2}\right)\left(-\frac{b_1}{b_2}\right) = -1 \Rightarrow a_1a_2 = -b_1b_2$

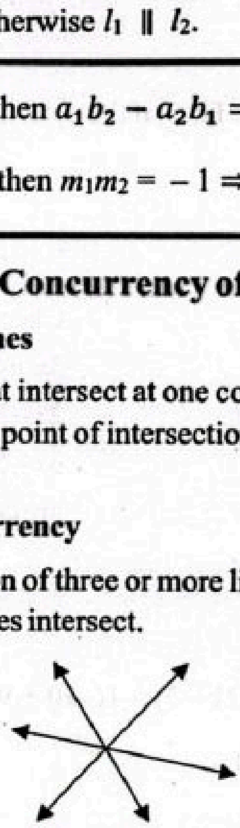
6.2 Condition for Concurrency of Three Lines

6.2.1 Concurrent Lines

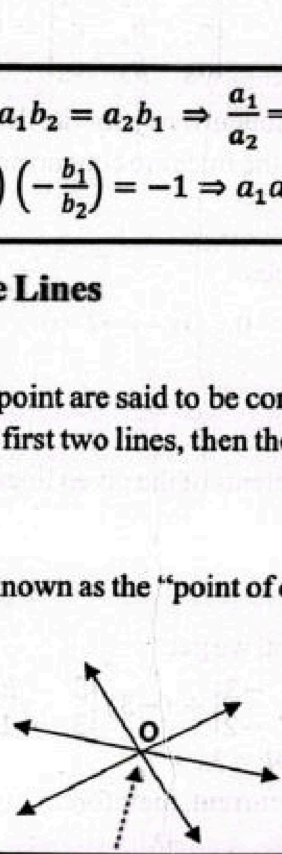
Three or more lines that intersect at one common point are said to be concurrent lines. When a third line passes through the point of intersection of the first two lines, then these three lines are known as the concurrent lines.

6.2.2 Point of Concurrency

The point of intersection of three or more lines is known as the "point of concurrency." It is the point where three or more lines intersect.



Concurrent Lines

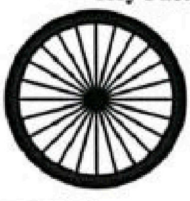


Point of Concurrency

Three lines are concurrent if the point of intersection of two lines, lies on the third line (i.e., satisfies the equation of the third line)

Key Facts

The diameters of a circle are concurrent at the center of the circle.



To check the concurrency of three lines, we use the following methods.

(a) Determinant Method

Consider three straight lines whose equations are:

$$a_1x + b_1y + c_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$a_2x + b_2y + c_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$
$$a_3x + b_3y + c_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

The system of homogenous equations (1)-(3) can be written in matrix form as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ldots\ldots\ldots\ldots (4)$$

If the lines (1) to (3) are concurrent then they must intersect at a point $O(x, y)$ which can be found by solving equations (1) to (3) simultaneously. The system (4) has a non-trivial solution if the determinant of coefficients of the three lines is zero. i.e.,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Which is the condition of concurrency for the three lines. Thus, if the determinant of the coefficients of the given lines is 0, then the lines are concurrent.

Example 1:

Find the value of k if the lines:

$3x+y-3=0, \quad 5x+ky-3=0, \quad 3x-y-2=0$

are concurrent.

Solution:

The determinant of coefficients of the given lines is:

$$D = \begin{vmatrix} 3 & 1 & -3 \\ 5 & k & -3 \\ 3 & -1 & -2 \end{vmatrix}$$

By solving the determinant, we get:

$$D = 3\begin{vmatrix} k & -3 \\ -1 & -2 \end{vmatrix} - 1\begin{vmatrix} 5 & -3 \\ 3 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 5 & k \\ 3 & -1 \end{vmatrix} = 3(-2k-3) - 1(-10+9) - 3(-5-3k)$$
$$= -6k - 9 + 1 + 15 + 9k = 3k + 7$$

As, the three lines are concurrent, therefore:

$$3k + 7 = 0 \Rightarrow 3k = -7 \Rightarrow k = \frac{-7}{3}$$

Check Point

Check whether the lines:

$3x+4y-7=0,$

$2x-3y+5=0,$

$3x-5y+8=0$

are concurrent or not.

(b) Direct Method

In this method, we first find the point of intersection of two lines and then check if the point lies on the third line. It ensures that all three lines are concurrent.

> **Check Point**
>
> What are collinear points?
>
> Are concurrent lines coplanar?

Consider equations of three lines as follows:

$$4x-2y-4=0 \quad \text{.........} \ (1)$$
$$y=x+2 \quad \text{.........} \ (2)$$
$$2x+3y=26 \quad \text{.........} \ (3)$$

Step 1: To find the point of intersection of line (1) and line (2), substituting the value of 'y' from equation (2) in equation (1) we get:

$$4x-2(x+2)-4=0$$
$$4x-2x-4-4=0$$
$$2x-8=0 \Rightarrow 2x=8 \Rightarrow x=4$$

Substituting the value of x in equation (2), we get:

$$y=4+2=6$$

Therefore, line (1) and line (2) intersect at the point (4, 6).

Step 2: Substituting the point of intersection of the first two lines in the equation of the third line, we get:

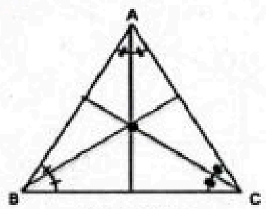
$$2(4)+3(6)=26 \Rightarrow 8+18=26 \Rightarrow 26=26$$

This implies that the line (3) also passes through the point (4, 6). Hence the three lines are concurrent.

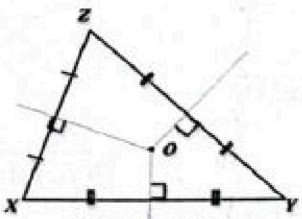
6.3 Concurrent Lines in Triangles

In a triangle, the three angle bisectors, perpendicular bisectors, medians and altitudes are examples of concurrent lines.

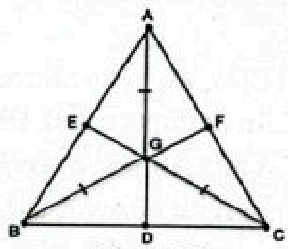
- **Incenter:** The point of intersection of three angle bisectors of a triangle is known as the incenter of a triangle.
- **Circumcenter:** The point of intersection of three perpendicular bisectors of a triangle is known as the circumcenter of a triangle.
- **Centroid:** The point of intersection of three medians of a triangle is known as the centroid of a triangle.
- **Orthocenter:** The point of intersection of three altitudes of a triangle is known as the orthocenter of a triangle.



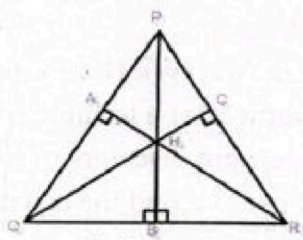
Incenter



Circumcenter



Centroid

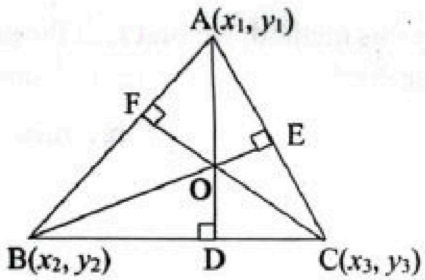


Orthocenter

Theorem 6.1:

Altitudes of a triangle are concurrent.

Proof:



Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the three vertices of a triangle ABC. In the figure, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are altitudes of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

First, we find the equation of altitude $\overline{AD}$.

