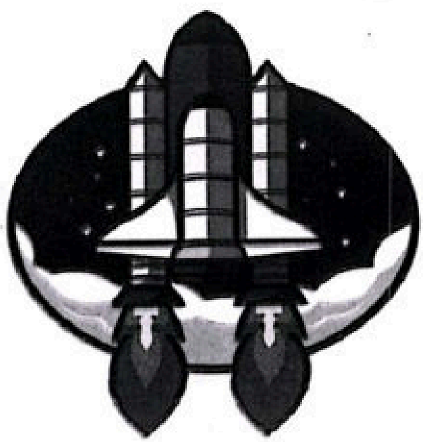
KINEMATICS OF MOTION IN A STRAIGHT LINE

After studying this unit, students will be able to:

- Recognize distance and speed as scalar quantities, displacement, velocity and acceleration as vector quantities.
- Sketch and interpret displacement-time graph and velocity time graph.
- Use differentiation and integration with respect to time to solve simple problems concerning displacement, velocity and acceleration.
- Use appropriate formula for motion with constant acceleration in a straight line.
- Apply the concept of mechanics to real life problems such as motion of vehicle on roads, projectile motion, free fall motion, relative motion animation.
- Explain the need for a vector valued function.
- Construct vector valued function.
- Identify domain and range of vector valued functions.
- Identify difference between scalar and vector valued functions.
- Explain derivative of a vector function of a single variable and elaborate the result. Apply vector differentiation to calculate velocity and acceleration of a position vector.
- Apply concepts of vector valued functions to real life word problems (such as engineering and transportation).

We see leaves falling from trees, movement of rockets and water flowing from the river. We walk, we run, we drive a car these are the activities that we carry out in our day-to-day life. These activities can be defined as motion. Motion is the change in position or orientation of a body. For example, a body on the surface of the Earth may appear to be at rest, but that is only because the observer is also on the surface of the Earth. The Earth itself, together with both the body and the observer, is moving in its orbit around the Sun and rotating on its own axis at all times in the same way.



5.1 Scalar and Vector Quantities

Physical quantities are divided into two categories, scalars and vectors. If only a magnitude is required to express a quantity, then the quantity is known as scalar and if both magnitude and direction are required to express a quantity then the quantity is known as vector quantity. e.g. time, distance, mass are scalars and velocity, acceleration, weight are vectors.

5.1.1 Distance

Distance is a scalar quantity since it only describes how much path is covered by a moving object regardless of direction. For example, you travel 3Km due north and then 2km due west then the total path covered is 5km.

5.1.2 Speed

Speed is also a scalar quantity since it represents how fast an object is moving regardless of its direction. For example, if you drive a car at the speed of 50km/hr it represents how fast you are moving no matter you are moving in which direction.

5.1.3 Displacement

Change in position of an object is known as displacement. It is a vector quantity because it includes both the magnitude i.e. distance between the initial position and final position and the direction i.e. straight path from the initial position to the final position. For example if you move along a curved path going from *A* to *B* then the straight path $\overrightarrow{AB}$ is the displacement.

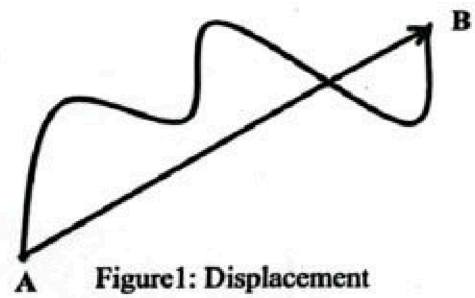


Figure1: Displacement

5.1.4 Velocity

Velocity is also a vector quantity, since it describes both how fast and in which direction an object is moving. In other words, rate of change of displacement with respect to time is called velocity. For example, a car moving 50km/h due west then its velocity is 50km/h westward.

5.2 Displacement-Time Graph

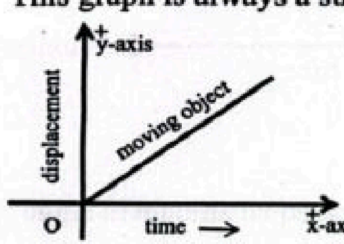
A moving object has different positions with respect to time. At any time 't' what is the position of object is represented on a graph known as displacement-time graph. While drawing the graph:

- Displacement values are drawn along y-axis (vertical axis)(time dependent quantity).
- Time is taken along x-axis (horizontal axis)(independant quantity).

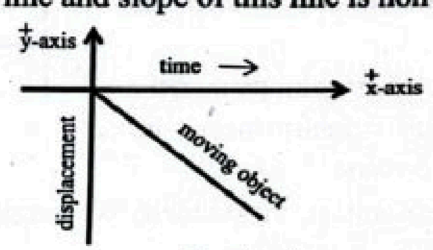
Note that independent values are taken along x-axis and the dependent values along y-axis.

5.2.1 Displacement-Time Graph for Uniform Motion

This is the graph between the displacement and the time when the motion of object is uniform. This graph is always a straight line and slope of this line is non-zero.



Positive slope



Negative slope

Here are some key points about the displacement-time graph.

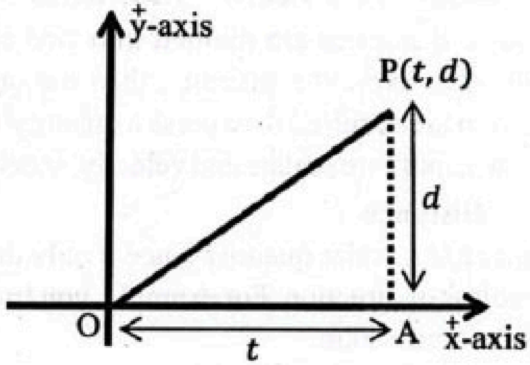
5.2.2 Slope of Displacement-time Graph

Let after time 't' the object moving with uniform speed is at point 'P' where its displacement is 'd'.

From figure slope of the graph $= \frac{d}{t}$.

By definition, ratio of displacement d with time 't' is velocity of the object. Therefore:

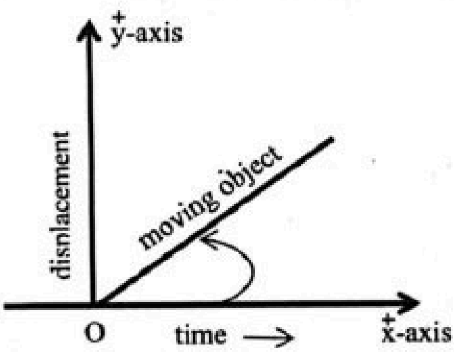
Slope of graph = velocity of the object



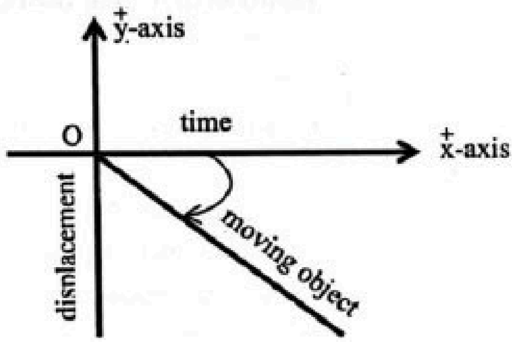
Since time is always non-negative and the displacement may be positive or negative. Therefore, slope may be positive or negative. Greater the value of slope means higher the velocity (+ve or -ve) of the object and smaller the value of slope means lower the velocity of object. If slope of the graph is zero (line is horizontal), then velocity of the object is zero.

