DIFFERENTIAL EQUATIONS

After studying this unit, students will be able to:

- Identify and construct first order differential equations from practical situations.
- Give the concept of solution of differential equation.
- Solve differential equations of first order and first degree of the form:
  - separable variables equations,
  - homogeneous equations,
- Solve real life problems related to differential equations such as population growth and decay, cooling/warming law, flow of electricity, series circuit, economics and finance, radioactive decay etc.

Differential equations are mathematical equations that describe relationships between a function and its derivatives. They are essential for modeling real-world phenomena in various fields. These equations are used to describe how quantities change over time and how they relate to each other. Differential equations are essential tools for modeling dynamic systems in real life.

These equations have wide-ranging applications across various fields of science and engineering, as they are fundamental in modeling dynamic systems and natural phenomena. In physics, they describe the motion of objects under forces, such as in Newton's laws and wave propagation. In biology, they are used to model population dynamics, the spread of diseases, and the interaction between species in ecosystems. In engineering, differential equations are crucial for analyzing electrical circuits, control systems, and mechanical vibrations. In chemistry, they help in understanding reaction rates and diffusion processes. In economics, they are used to model investment growth, market behavior, and optimization problems. Overall, differential equations serve as powerful tools for understanding and predicting the behavior of systems that change over time or space.

Introduction

Many problems in engineering and science can be formulated in terms of differential equations. The formulation of mathematical models is basically to address real-world problems which have been one of the most important aspects of applied mathematics. Differential equations arise in many areas of science and technology, specifically whenever a relation involving some continuously varying quantities and their rates of change in space and/or time (expressed as derivatives). This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's laws allow relating the position, velocity, acceleration and various forces acting on a body and state this relation as a differential equation for the unknown position of the body as a function of time. Such equations are called differential equations.

A mathematical model is a mathematical construction such as a differential equation, that simulates a natural engineering phenomenon. Most applications of differential equations take the form of mathematical models. For example, consider the problem of determining the velocity v of a falling object.

Newton's second law of motion tells us that the net force on the object is equal to the product of the mass, m and its acceleration, $\frac{dv}{dt}$

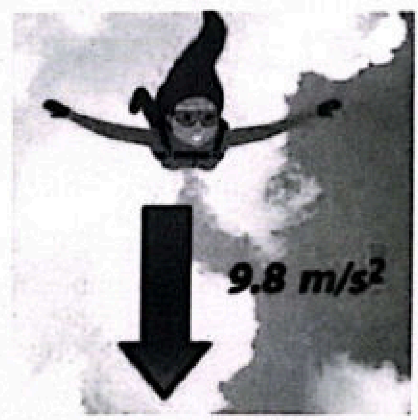
$$m\frac{dv}{dt} = F$$

This law is a differential equation as it contains derivative.

Ignoring air resistance, for an object falling close to the Earth's surface the force is $F = mg$, directed downward, where g is approximately 9.80 meters per second per second. Thus, the differential equation:

$$m\frac{dv}{dt} = mg$$

is a mathematical model corresponding to the free-falling object.



Key Facts

A good mathematical model has two important properties:

- It is sufficiently simple that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the behavior of the real problem.

4.1 Differential Equation

A differential equation is an equation containing one or more derivatives of an unknown function. A differential equation is an ordinary differential equation if it involves an unknown function of only one variable. For now, we will consider in this unit only ordinary differential equation just call them differential equation (DE). The word differential equation means involvement of derivative and equation are must.

Suppose we have function:

$$f(x) = y = x^4 + c, \text{ where } c \text{ is arbitrary constant.} \qquad \text{(i)}$$

Differentiating (i) with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 \qquad \text{(ii)}$$

This is called a differential equation.

The simplest differential equations are of the form:

$$\frac{dy}{dx} = f(x) \quad \text{or} \quad y' = f(x)$$

Where f is known function of x.

A mathematical equation containing the derivatives of one dependent variable, with respect to one independent variable, is said to be a differential equation (DE).

Some more examples of differential equations are:

$$\frac{dy}{dx} = 2x^2 - 1, \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3, \quad \frac{dy}{dx} = x + y, \quad xy\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx} + x^2$$

Where, y is a dependent and x is an independent variable.

4.2 Order and Degree of Differential Equation

The order of the differential equation is the order of the highest order derivative present in the equation. Here some examples for different orders of the differential equation are given.

- $\frac{dy}{dx} - 2x = 1$, the order of the DE is 1.
- $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 5 = 0$, the order of DE is 2.

4.2.1 Types of DE w.r.t. Order

(i) First Order Differential Equation

A differential equation containing first order derivatives is called first order differential equation.

$$\frac{dy}{dx} = f(x, y)$$

For example, $\frac{dy}{dx} = 3x$ is a first order differential equation.

(ii) Second Order Differential Equation

The equation which includes the second-order derivative is called the second-order differential equation. It is represented as:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f(x, y)$$

For example, $y'' + xy' + y = 0$ is a second order differential equation.

4.2.2 Degree of Differential Equation

The degree of the differential equation is the power of the highest order derivative in the equation.

order   degree

$$\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right) + y = x + 1$$

Examples:

- $\frac{dy}{dx} - 3 = 0$, degree is 1.
- $(y'')^2 + 6y' = 9$, degree is 2.
- $(y'')^3 - xy' + y = 0$, degree is 3.