Slope of $\overline{BC} = \dfrac{y_3 - y_2}{x_3 - x_2}$

Slope of altitude of $\overline{AD} = -\dfrac{x_3 - x_2}{y_3 - y_2}$

Equation of altitude $\overline{AD}$ is:

$$y - y_1 = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1) \quad \text{....... (Point-slope form)}$$

$$\Rightarrow (y - y_1)(y_3 - y_2) = -(x_3 - x_2)(x - x_1)$$

$$\Rightarrow (x_3 - x_2)x + (y_3 - y_2)y - x_1(x_3 - x_2) - y_1(y_3 - y_2) = 0 \quad \text{....... (1)}$$

By symmetry, equations of altitudes $\overline{BE}$ and $\overline{CF}$ are respectively as follows.

$$(x_1 - x_3)x + (y_1 - y_3)y - x_2(x_1 - x_3) - y_2(y_1 - y_3) = 0 \quad \text{....... (2)}$$

$$(x_2 - x_1)x + (y_2 - y_1)y - x_3(x_2 - x_1) - y_3(y_2 - y_1) = 0 \quad \text{....... (3)}$$

The determinant of coefficients of the three lines is:

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -x_1(x_3 - x_2) - y_1(y_3 - y_2) \\ x_1 - x_3 & y_1 - y_3 & -x_2(x_1 - x_3) - y_2(y_1 - y_3) \\ x_2 - x_1 & y_2 - y_1 & -x_3(x_2 - x_1) - y_3(y_2 - y_1) \end{vmatrix}$$

Adding R_2 and R_3 in R_1, we get:

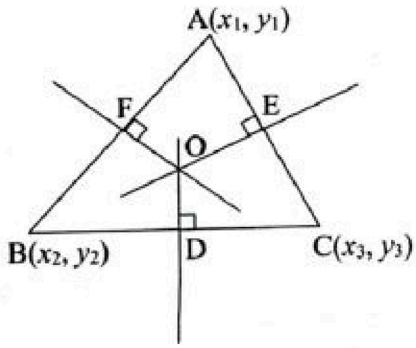
$$\begin{vmatrix} 0 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & -x_2(x_1 - x_3) - y_2(y_1 - y_3) \\ x_2 - x_1 & y_2 - y_1 & -x_3(x_2 - x_1) - y_3(y_2 - y_1) \end{vmatrix} = 0$$

Thus, the altitudes of a triangle are concurrent.

Theorem 6.2:

Right bisectors of a triangle are concurrent.

Proof:



Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the three vertices of a triangle ABC. In the figure, $\overline{OD}$, $\overline{OE}$ and $\overline{OF}$ are right bisectors of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

First, we find the equation of right bisector $\overline{OD}$.

Coordinates of mid point D of $\overline{BC}$ are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

Slope of $\overline{BC} = \frac{y_3 - y_2}{x_3 - x_2}$

Slope of right bisector $\overline{OD} = -\frac{x_3 - x_2}{y_3 - y_2}$

Equation of right bisector $\overline{OD}$ is:

$$y - \frac{y_2 + y_3}{2} = -\frac{x_3 - x_2}{y_3 - y_2}\left(x - \frac{x_2 + x_3}{2}\right) \quad \ldots\ldots \text{(Point-slope form)}$$

$$\Rightarrow \left(y - \frac{y_2 + y_3}{2}\right)(y_3 - y_2) = -(x_3 - x_2)\left(x - \frac{x_2 + x_3}{2}\right)$$

After simplification, we get:

$$(x_3 - x_2)x + (y_3 - y_2)y - \tfrac{1}{2}(x_3^2 - x_2^2) - \tfrac{1}{2}(y_3^2 - y_2^2) = 0 \quad \ldots\ldots (1)$$

By symmetry, equations of right bisectors $\overline{OE}$ and $\overline{OF}$ are respectively as follows.

$$(x_1 - x_3)x + (y_1 - y_3)y - \tfrac{1}{2}(x_1^2 - x_3^2) - \tfrac{1}{2}(y_1^2 - y_3^2) = 0 \quad \ldots\ldots (2)$$

$$(x_2 - x_1)x + (y_2 - y_1)y - \tfrac{1}{2}(x_2^2 - x_1^2) - \tfrac{1}{2}(y_2^2 - y_1^2) = 0 \quad \ldots\ldots (3)$$

The determinant of coefficients of the three lines is:

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -\frac{1}{2}(x_3^2 - x_2^2) - \frac{1}{2}(y_3^2 - y_2^2) \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) \end{vmatrix}$$

Adding R_2 and R_3 in R_1, we get:

$$\begin{vmatrix} 0 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & -\frac{1}{2}(x_1^2 - x_3^2) - \frac{1}{2}(y_1^2 - y_3^2) \\ x_2 - x_1 & y_2 - y_1 & -\frac{1}{2}(x_2^2 - x_1^2) - \frac{1}{2}(y_2^2 - y_1^2) \end{vmatrix} = 0$$

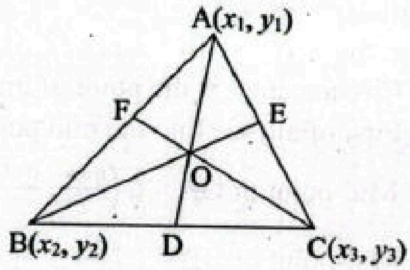
Thus, the right bisectors of a triangle are concurrent.

Theorem 6.3:

Medians of a triangle are concurrent.

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the three vertices of a triangle ABC. In the figure, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are medians of the triangle.



First, we find the equation of median $\overline{AD}$.

Coordinates of mid point D of $\overline{BC}$ are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

Using two-point formula, equation of median $\overline{AD}$ is:

$$\frac{y-y_1}{\frac{y_2+y_3}{2}-y_1} = \frac{x-x_1}{\frac{x_2+x_3}{2}-x_1}$$

$$\Rightarrow \left(\frac{x_2+x_3}{2}-x_1\right)(y-y_1) = \left(\frac{y_2+y_3}{2}-y_1\right)(x-x_1)$$

$$\Rightarrow \left(\frac{x_2+x_3}{2}-x_1\right)y - \left(\frac{x_2+x_3}{2}-x_1\right)y_1 = \left(\frac{y_2+y_3}{2}-y_1\right)x - \left(\frac{y_2+y_3}{2}-y_1\right)x_1$$

$$\Rightarrow \left(\frac{y_2+y_3}{2}-y_1\right)x - \left(\frac{x_2+x_3}{2}-x_1\right)y - \left(\frac{y_2+y_3}{2}-y_1\right)x_1 + \left(\frac{x_2+x_3}{2}-x_1\right)y_1 = 0 \qquad (1)$$

By symmetry, equations of medians $\overline{BE}$ and $\overline{CF}$ are respectively as follows.

$$\left(\frac{y_1+y_3}{2}-y_2\right)x - \left(\frac{x_1+x_3}{2}-x_2\right)y - \left(\frac{y_1+y_3}{2}-y_2\right)x_2 + \left(\frac{x_1+x_3}{2}-x_2\right)y_2 = 0 \qquad (2)$$

$$\left(\frac{y_1+y_2}{2}-y_3\right)x - \left(\frac{x_1+x_2}{2}-x_3\right)y - \left(\frac{y_1+y_2}{2}-y_3\right)x_3 + \left(\frac{x_1+x_2}{2}-x_3\right)y_3 = 0 \qquad (3)$$