5.2.3 Direction of Slope

If slope is positive, it indicates that particle is moving in the positive (anti-clockwise) direction and the negative slope indicates that particle is moving in the negative (clockwise) direction.



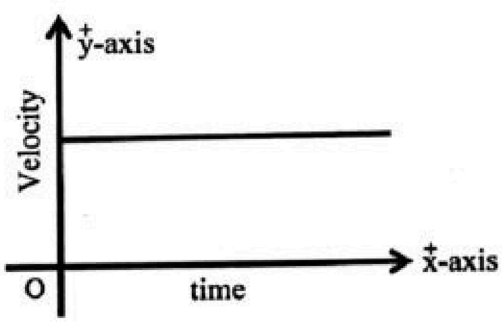
Positive slope



Negative slope

5.2.4 Velocity-Time Graph

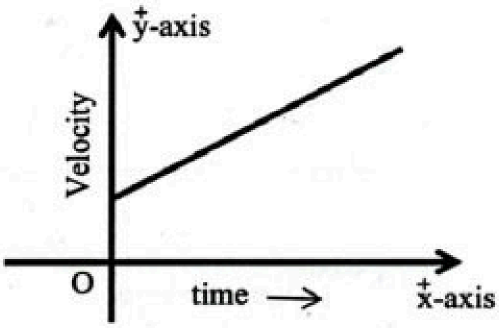
The velocity time graph is the graph plotted between the velocity of the particle with the passage of time. From the graph we can extract various kinds of information e.g. velocity and speed of particle at any time t, distance travelled by the particle, acceleration, average speed and average velocity etc. Velocity is plotted along y-axis and time along x-axis.



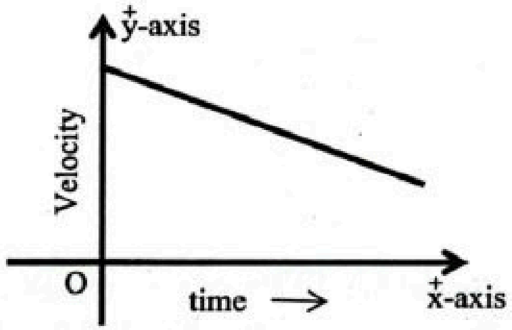
When the particle is moving with constant velocity then the graph is a horizontal line. Constant velocity means for different values of time the velocity is same.

A body is said to have a uniform acceleration if its velocity is changing by an equal amount in equal intervals of time.

In the case when body is moving with uniform acceleration the velocity-time graph is along a non-horizontal and non-vertical line.



Positive acceleration

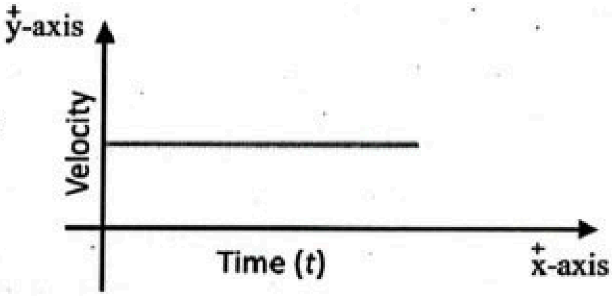


Negative acceleration

5.2.5 Slope of Velocity-Time Graph

Case I: When velocity is constant.

As we know that when velocity is constant then the graph is along a horizontal line so its slope is zero.



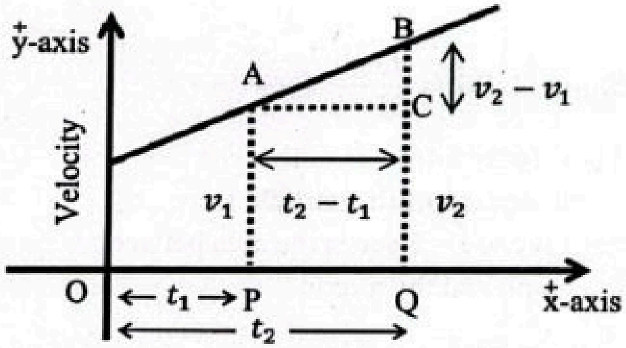
Case II: When acceleration is uniform.

For the particle which is moving with the uniform acceleration the graph is non-horizontal and non-vertical line. Consider any of the case with positive or negative acceleration.

Let at time t_1 the velocity is v_1 and at time t_2 velocity is v_2

From figure slope of line $= \frac{|BC|}{|AC|}$

$$= \frac{v_2 - v_1}{t_2 - t_1} = \frac{\text{Change in velocity}}{\text{Change in time}} = \text{acceleration}$$



Thus, slope of the graph of velocity-time graph is the acceleration of the particle.

Area between the Graph and the $\overset{+}{x}$-axis

The area between the graph and the x-axis gives the displacement of the particle in that interval of the time.

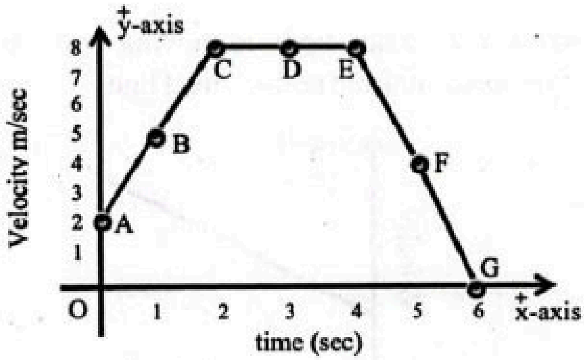
Example 1:

Draw the velocity-time graph of a moving particle with given data. Find its acceleration and retardation. Also find its displacement in the time interval 2 sec to 5 sec.