Key Facts

- Order and degree (if defined) of a differential equation are always positive integers.
- $y' - \log(y') + 3 = 0$, is not a polynomial equation in y' and the degree of such differential equation cannot be defined.

Example 1:

Determine the order and degree of the following differential equations.

(i) $\frac{dy}{dx} = -\frac{x}{y}$ **(ii)** $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + x = 0$ **(iii)** $\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 1$

Solution:

(i) order: 1, degree: 1 **(ii)** order: 2, degree: 1 **(iii)** order: 3, degree: 2

Some more types of differential equations are as follows.

- Ordinary differential equations (ODE)
- Partial differential equations (PDE)
- Linear differential equations
- Nonlinear differential equations
- Homogeneous differential equations
- Non-homogeneous differential equations

We will here discuss only ordinary differential equations.

4.2.3 Ordinary Differential Equation (ODE)

If in a differential equation, only one independent variable is involved, the equation is called an ordinary differential equation. It contains one or more of its derivatives with respect to the independent variable.

The general form of nth order ODE is a function F of x, y and derivatives of y.

$$F(x, y, y', \ldots, y^{(n-1)}) = y^{(n)}$$

Which is called an explicit ODE of order n.

Examples of ODE are:

(i) $y' - 4y + 2 = 0$

(ii) $(2x + 3y)dy = (x - 2y)dx = 0$

(iii) $xy'' - 5y' + 11 = y$

(iv) $(y'')^3 + 4y' + 2y = 1$

Key Facts

If a differential equation is not ODE, it is then PDE.

4.2.4 Linear and Non-Linear Differential Equations

Differential equations are classified into linear DEs or nonlinear DEs.

An n^{th} order differential equation is said to be linear if it can be written in the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

that is, it satisfies the following two conditions:

(a) the dependent variable (y) and all its derivatives in the equation are linear (i.e. of power one).

(b) all the coefficients $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ and the function $f(x)$ are either constants or depend only on the independent variable (x).

Note: If any one of these two conditions is not satisfied, then the DE is said to be nonlinear DE.

Example 2:

Identify linear and non-linear differential equations in the following.

(i) $\frac{dy}{dx} = a$ (ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 1$ (iii) $\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} - 2 = 0$

(iv) $\frac{dy}{dx} = \frac{x}{y}$ (v) $\frac{d^2y}{dx^2} + 12xy = 0$ (vi) $\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2x = 5$

Solution:

(i) linear (ii) linear (iii) non-linear

(iv) non-linear (v) linear (vi) non-linear

Note:

The differential equation $y^2(x-3)\frac{dy}{dx} = 2xy^2$ is not linear, but it can be reduced to linear differential equation if we divide both sides of it by y^2 as follows:

$$\frac{y^2}{y^2}(x-3)\frac{dy}{dx} = \frac{2xy^2}{y^2}$$

or $(x-3)\frac{dy}{dx} = 2x$ which is linear differential equation.

4.3 Concept of Solution of Differential Equation

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions as the set of functions that satisfy the equation.

A solution of a differential equation is any function f defined on some interval I that reduces the equation to an identity.

To solve a differential equation such as $\frac{dy}{dx} - x = 0$, we mean to find an unknown function $y = f(x)$ or $y = f(x, y)$.
Consider a simple first order differential equation:

$$\frac{dy}{dx} = f(x) \qquad \text{(i)}$$

Equation (i) can be solved by integration. If $f(x)$ is continuous function, then integrating both sides of (i) gives:

$$y = \int f(x)dx = F(x) + c$$

Where $F(x)$ is an anti-derivative of $f(x)$.

For example, the solution of differential equation $\frac{dy}{dx} = 1 + e^{2x}$ implies:

$$y = \int (1 + e^{2x})dx = x + \frac{1}{2}e^{2x} + c$$

Key Facts

Only the simplest differential equations admit solutions given by explicit formulas. However, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

Example 3:

Show that $x^2 + y^2 = c$ is a solution of the differential equation $y\frac{dy}{dx} + x = 0$. Also plot the graph of solution.

Solution: We have $x^2 + y^2 = c$

Differentiating w.r.t. x, we get:

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad y\frac{dy}{dx} + x = 0$$

Hence, $x^2 + y^2 = c$ is the solution of differential equation $y\frac{dy}{dx} + x = 0$.
We note that the solution, $x^2 + y^2 = c$ depends upon an arbitrary constant c.
By choosing different values of c, we get different solutions.
Let us take $c = 1, 4, 9, 16, \ldots$ then we get different solutions as:

$$x^2 + y^2 = 1 = 1^2$$
$$x^2 + y^2 = 4 = 2^2$$
$$x^2 + y^2 = 9 = 3^2$$
$$x^2 + y^2 = 16 = 4^2 \text{ and so on.}$$

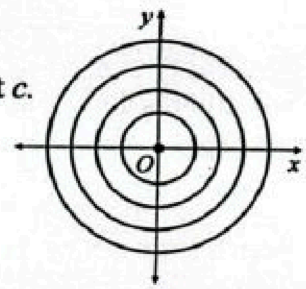


Fig. (i)

These solutions represent a family of circles with radii 1, 2, 3, 4, ...with centre (0, 0) in Fig. (i).
Thus, the solution, $x^2 + y^2 = c$ represents a family of infinite number of circles.
From this example, it has been observed that there are two types of solutions of a differential equation.

