UNIT
03

INTEGRATION

After studying this unit, students will be able to:

- Find the general antiderivative of a given function.
- Recognize and use the terms and notations for antiderivatives.
- State the power rule of integrals.
- State and apply the properties of indefinite integrals.
- Integrate functions involving the exponential and logarithmic functions.
- Identify when to use integration by parts to solve integration problems.
- Apply the integration-by-part formula for definite integrals.
- Solve integration problems involving trigonometric substitution.
- Integrate a rational function using the method of partial fraction.
- State the definition of definite integral.
- Explain the terms integrand, limits of integration and value of integration.
- State and apply the properties of definite integrals.
- State and apply fundamental theorem of calculus to evaluate the definite integrals.
- Describe the relation between the definite integral and net area.
- Find the area of a region bounded by a curve and lines parallel to axes, or between a curve and a line or between two curves.
- Find volume of the revolution about one of the axes.
- Demonstrate trapezium rule to estimate the value of a definite integral.
- Apply concept of integration to real world problems such as volume of a container, consumer and producer surplus, growth rate of a population, investment return time period, drug dosage required by integrating the concentration.

There is a lot of applications of integration in various fields. For example, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us. A view of Tarbela dam is shown below.

3.1 Integration

This unit examines the process by which we determine functions from their derivatives. We are already familiar with inverse operations. For example, addition and subtraction are inverse of each other. Similarly, multiplication and division are inverse of each other. In the same way, the inverse operation of differentiation is anti-differentiation or integration.

This unit provides two processes and their relationship to one another. One step is to find function from their derivatives. In the second step, we can determine things like area and volume through successive approximations. This process is called integration. This is very important area in mathematics and was discovered independently by Leibnitz and Newton.

The process of finding a function from one of its known values and its derivative $f(x)$ has two steps:

The first is to find a formula that gives us all the functions that could possibly have $f(x)$ as a derivative. If $f'(x)$ is defined as derivative, then $f(x)$ is called anti-derivative and the formula that gives them all is called the indefinite integral of $f(x)$. The reverse process of derivative or anti-differentiation is the main topic of this unit.

Definition 3.1:

A function $F'(x)$ is called an anti-derivative of another function $f(x)$ if:

$$F'(x) = f(x)$$

For example:

$$\tfrac{1}{4}x^4, \quad \tfrac{1}{4}x^4 + 3, \quad \tfrac{1}{4}x^4 - \pi, \quad \tfrac{1}{4}x^4 + c \quad (c \text{ is any constant.})$$

are anti-derivatives of x^3 since the derivative of each is x^3.

Above example shows that a function can have many anti-derivatives. In fact, if $F(x)$ is any anti-derivative of $f(x)$ and c is any constant, then $F(x) + c$ is also an anti-derivative of $f(x)$ since:

$$\frac{d}{dx}[F(x) + c] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[c] = f(x) + 0 = f(x)$$

Therefore, if $F(x)$ is any anti-derivative of $f(x)$ on a given interval, then for any value of c, the function $F(x) + c$ is also an anti-derivative of $f(x)$ on that interval.

Symbolically we write:

$$\int f(x)dx = F(x) + c$$

Where the symbol, "$\int$" is called 'integral sign' and $f(x)$ is called integrand. The symbol dx indicates that the integration is performed with respect to the variable x. The arbitrary constant c is called 'constant of integration'.

For Example,

As, $\quad \frac{d}{dx}(x^4) = 4x^3$

Therefore, $\int 4x^3 dx = x^4 + c$

As mentioned above, the constant c is arbitrary constant. Therefore,

$x^4, x^4+1, x^4-\sqrt{2}, x^4+\pi$ etc. all are anti-derivatives of $4x^3$.

Let us derive some basic and common integral formulae with the help of differentiation.

Key Facts

- The variable other than x, can also be used in indefinite integrals.
- A number of indefinite integral formulae are found by reversing derivative formulas.

Formula 3.1: $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

Derivation: We have,

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right] = \frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] + \frac{d}{dx}[c] = \frac{(n+1)\,x^n}{n+1} + 0 = x^n \qquad \text{(i)}$$

Integrating both sides of (i) with respect to x, we have:

$$\int \frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right] dx = \int x^n dx$$

$$\frac{x^{n+1}}{n+1} + c = \int x^n dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$$

In general,

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$$

Formula 3.2: $\int \frac{1}{x} dx = \ln x + c$

Derivation: We have,

$$\frac{d}{dx}[\ln x + c] = \frac{1}{x} \qquad \text{(ii)}$$

Integrating both sides of (ii) with respect to x, we have:

$$\int \frac{d}{dx}[\ln x + c] dx = \int \frac{1}{x} dx$$

$$\ln x + c = \int \frac{1}{x} dx$$

$$\int \frac{1}{x} dx = \ln x + c$$

In general,

$$\int \frac{f'(x)}{f(x)} dx = \ln[f(x)] + c$$

Formula 3.3: $\int e^x dx = e^x + c$

Derivation: As,

$$\frac{d}{dx}[e^x + c] = e^x \qquad \text{(iii)}$$

Integrating both sides of (iii) with respect to x, we have:

$$\int \frac{d}{dx}[e^x + c]dx = \int e^x dx$$

$$e^x + c = \int e^x dx$$

$$\int e^x dx = e^x + c$$

In general,

$$\int e^{f(x)} f'(x)dx = e^{f(x)} + c$$

Formula 3.4: $\int a^x dx = \frac{1}{\ln a} a^x + c, \; a > 0, \; a \neq 1$

Derivation: As,

$$\frac{d}{dx}\left[\frac{1}{\ln a} a^x + c\right] = a^x \qquad \text{(iv)}$$

Integrating both sides of (iv) with respect to x, we have:

$$\int \frac{d}{dx}\left[\frac{1}{\ln a} a^x + c\right] dx = \int a^x dx$$

$$\frac{1}{\ln a} a^x + c = \int a^x dx$$

$$\int a^x dx = \frac{1}{\ln a} a^x + c$$

In general,

$$\int a^{f(x)} f'(x)dx = \frac{1}{\ln a} a^{f(x)} + c$$

Theorem 3.1:

(i) A constant factor can be moved through an integral sign. That is:

$$\int c\, f(x)dx = c \int f(x)dx$$

(ii) An anti-derivative of a sum is the sum of anti-derivatives. That is:

$$\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$$

(iii) An anti-derivative of a difference is the difference of anti-derivatives. That is:

$$\int [f(x) - g(x)]\, dx = \int f(x)dx - \int g(x)dx$$

(iv) In general, $\int [af(x) \pm bg(x)]\, dx = a \int f(x)dx \pm b \int g(x)dx$

Example 1: Evaluate (i) $\int(4x^7 - 2x^3 + 9x + 3)dx$ (ii) $\int \frac{y^3-2y^6}{y^5}\, dy$

Solution: (i) $\int(4x^7 - 2x^3 + 9x + 3)dx$

$= 4\int x^7 dx - 2\int x^3 dx + 9\int x\, dx + 3\int dx$

Integrating term by term, we get:

$= 4\left(\frac{x^8}{8}\right) - 2\left(\frac{x^4}{4}\right) + 9\left(\frac{x^2}{2}\right) + 3x + c = \frac{x^8}{2} - \frac{x^4}{2} + \frac{9x^2}{2} + 3x + c$

(ii) $\int \frac{y^3-2y^6}{y^5}\, dy = \int\left(\frac{y^3}{y^5} - \frac{2y^6}{y^5}\right) dy = \int\left(\frac{1}{y^2} - 2y\right) dy$

$= \int(y^{-2} - 2y)\, dy = \int y^{-2}\, dy - 2\int y\, dy$

$= \frac{y^{-2+1}}{-2+1} - 2\left(\frac{y^2}{2}\right) + c = -\frac{1}{y} - y^2 + c$

Example 2: Evaluate (i) $\int \frac{ax + \frac{1}{2}b}{ax^2+bx+c}\, dx$ (ii) $\int e^{3x} dx$

Solution: (i) $\int \frac{ax + \frac{1}{2}b}{ax^2+bx+c}\, dx = \frac{1}{2}\int \frac{2ax + b}{ax^2+bx+c}\, dx$

$= \frac{1}{2} ln\,(ax^2 + bx + c) + C$

(ii) $\int e^{3x} dx = \frac{1}{3}\int e^{3x}(3)dx = \frac{1}{3}e^{3x} + c$

Example 3: Evaluate $\int \frac{e^{sin^{-1} x}}{\sqrt{1-x^2}}\, dx$

Solution: Here, $f(x) = sin^{-1} x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

So, by using formula:

$\int e^{f(x)} f'(x) dx = e^{f(x)} + c$

We have:

$\int \frac{e^{sin^{-1} x}}{\sqrt{1-x^2}}\, dx = e^{sin^{-1} x} + c$

Exercise 3.1

Evaluate the following integrals.