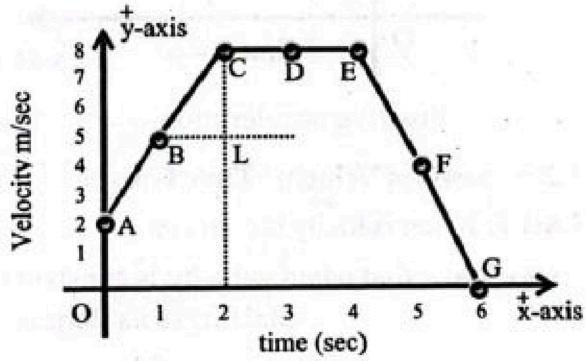
Time = t sec	0	1	2	3	4	5	6
Velocity = v m/s	2	5	8	8	8	4	0

Solution:

From the graph it is clear that particle is accelerating from point A to C as its velocity is increasing G with the passage of time.



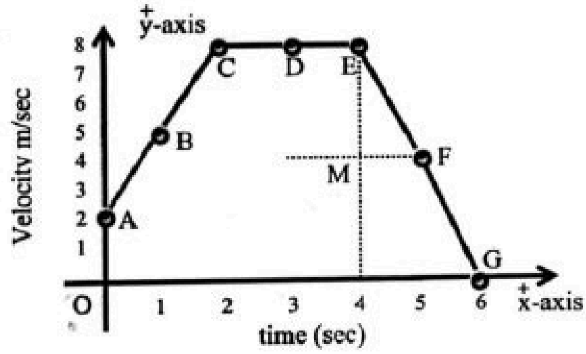
For the slope of line $|\overline{AC}|$ take any two points on the line say B and C and draw horizontal line through B and vertical line through C intersecting at point L.



acceleration = slope of $|\overline{AC}|$

$$= \frac{|CL|}{|BL|} = \frac{8-5}{2-1} = 3m/sec^2$$

From the graph it is clear that particle motion is retarding from E to G as the velocity of the particle is decreasing with the passage of time. Draw a horizontal line through F and a vertical line through E meeting at M.



Slope of $\overline{EG} = \frac{0-8}{6-4} = \frac{-8}{2} = -4$

Thus, retardation = $-4m/sec^2$

Displacement of the particle from t = 2 sec to t = 5 sec is the area between the graph and the x-axis.

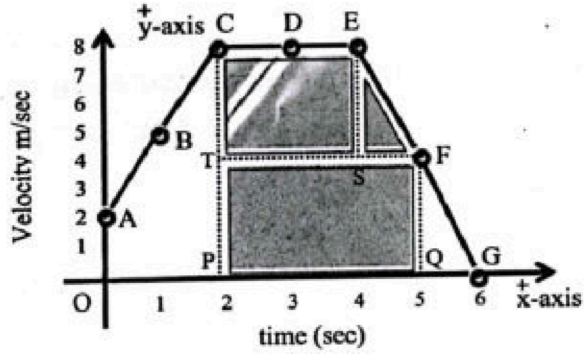
This is area of the region PCEFQ.

Displacement = area of the region PCEFQ

= Area of rectangle PQFT + Area of rectangle TSEC + Area of triangle FSE

$= (|PQ| \times |PT|) + (|TS| \times |TC|) + \frac{1}{2}(|SF| \times |SE|)$

$= (3 \times 4) + (2 \times 4) + \frac{1}{2}(1 \times 4) = 12 + 8 + 2 = 22$

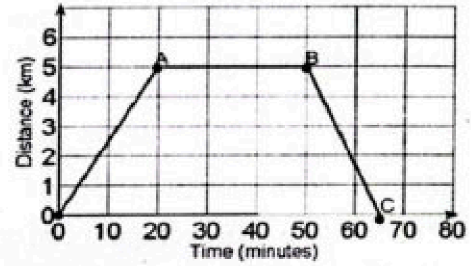


Displacement Function

A function which gives the displacement of moving particle at any time 't' is known as displacement function e.g. $S(t) = 3t^2 + 6t + 1$. At $t = 0$; $S(0) = 0 + 0 + 1 = 1$ is the position of the particle and at $t = 3$; $S(3) = 3(3)^2 + 6(3) + 1 = 46$ so the displacement of the particle at $t = 3$ is 46 - 1 = 45.

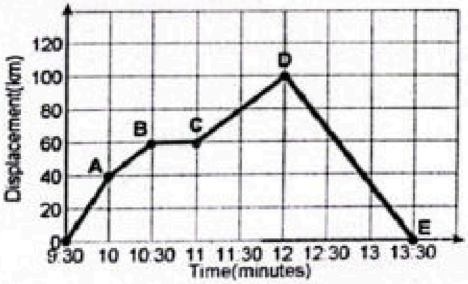
Exercise 5.1

1. A cyclist rides in a straight line for 20 minutes. He waits for half an hour, then returns in a straight line to his starting point in 15 minutes. This is a displacement-time graph for his journey.



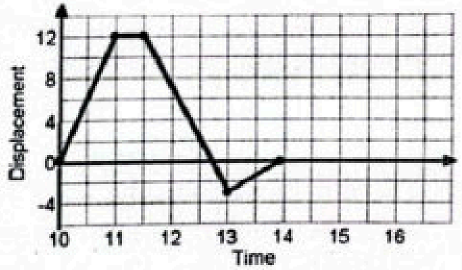
   (i) Work out the average velocity for each stage of the journey in km/h.
   (ii) Write down the average velocity for the whole journey.
   (iii) Work out the average speed for the whole journey.

2. This is a displacement-time graph for a car travelling along a straight road. The journey is divided into 5 stages labelled A to E.



   (i) Workout the average velocity for each stage of the journey.
   (ii) State the average velocity for the whole journey.
   (iii) Workout the average speed for the whole journey.

3. Fatima left home at 10:00 and cycled north-east in a straight line. The diagram shows a displacement time graph for her journey.



   (i) Find the Fatima's velocity between 10:00 and 11:00.

   On her return journey, Fatima continued passed her home before returning.

   (ii) Estimate the time that Fatima passed her home.
   (iii) Find Fatima's velocity for each of the last two stages for her journey.
   (iv) Calculate Fatima's average speed for her entire journey.

4. An electric train starts from the rest at a station A and moves along a straight level track. The train accelerates uniformly at 0.4 m/s^2 to a speed of 16 m/s. The speed is then maintained for a displacement of 2000 m. Finally the train retards uniformly for 20 s before coming to rest at a station B. For this journey from A to B,
   (i) Find the total time taken
   (ii) Find the displacement from A to B
   (iii) Sketch the displacement-time graph, showing clearly the shape of the graph for each stage of the journey.

5. Using the following data, draw time-displacement graph for a moving object.

Time(s)	0	2	4	6	8	10	12	14	16
Displacement(m)	0	2	4	4	4	6	4	2	0

Use the graph to find average velocity for first 4 *s*, for next 4 *s* and for last 6 *s* and the total displacement.

6. Ahmed leaves home at 11 *am*. He cycles at a speed of 16 *km/h* for 90 *minutes*. He stops for half an hour. Ahmed then cycles home and arrives at 3 *pm*.

(i) Draw a displacement-time graph to show Ahmed's journey.