(i) general solution (ii) particular solution

4.3.1 General Solution of DE

The solution that contains as many arbitrary constants as the order of the differential equation is called a general solution. It is the relation between the independent variables x and dependent variable y which is obtained after removing the derivatives (by integration) where the relation contains arbitrary constant to denote the order of an equation. In the above example, $x^2 + y^2 = c$ is the general solution.

4.3.2 Particular Solution of DE

The solution free from arbitrary constants is called a particular solution.

If particular values are given to the arbitrary constant, the particular solution of the differential equations is obtained. In the above example, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$ etc. are the particular solutions.

Key Facts

- The solution of a first-order differential equation contains one arbitrary constant whereas the second-order differential equation contains two arbitrary constants.
- The general solution of a differential equation represents a family of curves.
- The particular solution of a differential equation represents a particular curve for a particular value of constant from the family of curves.

Example 4:

Verify that $y = \frac{x^4}{16}$ is a solution of the differential equation $\frac{dy}{dx} - xy^{\frac{1}{2}} = 0$.

Solution: Given solution is:

$$y = \frac{x^4}{16} \qquad \text{(i)}$$

Differentiating (i) with respect to x, we get:

$$\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$$

Substituting for y and $\frac{dy}{dx}$ in the left side of given differential equation:

$$\frac{dy}{dx} - xy^{\frac{1}{2}} = \frac{x^3}{4} - x\left(\frac{x^4}{16}\right)^{\frac{1}{2}} = \frac{x^3}{4} - x\left(\frac{x^2}{4}\right) = \frac{x^3}{4} - \frac{x^3}{4} = 0$$

Which is true $\forall x \in \mathbb{R}$.

Thus, $y = \frac{x^4}{16}$ is a solution of the differential equation $\frac{dy}{dx} - xy^{\frac{1}{2}} = 0$.

Check Point

Find the DE corresponding to the equation $y = 3x^2 + c$. Of which type the family of curves does the solution represent?

Example 5:

Is the function $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$ on the interval $(-\infty, +\infty)$?

Solution: Given solution is:

$$y = xe^x \qquad \text{(i)}$$

Differentiating (i) with respect to x, we get:

$$y' = xe^x + e^x \qquad \text{(ii)}$$

Differentiating (ii) with respect to x, we get:

$$y'' = xe^x + 2e^x \qquad \text{(iii)}$$

Substituting the values in the left side of given differential equation:

$$y'' - 2y' + y = xe^x + 2e^x - 2(xe^x + e^x) + xe^x$$
$$= xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

Which is true $\forall x \in \mathbb{R}$. Thus, $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$ on the interval $(-\infty, +\infty)$.

Note: In examples (4) and (5), we notice that the constant function $y = 0$ for $(-\infty < x < +\infty)$ also satisfies the given differential equation.

Key Facts

- A solution of differential equation that is identically zero on any interval is often called a trivial solution.
- Every differential equation that we write necessarily has a solution either real or imaginary. For example, the differential equation $(y')^2 + 1 = 0$ has no real solution.

4.4 Formation of Differential Equation

We can form a differential equation by eliminating the constants appearing in an algebraic equation; the solution of differential equation.

Let us find the differential equation corresponding to the equation $y = e^x$.

Now, $y = e^x$ gives $y' = e^x$ and solving both equations, we get:

$$y' - y = 0,$$

which is a differential equation.

Example 6:

Find the DE corresponding to the equation $y = a\cos x + b\sin x$

Solution: Given that

$$y = a\cos x + b\sin x$$
$$y' = -a\sin x + b\cos x$$
$$y'' = -a\cos x - b\sin x = -(a\cos x + b\sin x)$$

$y'' = -y$ or $y'' + y = 0$ is required differential equation.

Note: $y = a\cos x + b\sin x$ is a solution of differential equation $y'' + y = 0$.

Example 6:

Eliminate the arbitrary constants from the following equation and form a differential equation of the lowest order: $y = A\sin(2x - B)$

Solution: Given that:

$$y = A\sin(2x - B) \qquad \text{(i)}$$

Taking derivative of equation (i) with respect to x.

$$\frac{dy}{dx} = 2A\cos(2x - B)$$

Again, differentiating w.r.t x, we get:

$$\frac{d^2y}{dx^2} = -4A\sin(2x - B) = -4[A\sin(2x - B)]$$

$$\frac{d^2y}{dx^2} = -4y \quad \Rightarrow \quad \frac{d^2y}{dx^2} + 4y = 0$$

Which is a second order differential equation (ODE). Its physical interpretation is that the acceleration varies as the distance varies. This essentially illustrates the differential equation governing the simple harmonic motion.

Check Point

Check whether equation of parabola

$$y^2 = 4a(x - b)$$

where a and b are arbitrary constants, is a solution of differential equation:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

4.4.1 Explicit and Implicit Solution

A solution of a differential equation that can be written in the form $y = f(x)$ is said to be an explicit solution while a solution of the form $f(x, y) = 0$ is said to be an implicit solution.