1. $\int(x^2 - 3x + 9)dx$
2. $\int(y^2 + 8y + \sqrt{2})dy$
3. $\int\left(\sqrt{y} + \frac{1}{y^2}\right) dy$
4. $\int(4 + x^2)^2 dx$
5. $\int(1 + x)(1 - x^2)dx$
6. $\int\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$
7. $\int(e^{4x} - e^{-1} + 1)dx$
8. $\int(e^{\frac{9}{2}x} + \frac{1}{x})dx$
9. $\int x\, e^{x^2} dx$
10. $\int 5^x dx$
11. $\int 7^{7y} dy$
12. $\int(x^3 + \frac{1}{2x} - \frac{1}{x^3})dx$

13. $\int \frac{2x+1}{x^2+3}\, dx$ 14. $\int \frac{e^{\tan^{-1}z}}{1+z^2}\, dz$ 15. $\int \left(x^{\frac{3}{2}} + e^{3x} + x^0\right) dx$

16. $\int (3x^2 + 2x)(x^3 + x^2 + 9)^5 dx$ 17. $\int (5e^{5x} - x^{-3} + 3^{2x}) dx$

18. $\int (z^{\frac{-1}{4}} + \sqrt{3z} + \frac{4}{z} - \frac{1}{e^z}) dz$.

3.2 Integration of Trigonometric Functions

While evaluating the integration of trigonometric functions, keep in mind the following formulae.

As, $\frac{d}{dx}(\sin x + c) = \cos x$ therefore, $\int \cos x\, dx = \sin x + c$

Similarly, $\frac{d}{dx}(\cos x + c) = -\sin x$ implies, $\int \sin x\, dx = -\cos x + c$

In the same way, $\frac{d}{dx}(\sin kx + c) = k\cos kx$ implies, $\int \cos kx\, dx = \frac{\sin kx}{k} + c$

And, $\frac{d}{dx}(\cos kx + c) = -k\sin kx$ implies, $\int \sin kx\, dx = -\frac{\cos kx}{k} + c$

Using above pattern, following formulae can be deduced easily.

$\int \sec^2 x\, dx = \tan x + c$ and $\int \text{cosec}^2 x\, dx = -\cot x + c$

$\int \sec x \tan x\, dx = \sec x + c$ and $\int \text{cosec}x \cot x\, dx = -\text{cosec}x + c$

Example 4: Evaluate:

(i) $\int \sin 2x\, dx$ (ii) $\int \cos \frac{3x}{5}\, dx$ (iii) $\int \sec^2 mx\, dx$

(iv) $\int 5\, \text{cosec}^2 \frac{7x}{5}\, dx$ (v) $\int 9 \sec 3x \tan 3x\, dx$

Solution:

(i) $\int \sin 2x\, dx = \int \frac{1}{2}\sin 2x\,(2)\, dx = \frac{1}{2}\int \sin 2x\,(2)\, dx$

$= \frac{1}{2}(-\cos 2x + c) = \frac{-\cos 2x}{2} + \frac{c}{2} = \frac{-\cos 2x}{2} + C$

(ii) $\int \cos\frac{3x}{5}\, dx = \int \frac{5}{3}\cos\frac{3x}{5}\left(\frac{3}{5}\right) dx = \frac{5}{3}\int \cos\frac{3x}{5}\left(\frac{3}{5}\right) dx$

$= \frac{5}{3}\left(\sin\frac{3x}{5} + c\right) = \frac{5}{3}\sin\frac{3x}{5} + \frac{5}{3}c = \frac{5}{3}\sin\frac{3x}{5} + C$

(iii) $\int \sec^2 mx\, dx = \int \frac{1}{m}\sec^2 mx\,(m)\, dx = \frac{1}{m}\int \sec^2 mx\,(m)\, dx$

$= \frac{1}{m}(\tan mx + c) = \frac{1}{m}(\tan mx) + \frac{c}{m} = \frac{1}{m}(\tan mx) + C$

(iv) $\int 5\, \text{cosec}^2\frac{7x}{5}\, dx = 5 \times \frac{5}{7}\int \text{cosec}^2\frac{7x}{5}\left(\frac{7}{5}\right) dx = \frac{25}{7}\left(-\cot\frac{7x}{5} + c\right)$

$$= -\frac{25}{7}\cot\frac{7x}{5} + \frac{25}{7}c \quad = \quad -\frac{25}{7}\cot\frac{7x}{5} + C$$

(v) $\int 9\sec 3x\tan 3x\,dx \;=\; \int 3\times 3\sec 3x\tan 3x\,dx \;=\; 3\int \sec 3x\tan 3x\,(3)dx$

$= 3(\sec 3x + c) \;=\; 3\sec 3x + 3c \;=\; 3\sec 3x + C$

Example 5: Prove that:

(i) $\int \sec x\,dx = ln|\sec x + \tan x| + c$

(ii) $\int \operatorname{cosec}x\,dx = ln|\operatorname{cosec}x - \cot x| + c$

(iii) $\int \tan x\,dx = -ln(\cos x) + c = ln(\sec x) + c$

Solution:

(i) $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$ [Multiplying and dividing by $(\sec x + \tan x)$]

$$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}dx = \int \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}dx$$

$$= \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}dx = ln|\sec x + \tan x| + c$$

$\int \frac{f'(x)}{f(x)}dx = ln[f(x)] + c$

(ii) $\int \operatorname{cosec}x\,dx = \int \frac{\operatorname{cosec}x(\operatorname{cosec}x - \cot x)}{\operatorname{cosec}x - \cot x}dx$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec}x\cot x}{\operatorname{cosec}x - \cot x}dx = \int \frac{-\operatorname{cosec}x\cot x + \operatorname{cosec}^2 x}{\operatorname{cosec}x - \cot x}dx$$

$$= \int \frac{\frac{d}{dx}(\operatorname{cosec}x - \cot x)}{\operatorname{cosec}x - \cot x}dx = ln|\operatorname{cosec}x - \cot x| + c$$

(iii) $\int \tan x\,dx \;=\; \int \frac{\sin x}{\cos x}dx \;=\; -\int \frac{-\sin x}{\cos x}dx \;=\; -\int \frac{\frac{d}{dx}(\cos x)}{\cos x}dx$

$= -ln(\cos x) + c \;=\; ln(\cos x)^{-1} + c$

$= ln\frac{1}{\cos x} + c \;=\; ln(\sec x) + c$

Check Point

Prove that

$\int \cot x\,dx = ln(\sin x) + c$

3.2.1 Integration of Sin²x and Cos²x

Sometimes it is difficult to evaluate integrals directly. Using trigonometric identities, we can easily evaluate integrals. For example, the integrals of $\sin^2 x$ and $\cos^2 x$ cannot be solved directly and can be handled using following relations.

$\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

Check Point

Evaluate $\int \cos^2 x\,dx$

Example 6: Evaluate $\int \sin^2 x\,dx$

Solution: $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Example 7: Integrate (i) $8\sec 9x - \tan 3x$ (ii) $\cos^2 7x$

Solution:

(i) $$\int(8\sec 9x - \tan 3x)dx = \int 8\sec 9x\, dx - \int \tan 3x\, dx$$
$$= \frac{8}{9}\int \sec 9x\,(9)dx - \frac{1}{3}\int \tan 3x\,(3)dx$$
$$= \frac{8}{9}\ln|\sec 9x + \tan 9x| - \ln(\sec 3x) + c$$

(ii) $$\int \cos^2 7x\, dx = \int \frac{1+\cos 14x}{2}\, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 14x\, dx$$
$$= \frac{1}{2}x + \frac{1}{2}\left(\frac{\sin 14x}{14}\right) + c = \frac{1}{2}x + \frac{1}{28}\sin 14x + c$$

Exercise 3.2

Evaluate the integrals and recheck your answer by differentiating.

1. $\int(\sin\pi x - 3\sin 3x)dx$
2. $\int -\sec^2\left(\frac{3}{2}y\right)dy$
3. $\int[1 - 8\operatorname{cosec}^2(2x)]dx$
4. $\int\frac{1}{2}(\operatorname{cosec}^2 x - \operatorname{cosec}x\cot x)dx$
5. $\int\frac{\cos^2 z}{7}\,dz$
6. $\int(1+\tan^2\theta)d\theta$
7. $\int\frac{1+\cos 4t}{2}dt$
8. $\int\sec^2(5x-1)\,dx$
9. $\int(\tan 5x + \cos 7x)dx$
10. $\int(\cot 9y - 3)\,dy$

Evaluate the integral.

11. $\int(\tan^2 2\theta + \cot^2 2\theta)d\theta$
12. $\int\sin^2\left(\frac{11}{2}y\right)dy$
13. $\int\operatorname{cosec}11x\tan 11x\,dx$
14. $\int\cos\theta(\tan\theta + \sec\theta)d\theta$
15. $\int\operatorname{cosec}^2\left(\frac{x-1}{3}\right)dx$
16. $\int(\cos x)^{\frac{1}{5}}\sin x\,dx$
17. $\int e^y \sin e^y dy$
18. $\int 9\tan(x+7)dx$

3.3 Integration by Substitution

There are many functions that cannot be integrated by simple techniques and can be integrated easily by using method of substitution. It is an integration technique which involves making a substitution to simplify the integral. In this method any given integral is transformed into a simple form of integral by substituting the independent variable by others. The exact substitution depends on the form of the given integral, as some substitutions are more appropriate for certain problems than others. The choice of substitution is not always immediately obvious. The ability to recognise an appropriate substitution comes from practising many different examples.

Mostly, we substitute trigonometric functions in place of variables to integrate algebraic functions. However, there is no hard and fast rule for selection of trigonometric functions to replace variables as some other substitutions are also used.

Key Facts

Usually, the method of integration by substitution is extremely useful when we make a substitution for a function whose derivative is also present in the integrand. Doing so, the function simplifies and then the basic formulas of integration can be used to integrate the function.