(ii) What is Ahmed's average speed on the return part of his cycle.

7. Dabeer leaves at 14:00. He drives at an average speed of 14 *km/h* for $3\frac{1}{2}$ *hours*. Dabeer stops the journey for 30 *minutes*. He then drives home at 70 *km/h*. Draw a displacement-time graph to show Dabeer's journey.

8. A helicopter leaves Islamabad at 09:00. It flies for 45 *minutes* at 80 *km/h*. It lands for 20 *minutes*. The helicopter then returns to its base in Islamabad, flying at 100 *km/h*. Draw a displacement-time graph to show the journey.

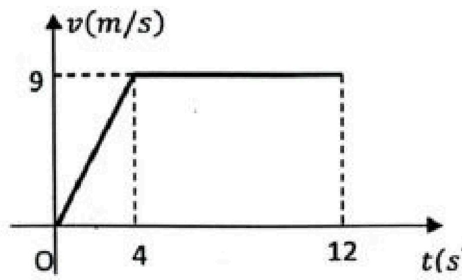
9. The diagram shows the velocity-time graph of the motion of an athlete running along a straight track.

For the first 4 *s*, he accelerates uniformly from rest to a velocity of 9 *m/s*.

This velocity is then maintained for a further 8*s*.

Find: (i) the rate at which the athlete accelerates.

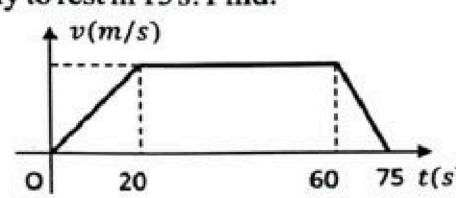
(ii) the displacement from the starting point of the athlete after 12 *s*.



10. The diagram shows the velocity-time of the motion of a cyclist riding along a straight road. He accelerates uniformly from rest to 8 *m/s* in 20 *s*. He then travels at a constant velocity of 8 *m/s* for 40 *s*. Then he decelerates uniformly to rest in 15 *s*. Find:

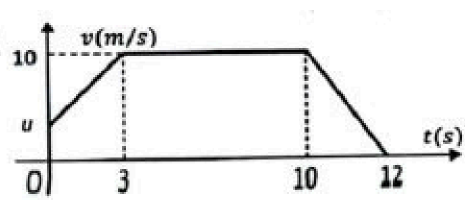
(i) The acceleration of the cyclist in the first 20 *s* of motion.

(ii) The deceleration of the cyclist in the last 15 *s* of motion.



(iii) The displacement from the starting point of the cyclist after 75 *s*.

11. A particle moves 100 *m* in a straight line. The diagram is a sketch of velocity-time graph of the motion of particle. The particle starts with velocity *u m/s* and accelerates to a velocity of 10 *m/s*. The velocity of 10 *m/s* is maintained for 7*s* and then



the particle decelerates to rest in a further 2 s. Find:

(i) The value of u.

(ii) The acceleration of the particle in the first part of the motion.

12. A car is moving along a straight road. When $t = 0\ s$, the car passes a point A with velocity 10 m/s and this velocity is maintained until $t = 30\ s$. The driver then applies the brakes and the car decelerates uniformly, coming to rest at point B when $t = 42\ s$.

(i) Sketch a velocity-time graph to illustrate the motion of the car.

(ii) Find the distance from A to B.

13. A particle moves along a straight line. The particle accelerates from rest to a velocity of 20 m/s in 15s. The particle then moves at a constant velocity of 20 m/s for a period of time. The particle then decelerates uniformly to rest. The period of time for which the particle is travelling at a constant velocity is 4 times the period of time for which it decelerating.

(i) Sketch a velocity-time graph to illustrate the motion of the particle.

Given that the displacement from the starting point of the particle after it comes to rest is 480 m.

(ii) Find the total time for which the particle is moving.

14. A motorcyclist M leaves a road junction at time $t = 0\ s$. He accelerates from rest at a rate of 3 m/s^2 for 8 s and then maintains the velocity he has reached. A car C leaves the same road junction as M at time $t = 0\ s$. The car accelerates from rest to 30 m/s in 20 s and then maintains a velocity of 30 m/s. C passes M as they both pass a pedestrian.

(i) On the same diagram, sketch velocity-time graphs to illustrate the motion of M and C.

(ii) Find the distance of the pedestrian from the road junction.

5.3 Velocity as Derivative of Displacement Function

Let a particle be moving and its position can be determined by the displacement function S. By definition velocity is the time rate of change of displacement. So we may write:

$$v = \text{velocity} = \frac{dS}{dt}$$

This gives us the instantaneous velocity of the particle at time 't'. For a particular value of 't' we will get the velocity of the particle at that particular time.

Example 2: A particle is moving such that its position can be determined by the function $S = t^2 + 2\sin t$. Find its velocity at any time 't'. Also find its initial velocity and velocity at time $t = \frac{\pi}{3}$ (S is in meters and t is in seconds).

Solution:

The position function (Displacement function) is

$$S = t^2 + 2\sin t$$

Differentiate it w.r.t 't'

$$\frac{dS}{dt} = 2t + 2\cos t$$

$$v(t) = 2t + 2\cos t \qquad \text{........(1)}$$

(1) shows the velocity of the particle at any time 't'. To find the initial velocity put $t = 0$ in (1).

$$v(0) = 2(0) + 2\cos 0 = 2m/sec$$

Now put $t = \frac{\pi}{3}$ in (1).

$$v\left(\frac{\pi}{3}\right) = 2\left(\frac{\pi}{3}\right) + 2\cos\frac{\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{1}{2}\right) = \frac{2\pi}{3} + 1 = \frac{2\pi + 3}{3} m/sec$$

Which is the velocity of the particle at $t = \frac{\pi}{3}$.

5.3.1 Acceleration as Derivative of Velocity and Displacement

Let the position of the moving particle at any time 't' be determined by the function $S(t)$.
By definition the acceleration of a particle is the rate of change of velocity w.r.t time; so

$$a = \text{acceleration} = \frac{dv}{dt}$$

Which is acceleration is a derivative of its velocity.

As we know that $v = \frac{dS}{dt}$. Hence $a = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d^2S}{dt^2}$ is the acceleration as derivative of its displacement.

Example 3: The position function of a moving particle is given by $S = \sqrt{t} + ln\,(t+1)$. Find the velocity and acceleration of the particle at any instant of time 't'. Also find its velocity and acceleration at $t = 1$ and $t = 4$. Here S is measured in meters and time in seconds.