Example 7:

Prove that, for $-2 < x < 2$, the relation $x^2 + y^2 - 4 = 0$ is an implicit solution of differential equation:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solution: Given equation is:

$$x^2 + y^2 - 4 = 0 \qquad \text{(i)}$$

By implicit differentiation, we have:

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

Note: The relation $x^2 + y^2 - 4 = 0$ in this example defines two explicit differentiable functions $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$ in the interval $(-2, 2)$.

4.4.2 Number of Solutions

A given differential equation usually possesses an infinite number of solutions. e.g.,

(i) For any value of c, the function $y = \frac{c}{x} + 1$ is a solution of the first order differential equation:

$$x\frac{dy}{dx} + y = 1 \quad \text{on the interval } (0, \infty)$$

The solution $y = \frac{c}{x} + 1$ represents infinite number of solutions for various values of c. In particular, for $c = 0$, we obtain a constant solution $y = 1$.

(ii) The functions $y = c_1 \cos 4x$ and $y = c_2 \sin 4x$ where c_1 and c_2 are arbitrary constants, are solutions of the differential equation:

$$y'' + 16y = 0$$

It is to be noted that the sum of solutions $y = c_1 \cos 4x$ and $y = c_2 \sin 4x$:

$$y = c_1 \cos 4x + c_2 \sin 4x$$

is also a solution of differential equation $y'' + 16y = 0$.

Exercise 4.1

1. Find the order and degree of each of differential equations.

(i) $(1 - x)y'' - 4xy' + 5y = \cos x$ (ii) $yy' + 2y = 1 + x^2$

(iii) $(y'')^3 - 3y' + 2y = x$ (iv) $(y')^2 - yy'' + 2 = 0$

(v) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ (vi) $\frac{dy}{dx} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$

(vii) $x^3\frac{d^4y}{dx^4} - x^2\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^5 + 4xy - 3y = 0$

2. Eliminate the arbitrary constants from the equations.

(i) $y = ae^x + be^{-x} + c$ (ii) $y = \cos(x + b)$

(iii) $y = mx + c$ (iv) $y = bx^2 + 2ax$

3. Verify that the indicated function is a solution of the given differential equation.

(i) $2y' + y = 0$; $y = e^{-\frac{x}{2}}$

(ii) $\frac{dy}{dx} - 2y = e^{3x}$; $y = e^{3x} + 10e^{2x}$

(iii) $y' = 25 + y^2$; $y = 5\tan 5x$

(iv) $y' + y = \sin x$; $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + 10e^{-x}$

(v) $x^3 dy - 2dx = 0$; $y = -\frac{1}{x^2} + 6$

(vi) $y' - \frac{1}{x}y = 1$; $y = x\,lnx, x > 0$

4. Find the order and degree, if defined, for the differential equation $dy - \sin x\,dx = 0$.
5. Verify that the function $y = a\cos x + b\sin x$, where, $a, b \in R$, is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.
6. Show that $y_1 = x^2$ and $y_2 = x^3$ are both solutions of:

$$x^2y'' - 4xy' + 6y = 0$$

(i) Are c_1y_1 and c_2y_2 also a solution? (ii) Check if $y_1 + y_2$ is also a solution.

4.5 Solution of Differential Equation

We know that the solution of an algebraic equation say, $x^2 - 5x + 6 = 0$, is the value of x that satisfies the given equation.

But the solution of a differential equation say, $y\,dy = x\,dx$, means to remove derivative from the given differential equation by using some technique/procedure. All these techniques involve integration. After removing derivative, the dependent variable y in terms of independent variable x (explicit: $y = f(x)$ or implicit: $g(x, y) = 0$) will be the solution of differential equation.

4.5.1 Solution of Differential Equations of First Order and First Degree

To solve the first-order differential equation of first degree, some standard forms are available to get the general solution. Some of them are:

- Variable separable differential equations
- Reducible into the variable separable differential equations
- Homogeneous differential equations
- Equations reducible to homogeneous differential equations

(i) Variable Separable Differential Equations

A first order differential equation of the form:

$$\frac{dy}{dx} = f(x)g(y)$$

where f is the function of x only and g is the function of y only, is said to be separable or to have separable variables.

For example, the equation $\frac{dy}{dx} = y^3xe^{2x+y}$ is separable. We can write it as:

$$\frac{dy}{dx} = (xe^{2x})(y^3e^y) = f(x)g(y)$$

The equation $\frac{dy}{dx} = y + \sin x$ is not separable. We cannot write $y + \sin x$ in the form of $f(x)g(y)$.

Example 8: Solve $\frac{dy}{dx} = \frac{x^2}{y}$

Solution: We first separate the variables of given equation as follows:

$$ydy = x^2dx$$

Integrating both sides, we have:

$$\int ydy = \int x^2dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + c_1$$

$$3y^2 = 2x^2 + 6c_1$$

$$\Rightarrow \quad 3y^2 = 2x^2 + c \qquad (c = 6c_1)$$

Note: To avoid lengthy process, we write constant at one side only.