The determinant of coefficients of the three lines is:

$$\begin{vmatrix} \left(\frac{y_2+y_3}{2}-y_1\right) & -\left(\frac{x_2+x_3}{2}-x_1\right) & -\left(\frac{y_2+y_3}{2}-y_1\right)x_1 + \left(\frac{x_2+x_3}{2}-x_1\right)y_1 \\ \left(\frac{y_1+y_3}{2}-y_2\right) & -\left(\frac{x_1+x_3}{2}-x_2\right) & -\left(\frac{y_1+y_3}{2}-y_2\right)x_2 + \left(\frac{x_1+x_3}{2}-x_2\right)y_2 \\ \left(\frac{y_1+y_2}{2}-y_3\right) & -\left(\frac{x_1+x_2}{2}-x_3\right) & -\left(\frac{y_1+y_2}{2}-y_3\right)x_3 + \left(\frac{x_1+x_2}{2}-x_3\right)y_3 \end{vmatrix}$$

Adding R_2 and R_3 in R_1, we get:

$$\begin{vmatrix} 0 & 0 & 0 \\ \left(\frac{y_1+y_3}{2}-y_2\right) & -\left(\frac{x_1+x_3}{2}-x_2\right) & -\left(\frac{y_1+y_3}{2}-y_2\right)x_2 + \left(\frac{x_1+x_3}{2}-x_2\right)y_2 \\ \left(\frac{y_1+y_2}{2}-y_3\right) & -\left(\frac{x_1+x_2}{2}-x_3\right) & -\left(\frac{y_1+y_2}{2}-y_3\right)x_3 + \left(\frac{x_1+x_2}{2}-x_3\right)y_3 \end{vmatrix}$$

$= 0$

Thus, the medians of a triangle are concurrent.

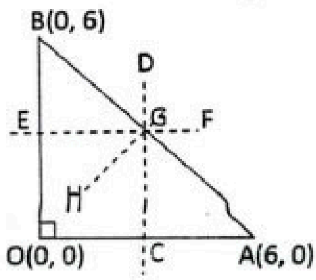
Example 2:

Find (i) circumcenter (ii) centroid and (iii) orthocenter of a triangle ABC with A(0, 0), B(6, 0) and C(0, 6). Also prove that circumcenter, centroid and orthocenter are collinear.

Solution:

Given that A(0, 0), B(6, 0) and C(0, 6) are vertices of triangle ABC.

(i) Circumcenter is the point of intersection of three perpendicular bisectors of triangle. First of all, we find the mid points of three sides.



Mid-point of OA $= C\left(\frac{0+6}{2}, \frac{0+0}{2}\right) = C(3, 0)$

Mid-point of OB $= E\left(\frac{0+0}{2}, \frac{0+6}{2}\right) = E(0, 3)$

Mid-point of AB $= G\left(\frac{6+0}{2}, \frac{0+6}{2}\right) = G(3, 3)$

Slope of OA $= \frac{0-0}{6-0} = 0$, Slope of OB $= \frac{6-0}{0-0} = \infty$ (undefined)

Slope of AB $= \frac{6-0}{0-6} = -1$

Now, we find slopes of right bisectors.

Slope of right bisector CD $= \frac{-1}{\text{Slope of OA}} = \frac{-1}{0} = \infty$

Slope of right bisector EF $= \frac{-1}{\text{Slope of OB}} = \frac{-1}{\infty} = 0$

Slope of right bisector GH $= \frac{-1}{\text{Slope of AB}} = \frac{-1}{-1} = 1$

Equation of right bisector CD is:

$y - 0 = \infty\,(x-3) \quad \Rightarrow \quad x - 3 = 0$ .... (1)

Equation of right bisector EF is:

$y - 3 = 0\,(x-0) \quad \Rightarrow \quad y - 3 = 0$ .... (2)

Equation of right bisector GH is:

$y - 3 = 1\,(x-3) \quad \Rightarrow \quad y - x = 0$ .... (3)

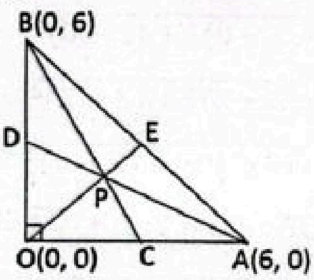
Equations (1), (2) and (3) are equations of right bisectors of sides of triangle ABC.

Solving (1) and (2), we see that: $x = 3, y = 3$

$\therefore$ Circumcenter = G(3, 3) .... (A)

(ii) Centroid is the point of intersection of three medians of a triangle.

First of all, we find the mid points of three sides.



Mid-point of OA = C(3, 0)

Mid-point of OB = D(0, 3)

Mid-point of AB = E(3, 3)

Equation of median OE is:

$\frac{y-0}{3-0} = \frac{x-0}{3-0}$ (two-point formula)

$\Rightarrow y = x$ .... (4)

Equation of median AD is:

$\frac{y-0}{3-0} = \frac{x-6}{0-6}$ (two-point formula)

$\Rightarrow -6y = 3x - 18 \Rightarrow 3x + 6y - 18 = 0$

$\Rightarrow x + 2y - 6 = 0$ .... (5)

> **Key Facts**
> Medians of a triangle trisect each other (i.e., bisect each other in the ratio 2 : 1)

Equation of median BC is:

$\frac{y-6}{0-6} = \frac{x-0}{3-0}$ (two-point formula)

$\Rightarrow 3y - 18 = -6x \Rightarrow 6x + 3y - 18 = 0$

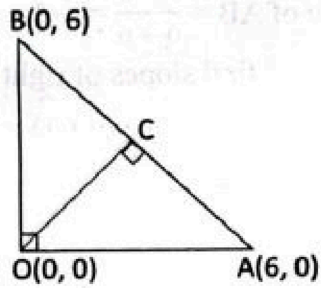
$\Rightarrow 2x + y - 6 = 0$ .... (6)

Solving (4) and (5), we see that: $x = 2, y = 2$

$\therefore$ Centroid = P(2, 2) .... (B)

(iii) Orthocenter is the point of intersection of three altitudes of a triangle.

As the triangle is right angled, therefore two of its sides OA and OB are also altitudes. The third altitude is OC.



Slope of OA $= 0$, Slope of OB $= \infty$ (undefined)

Slope of AB $= \dfrac{6-0}{0-6} = -1$

Slope of altitude OC $= \dfrac{-1}{\text{Slope of AB}} = \dfrac{-1}{-1} = 1$

Equation of altitude OA is:

$y - 0 = 0\,(x - 0) \quad \Rightarrow \quad y = 0 \quad \ldots\ldots\ldots (7)$

Equation of altitude OB is:

$y - 0 = \infty\,(x - 0) \quad \Rightarrow \quad x = 0 \quad \ldots\ldots\ldots (8)$

Equation of altitude OC is:

$y - 0 = 1\,(x - 0) \quad \Rightarrow \quad y - x = 0 \quad \ldots\ldots\ldots (9)$

Key Facts

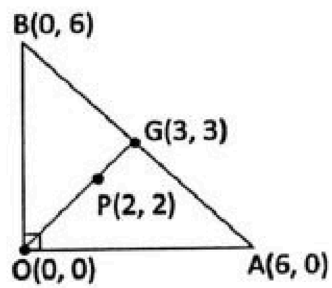
In a right triangle, the orthocentre is the vertex containing right angle.

Equations (1), (2) and (3) are equations of right bisectors of sides of triangle ABC.