Example 8: Evaluate $\int 3x^2 \cos(x^3)\,dx$

Solution:
In the equation given above the independent variable can be transformed into another variable say t by substituting:

$$x^3 = t \qquad \text{(i)}$$

Differentiation of (i) gives:

$$3x^2\,dx = dt \qquad \text{(ii)}$$

Substituting the values of (i) and (ii) in the given integral.

$$\int 3x^2 \cos(x^3)\,dx = \int \cos t\,dt = \sin t + c$$

Again, substituting back the value of t, we get:

$$\int 3x^2 \cos(x^3)\,dx = \sin(x^3) + c$$

Key Facts

The method of substitution to find an integral is used when it is set up in the special form.
$\int f(g(x)).g'(x).dx = \int f(t).dt$
where $t = g(x)$

Check Point

Integrate:
$x\sin(x^2-3)$ with respect to x.

Example 9: Integrate: $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

Solution: Let $u = \tan^{-1}x$ then $du = \frac{1}{1+x^2}dx$

Therefore, $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int e^u du = e^u + c = e^{\tan^{-1}x} + c$

Formula 3.5: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Derivation: Substituting $x = a\sin\theta$, we have $dx = a\cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-(a\sin\theta)^2}}\,a\cos\theta\,d\theta = \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\,a\cos\theta\,d\theta$$

$$= \int \frac{1}{a\sqrt{1-\sin^2\theta}}\,a\cos\theta\,d\theta = \int \frac{1}{\cos\theta}\,\cos\theta\,d\theta$$

$$= \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

Note: We can apply the formula directly too.

Formula 3.6: $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + c$

Derivation: Substituting $x = a\sin\theta$, we have $dx = a\cos\theta\,d\theta$

$$\therefore \int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-(a\sin\theta)^2}\,a\cos\theta\,d\theta = \int \sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta$$

$$= \int a\sqrt{1-\sin^2\theta}\,a\cos\theta\,d\theta = \int a\cos\theta\,a\cos\theta\,d\theta = a^2\int \cos^2\theta\,d\theta$$

$$= a^2 \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{\sin 2\theta}{2}\right) + c = \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{2\sin\theta\cos\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\sin\theta\sqrt{1-\sin^2\theta}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + c$$

Example 10: Evaluate: $\frac{1}{\sqrt{5-4x-x^2}}$

Solution: $\int \frac{1}{\sqrt{5-4x-x^2}}\, dx = \int \frac{1}{\sqrt{5+4-4-4x-x^2}}\, dx = \int \frac{1}{\sqrt{9-(4+4x+x^2)}}\, dx$

$$= \int \frac{1}{\sqrt{(3)^2-(2+x)^2}}\, dx = \sin^{-1}\left(\frac{2+x}{3}\right) + c \qquad \text{(Using direct formula)}$$

Note: We can also solve by substituting $x+2 = 3\sin\theta$

Formula 3.7: $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left(x+\sqrt{x^2-a^2}\right) + C$

Derivation: Substituting $x = a\sec\theta$, we have $dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}\, dx = \int \frac{1}{\sqrt{(a\sec\theta)^2-a^2}}\, a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2\sec^2\theta-a^2}}\, a\sec\theta\tan\theta\, d\theta = \int \frac{1}{\sqrt{a^2(\sec^2\theta-1)}}\, a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1}{a\tan\theta}\, a\sec\theta\tan\theta\, d\theta = \int \sec\theta\, d\theta = \ln[\sec\theta+\tan\theta] + c$$

$$= \ln\left[\sec\theta+\sqrt{\sec^2\theta-1}\right] + c = \ln\left[\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right] + c = \ln\left[\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right] + c$$

$$= \ln\left[\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right] + c = \ln\left[\frac{x+\sqrt{x^2-a^2}}{a}\right] + c = \ln\left(x+\sqrt{x^2-a^2}\right) - \ln a + c$$

$$= \ln\left(x+\sqrt{x^2-a^2}\right) + (c-\ln a) = \ln\left(x+\sqrt{x^2-a^2}\right) + C$$

Note: Expression $\frac{1}{\sqrt{x^2-a^2}}$ can also be integrated by making the substitution $x = a\cosh\theta$.

Formula 3.8: $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left(x+\sqrt{a^2+x^2}\right) + C$

Derivation: Substituting $x = a\tan\theta$, we have $dx = a\sec^2\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \int \frac{1}{\sqrt{(a\tan\theta)^2+a^2}}\, a\sec^2\theta\, d\theta = \int \frac{1}{\sqrt{a^2\tan^2\theta+a^2}}\, a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\tan^2\theta+1)}}\, a\sec^2\theta\, d\theta = \int \frac{1}{a\sec\theta}\, a\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$$= ln[\sec\theta + \tan\theta] + c = ln[\tan\theta + \sec\theta] + c = ln\left[\tan\theta + \sqrt{1+\tan^2\theta}\right] + c$$

$$= ln\left[\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right] + c = ln\left[\frac{x}{a} + \sqrt{\frac{a^2+x^2}{a^2}}\right] + c = ln\left[\frac{x}{a} + \frac{\sqrt{a^2+x^2}}{a}\right] + c$$

$$= ln\left[\frac{x+\sqrt{a^2+x^2}}{a}\right] + c = ln\left(x+\sqrt{a^2+x^2}\right) - lna + c$$

$$= ln\left(x+\sqrt{a^2+x^2}\right) + (c - lna) = ln\left(x+\sqrt{a^2+x^2}\right) + C$$

Formula 3.9: $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

This formula can easily be proved by substituting $x = a\tan\theta$.

Example 11: Evaluate: $\int \frac{1}{x^2+4x+5}\,dx$

Solution: $\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{x^2+4x+4+1}\,dx = \int \frac{1}{(x+2)^2+(1)^2}\,dx$

$= \frac{1}{1}\tan^{-1}\left(\frac{x+2}{1}\right) + c = \tan^{-1}(x+2) + c$ (Using direct formula)

Note: We can also solve by substituting $x + 2 = \tan\theta$

Example 12: Evaluate: $\int x(x^2-a^2)^{\frac{3}{2}}\,dx$

Solution: Putting $x^2 - a^2 = u$

$$\Rightarrow 2xdx = du \Rightarrow xdx = \frac{du}{2}$$

$$\therefore \int x(x^2-a^2)^{\frac{3}{2}}\,dx = \int (u)^{\frac{3}{2}}\frac{du}{2} = \frac{1}{2}\int (u)^{\frac{3}{2}}\,du$$

$$= \frac{1}{2}\frac{(u)^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{1}{2}\times\frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{1}{5}(x^2-a^2)^{\frac{5}{2}} + c$$

Exercise 3.3

Use suitable substitution, to evaluate the integrals.

1. $\int \frac{dx}{x^2+9}$
2. $\int \frac{dx}{\sqrt{5-x^2}}$
3. $\int (2x+7)(x^2+7x+3)^{\frac{4}{5}}\,dx$
4. $\int \frac{x^2}{x^3+1}\,dx$
5. $\int \frac{dy}{y^2+8y+20}$
6. $\int \frac{dx}{\sqrt{20-x^2-4x}}$
7. $\int \frac{x\,dx}{(4x^2+1)^3}$
8. $\int x^4\sqrt{3x^5-5}\,dx$
9. $\int \frac{2ax+b}{ax^2+bx+c}\,dx$
10. $\int \frac{dx}{(1-3x)^2}$
11. $\int \frac{z^3}{1+z^4}\,dz$
12. $\int \frac{\cot^{-1}x}{1+x^2}\,dx$

3.4 Integration by Parts

Integration by parts is a special method of integration that is very helpful technique to evaluate a wide variety of integrals that sometimes do not fit any of the basic integration formula. This method is used to find the integrals by reducing them into standard forms.

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int[f'(x)\int g(x)dx]\,dx \qquad (1)$$

Formula (1) is called the formula for integration by parts. Using this formula, we integrate the product of two functions. The important thing to use this formula is the selection of given functions given in the product as a first or second function. The function whose integration can easily be found is considered as the second function while the first function is chosen whose derivative could be easily found. In formula (1), $f(x)$ is treated as first function while $g(x)$ as a second function.

Key Facts

- Integration by parts is not applicable for functions such as $\int \sqrt{x}\sin x\, dx$.
- We do not add any constant while finding the integral of the second function.
- Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, if the other function is an inverse trigonometric function or logarithmic function, then we take them as first function.
- If the product of functions contains exponential and trigonometric functions, then we can select any one of the two as a first function.