Example 9: Solve $(1+x)dy - y\,dx = 0$

Solution: The given equation can be written as:

$$(1+x)dy = y\,dx$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrating both sides, we have:

$$\int \frac{dy}{y} = \int \frac{dx}{1+x} \quad \Rightarrow \quad \ln y = \ln(x+1) + \ln c$$

$$\Rightarrow \quad \ln y = \ln[c(x+1)]$$

Taking antilog on both sides, we have:

$$y = c(x+1)$$

Example 10: Solve $\frac{dy}{dx} = \frac{1}{x\tan y}$

Solution: The given equation is:

$$\frac{dy}{dx} = \frac{1}{x\tan y} \quad \Rightarrow \quad \tan y\,dy = \frac{1}{x}dx$$

$$\int \tan y\,dy = \int \frac{1}{x}dx \quad \Rightarrow \quad -\ln(\cos y) = \ln x + \ln c$$

$$\Rightarrow \quad 0 = \ln(\cos y) + \ln x + \ln c \quad \Rightarrow \quad \ln(cx\cos y) = 0$$

$$\Rightarrow \quad e^{\ln(cx\cos y)} = e^0 \quad \Rightarrow \quad cx\cos y = 1$$

$$\Rightarrow \quad x\cos y \quad \Rightarrow \quad x\cos y = C \qquad \left(\frac{1}{c} = C\right)$$

(ii) Initial Condition and Initial Value Problem (IVP)

We have observed that general solution of differential equation contains the same number of arbitrary constants as is the order of differential equation. Sometimes we need to find the solution of DE subject to the supplementary conditions.

Suppose we want to find the solution of differential equation $\frac{dy}{dx} = f(x,y)$ subject to conditions $y = y_0$ at $x = x_0$. If we substitute $x = x_0$ and $y = y_0$, in the solution of $\frac{dy}{dx} = f(x,y)$ then we get a particular value of constant obtained in the general solution. Thus, a particular solution is obtained with the choice of some values of variables given in the differential equation. We call $y(x_0) = y_0$ as initial condition and the differential equation of $\frac{dy}{dx} = f(x,y)$ becomes an initial value problem as follows:

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$$

Example 11: Solve the initial value problem: $\frac{dy}{dx} = -\frac{x}{y},\ y(1) = 3$

Solution: The given equation can be written as:

$$ydy = -x\,dx$$

Integrating both sides, we get:

$$\int ydy = \int -x\,dx \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + c \quad \Rightarrow \quad x^2 + y^2 = 2c \qquad \text{(i)}$$

Using the initial condition i.e. $x = 1, y = 3$ in equation (i), we get:

$$1^2 + 3^2 = 2c \quad \Rightarrow \quad 2c = 10 \quad \Rightarrow \quad c = 5$$

Substituting the value of c in equation (i), we get:

$$x^2 + y^2 = 2 \times 5 \quad \Rightarrow \quad x^2 + y^2 = 10$$

Which is a solution of initial value problem representing a circle with centre (0, 0) and radius $\sqrt{10}$.

Example 12: Solve $\frac{dy}{dx} = 2x$ such that $y(2) = 4$.

Solution: Given equation can be written as:

$$dy = 2x\,dx$$

Integrating both sides, we get:

$$\int dy = \int 2x\,dx \quad \Rightarrow \quad y = 2\left(\frac{x^2}{2}\right) + c \quad \Rightarrow \quad y - x^2 = c \qquad \text{(i)}$$

Using $x = 2, y = 4$ in equation (i), we get:

$$4 - 2^2 = c \quad \Rightarrow \quad c = 0$$

Substituting the value of c in equation (i), we get:

$$y - x^2 = 0 \qquad \text{(ii)}$$

Which is a solution of initial value problem.

Note: The general solution (i) represents a family of parabolas for different values of c whereas the particular solution (ii) represents a member of family that passes through (2, 4).

Check Point

Show that the solution of differential equation $\frac{dy}{dx} = 2$ represnts a family of parallel lines. Draw some of parallel lines. Also solve differential equation for initial condition $y(0) = 1$.

Exercise 4.2

Slove the differential equations by separating the variables.

1. $\frac{dy}{dx}=-\frac{1}{e^{3x}}$
2. $x\frac{dy}{dx}=4y$
3. $\frac{dy}{dx}=\frac{y^3}{x^2}$
4. $\frac{dy}{dx}=e^{2x+3y}$
5. $\frac{dy}{dx}=\frac{x^2y^2}{1+x}$
6. $2y(x+1)dy=xdx$
7. $\frac{dy}{dx}+y^2\sin x=0$
8. $(\sin x+\cos x)dx=\cot y\cos x\,dy$

Solve the initial value problems.

9. $\frac{dy}{dx}=\cos x\ ;\ y(0)=1$
10. $2\frac{dy}{dx}=4x\,e^{-x}\ ;\ y(0)=2$
11. $\frac{dy}{dx}+\left(\frac{1+x}{x}\right)y=0\ ;\ y(1)=1$
12. $\frac{dy}{dx}+y\tan 2x=0\ ;\ y(0)=2$
13. $\frac{dy}{dx}=y^2+4\ ;\ y(0)=-2$
14. $(1-x)dy+y^{-1}dx=0\ ;\ y(0)=2$
15. $2(y-1)dy=(3x^2+4x+2)dx\ ;\ y(0)=-1$

4.6 Homogeneous First order Differential Equations

Before considering a homogeneous differential equation of first order, we need to recall a homogeneous function.

4.6.1 Homogeneous Function

If a function f has the property that:

$$f(tx,ty)=t^n f(x,y)$$

where $t\in R^+$, $n\in R$. Then f is said to be a homogenous function of degree n.