Solving (7) and (8), we see that: $x = 0, y = 0$

$\therefore$ Orthocenter $= O(0, 0)$ ……… (C)

Now we prove that circumcenter G(3, 3), centroid P(2, 2) and orthocenter O(0, 0) are collinear.



$OP = \sqrt{(2-0)^2 + (2-0)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$PG = \sqrt{(3-2)^2 + (3-2)^2} = \sqrt{1+1} = \sqrt{2}$

$OG = \sqrt{(3-0)^2 + (3-0)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

Now, $OP + PG = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2} = OG$

Which shows that circumcenter, centroid and orthocenter are collinear in any triangle.

Example 3:

The points P(– 1, 2), Q(3, – 2) and R(6, 3) are vertices of a triangle PQR. Show that altitudes, right bisectors and medians of the triangle are concurrent.

Solution:

Let $(x_1, y_1) = (-1, 2)$, $(x_2, y_2) = (3, -2)$ and $(x_3, y_3) = (6, 3)$, then:

We know that the determinant of coefficients of the three altitudes is:

$$\begin{vmatrix} x_3 - x_2 & y_3 - y_2 & -x_1(x_3 - x_2) - y_1(y_3 - y_2) \\ x_1 - x_3 & y_1 - y_3 & -x_2(x_1 - x_3) - y_2(y_1 - y_3) \\ x_2 - x_1 & y_2 - y_1 & -x_3(x_2 - x_1) - y_3(y_2 - y_1) \end{vmatrix}$$

Substituting the values $x_1 = -1, y_1 = 2, x_2 = 3, y_2 = -2, x_3 = 6$ and $y_3 = 3$ in above determinant, we get:

$$\begin{vmatrix} 6-3 & 3+2 & 1(6-3)-2(3+2) \\ -1-6 & 2-3 & -3(-1-6)+2(2-3) \\ 3+1 & -2-2 & -6(3+1)-3(-2-2) \end{vmatrix} = \begin{vmatrix} 3 & 5 & -7 \\ -7 & -1 & 19 \\ 4 & -4 & -12 \end{vmatrix}$$

Adding R_3 in R_2, we get:

$$\begin{vmatrix} 3 & 5 & -7 \\ -3 & -5 & 7 \\ 4 & -4 & -12 \end{vmatrix} = \begin{vmatrix} 3 & 5 & -7 \\ 3 & 5 & -7 \\ 4 & -4 & -12 \end{vmatrix} \text{ (Multiplying } R_2 \text{ by } -1)$$

$$= 0 \text{ (} R_1 \text{ and } R_2 \text{ are identical.)}$$

Hence altitudes of triangle are concurrent.

Now, the determinant of coefficients of the three right bisectors is:

$$\begin{vmatrix} x_3-x_2 & y_3-y_2 & -\frac{1}{2}(x_3^2-x_2^2)-\frac{1}{2}(y_3^2-y_2^2) \\ x_1-x_3 & y_1-y_3 & -\frac{1}{2}(x_1^2-x_3^2)-\frac{1}{2}(y_1^2-y_3^2) \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}(x_2^2-x_1^2)-\frac{1}{2}(y_2^2-y_1^2) \end{vmatrix}$$

Substituting the values $x_1 = -1, y_1 = 2, x_2 = 3, y_2 = -2, x_3 = 6$ and $y_3 = 3$ in above determinant, we get:

$$\begin{vmatrix} 6-3 & 3+2 & -\frac{1}{2}(36-9)-\frac{1}{2}(9-4) \\ -1-6 & 2-3 & -\frac{1}{2}(1-36)-\frac{1}{2}(4-9) \\ 3+1 & -2-2 & -\frac{1}{2}(9-1)-\frac{1}{2}(4-4) \end{vmatrix} = \begin{vmatrix} 3 & 5 & -16 \\ -7 & -1 & 20 \\ 4 & -4 & -4 \end{vmatrix}$$

Adding R_3 in R_2, we get:

$$\begin{vmatrix} 3 & 5 & -16 \\ -3 & -5 & 16 \\ 4 & -4 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ -3 & -5 & 16 \\ 4 & -4 & -4 \end{vmatrix} \text{ (Adding } R_2 \text{ in } R_1)$$

$$= 0$$

Hence right bisectors of triangle are concurrent.

Again, the determinant of coefficients of the three medians is:

$$\begin{vmatrix} \left(\frac{y_2+y_3}{2}-y_1\right) & -\left(\frac{x_2+x_3}{2}-x_1\right) & -\left(\frac{y_2+y_3}{2}-y_1\right)x_1+\left(\frac{x_2+x_3}{2}-x_1\right)y_1 \\ \left(\frac{y_1+y_3}{2}-y_2\right) & -\left(\frac{x_1+x_3}{2}-x_2\right) & -\left(\frac{y_1+y_3}{2}-y_2\right)x_2+\left(\frac{x_1+x_3}{2}-x_2\right)y_2 \\ \left(\frac{y_1+y_2}{2}-y_3\right) & -\left(\frac{x_1+x_2}{2}-x_3\right) & -\left(\frac{y_1+y_2}{2}-y_3\right)x_3+\left(\frac{x_1+x_2}{2}-x_3\right)y_3 \end{vmatrix}$$

Substituting the values $x_1 = -1, y_1 = 2, x_2 = 3, y_2 = -2, x_3 = 6$ and $y_3 = 3$ in above determinant, we get:

$$\begin{vmatrix} \left(\frac{-2+3}{2}-2\right) & -\left(\frac{3+6}{2}+1\right) & -\left(\frac{-2+3}{2}-2\right)(-1)+\left(\frac{3+6}{2}+1\right)(2) \\ \left(\frac{2+3}{2}+2\right) & -\left(\frac{-1+6}{2}-3\right) & -\left(\frac{2+3}{2}+2\right)(3)+\left(\frac{-1+6}{2}-3\right)(-2) \\ \left(\frac{2-2}{2}-3\right) & -\left(\frac{-1+3}{2}-6\right) & -\left(\frac{2-2}{2}-3\right)(6)+\left(\frac{-1+3}{2}-6\right)(3) \end{vmatrix}$$

$$\begin{vmatrix} -1.5 & -5.5 & -1.5+11 \\ 4.5 & 0.5 & -13.5+1 \\ -3 & 5 & 18-15 \end{vmatrix} = \begin{vmatrix} -1.5 & -5.5 & 9.5 \\ 4.5 & 0.5 & -12.5 \\ -3 & 5 & 3 \end{vmatrix}$$

Adding R_3 in R_2, we get:

$$\begin{vmatrix} -1.5 & -5.5 & 9.5 \\ 1.5 & 5.5 & -9.5 \\ -3 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ -3 & -5 & 16 \\ 4 & -4 & -4 \end{vmatrix} \quad \text{(Adding } R_2 \text{ in } R_1\text{)}$$

$$= 0$$

Hence medians of triangle are concurrent.

6.4 Area of Triangular Region

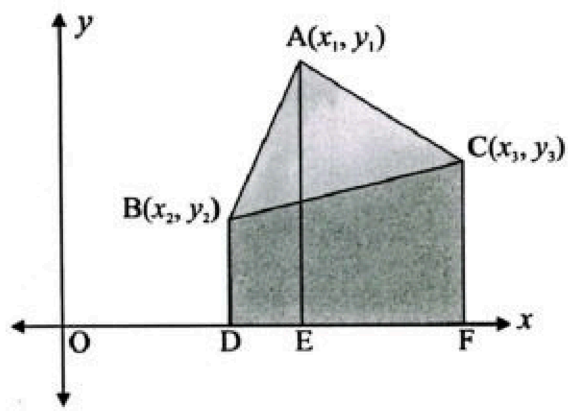
The area of a plane figure is the space covered by it.