Example 13: Evaluate the integral: $\int xe^x\, dx$

Solution: In the integral $\int xe^x\, dx$, we take 'x' as a first function as its derivative will reduce it and 'e^x' as second function.

$$\therefore \quad \int xe^x\, dx = x\int e^x dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx$$

$$= xe^x - \int 1.e^x dx = xe^x - e^x + c$$

Example 14: Evaluate: (i) $\int x^2\, lnx\, dx$ (ii) $\int x\tan^{-1}x\, dx$

Solution:

(i) In the integral $\int x^2\, lnx\, dx$, we take 'lnx' as first function and 'x^2' as second function.

$$\therefore \quad \int x^2\, lnx\, dx = \int(lnx)(x^2)dx = lnx\int x^2 dx - \int\left[\frac{d}{dx}(lnx)\int x^2 dx\right]dx$$

$$= lnx\,.\frac{x^3}{3} - \int\frac{1}{x}.\frac{x^3}{3}dx = \frac{x^3\, lnx}{3} - \frac{1}{3}\int x^2\, dx$$

$$= \frac{x^3\, lnx}{3} - \frac{1}{3}.\frac{x^3}{3} + c \quad = \frac{x^3\, lnx}{3} - \frac{x^3}{9} + c$$

(ii) In the integral $\int x\tan^{-1}x\, dx$, we take '$\tan^{-1}x$' as first function and 'x' as second function.

$$\therefore \quad \int x\tan^{-1}x\, dx = \int(\tan^{-1}x)(x)\,dx$$

$$= \tan^{-1}x \int x dx - \int \left[\frac{d}{dx}(\tan^{-1}x) \int x dx\right] dx$$

$$= \tan^{-1}x . \frac{x^2}{2} - \int \frac{1}{x^2+1} . \frac{x^2}{2} dx = \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{x^2+1} dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx = \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}(x - \tan^{-1}x) + c$$

Example 15: Apply integration by parts to evaluate:

(i) $\int \sqrt{a^2 - x^2}\, dx$ (ii) $\int \sqrt{a^2 + x^2}\, dx$

Solution:

(i) $\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\, (1)\, dx,$

Here, we take '$\sqrt{a^2 - x^2}$' as first function and '1' as second function.

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\, (1) dx$$

$$= \sqrt{a^2 - x^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{a^2 - x^2}) \int 1 dx\right] dx$$

$$= \sqrt{a^2 - x^2}\, (x) - \int \frac{-2x}{2\sqrt{a^2 - x^2}} (x) dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx + \int \frac{a^2}{\sqrt{a^2 - x^2}} dx$$

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\, dx + \int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$2\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{c}{2} = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$$

(ii) $\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 + x^2}\, (1)\, dx,$

Here, we take '$\sqrt{a^2 + x^2}$' as first function and '1' as second function.

$$\therefore \int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 + x^2}\, (1) dx$$

$$= \sqrt{a^2 + x^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{a^2 + x^2}) \int 1 dx\right] dx$$

$$= \sqrt{a^2 + x^2}\, (x) - \int \frac{2x}{2\sqrt{a^2 + x^2}} (x) dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}} dx + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$\int \sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\, dx + a^2 \times \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int\sqrt{a^2+x^2}\,dx+\int\sqrt{a^2+x^2}\,dx=x\sqrt{a^2+x^2}+a\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$2\int\sqrt{a^2+x^2}\,dx=x\sqrt{a^2+x^2}+a\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a}{2}\tan^{-1}\left(\frac{x}{a}\right)+\frac{c}{2}=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a}{2}\tan^{-1}\left(\frac{x}{a}\right)+C$$

Check Point

Using integration by parts, prove that:

$$\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+c$$

Example 16: Apply integration by parts to evaluate:

$$\int e^{ax}\sin bx\,dx$$

Solution: Let,

$$\begin{aligned}
I&=\int e^{ax}\sin bx\,dx=\int(\sin bx)(e^{ax})\,dx\\
&=\sin bx\int e^{ax}dx-\int\left[\frac{d}{dx}(\sin bx)\int e^{ax}dx\right]dx\\
&=\sin bx\left(\frac{e^{ax}}{a}\right)-\int\left[(b\cos bx)\left(\frac{e^{ax}}{a}\right)\right]dx\\
&=\sin bx\left(\frac{e^{ax}}{a}\right)-\frac{b}{a}\int[(\cos bx)(e^{ax})]\,dx\\
&=\sin bx\left(\frac{e^{ax}}{a}\right)-\frac{b}{a}\left[\cos bx\int e^{ax}\,dx-\int\left\{\frac{d}{dx}(\cos bx)\int e^{ax}dx\right\}dx\right]
\end{aligned}$$

$$I=\sin bx\left(\frac{e^{ax}}{a}\right)-\frac{b}{a}\left[\cos bx\left(\frac{e^{ax}}{a}\right)-\int\left\{(-b\sin bx)\left(\frac{e^{ax}}{a}\right)\right\}dx\right]$$

$$I=\sin bx\left(\frac{e^{ax}}{a}\right)-\frac{b}{a}\cos bx\left(\frac{e^{ax}}{a}\right)-\frac{b^2}{a^2}\int e^{ax}\sin bx\,dx+c$$

$$I=\sin bx\left(\frac{e^{ax}}{a}\right)-\frac{b}{a}\cos bx\left(\frac{e^{ax}}{a}\right)-\frac{b^2}{a^2}I+c$$

$$I+\frac{b^2}{a^2}I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx+c$$

$$\left(\frac{a^2+b^2}{a^2}\right)I=e^{ax}\left[\frac{1}{a}\sin bx-\frac{b}{a^2}\cos bx\right]+c$$

$$I=e^{ax}\left[\frac{1}{a}\times\frac{a^2}{a^2+b^2}\sin bx-\frac{b}{a^2}\times\frac{a^2}{a^2+b^2}\cos bx\right]+c\times\frac{a^2}{a^2+b^2}$$

$$I=e^{ax}\left[\frac{a}{a^2+b^2}\sin bx-\frac{b}{a^2+b^2}\cos bx\right]+C$$

Exercise 3.4

Evaluate the integrals using integration by parts.

1. $\int \ln x\,dx$
2. $\int(\ln x)^2\,dx$
3. $\int\sin(\ln x)\,dx$
4. $\int x^3\ln x\,dx$
5. $\int y\sin 2y\,dy$
6. $\int e^x\cos x\,dx$
7. $\int x\sec^{-1}x\,dx$
8. $\int \ln(2x+3)\,dx$
9. $\int x^2e^x dx$
10. $\int x\cos x\,dx$
11. $\int\cos^{-1}x\,dx$
12. $\int\tan^{-1}x\,dx$
13. $\int x\sec^2x\,dx$
14. $\int x^2\sin^{-1}x\,dx$
15. $\int \ln\left[x+\sqrt{1+x^2}\right]dx$
16. $\int x^3e^{x^2}dx$
17. $\int x^2\sin x\,dx$
18. $\int\frac{\ln x}{\sqrt{x}}\,dx$

3.5 Integration by Partial Fraction

When the terms in the sum:

$$\frac{3}{x+4}+\frac{4}{x+2} \qquad \text{(i)}$$

are combined by means of a common denominator, we obtain a single rational expression:

$$\frac{7x+22}{(x+4)(x+2)} \qquad \text{(ii)}$$

Suppose that we are faced with the problem of evaluating the integral:

$$\int \frac{7x+22}{(x+4)(x+2)}dx$$

From (i) and (ii), we have:

$$\int \frac{7x+22}{(x+4)(x+2)}dx = \int \left[\frac{3}{(x+4)}+\frac{4}{(x+2)}\right]dx = \int \frac{3}{(x+4)}dx + \int \frac{4}{(x+2)}dx$$

$$= 3\int \frac{1}{(x+4)}dx + 4\int \frac{1}{(x+2)}dx = 3\ln(x+4)+4\ln(x+2)+c$$

This example illustrates a procedure for integrating certain rational fractions $\frac{P(x)}{Q(x)}$, where the degree of $P(x)$ is less than the degree of $Q(x)$. This method, known as partial fractions consists of decomposing such rational fractions into simplest component fractions and then evaluating the integral term by term.

Example 17: Evaluate: $\int \frac{x^3-2x}{x^2+3x+2}dx$

Solution: We observe that degree of numerator is greater than that of denominator.

$$\therefore \int \frac{x^3-2x}{x^2+3x+2}dx = \int \left[x-3+\frac{5x+6}{x^2+3x+2}\right]dx \qquad \text{(i)}$$

Now, $\frac{5x+6}{x^2+3x+2} = \frac{5x+6}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2}$

> **Check Point**
>
> Evaluate $\int \frac{2x+1}{(x-1)(x+3)}dx$

By equating numerator, we get:

$$5x+6 = A(x+2)+B(x+1) \qquad \text{(ii)}$$

If we set $x=-2$ and $x=-1$, we get $B=4$ and $A=1$, repectively.

$$\therefore \int \frac{x^3-2x}{x^2+3x+2}dx = \int\left[x-3+\frac{1}{x+1}+\frac{4}{x+2}\right]dx = \int x\,dx - 3\int dx + \int \frac{1}{x+1}dx + 4\int \frac{1}{x+2}dx$$

$$= \frac{x^2}{2} - 3x + ln(x+1) + 4\,ln(x+2) + c$$

Example 18: Evaluate: $\int \frac{x^2+2x+4}{(x+1)^3}dx$

Solution: Given fraction can be written as:

$$\frac{x^2+2x+4}{(x+1)^3} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

By equating numerator, we get:

$$x^2 + 2x + 4 = A(x+1)^2 + B(x+1) + C$$
$$x^2 + 2x + 4 = Ax^2 + (2A+B)x + (A+B+C)$$

Comparing coefficients of like powers of x from both sides, we gat:

$$A = 1, 2A + B = 2 \text{ and } A + B + C = 4$$

Solving the equations, we have:

$$A = 1, B = 0 \text{ and } C = 3$$

$$\therefore \int \frac{x^2+2x+4}{(x+1)^3}dx = \int \left[\frac{1}{x+1} + \frac{0}{(x+1)^2} + \frac{3}{(x+1)^3}\right]dx = \int \frac{1}{x+1}dx + 3\int \frac{1}{(x+1)^3}dx$$

$$= \int \frac{1}{x+1}dx + 3\int (x+1)^{-3}dx = ln(x+1) - \frac{3}{2}(x+1)^{-2} + c$$

$$= ln(x+1) - \frac{3}{2(x+1)^2} + c$$

Example 19: Evaluate: $\int \frac{3x^2+5x+3}{(x+2)(x^2+1)}dx$

Solution: Given fraction can be written as:

$$\frac{3x^2+5x+3}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$
$$3x^2 + 5x + 3 = A(x^2+1) + (Bx+C)(x+2)$$
$$3x^2 + 5x + 3 = (A+B)x^2 + (2B+C)x + (A+2C)$$

Equating coefficients:

$$A + B, \quad 2B + C = 5, \quad A + 2C = 3$$

Solving the equations, we have:

$$A = 1, B = 2, C = 1$$

$$\therefore \int \frac{3x^2+5x+3}{(x+2)(x^2+1)}dx = \int \left(\frac{1}{x+2} + \frac{2x+1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x+2}dx + \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= ln(x+1) + ln(x^2+1) + \tan^{-1}x + c$$

Exercise 3.5

Evaluate the integrals using partial fractions.