Example 13: Check whether the function

(i) $f(x,y)=\sqrt{x^3+y^3}$ (ii) $f(x,y)=x^2+y^2+2$ (iii) $(x,y)=\frac{x}{2y}+4$

are homogeneous or not. If homogeneous then find degree.

Solution:

(i) $f(x,y)=\sqrt{x^3+y^3}$

Replacing x with tx and y with ty, we have:

$$f(tx,ty)=\sqrt{(tx)^3+(ty)^3}=\sqrt{t^3x^3+t^3y^3}$$

$$=t^{\frac{3}{2}}\sqrt{x^3+y^3}=t^{\frac{3}{2}}f(x,y)$$

$\therefore f(x,y)$ is homogeneous function of degree $\frac{3}{2}\in R$

Check Point

Check whether the functions are homogeneous or not. If homogeneous then find degree.

(a) $f(x,y)=x^2-3xy+y^2$

(b) $f(x,y)=x-\sqrt{xy}+5y$

(ii) $f(x,y) = x^2 + y^2 + 2$

$$f(tx,ty) = (tx)^2 + (ty)^2 + 2 = t^2x^2 + t^2y^2 + 2 \neq t^2 f(x,y)$$

$\therefore f(x,y)$ is not homogeneous function.

(iii) $f(x,y) = \frac{x}{2y} + 4$

$$f(tx,ty) = \frac{tx}{2ty} + 4 = \frac{x}{2y} + 4 = t^0 f(x,y)$$

$\therefore f(x,y)$ is homogeneous function of degree 0.

4.6.2 Homogeneous Differential Equations

A differential equation of the form:

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

is said to be homogeneous if both M and N are homogeneous functions of the same degree. In other words, differential equation (1) is homogeneous if

$$M(tx,ty) = t^n M(x,y) \quad \text{and} \quad N(tx,ty) = t^n N(x,y)$$

have the same degree n. The differential equation can be reduced to separable variables by either substituting $y = ux$ or $x = vy$, where u and v are new dependent variables. In particular, if we choose $y = ux$, then:

$$\frac{dy}{dx} = u + x\frac{du}{dx} \quad \text{or} \quad dy = udx + xdu$$

Hence the differential equation becomes:

$$P(x,ux)dx + Q(x,ux)[udx + xdu] = 0$$

$$\Rightarrow \quad x^n P(1,u)dx + x^n Q(1,u)[udx + xdu] = 0$$

$$\Rightarrow \quad [P(1,u)dx + uQ(1,u)]dx + xQ(1,u)du = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{Q(1,u)du}{P(1,u)dx + uQ(1,u)} = 0 \qquad (2)$$

Key Facts

- To solve homogeneous differential equations, we have to write out whole procedure for each problem. Therefore, it is not recommended to follow the equation (2) as a formula.
- The substitution $x = vy$ also leads to a separable differential equation.

Example 14: Solve $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

Solution: Putting $y=ux$ in the given differential equation, we have $\frac{dy}{dx}=u+x\frac{du}{dx}$

Therefore, given differential equation leads to:

$$u+x\frac{du}{dx}=\frac{x^2+u^2x^2}{2x\,ux} \quad \Rightarrow \quad u+x\frac{du}{dx}=\frac{1+u^2}{2u}$$

$$\Rightarrow \quad x\frac{du}{dx}=\frac{1+u^2}{2u}-u \quad \Rightarrow \quad x\frac{du}{dx}=\frac{1-u^2}{2u} \quad \Rightarrow \quad \frac{2u}{1-u^2}du=\frac{dx}{x}$$

Integrating, we get:

$$\int\frac{2u}{1-u^2}du=\int\frac{dx}{x}$$

$$-ln(1-u^2)=lnx+lnc \quad \Rightarrow \quad ln(1-u^2)+lnx+lnc=0$$

$$\Rightarrow \quad ln[cx(1-u^2)]=0$$

Taking antilog, we have:

$$cx(1-u^2)=1 \quad \Rightarrow \quad cx\left(1-\frac{y^2}{x^2}\right)=1 \quad \text{..... (replacing } u \text{ by } \tfrac{y}{x})$$

$$\Rightarrow \quad cx\left(\frac{x^2-y^2}{x^2}\right)=1 \quad \Rightarrow \quad x^2-y^2=\frac{x}{c} \quad \Rightarrow \quad x^2-y^2=Cx \qquad \left(\frac{1}{c}=C\right)$$

Example 15: Solve the initial value problem:

$$x\frac{dy}{dx}=y+x\,e^{\frac{y}{x}} \;;\quad y(1)=1$$

Solution: Given equation is homogeneous of degree zero and can be rewritten as:

$$\frac{dy}{dx}=\frac{y}{x}+e^{\frac{y}{x}} \;;\quad y(1)=1$$

Substituting $y=ux$ in the given differential equation, we have $\frac{dy}{dx}=u+x\frac{du}{dx}$

Therefore, given differential equation becomes:

$$u+x\frac{du}{dx}=\frac{ux}{x}+e^{\frac{ux}{x}} \quad \Rightarrow \quad u+x\frac{du}{dx}=u+e^u$$

$$\Rightarrow \quad e^{-u}\,du=\frac{dx}{x} \quad \Rightarrow \quad \int e^{-u}\,du=\int\frac{dx}{x}$$

$$\Rightarrow \quad -e^{-u}=lnx+c \quad \Rightarrow \quad -e^{-\frac{y}{x}}=lnx+c \qquad \text{(i)}$$

Substituting, $x=1, y=1$ in equation (i), we have:

$$-e^{-1}=ln1+c \quad \Rightarrow \quad c=-e^{-1}$$

Therefore, (i) leads to:

$$-e^{-\frac{y}{x}}=lnx-e^{-1} \quad \Rightarrow \quad e^{-1}-e^{-\frac{y}{x}}=lnx$$

Exercise 4.3

Check whether the functions are homogeneous or not. If homogeneous then find degree.