Consider ΔABC as given in the adjoining figure with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

In the figure, we have drawn perpendiculars BD, AE and CF from the vertices of the triangle to the x-axis.

Notice that three trapeziums are formed: ABDE, AEFC and BCFD.

We can express the area of triangle ABC in terms of the areas of these three trapeziums as follows.



Area of ΔABC = Area of Trap.ABDE + Area of Trap.AEFC – Area of Trap.BCFD .... (1)

Now, Area of Trap.ABDE $= \frac{1}{2} \times (BD + AE) \times DE$

$= \frac{1}{2} \times (y_2 + y_1) \times (x_1 - x_2)$

Area of Trap.AEFC $= \frac{1}{2} \times (AE + CF) \times EF$

$= \frac{1}{2} \times (y_1 + y_3) \times (x_3 - x_1)$

Area of Trap.BCFD $= \frac{1}{2} \times (BD + CF) \times DF$

$= \frac{1}{2} \times (y_2 + y_3) \times (x_3 - x_2)$

Recall

Area of trapezium is:

$A = \frac{1}{2} \times$ (sum of lengths of parallel sides) × distance between parallel sides (altitude)

Substituting these values in equation (1), we can find area A of triangle ABC as follows.

$$A = \frac{1}{2} \times (y_2 + y_1) \times (x_1 - x_2) + \frac{1}{2} \times (y_1 + y_3) \times (x_3 - x_1) - \frac{1}{2} \times (y_2 + y_3) \times (x_3 - x_2)$$

$$= \frac{1}{2} \times [(y_2 + y_1) \times (x_1 - x_2) + (y_1 + y_3) \times (x_3 - x_1) - (y_2 + y_3) \times (x_3 - x_2)]$$

$$= \frac{1}{2} \times [x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3 y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3]$$

$$= \frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2} \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Check Point

Find the area of a triangle whose vertices are given as (1, -1), (-4, 6) and (-3, -5).

Key Facts

- If points A, B and C are collinear, then area is zero.
- If the sign of value of area obtained is negative, ignore it as the area cannot be negative.
- Area of a triangle can also be found by finding the length of three sides of a triangle using the distance formula and then applying Heron's formula.

Example 4:

Find the area of triangle if points (4, – 2), (– 2, 4) and (5, 5) are vertices of a triangle.

Solution:

Here, $x_1 = 4, y_1 = -2, x_2 = -2, y_2 = 4, x_3 = 5$ and $y_3 = 5$

$$\text{Area of triangle} = \frac{1}{2} \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \times \begin{vmatrix} 4 & -2 & 1 \\ -2 & 4 & 1 \\ 5 & 5 & 1 \end{vmatrix}$$

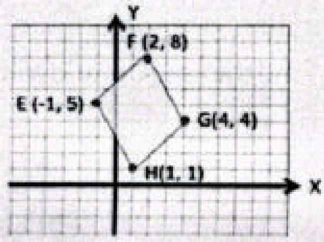
$$= \frac{1}{2} \times [4(4 - 5) + 2(-2 - 5) + 1(-10 - 20)]$$

$$= \frac{1}{2} \times [-4 - 14 - 30] = \frac{1}{2} \times [-48] = -24$$

∴ Area of triangle = 24 square units

Challenge

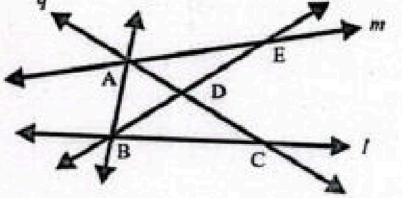
Find the area of parallelogram shown.



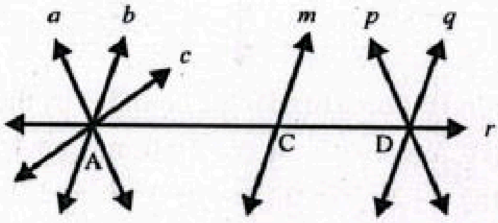
Exercise 6.1

1. Which sets of lines are concurrent in the given figure? Also, tell the point of concurrency.

(i)

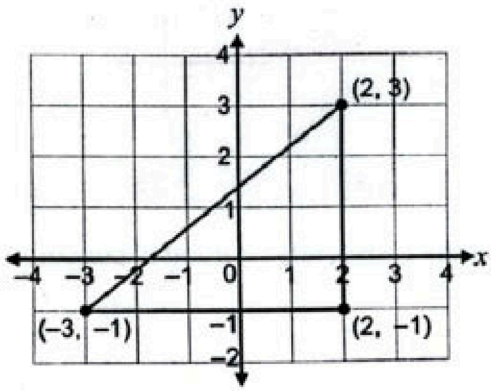


(ii)

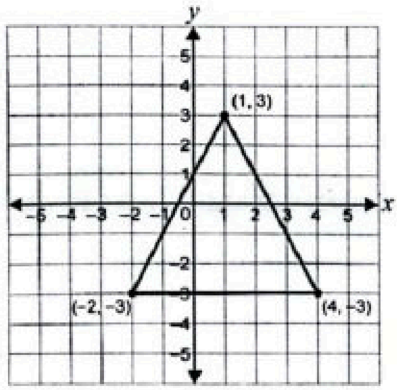


2. Check whether the lines are concurrent or not.
   (i) $3x-4y-13=0,\ \ 8x-11y-33=0,\ \ 2x-3y-7=0$
   (ii) $x+2y-4=0,\ \ x-y-1=0,\ \ 4x+5y-13=0$
3. Determine the value of a if the line $2x-y+3=0,\ \ x-y=0,\ \ 3x+ay+1=0$ are concurrent.
4. Show that the lines $x+y+1=0, x-y+1=0$ and x-axis are concurrent. Also prove that x-axis bisects the angle between $x+y+1=0$ and $x-y+1=0$.
5. Find the equations of altitudes and their point of concurrency in the triangle ABC when A(4, –2), B(5, 5) and C(–1, 3). What is the name of point of concurrency?
6. Find the equations of right bisectors and their point of concurrency in the triangle XYZ when X(0, 0), Y(7, 0) and Z(7, 4). What is the name of point of concurrency?
7. Find the equations of medians and their point of concurrency in the triangle DEF when D(–6, –4), E(6, –4) and F(–2, 4). What is the name of point of concurrency?
8. Prove that (a) altitudes, (b) right bisectors and (c) medians of the following triangles are concurrent.
   (i) A(4, 6), B(7, 2), C(2, 3)   (ii) P(–4, 0), Q(2, 0), R(0, 3)
9. Find the area of triangles.
   (i) A(1, 1), B(4, 5), C(12, –1)   (ii) D(3, 1), E(2, 3), F(2, 2)
10. By finding area of triangle, show that points A(6, 0), B(–3, 6) and C(3, 2) are collinear.
11. Using the formula of area, find x if the points P(3, 2), Q(–1, x), R(7, 3) are collinear.
12. Vertices of a triangle are (3, –4), (4, h) and (2, 6). Find h if area of the triangle is 10 square units.
13. Find the area of following figures.

(i)



(ii)



14. Find the area of triangle bounded by the lines:
    (i) $4x-5y+7=0,\ x-2=0$ and $y+1=0$
    (ii) $x-2y-6=0,\ 3x-y+3=0$ and $2x+y-4=0$

6.5 Homogeneous Equations in two Variables

Let $f(x, y) = 0$ be any equation in two variables x and y. The equation $f(x, y) = 0$ is called a homogeneous equation of degree n if

$$f(kx, ky) = k^n f(x, y) \quad \ldots\ldots\ldots (1)$$

where k is any real number and n is a positive integer.