1. $\int \frac{3x+7}{(x+2)(x+3)}dx$
2. $\int \frac{4x+9}{x^2+x-12}dx$
3. $\int \frac{21-8x}{x^2+x-6}dx$
4. $\int \frac{3x+7}{(x+2)^2}dx$
5. $\int \frac{5x^2-5x+2}{(x+1)(x-1)^2}dx$
6. $\int \frac{9x^2+3x+29}{(x+1)(x^2+4)}dx$
7. $\int \frac{7x^2+7x+4}{(2x+1)(x^2+x+1)}dx$
8. $\int \frac{x^3+4x^2+9x+14}{x^2+4x+3}dx$
9. $\int \frac{1}{x^2-9}dx$
10. $\int \frac{1}{x^3+2x^2+x}dx$
11. $\int \frac{e^x}{(e^x+1)^2(e^x-2)}dx$
12. $\int \frac{x}{(x+1)^2(x^2+1)}dx$

3.6 The Definite Integral

This section introduces the definite integral, a fundamental mathematical tool that establishes relationships between area and other essential quantities including length, volume, density, probability, and work.

3.6.1 Partition of the Interval

A partition of the interval $[a, b]$ is a collection of points:

$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$

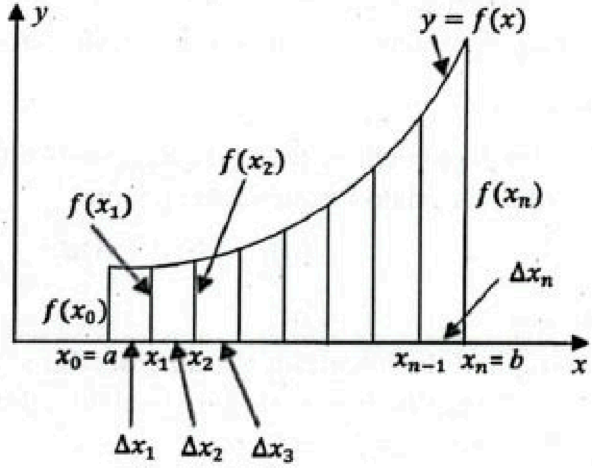
that divides $[a, b]$ into n subintervals of lengths:

$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1,$

$\Delta x_3 = x_3 - x_2, \ldots, \Delta x_n = x_n - x_{n-1}$

The partition is said to be regular provided all subintervals have the same length:

$$\Delta x = \Delta x_1 = \frac{b-a}{n}$$



In the figure, each partition looks like a rectangle. For a regular partition, widths of the rectangles approach to zero as n is made large.

Area of first (left most) rectangle = length × width = $f(x_1) \times \Delta x_1$

Area under the curve = sum of areas of n rectangles

$= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_n)\Delta x_n$

$$= \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots\ldots (i)$$

Expression (i) represents approximation of sum of areas of n rectangles.

Based on our inductive concept, the area under the curve and between the interval $[a, b]$ is:

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots\ldots (ii)$$

Expression (ii) provides the fundamental concept of integral calculus and form the basis of the following definition.

Definition 3.2: A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit:

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

exists and does not depend upon the choice of partitions or on the choice of the points x_k in the subintervals. In the such case, we denote the limit by the symbol:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k \quad \ldots\ldots (iii)$$

Expression (iii) is called the definite integral of f from a to b. The numbers a and b are called lower limit and upper limit of integration respectively and $f(x)$ is called the integrand.

Theorem 3.2: If a function f is continuous on an interval $[a, b]$ then f is integrable on $[a, b]$ and the net signed area under the curve between the interval $[a, b]$ is:

$$A = \int_a^b f(x)dx$$

In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute the shaded area.

Example 20:

Sketch the region where area is represented by the definite integral and evaluate the integral using an appropriate formula from geometry.

(i) $\int_1^5 3dx$ (ii) $\int_{-2}^2 (x+3)dx$ (iii) $\int_0^1 \sqrt{1-x^2}dx$

Solution:

(i) Graph of the integral is the horizontal line $y = 3$.
So, the region is a rectangle of height 3 drawn over the interval from1 to 5.
From figure (1), we have:
$\int_1^5 3dx$ = area of rectangle $= 4 \times 3 = 12$ sq. units

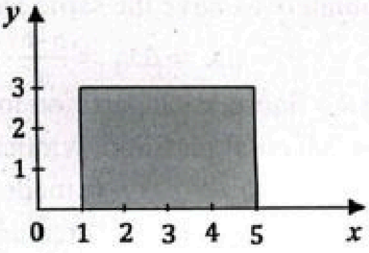


Fig. (1)

(ii) Graph of the integral is the line $y = x + 3$.
When $x = -2,\ y = -2 + 3 = 1$
When $x = 2,\ y = 2 + 3 = 5$
So, the region is trapezoid where base ranges from $x = -2$ to $x = 2$.
From figure (2), we have:
$\int_{-2}^2 (x+3)dx$ = area of trapezoid
$= \frac{1}{2}(1+5)(4) = 12$ sq. units

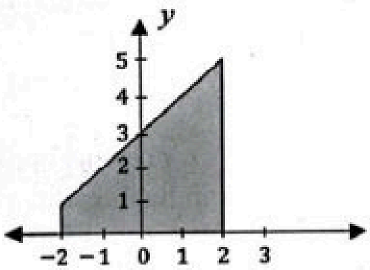


Fig. (2)

(iii) Graph of the function $y = \sqrt{1-x^2}$ is the upper semi-circle of radius 1 centred at the origin.
So, the region is upper right quarter-circle of radius 1 centred at origin.
From figure (3), we have:
$\int_0^1 \sqrt{1-x^2}dx$ = area of quarter circle
$= \frac{1}{4} \times \pi(1)^2 = \frac{\pi}{4}$ sq. units

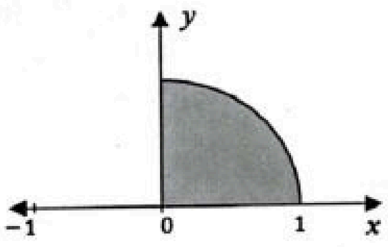


Fig. (3)

Example 21: Evaluate the following.

(i) $\int_0^1 (x-1)dx$ (ii) $\int_0^2 (x-1)dx$

Solution:

(i) The graph of the integral is the line $y = x - 1$.

When $x = 0$, $y = 0 - 1 = -1$

When $x = 1$, $y = 1 - 1 = 0$

The region is a triangle from $x = 0$ to $x = 1$.

From figure (4), we get:

$\int_0^1 (x-1)dx$ = area of triangle $= \frac{1}{2}(1)(1) = \frac{1}{2}$ sq. units

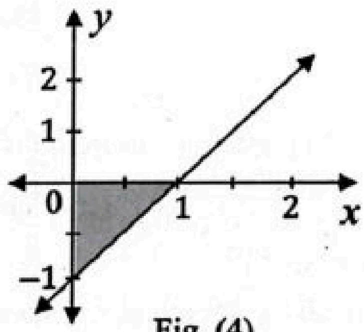


Fig. (4)

(ii) The graph of the integral is the line $y = x - 1$.

When $x = 0$, $y = 0 - 1 = -1$

When $x = 1$, $y = 1 - 1 = 0$

When $x = 2$, $y = 2 - 1 = 1$

The regions are two triangles from $x = -1$ to $x = 0$ and $x = 1$ to $x = 2$. From figure (5), we get:

$\int_0^2 (x-1)dx = \int_0^1 (x-1)dx + \int_1^2 (x-1)dx$

= area of triangle A_1 + Area of triangle A_2

$= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1$ sq. units

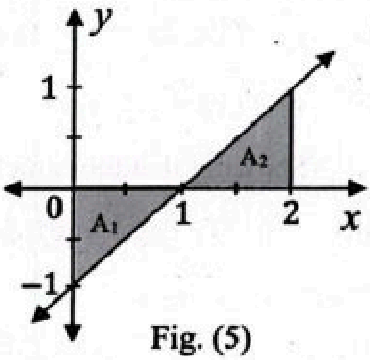


Fig. (5)

Note: In the figure (5), the area of triangle A_1 is below the x-axis and the area of triangle A_2 is above x-axis, therefore:

$A_1 = -\frac{1}{2}$ and $A_2 = \frac{1}{2}$ which implies $A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$

But area cannot be negative, therefore in such cases, we take net area as:

$$A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$$

3.7 Properties of The Definite Integral

In the finite closed interval $[a, b]$, when upper limit of integration in the definite integral is greater than the lower limit of integration ($a < b$), the following facts are true.