1. $f(x, y) = 6xy^3 - x^2y^2$
2. $f(x, y) = x^2 - y$
3. $f(x, y) = \frac{2y^3}{x^2y} - 7$

Solve the homogeneous differential equations.

4. $(x - y)dx + xdy = 0$
5. $ydx - (y - x)dy = 0$
6. $\frac{dy}{dx} = \frac{y - x}{x + y}$
7. $\frac{dy}{dx} = \frac{y^2 + yx}{x^2}$
8. $\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$
9. $\frac{dy}{dx} = \frac{3x^3 + y^3}{xy^2}$
10. $\frac{dy}{dx} = \frac{x + 3y}{3x + y}$
11. $\left[y + x\cot\left(\frac{y}{x}\right)\right] dx - xdy = 0$

Solve the initial value problems.

12. $xy^2\frac{dy}{dx} = y^3 - x^3$; $y(1) = 2$
13. $(x^2 + 2y^2)dx = xydy$; $y(1) = 1$
14. $2x^2\frac{dy}{dx} = 3xy + y^2$; $y(1) = -2$
15. $\left(x + ye^{\frac{y}{x}}\right) dx - x\,e^{\frac{y}{x}}dy = 0$; $y(1) = 0$

4.7 Applications of Differential Equations

Differential equations can be applied in solving problems arising in engineering and physical sciences such as physics, chemistry, biology and economics etc.

First order ordinary differential equations are used to calculate the movement or flow of electricity, motion of a pendulum and a falling object, to explain thermodynamics concepts and population growth etc. Also, in medical terms, they are used to check the growth of diseases in graphical representation.

Example 16:

A ball is thrown downward from a tower having height 20m. Develop a differential equation representing the flow phenomenon and find the velocity of the ball after 1 second. Also find the velocity with which the ball hits the ground. Neglect the air resistance.

Solution: For downward motion, acceleration $a = \frac{dv}{dt}$ can be written as:

$$\frac{dv}{dt} = g \quad \text{or} \quad dv = gdt$$

That is required differential equation.

Now, integrating both sides, we get

$$v = gt + c_1$$

At $t = 0, v = 0$, the constant $c_1 = 0$.

Thus, $v = gt$ ......(i)

Substituting the values,

$$v = 9.8 \times 1 = 9.8m/s$$

Which is velocity of the ball after 1 second.

Now from (i)

$$\frac{dS}{dt} = gt \text{ where } S \text{ is the distance covered by the ball.}$$

$$dS = gtdt$$

Integrating both sides, se have:

$$S = g\frac{t^2}{2} + c_2 \quad .... \text{(ii)}$$

At $t = 0, S = 0$, therefore from (ii), $c_2 = 0$.

and, $S = g\frac{t^2}{2}$ implies

$$20 = 9.8 \times \frac{t^2}{2} \quad \text{or} \quad t = 2.02sec$$

Now from (i)

$$v = 9.8 \times 2.02 = 19.8m/s$$

Thus, the velocity with which the ball hits the ground is $19.8m/s$.

Check Point

A thermometer showing the temperature of 20°C indoors is placed outdoors. After 8 minutes it reads 25°C and after another 8 minutes it reads 30°C. Using Newton's law of cooling, find the outdoors temperature.

Example 17:

According to Newton, cooling of a hot body is proportional to the temperature difference between its temperature T and the temperature T_0 of its surrounding medium. If a body at 90°C is allowed to cool in air with temperature 30°C and if it is observed after 5 min the body has cooled to 70°C, find the temperature of the body as a function of time.

Solution: The mathematical formulation of Newton's law of cooling in this problem is:

$$\frac{dT}{dt} \propto (T - T_0) \text{ ....(1)}$$

Introducing a proportionality constant $k > 0$, the above equation can be written as:

$$\frac{dT}{dt} = k(T - T_0) \text{ ....(2)}$$

Here, T is the temperature of the body and t is the time, T_0 is the temperature of the surrounding and $\frac{dT}{dt}$ is the rate of cooling of the body. Substituting, $T_0 = 30°$ in equation (2), we get:

$$\frac{dT}{dt} = k(T - 30) \quad \Rightarrow \quad \frac{dT}{T-30} = kdt$$

Integrating both sides, we get:

$$ln(T-30) = kt + lnc \quad \text{where } c \text{ is the constant of integration.}$$

$$ln(T-30) = lne^{kt} + lnc \quad \Rightarrow \quad ln(T-30) = ln(ce^{kt})$$

Taking antilog, we get:

$$T - 30 = ce^{kt} \qquad \text{.......... (3)}$$

Imposing the initial condition $T(0) = 90^o$, we find:

$$90 - 30 = c \quad \Rightarrow \quad c = 60$$

Therefore (3) implies:

$$T - 30 = 60e^{kt} \quad \Rightarrow \quad T = 60e^{kt} + 30 \quad \text{.......... (4)}$$