6.5.1 Homogeneous Linear Equations in two Variables

An equation of the form $ax + by = 0$ is called a homogeneous linear equation in two variables and always passes through origin.

If we take $n = 1$ in equation (1), then we get:

$$f(kx, ky) = kf(x, y) \quad \ldots\ldots\ldots (2)$$

Equation (2) is called homogeneous linear equation in two variables. Consider:

$$f(x, y) = ax + by = 0 \quad \ldots\ldots\ldots (3)$$

Key Facts

In the homogeneous system of equations, no equation has a constant term. A homogeneous linear system may have one or infinitely many solutions. But it has at least one trivial solution always.

Replacing x by kx and y by ky, we have:

$$f(kx, ky) = akx + bky = k(ax + by) = kf(x, y) \quad \ldots\ldots (4)$$

From (4), it is clear that equation (3) is homogeneous linear equation in two variables.

For example, $x - 4y = 0$ is a homogeneous linear equation in two variables x and y.

6.5.2 Non-Homogeneous Linear Equation

An equation of the form $ax + by + c = 0$ is called a non-homogeneous linear equation in two variables.

For example, $2x - 5y + 7 = 0$ is a non-homogeneous linear equation in two variables x and y.

6.5.3 Joint Equation

Consider two straight lines represented by:

$$a_1x + b_1y + c_1 = 0 \quad \ldots\ldots (5)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots\ldots (6)$$

Check Point

Show that $2x + 3y = 0$ is homogeneous linear equation in two variables.

Multiplying equations (5) and (6), we have:

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0 \quad \ldots\ldots (7)$$

Equation (7) is called joint equation and can be re-written as:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots (8)$$

Equation (8) represents a pair of lines and can be resolved back into two linear equations.

If we put $c_1 = c_2 = 0$ in equations (5) and (6), then:

$$a_1x + b_1y = 0 \quad \ldots\ldots (9)$$

$$a_2x + b_2y = 0 \quad \ldots\ldots (10)$$

Equations (9) and (10) represent a system of homogeneous linear equations in two variables.

Multiplying (9) and (10), we get:

$$(a_1x + b_1y)(a_2x + b_2y) = 0 \quad \ldots\ldots (11)$$

Equation (11) is special joint equation and can be re-written as:

$$a_1a_2x^2 + a_1b_2xy + a_2b_1xy + b_1b_2y^2 = a_1a_2x^2 + (a_1b_2 + a_2b_1)\,xy + b_1b_2y^2$$

If we put $a_1a_2 = a$, $a_1b_2 + a_2b_1 = 2h$ and $b_1b_2 = b$, then we get:

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{........(12)}$$

where a, h and b are not simultaneously zero.

Equation (12) represents a special pair of lines passing through origin and can be resolved back into two homogeneous linear equations. This equation is called general second-degree homogeneous equation.

Any point P(x, y) that satisfies $a_1x + b_1y = 0$ or $a_2x + b_2y = 0$ will also satisfy equation (12). For example, $6x^2 - 4xy + 8y^2 = 0$ is a homogeneous quadratic equation in two variables x and y.

Equations (9) and (10) can also be written as:

$y = m_1x$ and $y = m_2x$ where m_1 and m_2 are slopes of lines passing through origin.

Their joint equation is:

$$(y - m_1x)(y - m_1x) = 0 \;\Rightarrow\; y^2 - (m_1 + m_2)\,xy + m_1 m_2 x^2 \quad \text{....... (13)}$$

Equation (13) is another special type of second-degree homogeneous equation.

Comparing equations (12) and (13), we have:

$$\frac{m_1m_2}{a} = \frac{-(m_1+m_2)}{2h} = \frac{1}{b} \qquad \Rightarrow \qquad m_1m_2 = \frac{a}{b} \text{ and } m_1 + m_2 = \frac{-2h}{b}$$

6.5.4 Homogeneous Quadratic Equations in two Variables

If we take $n = 2$ in equation (1), then we get:

$$f(kx, ky) = k^2 f(x, y) \qquad \text{.......... (14)}$$

Equation (14) is called homogeneous quadratic equation in two variables. Consider:

$$f(x, y) = ax^2 + 2hxy + by^2 = 0 \text{.......... (15)}$$

Replacing x by kx and y by ky, we have:

$$f(kx, ky) = a(kx)^2 + 2h(kx)(ky) + b(ky)^2 = k^2(ax^2 + 2hxy + by^2) = k^2 f(x, y) \quad \text{....... (16)}$$

From (16), it is clear that equation (15) is homogeneous quadratic equation in two variables.

Key Facts

The most general equation of second degree: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Represents a pair of lines if:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Theorem 6.4:

Every homogeneous second-degree equation:

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{.......... (1)}$$

represents a pair of lines passing through the origin. The lines are:

(i) Real and distinct if $h^2 > ab$

(ii) Real and coincident if $h^2 = ab$

(iii) Imaginary if $h^2 < ab$

Proof: Multiplying equation (1) by a, we have:

$$a^2x^2 + 2ahxy + aby^2 = 0$$
$$a^2x^2 + 2ahxy + h^2y^2 - h^2y^2 + aby^2 = 0$$
$$(ax + hy)^2 - y^2(h^2 - ab) = 0$$
$$(ax + hy)^2 - (y\sqrt{h^2 - ab})^2 = 0$$
$$(ax + hy + y\sqrt{h^2 - ab})(ax + hy - y\sqrt{h^2 - ab}) = 0 \quad \text{.... (2)}$$

Which shows that equation (1) represents a pair of lines through origin. From equation (2):

$$ax + hy + y\sqrt{h^2 - ab} = 0 \quad \text{and} \quad ax + hy - y\sqrt{h^2 - ab} = 0$$

or
$$ax + y(h + \sqrt{h^2 - ab}) = 0 \quad \text{...... (3)}$$
$$ax + y(h - \sqrt{h^2 - ab}) = 0 \quad \text{...... (4)}$$

From Equations (3) and (4), it is clear that the lines are:

(i) Real and distinct if $h^2 > ab$ (ii) Real and coincident if $h^2 = ab$
(iii) Imaginary if $h^2 < ab$

Note: It is interesting to note that even the lines are imaginary, they pass through the real point (0, 0) as this point lies on the joint equation.

Example 5:

Find the straight lines represented by $x^2 - 7xy + 12y^2 = 0$

Solution:

$$x^2 - 7xy + 12y^2 = 0 \quad \Rightarrow \quad x^2 - 3xy - 4xy + 12y^2 = 0$$
$$\Rightarrow \quad x(x - 3y) - 4y(x - 3y) = 0 \quad \Rightarrow \quad (x - 3y)(x - 4y) = 0$$
$$\Rightarrow \quad x - 3y = 0 \quad \text{or} \quad x - 4y = 0$$

Which are required straight lines.