Solution:

Given that $S = \sqrt{t} + ln\,(t+1)$

Differentiate w.r.t 't'

$$\frac{dS}{dt} = \frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{t+1}$$

$$v(t) = \frac{1}{2\sqrt{t}} + \frac{1}{t+1} \qquad \text{.......(1)}$$

Equation (1) shows the velocity of the particle at any time 't'. Differentiate (1) w.r.t 't'.

$$\frac{dv}{dt} = -\frac{1}{4}t^{-\frac{3}{2}} - \frac{1}{(t+1)^2} \qquad \text{........(2)}$$

Which is the acceleration of the particle at any time 't'.

Velocity at $t = 1$

Put $t = 1$ in Eq. (1).

$$v(1) = \frac{1}{2} + \frac{1}{2} = 1m/sec$$

Velocity at $t = 4$

Put $t = 4$ in Eq. (1).

$$v(4) = \frac{1}{2\sqrt{4}} + \frac{1}{4+1} = \frac{1}{4} + \frac{1}{5} = \frac{9}{20} m/sec$$

Acceleration at $t = 1$
Put $t = 1$in Eq. (2).

$$a(1) = -\left[\frac{1}{4(1)^{\frac{3}{2}}} + \frac{1}{(1+1)^2}\right] = -\frac{1}{2} m/sec^2$$

Acceleration at $t = 4$
Put $t = 4$ in Eq. (2).

$$a(4) = -\left[\frac{1}{4(4)^{\frac{3}{2}}} + \frac{1}{(4+1)^2}\right] = -\left(\frac{1}{32} + \frac{1}{25}\right) = -\frac{37}{800} m/sec^2$$

5.3.2 Displacement as an Integral of Velocity

As we know that for a moving particle

$$\frac{dS}{dt} = v$$

Integrating both sides w.r.t 't'

$$\int \frac{dS}{dt} dt = \int v\, dt$$

$$S = \int v\, dt$$

5.4 Velocity as Integral of Acceleration

For a moving particle we have:

$$\frac{dv}{dt} = a$$

Integrating both sides w.r.t 't', we have:

$$\int \frac{dv}{dt} dt = \int a$$

$$v = \int a\, dt$$

Example 4: The acceleration of a moving particle at any time 't' is given by

$$a = 24t + \cos t$$

Find its velocity and position at any time 't'. Given that at $t = 0; v = 0$ and at $t = \frac{\pi}{2}; S = \frac{\pi^3}{2}$.

Solution:

Given that $a = 24t + \cos t$

$$v = \int a\, dt$$

$\Rightarrow \quad v = \int (24t + \cos t)dt + A = 12t^2 + \sin t + A$

Where A is constant of integration.

It is given that at $t = 0$; $v = 0$, so

$$0 = 12(0)^2 + \sin(0) + A$$

$\Rightarrow$ $A = 0$

Hence $v = 12t^2 + \sin t$

Which is the velocity of the particle at any time 't'.

Also $S = \int v(t)dt$

$\Rightarrow$ $S = \int (12t^2 + \sin t)dt$

$$S = 4t^3 - \cos t + B$$

Where B is constant of integration.

Given that at $t = \frac{\pi}{2}; S = \frac{\pi^3}{2}$, therefore

$$\frac{\pi^3}{2} = 4\left(\frac{\pi}{2}\right)^3 - \cos\left(\frac{\pi}{2}\right) + B$$

$\Rightarrow$ $B = 0$

Hence $S = 4t^3 - \cos t$

Which is the position of particle at any time 't'.

5.5 Application of Mechanics in Real Life Situation

Example 5:

A stone is projected vertically upwards with a velocity of $10m/sec$. Find the height attained by the stone at any time 't'. When will it attain its maximum height? Also find the maximum height attained by the stone and the total distance travelled when it hits the ground.

Solution:

Since the stone is thrown vertically upwards. So $a = -g \ m/sec^2$.

Since, $v = \int a \, dt$

$$v = \int -g dt = -gt + A$$

Given that at $t = 0; V = 10m/sec$ therefore

$$10 = -g(0) + A$$

$\Rightarrow$ $A = 10$

Thus, $v = -gt + 10$

Which is the velocity of the stone at any time 't'.

Also $S = \int v \, dt$

$\Rightarrow$ $S = \int (-gt + 10)dt$

$\Rightarrow$ $S = -\frac{1}{2}gt^2 + 10t + B$

At $t = 0; S = 0$, therefore:

$$0 = 0 + 0 + B \Rightarrow B = 0$$

Hence, $S = -\frac{1}{2}gt^2 + 10t$

Which is the height attained by the stone at any time 't'. Now when stone attains its maximum height its velocity becomes zero. i.e.

$$-gt + 10 = 0$$

$$\Rightarrow \quad t = \frac{10}{g}\text{sec}$$

Which is the time after that stone attains its maximum height. Put the value of t in

$$S = -\frac{1}{2}gt^2 + 10t.$$

$$S = -\frac{1}{2}g\left(\frac{10}{g}\right)^2 + 10\left(\frac{10}{g}\right)$$

$$= -\frac{50}{g} + \frac{100}{g} = \frac{50}{g}\,m$$

Thus, stone will attain a maximum height of $\frac{50}{g}$ meters.

Total distance travelled by the stone = distance travelled in upward direction + distance travelled in downwards direction

$$= \frac{50}{g} + \frac{50}{g} = \frac{100}{g}m$$

Exercise 5.2

1. A projectile is launched vertically upward from an initial height of $129\ ft$ with an initial velocity of $87\ ft/s$.
   (i) What are the position, velocity, and acceleration functions?
   (ii) When will the projectile hit the ground?
   (iii) What is its impact velocity?
   (iv) When will the projectile reach its maximum height?
   (v) What is the maximum height of projectile?

2. An object has its position defined by $S = t^3 - 9t^2 + 24t + 20$ in feet.
   (i) What are the velocity and acceleration functions?
   (ii) What are the position and velocity of the object when its acceleration is $-6.5ft/s^2$?
   (iii) Find the displacement and the total distance travelled by the particle from $t = 1.5s$ to $t = 7s$.

3. A person is standing on top of the Meinar-e-Pakistan and throws a ball directly upward with an initial velocity of $96\ ft/s$. The Meinar-e-Pakistan is $176\ ft$ high.
   (i) What are the functions for position, velocity, and acceleration of the ball?
   (ii) When does the ball hit the ground and with what velocity?
   (iii) How far does the ball travel during its flight?

4. A particle moves along a line such that its position is:
$$S = 2t^3 - 9t^2 + 12t - 4, \text{ for } t \geq 0.$$
   (i) Find t for which the distance S is increasing
   (ii) Find t for which the velocity is increasing.
   (iii) Find t for which the speed of the particle is increasing.
   (iv) Find the speed when $t = \frac{3}{2}s$.
   (v) Find the total distance travelled in the time interval $[0, 4]$.