(i) If lower and upper limits of integration are equal, then area is zero. i.e.,

$$\int_a^a f(x)dx = 0$$

For example,

$$\int_2^2 x\,dx = 0$$

(ii) If the lower limit of integration is greater than the upper limit of integration, then:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Which states that interchanging the limits of integral reverses the sign of integral.

For example,

$$\int_1^0 (x-1)dx = -\int_0^1 (x-1)dx = \frac{1}{2}$$

(iii) If c is the point between a and b then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b dx$$

For example, in figure (5), we have:

$$\int_0^2 (x-1)dx = \int_0^1 (x-1)dx + \int_1^2 (x-1)dx$$

Theorem 3.3:

If f and g are integrable on $[a, b]$ and c is a constant, then cf, $f+g$ and $f-g$ are integrable on $[a, b]$ and the following statements are true.

(i) $\int_a^b c\, f(x)dx = c\int_a^b f(x)dx$ (The constant has no effects of limits on it.)

(ii) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Showing that the limit of a sum or difference is the sum or difference of the limits.

(iii) $\int_a^b [c\, f(x) \pm d\, g(x)]dx = c\int_a^b f(x)dx \pm d\int_a^b g(x)dx$

Example 22: Find:

(i) $\int_{-1}^4 [2f(x) + 5g(x)]dx$ if $\int_{-1}^4 f(x)dx = 2$ and $\int_{-1}^4 g(x)dx = 4$

(ii) $\int_{-1}^3 4f(x)dx$ if $\int_{-1}^2 f(x)dx = 3$ and $\int_2^3 f(x)dx = 1$

Solution:

(i) $\int_{-1}^4 [2f(x) + 5g(x)]dx = \int_{-1}^4 2f(x)dx + \int_{-1}^4 5g(x)dx = 2\int_{-1}^4 f(x)dx + 5\int_{-1}^4 g(x)dx$

$= 2(2) + 5(4) = 24$

(ii) $\int_{-1}^3 4f(x)dx = 4\int_{-1}^3 f(x)dx = 4\left[\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx\right]$

$= 4(3+1) = 4 \times 4 = 16$

Exercise 3.6

1. Sketch the region where area is represented by the definite integral and evaluate the integral using an appropriate formula from geometry.

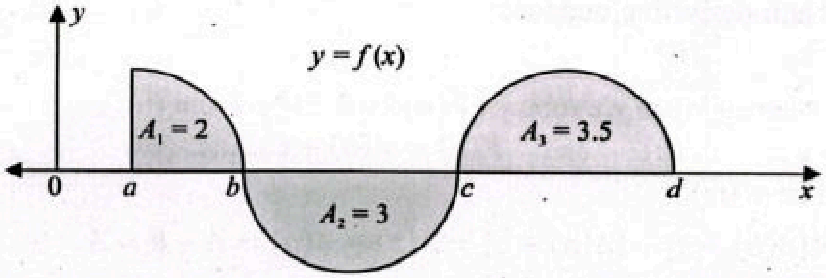
   (i) $\int_0^4 x dx$ (ii) $\int_{-3}^0 x dx$ (iii) $\int_0^2 (x-1)dx$

   (iv) $\int_0^2 (x+1)dx$ (v) $\int_{-3}^3 2dx$

2. Evaluate the integrals in each part when $f(x) = \begin{Bmatrix} x; & x \le 1 \\ 3; & x > 1 \end{Bmatrix}$.

   (i) $\int_0^1 f(x)dx$ (ii) $\int_{-1}^1 f(x)dx$ (iii) $\int_1^4 f(x)dx$ (iv) $\int_{-1}^2 f(x)dx$

3. Using the area shown below in the figure, evaluate the integrals.



(i) $\int_a^b f(x)dx$ (ii) $\int_b^c f(x)dx$ (iii) $\int_c^d f(x)dx$

(iv) $\int_a^c f(x)dx$ (v) $\int_b^d f(x)dx$ (vi) $\int_a^d f(x)dx$

4. Find:
$\int_1^5 [3f(x) - 2g(x)]dx$ if $\int_1^5 f(x)dx = 4$ and $\int_1^5 g(x)dx = 5$

5. Find:
$\int_1^4 f(x)dx$ if $\int_1^2 f(x)dx = 1$ and $\int_2^4 f(x)dx = 2$

6. Find:
$\int_3^{-2} f(x)dx$ if $\int_{-2}^1 f(x)dx = 1$ and $\int_1^3 f(x)dx = -5$

7. Use appropriate formula from geometry to evaluate integrals.
(i) $\int_{-1}^4 (3 - x)dx$ (ii) $\int_0^1 [2 + \sqrt{1 - x^2}\,]dx$ (iii) $\int_2^2 \sqrt{x^3 - 4}\,dx$

3.8 Fundamental Theorem of Calculus

In this section, we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the 'Fundamental Theorem of Calculus'. We will provide a powerful method for evaluating definite integrals using anti-derivatives.

We consider a non-negative and continuous function f on an interval $[a, b]$. The area A under the graph f over the interval $[a, b]$ is represented by the definite integral:

$$A = \int_a^b f(x)dx \;\ldots\ldots\ldots (i)$$

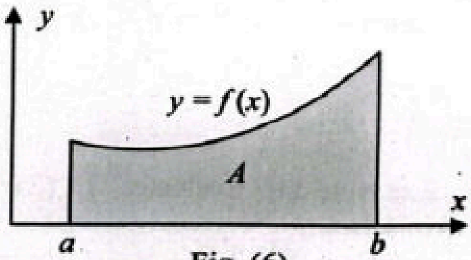


Fig. (6)

From (i), we have:

$A(a) = 0$ [The area under the curve from a to a is the area above the single point a and hence is zero.]

Similarly, $A(b) = A$ [The area under the curve from a to b is A.]

The formula $A'(x) = f(x)$ provides that $A(x)$ is an anti-derivative of $f(x)$ which implies that every other anti-derivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$.

By definition of anti-derivative, suppose:

$F(x) = A(x) + c \ldots\ldots\ldots (ii)$

We check what happens when we subtract $F(a)$ from $F(b)$. From (ii):

$F(a) = A(a) + c \ldots\ldots\ldots (iii)$ and $F(b) = A(b) + c \ldots\ldots\ldots (iv)$

Subtracting (iii) from (iv):

$$F(b) - F(a) = [A(b) + c] - [A(a) + c] = A(b) - A(a) = A - 0 = A$$

Therefore, from (i), we have:

$$A = \int_a^b f(x)dx = F(b) - F(a) \ldots\ldots\ldots (v)$$

Statement: The Fundamental Theorem of Calculus states that if f is continuous on $[a, b]$ ad F is antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

This can be written as:

$$\int_a^b f(x)dx = |F(x)|_{x=a}^{x=b} = F(b) - F(a)$$

We can emphasise that a and b are values for the variable x.

Thus, the definite integral can be evaluated by finding any anti-derivative of the integral and then subtracting the value of this anti-derivative at the lower limit of integration from its value at the upper limit of integration.

Example 23: Evaluate: $\int_1^3 x\,dx$

Solution: $\int_1^3 x\,dx = \left|\frac{x^2}{2}\right|_1^3 = \frac{3^2}{2} - \frac{1^2}{2}$ → First, we apply upper limit and then lower limit.

$$= \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$$

Example 24: Evaluate: $\int_{-2}^{2}(3x^2 - x + 1)dx$

Solution: $\int_{-2}^{2}(3x^2 - x + 1)dx = \left|x^3 - \frac{x^2}{2} + x\right|_{-2}^{2} = \frac{3^2}{2} - \frac{1^2}{2}$

$$= \left(2^3 - \frac{2^2}{2} + 2\right) - \left((-2)^3 - \frac{(-2)^2}{2} + (-2)\right)$$

$$= (8 - 2 + 2) - (-8 - 2 - 2) = 8 + 12 = 20$$

Example 25: Evaluate: $\int_0^2 \sqrt{2x^2+1}\,xdx$

Check Point

Evaluate: $\int_{\frac{\pi}{6}}^{\pi} \cos x\, dx$

Solution: We can apply two methods.

Method-1: By substitution but without changing the limits.

Let $u = 2x^2 + 1$ which implies $du = 4x\,dx$

Thus, $\int_0^2 \sqrt{2x^2+1}\,xdx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1} \times 4xdx$

$= \frac{1}{4}\int_0^2 \sqrt{u} \times du = \frac{1}{4} \times \left|\frac{2}{3}u^{\frac{3}{2}}\right|_0^2$ (Substituting for u)

$= \left|\frac{1}{6}(2x^2+1)^{\frac{3}{2}}\right|_0^2$ (Resubstituting for x)

Applying limits, we get:

$= \frac{1}{6}[2(2)^2+1]^{\frac{3}{2}} - \frac{1}{6}[2(0)^2+1]^{\frac{3}{2}} = \frac{1}{6}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$

Method-2: By substitution with changing the limits.