6.5.5 Angle between Lines Represented by $ax^2 + 2hxy + by^2 = 0$

We have already proved that $ax^2 + 2hxy + by^2 = 0$ represents two straight lines:

$$ax + y(h + \sqrt{h^2 - ab}) = 0 \quad \text{...... (1)}$$
$$ax + y(h - \sqrt{h^2 - ab}) = 0 \quad \text{...... (2)}$$

Slopes of (1) and (2) respectively are:

$$m_1 = \frac{-(h + \sqrt{h^2 - ab})}{b} \quad \text{and} \quad m_2 = \frac{-(h - \sqrt{h^2 - ab})}{b}$$

$$\Rightarrow \quad m_1 + m_2 = \frac{-2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b} \quad \text{and}$$

If θ is the measure of acute angle between (1) and (2), then:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{2\sqrt{h^2 - ab}}{a + b} \quad \text{........ (3)}$$

Special cases:

(i) When $\theta = 0$, then $\tan\theta = 0$ and equation (3) implies $h^2 - ab = 0$ or $h^2 = ab$ which is the condition for the lines to be coincident.

(ii) When $\theta = 90^\circ$, then $\tan\theta =$ undefined and equation (3) implies $a + b = 0$ which is condition for the lines to be orthogonal. i.e., sum of coefficients of x^2 and y^2 is 0.

Example 6:

Find measure of acute angle between lines represented by $6x^2 - xy - y^2 = 0$.

Solution:

Given equation is $6x^2 - xy - y^2 = 0$

Here, $a = 6$, $h = \frac{-1}{2}$, $b = -1$

If θ is the measure of acute angle between lines, then:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{\frac{1}{4} + 6}}{6 - 1} = \frac{5}{5} = 1 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$$

Example 7:

Find a joint equation of straight lines through the origin and perpendicular to the lines represented by $3x^2 + 5xy + 2y^2 = 0$

Solution:

$$3x^2 + 5xy + 2y^2 = 0 \quad \Rightarrow \quad 3x^2 + 3xy + 2xy + 2y^2 = 0$$

$$\Rightarrow \quad 3x(x + y) + 2y(x + y) = 0 \quad \Rightarrow \quad (x + y)(3x + 2y) = 0$$

Thus, the lines represented by the given equation are:

$$x + y = 0 \quad \text{and} \quad 3x + 2y = 0$$

$$\Rightarrow \quad y = -x \quad \ldots\ (1) \quad \text{and} \quad y = \frac{-3x}{2} \quad \ldots\ (2)$$

The line through (0, 0) and perpendicular to (1) is:

$$y = x \quad \text{or} \quad y - x = 0 \quad \ldots\ (3)$$

The line through (0, 0) and perpendicular to (2) is:

$$y = \frac{2x}{3} \quad \text{or} \quad 3y - 2x = 0 \quad \ldots\ (4)$$

Joint equation of lines (3) and (4) is:

$$(y - x)(3y - 2x) = 0 \quad \Rightarrow \quad 3y^2 - 2xy - 3xy + 2x^2 = 0$$

$$\Rightarrow \quad 3y^2 - 5xy + 2x^2 = 0 \quad \text{or} \quad 2x^2 - 5xy + 3y^2 = 0$$

Key Facts

If $y = mx$ .... (i)

is equation of line passing through origin, then the line perpendicular to (i) through origin is $y = -\frac{1}{m}x$.

Exercise 6.2

1. Find the lines represented by each of the following joint equations.

(i) $x^2 + 5xy + 6y^2 = 0$ (ii) $7x^2 - 2xy - 9y^2 = 0$

(iii) $x^2 + 6xy = 0$ (iv) $x^2 - 8xy + 12y^2 = 0$

(v) $5x^2 + 3xy - 2y^2 = 0$ (vi) $x^2 - 3xy - y^2 = 0$

2. Find measure of acute angle between lines represented by the following equations.
   (i) $x^2 - y^2 = 0$ (ii) $x^2 + 5xy + 4y^2 = 0$
   (iii) $15x^2 - 19xy + 6y^2 = 0$ (iv) $10x^2 - xy - 9y^2 = 0$
   (v) $5x^2 - 3xy - 2y^2 = 0$ (vi) $7x^2 + 2xy - 9y^2 = 0$
3. Show that the lines represented by following equations are real coincident or real distinct or imaginary.
   (i) $x^2 + 4xy - 21y^2 = 0$ (ii) $4x^2 - 12xy + 9y^2 = 0$
   (iii) $x^2 + xy + y^2 = 0$ (iv) $x^2 - 9y^2 = 0$
4. Show that the angle between lines represented by the following equations is right angle.
   (i) $x^2 + 5xy - y^2 = 0$ (ii) $x^2 - 2(\tan\theta)xy - y^2 = 0$
   Explain the reason of right angle between lines.
5. Find a joint equation of the lines through the origin and perpendicular to the lines:
   (i) $2x^2 - 7xy + 6y^2 = 0$ (ii) $x^2 + 17xy + 60y^2 = 0$
   (iii) $3x^2 - 13xy - 10y^2 = 0$ (iv) $x^2 + 3xy - 28y^2 = 0$

6.6 Application of Analytic Geometry

Analytic geometry is used in physics and engineering, and also in aviation, rocketry, space science, and spaceflight. It is the foundation of most modern fields of geometry, including algebraic, differential, discrete and computational geometry. See the following examples for understanding.

Example 8:

An engineer needs to design a traffic intersection where three straight roads meet. The roads are represented by the equations of lines. The roads are given by the following equations:

$2x - 4y + 5 = 0, 7x - 8y + 5 = 0$ and $4x + 5y = 45$

Determine if the roads are concurrent and if so, find the point of intersection of roads.

Solution:

$2x - 4y + 5 = 0$ (1)

$7x - 8y + 7 = 0$ (2)

$4x + 4y - 11 = 0$ (3)

Multiplying equation (1) by -2 and adding with (2), we get:

$-4x + 8y - 10 = 0$ (4)

$7x - 8y + 7 = 0$ (2)

$3x - 3 = 0 \Rightarrow 3x = 3 \Rightarrow x = 1$

Substituting in equation (1), we have:

$2(1) - 4y + 5 = 0 \Rightarrow -4y = -7 \Rightarrow y = \frac{7}{4}$

Showing that two roads meet at $\left(1, \frac{7}{4}\right)$.

Putting $\left(1, \frac{7}{4}\right)$ in equation (3), we get:

$4(1) + 4\left(\frac{7}{4}\right) - 11 = 0 \quad \Rightarrow \quad 4 + 7 - 11 = 0 \quad \Rightarrow \quad 0 = 0$

Therefore, third road also pass through $\left(1, \frac{7}{4}\right)$. Hence the three roads are concurrent.

Example 9:

A triangular park is bounded by three straight roads that meet at points A(2, 3), B(8, 5) and C(5, 10). A city planner needs to determine the area of the park for landscaping purpose. How can he determine the area of the park?

Solution:

Here, $x_1 = 2, y_1 = 3, x_2 = 8, y_2 = 5, x_3 = 5$ and $y_3 = 10$

$$\text{Area of triangular park} = \frac{1}{2} \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \times \begin{vmatrix} 2 & 3 & 1 \\ 8 & 5 & 1 \\ 5 & 10 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times [2(5-10) - 3(8-5) + 1(80-25)]$$

$$= \frac{1}{2} \times [-10 - 9 + 55] = \frac{1}{2} \times [36] = 18$$

$\therefore$ Area of the plot = 18 square units

Example 10:

A railway engineer is designing an intersection where two rail tracks meet at a junction. The joint equation representing the tracks is given by $x^2 - 4xy - 12y^2 = 0$. Determine the angle between the individual lines representing the tracks.