5. The position of an object moving on a line is given by $S = 6t^2 - t^3, t \geq 0$, where S is in metres and t is in seconds.
   (i) Determine the velocity and acceleration of the object at $t = 2$.
   (ii) At what time is the object at rest?
   (iii) In which direction is the object moving at $t = 5s$?
   (iv) When is the object moving in a positive direction?
   (v) When does the object return to its initial position?

6. A particle P moves along the x^+-axis. The acceleration of P in time t seconds, when $t \geq 0$, is $a = (3t + 5) m/s^2$ in the positive x-direction. When $t = 0$, the velocity of P is $2\ m/s$ in the positive x-direction. When $t = T$, the velocity of P is $6\ m/s$. Find the value of T.

7. A particle P moves along the x^+-axis. At time t seconds the velocity of P is $v = (3t^2 - 4t + 3) m/s$. When $t = 0$, P is at the origin O. Find the distance of P from O when P is moving with minimum velocity.

8. A particle P moves along the x-axis in a straight line so that, at time t seconds, the velocity of P is $v\ m/s$, where $v = \begin{cases} 10t - 2t^2, & 0 \leq t \leq 6, \\ -\frac{432}{t^2}, & t > 6. \end{cases}$
   At $t = 0, P$ is at the origin O. Find the displacement of P from O when:
   (i) $t = 6s$,
   (ii) $t = 10s$.

9. A particle P moves along the x^+-axis. At time t seconds the velocity v of P is increasing in the direction of x-axis given by $v = \begin{cases} 8t - \frac{3}{2}t^2, & 0 \leq t \leq 4, \\ 16 - 2t, & t > 4. \end{cases}$
   When $t = 0, P$ is at the origin O. Find:
   (i) the greatest speed of P in the interval $0 \leq t \leq 4$,
   (ii) the distance of P from O when $t = 4$,
   (iii) the time at which P is instantaneously at rest for $t > 4$,
   (iv) the total distance travelled by P in the first 10 s of its motion.

10. A particle P moves along the x^+-axis. The acceleration of P at time t seconds is $a = (4t - 8)\ m/s^2$, measured in the increasing direction of x. The velocity of P at time t seconds is $v\ m/s$. Given that $v = 6$ when $t = 0$, find:
    (i) v in terms of t.
    (ii) the distance between the two points where P is instantaneously at rest.
11. A particle P moves along the x^+-axis and its acceleration after a given instant t is given by $a = (4t - 9)m/s^2$, $t \geq 0$. When $t = 1$, P is moving with velocity of $-3\ m/s$.
    (i) Find the minimum velocity of P.
    (ii) Determine the times when P is instantaneously at rest.
    (iii) Find the distance travelled by P in the first $4\frac{1}{2}$ seconds of its motion.
12. A car moving on a straight road is modelled as a particle moving along the x^+-axis, and its acceleration $a\ m/s^2$, after a given instant t, is given by

$$a = \begin{cases} 4 - \frac{1}{2}t & 0 \leq t \leq 8 \\ 0 & t > 0 \end{cases}$$

The car starts from rest.
    (i) Find a similar expression for the velocity of the car, as that of its acceleration.
    (ii) Find the time it takes for the car to reach its maximum speed.
    (iii) Show that the displacement of P form O is given by:

$$S = \begin{cases} 2t^2 - \frac{1}{12}t^3 & 0 \leq t \leq 8 \\ 16t - \frac{128}{3} & t > 8 \end{cases}$$

    (iv) Calculate the time it takes the car to cover the first $1000\ m$.
13. A particle is moving with constant acceleration a with an initial velocity v_i and after time t it covers a distance S. Prove that:
    (i) $v_f = v_i + at$
    (ii) $S = v_i t + \frac{1}{2}at^2$
    (iii) $2aS = v_f^2 - v_i^2$

5.6 Vector Valued Function

Vector valued functions provide a useful method for studying various curves both in plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length and curvature of an object's trajectory.

Definition: A vector valued function is a function where the domain is the subset of real numbers and the range is a vector. In three dimensions $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, where x, y, and z are the functions of t or may be constants.

The example of the vector valued functions is

$r(t) = 3\hat{\imath} + t\hat{\jmath} + (\sin t)\hat{k}$

5.6.1 Domain and Range of Vector valued Function

The **domain** of a vector-valued function $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is the set of all t's for which all the component functions $x(t), y(t)$ and $z(t)$ are defined. If r(t) is defined in terms of component functions and the domain is not specified explicitly, then the domain is the intersection of the domains of the component functions. The **range** of a vector-valued function is the set of all possible output values that the function can produce, which are the vectors.

Example 6: Find the domain of $r(t) = ln|t-1|\hat{\imath} + e^t\hat{\jmath} + \sqrt{t}\hat{k}$

Solution: Here $x(t) = ln|t-1|$, $y(t) = e^t$ and $z(t) = \sqrt{t}$

Domain of $x(t)$ is $(-\infty, 1) \cup (1, \infty)$

Domain of $y(t)$ is $(-\infty, \infty)$

Domain of $z(t)$ is $[0, \infty)$

Thus, the domain of the function is

$$\{(-\infty, 1) \cup (1, \infty)\} \cap \{(-\infty, \infty)\} \cap \{[0, \infty)\} = [0,1) \cup (1, \infty)$$

5.6.2 Construction of Vector Valued Function

Consider a particle is moving in space then its vector function can be constructed by considering it motion along x-, y- and z-axes. Suppose that the motion of particle along x- and y-axes is in circular shape and z-axis is changing linearly 3 times with time, then its vector function is given by

$$r = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + 3t\hat{k}$$

5.6.3 Scalar Valued Function

A function from a vector to some constant is known as scalar valued function.

For example, when we take the magnitude of the vector or the dot product of two same vectors then the vector valued function becomes a scalar valued function. Let us consider the vector:

$$r = 3\hat{\imath} + t\hat{\jmath} + (\sin t)\hat{k}$$

Its magnitude is $|r| = |3\hat{\imath} + t\hat{\jmath} + (\sin t)\hat{k}|$

$$= \sqrt{3^2 + t^2 + \sin^2 t} = \sqrt{9 + t^2 + \sin^2 t}$$

The value the function $\sqrt{9 + t^2 + \sin^2 t}$ also depends on the scalar variable t.

In this case the domain of the function ranges 0 to infinity (as we considering t as time) and the range of the function is from 9 to infinity.

Key Facts

A vector valued function has three coordinates along three different axes whereas scalar valued function gives one scalar value.

5.6.4 Derivative of a Vector Valued Function

An instantaneous rate of change is known as the derivative of vector valued function.