To find the value of constant k, we use the second condition $T(5) = 70^o$ in (4).

$$70 = 60e^{5k} + 30 \Rightarrow \quad 60e^{5k} = 40 \quad \Rightarrow \quad e^{5k} = \frac{2}{3}$$

$$5k = ln\left(\frac{2}{3}\right) \quad \Rightarrow \quad k = \frac{1}{5}ln\left(\frac{2}{3}\right) = -0.081$$

Substituting the value of k in relation (4), we have:

$$T = 60e^{-0.081t} + 30$$

Which shows the temperature of the body as a function of time.

Exercise 4.4

1. Thomas Malthus in 1798 proved that increase in population of a country or a city at a certain time is proportional to the total population of the country at that time $\left(\frac{dP}{dt} \propto P\right)$. If at present the population of city A is 20 million and after 4 years, it is expected to be 25 million, what would be the population of that city after 12 years?
2. Ayesha was preparing a pizza in a baking oven. She observed that temperature of the cooked pizza was 150°C. Four minutes after removing from the oven, the temperature of pizza was 90°C. How long will it take to cool off to a temperature of 40°C if room temperature is 20°C?
3. In a culture, the rate of growth of bacteria is proportional to the population present. If the population of bacteria becomes four times in two days, how much the population would be after ten days at the same rate if the initial population was 20?
4. Most of the radioactive substances disintegrate at the rate proportional to the amount present. If the amount of a radioactive substances is 50 grams and its half life is 1000 years, find the amount of substance present after 800 years.
5. A thermometer showing room temperature of 80°F is placed on a block of ice with a temperature of 30°F. After one minute the temperature of thermometer is 40°F. How long will it take for the thermometer to have a temperature of 70°F?
6. A ball is thrown upward with a velocity of 40m/s. Develop a differential equation representing the flow phenomenon and find the velocity of the ball after 1 second. Also find the maximum height attained by the ball. Neglect the air resistance.

Review Exercise

1. Select the correct option in the following.

(i) The order of differential equation $x\frac{d^3y}{dx^3} - 2\left(\frac{dy}{dx}\right)^4 + y = 0$, is:

(a) 1 (b) 2 (c) 3 (d) 4

(ii) The degree of differential equation $\frac{d^2y}{dx^2} + 9y^3 = \sin x$, is:

(a) 0 (b) 1 (c) 2 (d) 3

(iii) $y = 8$ is a solution of the differential equation:

(a) $\frac{dy}{dx} + 8y = 32$ (b) $\frac{dy}{dx} + 6y = 32$

(c) $\frac{dy}{dx} + 5y = 32$ (d) $\frac{dy}{dx} + 4y = 32$

(iv) $f(x,y) = \frac{x^3 - y^3}{x - y}$ is a homogeneous function of degree:

(a) 1 (b) 2 (c) 3 (d) 4

(v) The solution of the differential equation $dy = dx$ is:

(a) $y = x + c$ (b) $y = x^2 + c$ (c) $y^2 = x^2 + c$ (d) $y^2 = x + c$

(vi) The number of arbitrary constants present in the general solution of a differential equation of first order is:

(a) 1 (b) 2 (c) 3 (d) 0

(vii) The differential equation $\frac{dy}{dx} = e^{x+y}$ has solution:

(a) $e^{-x-y} = c$ (b) $e^{-x} + e^{y} = c$ (c) $e^{x} + e^{y} = c$ (d) $e^{x} + e^{-y} = c$

(viii) The general solution of $y^2dy - x^2dx = 0$ is:

(a) $x^2 - y^2 = c$ (b) $x^3 + y^3 = c$ (c) $x^3 - y^3 = c$ (d) $x^2 + y^2 = c$

(ix) The solution of $\cos x \sin y\, dx + \sin x \cos y\, dy = 0$ is:

(a) $\sin x \cos y = c$ (b) $\cos x \cos y = c$ (c) $\cos x \sin y = c$ (d) $\sin x \sin y = c$

(x) Which of the following cannot be the order of differential equation?

(a) -1 (b) 1 (c) 10 (d) 100

2. Find the order and degree of the differential equations.

(i) $x\left(\frac{dy}{dx}\right)^2 + 2\sqrt{xy}\frac{dy}{dx} + y = 0$ (ii) $\frac{dy}{dx} = \sqrt{1 + \left(\frac{d^2y}{d^2x}\right)^4}$

3. Verify that the indicated function is a solution of the given differential equation.

(i) $2\frac{dy}{dx} + y = 0$; $y = e^{-\frac{x}{2}}$ (ii) $\frac{dy}{dx} + 20y = 24$; $y = \frac{6}{5} - \frac{6}{5}e^{-20x}$

(iii) $\frac{dy}{dx} = 25 + y^2$; $y = 5\tan 5x$ (iv) $x^2dy + 2xydx = 0$; $y = \frac{1}{x^2}$

4. Solve the differential equations.

(i) $\frac{dy}{dx} = x \ln x$ (ii) $(y+1)\frac{dy}{dx} + x\sin x$ (iii) $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$; $y(1) = 2$