Solution:

Given joint equation of the tracks is:

$x^2 - 4xy - 12y^2 = 0 \quad \Rightarrow \quad x^2 + 2xy - 6xy - 12y^2 = 0$

$\Rightarrow x(x + 2y) - 6y(x + 2y) = 0 \quad \Rightarrow \quad (x + 2y)(x - 6y) = 0$

The lines representing the tracks are:

$$x + 2y = 0 \qquad (i)$$

$$x - 6y = 0 \qquad (ii)$$

From (i) and (ii), we have:

$$y = -\frac{1}{2}x \qquad (iii)$$

$$y = \frac{1}{6}x \qquad (iv)$$

Slopes of lines (iii) and (iv) are:

$$m_1 = -\frac{1}{2} \quad \text{and} \quad m_2 = \frac{1}{6}$$

If θ is angle between both tracks, then:

$$\text{Tan}\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{1}{6}+\frac{1}{2}}{1+\frac{1}{6}\times(-\frac{1}{2})} = \frac{8}{13} = 0.727$$

$$\theta = \text{Tan}^{-1}(0.727) = 36.03^\circ$$

Therefore, angle between the tracks is 36.03°.

Exercise 6.3

1. Three jet fighters are flying straight in different directions along the lines $2x - y - 14 = 0,\ x + y - 1 = 0$ and $3x + 2y - 7 = 0$
   Check whether the jet fighters will pass through a single point or not. If yes, find that point.

2. A farmer owns a triangular shaped piece of land with corners at points X(3, 7), Y(6, 2) and Z(10, 8). Calculate the cost of planting maize crop @ Rs. 300 per square unit.

3. Hira is designing a triangular section of a roof with vertices at points P(4, 1), Q(9, 5) and R(7,10). She needs to calculate the area of the section to determine how much roofing material is required. Find the area calculated by her.

4. A landscaper is designing a triangular garden bed with vertices A(1, 4), B(5, 1) and C(8, 6). Calculate the cost of planting mango trees in the garden @ Rs.70 per square unit.

5. A civil engineer is tasked with designing a roundabout where three main roads converge. The equations of roads are $2x + y - 11.5 = 0,\ x - 4y + 1 = 0$ and $3x - 2y - 12 = 0$.
   Find the coordinates of the point to design the roundabout.

6. Asad is arranging a flashlight in a marriage ceremony. The position of the flashlight is at the intersection of lines $2x + y - 23 = 0,\ 0.5x - y + 3 = 0$ and $x - y = 1$. Find the position of the flashlight.

7. A surveyor is mapping out a triangular park where three straight walkways meet. The walkways are represented by the equations of lines $x + y - 4 = 0,\ 2x - y - 2 = 0$ and $x - 2y + 2 = 0$. Find the coordinates of point of intersection of the walkways.

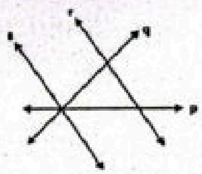
8. An astronomer is studying the behavior of light rays passing through a converging lens. The lens focuses the rays at the origin. The equation of rays is given by $3x^2 - 4xy + y^2 = 0$. Find the path of individual light rays and angel between rays.

9. A welder is designing a support structure for a building. The structure is made up of beams intersecting at origin. The equations of the lines representing the beams are shown by the joint equation $2x^3 - 8xy^2 = 0$. Find the equations of iron beams.

Review Exercise

1. Select the correct option.

(i) *The lines* p, q and s are:



(a) parallel (b) concurrent
(c) perpendicular (d) collinear

(ii) If the determinant of coefficients the three lines is 0, then the lines are:
(a) concurrent (b) intersecting (c) parallel (d) perpendicular

(iii) Which of the following equations is homogeneous?
(a) $x^2 + 5x = 0$ (b) $2x + 3y + 1 = 0$
(c) $x^2 + 5xy = 0$ (d) $4y + 8 = 0$

(iv) Which of the following equations is not homogeneous?
(a) $x^2 + 3xy = 0$ (b) $2x - 3y = 0$
(c) $xy^2 + y^3 = 0$ (d) $5x - 5 = 0$

(v) Equation $ax + by + c = 0$ passes through origin if:
(a) $a = 0$ (b) $c = 0$ (c) $b = 0$ (d) $a = b = c = 0$

(vi) The lines represented by $ax^2 + 2hxy + by^2 = 0$ are real and distinct if:
(a) $h^2 > ab$ (b) $h^2 = ab$ (c) $h^2 < ab$ (d) $h^2 \leq ab$

(vii) The lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if:
(a) $h^2 > ab$ (b) $h^2 = ab$ (c) $h^2 < ab$ (d) $h^2 \leq ab$

(viii) The lines represented by $ax^2 + 2hxy + by^2 = 0$ are imaginary if:
(a) $h^2 > ab$ (b) $h^2 = ab$ (c) $h^2 < ab$ (d) $h^2 \geq ab$

(ix) The lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular if:
(a) $a - b = 0$ (b) $b - a = 0$ (c) $a + b = 1$ (d) $a + b = 0$

(x) Half the determinant of vertices of triangle gives its:
(a) perimeter (b) area (c) volume (d) both (a) and (b)

(xi) If the determinant of three points is zero then the points are:
(a) collinear (b) non-collinear (c) imaginary (d) concurrent

(xii) The point of intersection of three angle bisectors of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

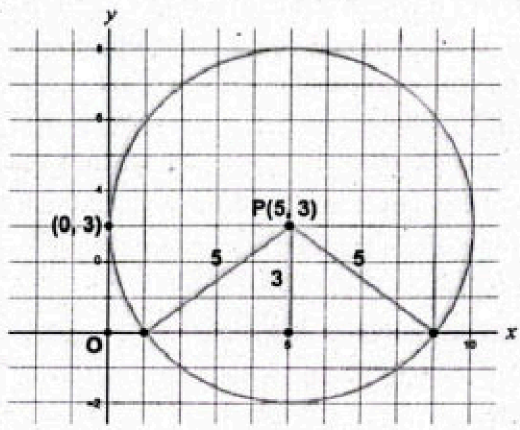
(xiii) The point of intersection of right bisectors of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

(xiv) The point of intersection of three medians of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

(xv) The point of intersection of three altitudes of a triangle is called:
(a) incenter (b) circumcenter (c) centroid (d) orthocenter

(xvi) Which of the following is perpendicular to $y = -\frac{1}{2}x$?
(a) $y = 2x$ (b) $y = -2x$ (c) $2y = x$ (d) $2y = -x$

2. Prove that altitudes, right bisector and medians of triangle ABC are concurrent when: A(0, 0), B(a, 0), C(b, c) where a, b and c are not equal.
3. Find the value of m if the points (6, 1), (–2, –3) and (8, 2m) are collinear.
4. Find a relation between x and y if the points A(x, y), B(– 4, 6) and C(– 2, 3) are collinear.
5. The points A (0, 3), B (a, 0) and C (0, –3) are the vertices of a triangle ABC right angled at B. Find the values of a and hence the area of ΔABC.
6. A circle has a centre at point P(5, 3) and radius r = 5. This circle intersects the y-axis at one intercept and the x-axis at two intercepts. What is the area of the triangle formed by these three intercepts?



7. Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, – 1), (2, 1) and (0, 3). Find the ratio of this area to the area of the given triangle ABC.
8. Find the value of m, such that l_1, l_2, l_3 intersect each other at one point.
   $l_1: x - y = 1;\quad l_2: 2x + y = 5;\quad l_3: (2m - 5)x - my = 3$
9. Find the area of triangle bounded by the line $2x - y + 10 = 0$ and the coordinate axes.
10. Find the area of triangle bounded by the lines:
    $$x^2 - xy - 6y^2 = 0,\ x - y + 3 = 0$$
11. Find the area of the figure shown below.