Let $u = 2x^2 + 1$ which implies $du = 4x\,dx$

When $x = 0, u = 2(0)^2 + 1 = 1$ and when $x = 2, u = 2(2)^2 + 1 = 9$

Thus, $\int_0^2 \sqrt{2x^2+1}\,xdx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1} \times 4xdx$

$= \frac{1}{4}\int_1^9 \sqrt{u} \times du = \frac{1}{4} \times \left|\frac{2}{3}u^{\frac{3}{2}}\right|_1^9$ (Substituting for u)

$= \frac{1}{6}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$

Example 26: Evaluate: $\int_a^b \frac{1}{1-\cos x}dx$ when $a = \frac{\pi}{4}, b = \frac{\pi}{3}$

Solution: $\int_a^b \frac{1}{1-\cos x}dx = \int_a^b \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x}dx = \int_a^b \frac{1+\cos x}{1-\cos^2 x}dx$

$= \int_a^b \frac{1+\cos x}{\sin^2 x}dx = \int_a^b \left[\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right]dx$

$= \int_a^b [\text{cosec}^2 x + \cot x\, \text{cosec} x]dx$

$= |-\cot x|_a^b + |-\text{cosec} x|_a^b$

Check Point

Evaluate:

$\int_0^1 \sin^{-1}x\, dx$

Applying limits and substituting values of a and b, we get:

$\int_a^b \frac{1}{1-\cos x}dx = -\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right) - \left(\text{cosec}\frac{\pi}{3} - \text{cosec}\frac{\pi}{4}\right)$

$= -\left(\frac{1}{\sqrt{3}} - 1\right) - \left(\frac{2}{\sqrt{3}} - \sqrt{2}\right) = \frac{1}{\sqrt{3}} + 1 - \frac{2}{\sqrt{3}} + \sqrt{2}$

$= 1 + \sqrt{2} - \sqrt{3}$

Example 27: Evaluate: $\int_1^e x\, lnx\, dx$

Solution: Taking lnx as first function and integrating by parts, we get:

$$\int_1^e x\, lnx\, dx = \int_1^e (lnx)(x)\, dx = \left|lnx \times \frac{x^2}{2}\right|_1^e - \int_1^e \frac{1}{x} \times \frac{x^2}{2} dx$$

$$= \left|lnx \times \frac{x^2}{2}\right|_1^e - \frac{1}{2}\int_1^e x\, dx = \left|lnx \times \frac{x^2}{2}\right|_1^e - \frac{1}{2} \times \left|\frac{x^2}{2}\right|_1^e$$

$$= \left(lne \times \frac{e^2}{2} - ln1 \times \frac{1^2}{2}\right) - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1^2}{2}\right) = \left(1 \times \frac{e^2}{2} - 0 \times \frac{1}{2}\right) - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{2} - 0 - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

Exercise 3.7

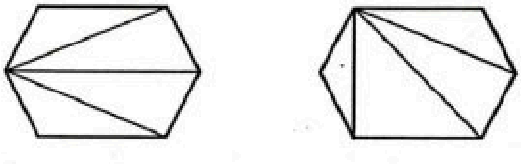
Evaluate the definite integrals.

1. $\int_{-1}^{2}(2x+3)\, dx$
2. $\int_{-4}^{12}\sqrt{y+4}\, dy$
3. $\int_0^{\frac{1}{2}}(2x+1)^{-\frac{1}{3}}\, dx$
4. $\int_0^3(6x^2-4x+5)\, dx$
5. $\int_{-2}^{1}(12x^5-36)\, dx$
6. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \cos\theta\, d\theta$
7. $\int_0^{\frac{\pi}{4}} \sec^2 2\theta\, d\theta$
8. $\int_2^4 \frac{x^2+8}{x^2}\, dx$
9. $\int_{-\frac{1}{2}}^{\frac{3}{2}} x - \cos\pi x\, dx$
10. $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx$
11. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \cos x\, dx$
12. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos\theta}{(\theta+\sin\theta)^2}\, d\theta$
13. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec x + \tan x)^2\, dx$
14. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx$
15. $\int_1^3 lnx\, dx$
16. $\int_2^4 \left(e^{\frac{x}{2}} - e^{\frac{x}{4}}\right) dx$
17. $\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\, dx$
18. $\int_0^{\frac{\pi}{4}} \tan^{-1} y\, dy$
19. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(2+\cos x)(5+\cos x)}\, dx$
20. $\int_2^5 \frac{1}{x(x+1)}\, dx$

3.9 Area and Volume

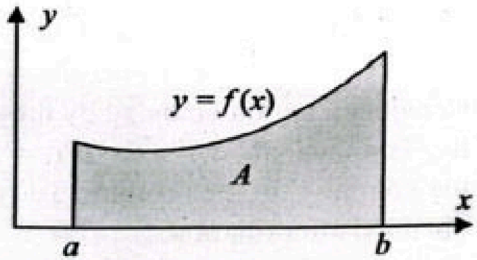
The definite integrals have applications that extend far beyond the area problems. In this section, we will also apply definite integrals for finding the volume. We have an inductive idea of what is meant by the area of certain geometrical figures. It is a number that in same way measures the size of the region enclosed by the figure. The area of a rectangle is the product of its length and width likewise the area of a triangle is half the product of lengths of the base and the altitude.

The area of a polygon may be defined as the sum of the areas of triangles into which it is decomposed and it can be proved that the area thus obtained is independent of how the polygon is decomposed into triangles.



However, how do we define the area of a region in a plane if the region is bounded by a curve? We even certain that such a region has an area? In the same way volume of solids can be found by using definite integration.

3.10 Area of Bounded Region

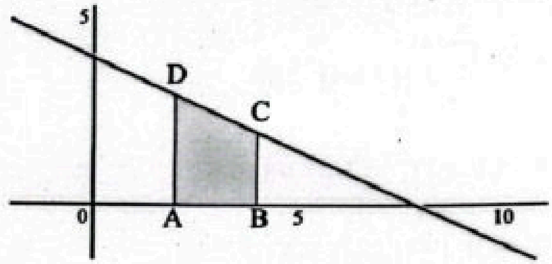


3.10.1 Area Between a Curve and the X-axis

If f is a non-negative continuous function on $[a, b]$, then the area under the graph of f from a to b is:

$$A = \int_a^b f(x)dx$$

Example 28: Find the area of the region bounded by the line $2y + x = 8$, the x-axis and, the lines $x = 2$ and $x = 4$.



Solution: In the graph, CD is the given line.

$2y + x = 8 \;\Rightarrow\; y = \frac{8-x}{2} \;\Rightarrow\; y = 4 - \frac{x}{2}$

Required area = area of trapezium ABCD

= area between line CD and x-axis from $x = 2$ to $x = 4$

$= \int_2^4 ydx = \int_2^4 \left(4 - \frac{x}{2}\right) dx = \left|4x - \frac{x^2}{4}\right|_2^4$

$= \left[4(4) - \frac{4^2}{4}\right] - \left[4(2) - \frac{2^2}{4}\right] = (16 - 4) - (8 - 1) = 5$ sq. units

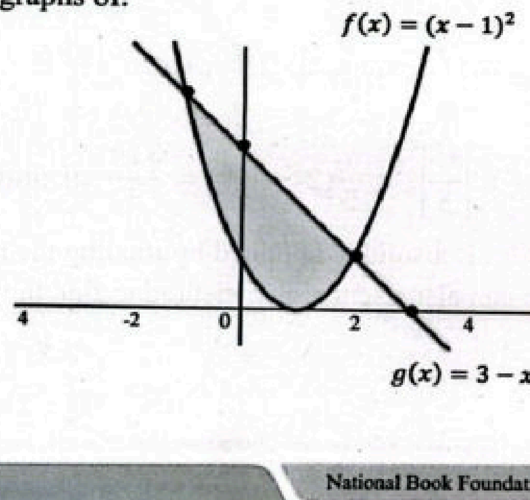
3.10.2 Area Between Curves

If the function $f(x)$ is greater than the function $g(x)$ for all x between a and b, then the area under the graph of $f(x)$ minus the area under the graph of $g(x)$ is the area between the curves. Thus, the area between the curves $f(x)$ and $g(x)$ is:

$$A = \int_a^b [f(x) - g(x)]dx \;;\; f(x) > g(x)$$

Example 29: Find the area of the region bounded by graphs of:

$$f(x) = (x - 1)^2 \text{ and } g(x) = 3 - x$$



Solution: To find the limits of integration, we find common points of both functions by solving

$f(x) = g(x) \Rightarrow (x - 1)^2 = 3 - x$

$\Rightarrow x^2 - x - 2 = 0$

After solving, we get:

$x = -1$ and $x = 2$

For $-1 < x < 2,\; g(x) > f(x)$

(Also clear from the graph of both curves.)

Thus, the area of region bounded is:

$$A = \int_{-1}^{2} [g(x) - f(x)]dx = \int_{-1}^{2} [(3 - x) - (x - 1)^2]dx \int_{-1}^{2} (2 + x - x^2)dx$$

$$= \left|2x + \frac{x^2}{2} - \frac{x^3}{3}\right|_{-1}^{2} = \left[2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right] - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right]$$

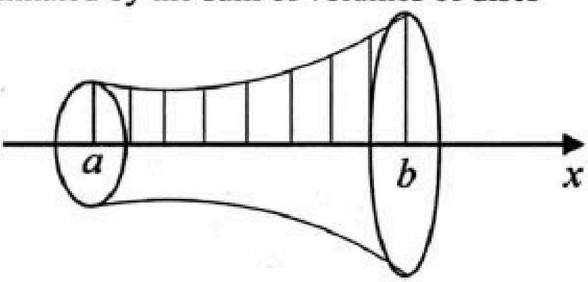
$$= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = 8 - 3 - 0.5 = 4.5 \text{ sq. units}$$

3.11 Volume of Solids of Revolution

3.11.1 Disc Method

Consider a region bounded by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ that is rotated about x-axis. If $a = x_0 < x_1 < x_2 ... < x_n = b$ is partition of the interval $[a, b]$, the volume V of the resulting 3-D region can approximated by the sum of volumes of discs obtained after rotation.