For example, the function represents the position of an object at a point in time t, the derivative of that function represents its velocity at that time on the same point. Consider a function $f(t)$ which has three components that is $f_1(t), f_2(t)$ and $f_3(t)$. The function $f(t)$ is said to be differentiable if all of its three components are differentiable.

$$f(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$$

Then $f(t)$ is differentiable at $t = t_0$ and its derivative is given as

$$f'(t_0) = f'_1(t_0)\hat{\imath} + f'_2(t_0)\hat{\jmath} + f'_3(t_0)\hat{k}$$

Or we may write it as

$$\left.\frac{df}{dt}\right|_{t=t_0} = \frac{d}{dt}f_1(t_0)\hat{\imath} + \frac{d}{dt}f_2(t_0)\hat{\jmath} + \frac{d}{dt}f_3(t_0)\hat{k}$$

Example 7: Find the derivative of the vector function $f(t) = 3t^2\hat{\imath} + 8t\hat{\jmath} - \frac{1}{t^3}\hat{k}$ at t = 5.

Solution: $f(t) = 3t^2\hat{\imath} + 8t\hat{\jmath} - \frac{1}{t^3}\hat{k}$

$$\frac{df}{dt} = \frac{d}{dt}(3t^2)\hat{\imath} + \frac{d}{dt}(8t)\hat{\jmath} - \frac{d}{dt}\left(\frac{1}{t^3}\right)\hat{k} = 3\frac{d}{dt}(t^2)\hat{\imath} + 8\frac{d}{dt}(t)\hat{\jmath} - \frac{d}{dt}(t^{-3})\hat{k}$$

$$\frac{df}{dt} = 3 \times 2t^{2-1}\hat{\imath} + 8 \times 1\hat{\jmath} - (-3t^{-3-1})\hat{k} = 6t\hat{\imath} + 8\hat{\jmath} + -3t^{-4}\hat{k}$$

$$\frac{df}{dt} = 6t\hat{\imath} + 8\hat{\jmath} + \frac{-3}{t^4}\hat{k}$$

$$\left.\frac{df}{dt}\right|_{t=5} \frac{df}{dt} = 6(5)\hat{\imath} + 8\hat{\jmath} + \frac{-3}{(5)^4}\hat{k} = 30\hat{\imath} + 8\hat{\jmath} + \frac{-3}{625}\hat{k}$$

5.6.5 Velocity and Acceleration of a Vector Valued Function

The derivative of the vector valued function gives the velocity of the function at the particular point and if we take again the derivative of the velocity function then it will be acceleration of the function at the particular point.

Consider the function in space

$$f(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$$

Then $f(t)$ is differentiable at $t = t_0$ and its derivative is given as

$$f'(t_0) = v(t_0) = f'_1(t_0)\hat{\imath} + f'_2(t_0)\hat{\jmath} + f'_3(t_0)\hat{k} \qquad \text{where } f'(t_0) = v(t_0)$$

Which gives the velocity of the function at $t = t_0$ and for the acceleration of the function again we differentiate the velocity function to get the acceleration

$$a(t_0) = \frac{d}{dt}v(t_0) = f''_1(t_0)\hat{\imath} + f''_2(t_0)\hat{\jmath} + f''_3(t_0)\hat{k}$$

Example 8: Find the velocity and acceleration of function $f(t) = 2t^2\hat{\imath} + 3t^4\hat{\jmath} - t^3\hat{k}$ at t = 1.

Solution: $v(t)\ \frac{d}{dt}f(t) = \frac{d}{dt}(2t^2)\hat{\imath} + \frac{d}{dt}(3t^4)\hat{\jmath} - \frac{d}{dt}(t^3)$

$$v(t) = 2 \times 2t\hat{\imath} + 3 \times 4t^3\hat{\jmath} - 3t^2\hat{k} = 4t\hat{\imath} + 12t^3\hat{\jmath} - 3t^2\hat{k}$$

$$v(t) = 4t\hat{\imath} + 12t^3\hat{\jmath} - 3t^2\hat{k}$$

$$\therefore\ v(1) = 4(1)\hat{\imath} + 12(1)^3\hat{\jmath} - 3(1)^2\hat{k} = 4\hat{\imath} + 12\hat{\jmath} - 3\hat{k}$$

For acceleration we differentiate the velocity:

$$\boldsymbol{a}(t) = \frac{d}{dt}\boldsymbol{v}(t) = \frac{d}{dt}(4t)\hat{\imath} + \frac{d}{dt}(12t^3)\hat{\jmath} - \frac{d}{dt}(3t^2)\hat{k}$$

$$\boldsymbol{a}(t) = 4 \times 1\hat{\imath} + 12 \times 3t^2\hat{\jmath} - 3 \times 2t\hat{k} = 4\hat{\imath} + 36t^2\hat{\jmath} - 6t\hat{k}$$

$$\therefore \quad \boldsymbol{a}(1) = 4\hat{\imath} + 36(1)^2\hat{\jmath} - 6(1)\hat{k} = 4\hat{\imath} + 36\hat{\jmath} - 6\hat{k}$$

Exercise 5.3

1. If $\boldsymbol{r}(t)$ is the position of the particle. Find its domain and the range at given point. Also find its first and second derivative.
   i. $\boldsymbol{r}(t) = (t+1)\boldsymbol{i} + (t^2-1)\boldsymbol{j} + 5t\boldsymbol{k}, \quad t = 2$
   ii. $\boldsymbol{r}(t) = \frac{t}{t+1}\boldsymbol{i} + \frac{1}{t}\boldsymbol{j} + t^3\boldsymbol{k}, \quad t = -\frac{1}{2}$
   iii. $\boldsymbol{r}(t) = e^t\boldsymbol{i} + \frac{2}{9}e^{2t}\boldsymbol{j} + 5e^{-t}\boldsymbol{k}, \quad t = ln2$
   iv. $\boldsymbol{r}(t) = (\cos 2t)\boldsymbol{i} + (3\sin 2t)\boldsymbol{j} + 5t\boldsymbol{k}, \quad t = 0$
2. Find the velocity and acceleration of the function at $t = 0$
   i. $\boldsymbol{r}(t) = (3t+1)\boldsymbol{i} + \sqrt{3}t\boldsymbol{j} + t^2\boldsymbol{k}$
   ii. $\boldsymbol{r}(t) = \left(\frac{t}{\sqrt{2}}\right)\boldsymbol{i} + \left(\frac{t}{\sqrt{2}} - 16t^2\right)\boldsymbol{j} - 2t\boldsymbol{k}$
   iii. $\boldsymbol{r}(t) = (\ln(t^2+1))\boldsymbol{i} + (\tan^{-1} t)\boldsymbol{j} + \sqrt{t^2+1}\boldsymbol{k}$
   iv. $\boldsymbol{r}(t) = \frac{4}{9}(t+1)^{3/2}\boldsymbol{i} + \frac{4}{9}(1-t)^{3/2}\boldsymbol{j} + \frac{1}{3}t\boldsymbol{k}$
3. Find the scalar valued function in term of the magnitude
   i. $\boldsymbol{r}(t) = (3t-7)\boldsymbol{i} + t\boldsymbol{j} - t^2\boldsymbol{k}$
   ii. $\boldsymbol{r}(t) = \left(\frac{t}{\sqrt{2}}\right)\boldsymbol{i} + \left(\frac{t}{\sqrt{2}} + 16t^2\right)\boldsymbol{j} - 2t\boldsymbol{k}$
   iii. $\boldsymbol{r}(t) = (\ln(t^2+1))\boldsymbol{i} + (\tan t)\boldsymbol{j} + \sec t\,\boldsymbol{k}$
   iv. $\boldsymbol{r}(t) = \frac{4}{9}(t+1)^{3/2}\boldsymbol{i} + \frac{4}{9}(1-t)^{3/2}\boldsymbol{j} + \frac{1}{3}t^{3/2}\boldsymbol{k}$

Review Exercise

1. Choose the right option.
   (i) Which of the following quantities is a vector?
   (a) Charge (b) Mass (c) Momentum (d) Time

(ii) Which of following is scalar quantity?

(a) Displacement (b) Weight (c) Force (d) Work

(iii) Which of following can be used to determine the magnitude of velocity?

(a) Area under acceleration-time graph
(b) Area under velocity-time graph
(c) Gradient of an acceleration-time graph
(d) Gradient of a velocity-time graph

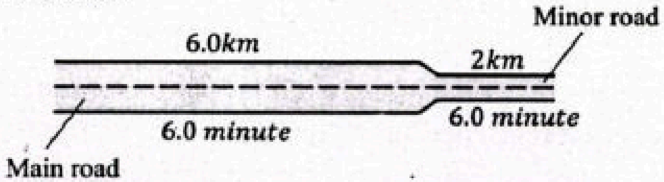
(iv) The winner of 400 metre race must have the greatest:

(a) acceleration (b) average speed
(c) instantaneous speed (d) maximum speed

(v) A car travels $100km$. The journey takes two hours. The highest speed of the car is $80km/h$, and the lowest speed $40km/h$. What is the average speed for the journey?

(a) $40km/h$ (b) $50km/h$ (c) $60km/h$ (d) $120km/h$

(vi) A car travels $6.0km$ along a main road in 6.0 minutes. It then travels $2km$ along minor road in 6.0 minutes. Which calculation of average speed for the whole journey is correct?



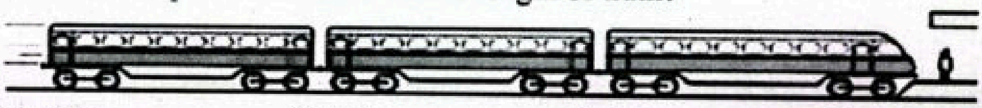
(a) $\frac{8.0}{12.0} = 0.67\frac{km}{min}$ (b) $\frac{12.0}{8.0} = 1.5\frac{km}{min}$

(c) $8.0 + 12.0 = 20\frac{km}{min}$ (d) $8.0 \times 12.0 = 96\frac{km}{min}$

(vii) Which person is experiencing an acceleration?

(a) A driver of a car that is braking to stop at traffic light.
(b) A passenger in a train that is stationary in a railway station.
(c) A shopper in a large store ascending an escalator (moving stairs) at a uniform speed.
(d) A skydiver that is falling at a constant speed towards the Earth.

(viii) A child is standing on the platform of station. A train travelling at $30m/s$ takes $3.0s$ to pass the child. What is length of train?

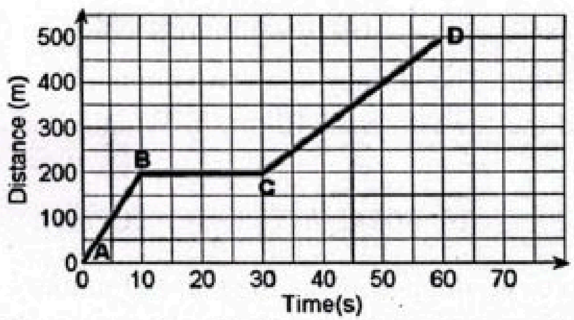


(a) $10m$ (b) $27m$ (c) $30m$ (d) $90m$

(ix) A heavy object is released near the surface of earth and falls freely. Air resistance can be ignored. Which statement about the acceleration of the object due to gravity is correct?

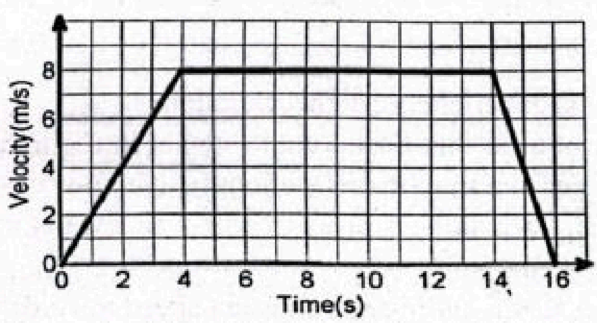
(a) The acceleration depends on mass of the object.
(b) The acceleration depends on volume of the object.
(c) The acceleration is constant.
(d) The acceleration is initially zero and increases as the object falls.

2. The graph shows the movement of a car from point A to point D, D is $500m$ from A, note that there are two slant lines AB and CD. The slant lines indicate that car is moving. The flat line BC indicates that car is stopped or is at rest.



(i) Calculate the speed of car during the first 10 seconds.
(ii) For how long did the car stop?
(iii) What is speed of car on its journey from C to D?
(iv) On which part of the journey did the car travel faster?
(v) What is average speed of the car for whole journey?
(vi) What is the average speed of the car for the time it was moving?

3. The graph below shows how the speed of an athlete varies during a race.



(i) Calculate the acceleration of athlete during the first 4 second.
(ii) What was the athlete doing between the 4^{th} and 14^{th} second?
(iii) Calculate the deceleration of the athlete in final stage of the race.
(iv) How far has the athlete moved in the first 4 seconds?
(v) What is the total distance travelled by the athlete?

4. Let $v(t) = \frac{1}{\pi} + \sin 3t$ represent the velocity of an object moving on a line. At $t = \frac{\pi}{3}$ the position is 4.
   (i) Write the acceleration function.
   (ii) Write the position function.
   (iii) At $t = \frac{\pi}{4}$, is the object speeding up or slowing down? Explain your answer.
   (iv) On the interval $\left[\frac{\pi}{2}, \pi\right]$, what is the velocity of object when acceleration is 3?

5. A particle moves along y-$axis$ so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - tan^{-1}(e^t)$. At time $t = 0$ the particle is at $y = -1$.
   (a) Find acceleration of the particle at time $t = 2$.
   (b) Is the speed of particle increasing or decreasing at time $t = 2$? Give a reason for your answer.
   (c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.