The radius and height of discs D_i are $f(x)_i$ and Δx_i Respectively. Thus:



$$V = \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x_i$$

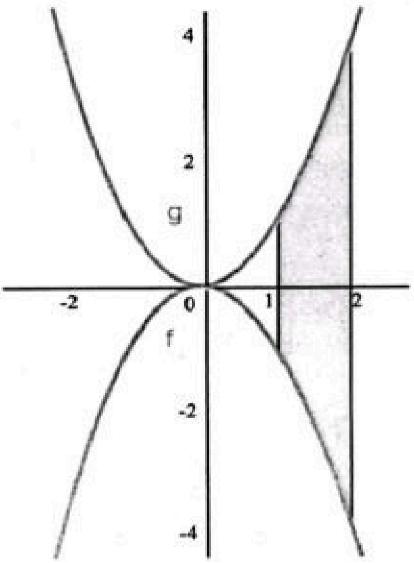
Volume of disc = area of base × height = $(\pi r^2)(h)$

Letting $\Delta x_i \to 0$, we have:

$$V = \pi \int_a^b [f(x)]^2 dx$$

Example 30:

Find the volume of the solid obtained by rotating the graph $y = x^2$ between $x = 1$ and $x = 2$ about x-axis.



Solution:

$$V = \pi \int_1^2 [f(x)]^2 dx$$

$$V = \pi \int_1^2 (x^2)^2 dx = \pi \int_1^2 x^4 dx$$

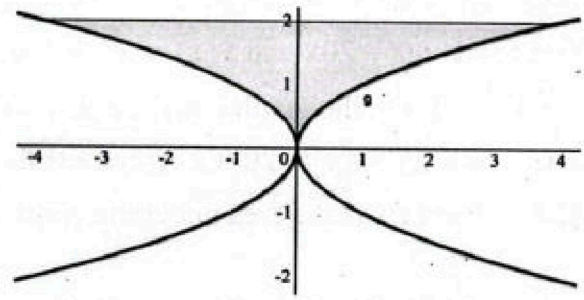
$$V = \pi \left|\frac{x^5}{5}\right|_1^2 = \frac{\pi}{5}(2^5 - 1^5) = \frac{31\pi}{5} \text{ cu.units}$$

Note: If a solid is obtained by rotating the regions bounded by the graph $x = g(y)$ about y-axis, we can also use the disc method to find the volume as follows.

$$V = \pi \int_a^b [g(x)]^2 dx$$

Example 31:

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}, y = 0$ and $y = 2$ is revolved about the y-axis.



Solution:

First sketch the region and the solid. The cross section taken perpendicular to the y-axis and disk suggests that we can rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, $g(y) = y^2$ and the volume is:

$$V = \pi \int_a^b [g(y)]^2 dy = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy$$

$$V = \pi \left|\frac{y^5}{5}\right|_0^2 = \frac{\pi}{5}(2^5 - 0^5) = \frac{32\pi}{5} \text{ cu. units}$$

3.12 Applications

3.12.1 Consumer and Producer Surpluses

Economists use the definite integral to define the concept of consumer and producer surpluses.

The demand for a commodity by consumers as well as the amount supplied to the market by the manufacturers can often be expressed as a function of the per unit price. Let $D(x)$ and $S(x)$ be the number of units demanded and the number of units supplied, respectively, when the commodity sells at a price x per unit.

If the demand equals the supply:

$$D(x) = S(x)$$

The market is said to be in equilibrium and the corresponding price of the commodity is called the equilibrium price. If p is the equilibrium price and b is the price at which the demand of the commodity is zero ($b(s) = 0$), the integral:

$$Cs = \int_p^b D(x)dx$$

is called the consumer surplus. Similarly, the integral:

$$Ps = \int_c^p S(x)dx$$

where $S(c) = 0$, is called the producer surplus.

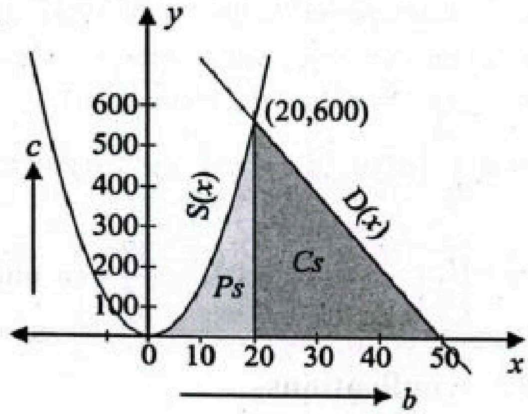
Example 32:

Suppose the demand and supply of a commodity selling for x dollars a unit and $D(x) = 1000 - 20x$ and $S(x) = x^2 + 10x$, respectively. Find the consumer and producer surplus.

Solution: From the graph it is clear that $D(x) = 0$ when $b = 50$, $S(x) = 0$ when $c = 0$ and $D(s) = S(x)$ for $p = 20$. Cs represents the area under the graph of $D(x)$ on the interval [20, 50] and Ps is the area under the graph of $S(x)$ on [0, 20]. We have:

$$Cs = \int_p^b D(x)dx = \int_{20}^{50} (1000 - 20x)dx$$

$$Cs = \left|1000x - \frac{20}{2}x^2\right|_{20}^{50} = \$\,9000$$



And,

$$Ps = \int_c^p S(x)dx = \int_0^{20} (x^2 + 10x)dx$$

$$Ps = \left|\frac{1}{3}x^3 + 5x^2\right|_0^{20} = \$\,4666.67$$

3.12.2 Rectilinear Motion

If $f(t)$ is the position function of an object moving in the straight line, then we have:

velocity $= v(t) = \frac{dS}{dt}$ and acceleration $= a(t) = \frac{dv}{dt}$

By using the definition of anti-derivative, the quantities S and v can be written as indefinite integrals.

$S(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$

By knowing the initial position $S(0)$ and the initial velocity $v(0)$, we can find specific values of the constants of integration.

Key Facts

(i) For upward motion:

$S(0) = 0,\quad v(0) > 0,\quad a = g = -98m/s^2 = -32ft/s^2$

(ii) For downward motion:

$S(0) = h,\quad v(0) = 0,\quad a = g = 98m/s^2 = 32ft/s^2$

Example 33:

The position function of an object that moves on a coordinate line is $S(t) = t^2 - 6t$. Where S is measured in centimetres and t in seconds. Find the distance travelled in the time interval [3, 9].

Solution: The velocity function:

$$v(t) = \frac{dS}{dt} = 2t - 6$$

implies that $v \geq 0$ for $3 \leq t \leq 9$. Hence the distance travelled is:

$$S(t) = \int_3^9 v(t)dt = \int_3^9 (2t - 6)dt$$

$$= |t^2 - 6t|_3^9 = (81 - 54) - (9 - 18) = 4\ cm$$

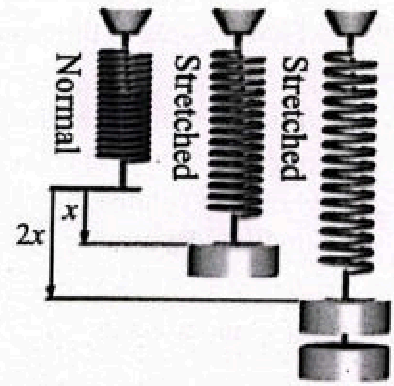
3.12.3 Work

In physics when a constant force F moves an object a distance d in the same direction, the work done is defined as $W = Fd$.

Definition: Let $F(x)$ be a continuous force acting at a point in the interval $[a, b]$, then the work done W by the force on moving an object from a to b is:

$$W = \int_a^b F(x)dx$$

3.12.4 Motion of Spring



Hook's law states that "when a spring is stretched (or compressed) beyond its natural length, the restoring force exerted by the spring is directly proportional to the amount of elongation (or compression)". Thus, in order to stretch a spring, x units beyond its natural length, we need to apply the force:

$F(x) = kx$; k is spring constant.

Example 34:

A force of 130 N is required to stretch a spring 50 cm. Find the work done in stretching the spring 20 cm beyond its natural (unstretched) length.

Solution:

$x = 50\ cm = 0.5m$ and $F = 130N$

Substituting values of x and F in $F = kx$, we have:

$130 = k \times 0.5 \Rightarrow k = 260\ N/m$

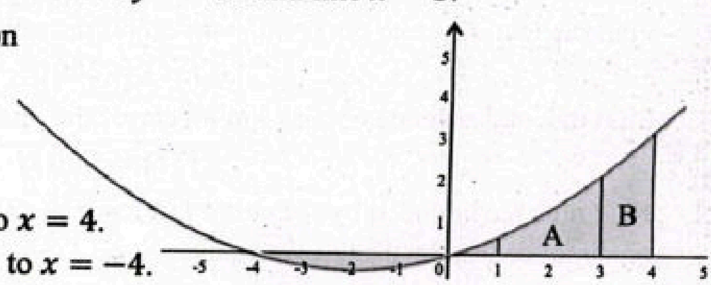
Thus, $F = kx \Rightarrow F = 260\,x$

Now, $x = 20\ cm = 0.2m$, so that the work done in stretching the spring by this amount is:

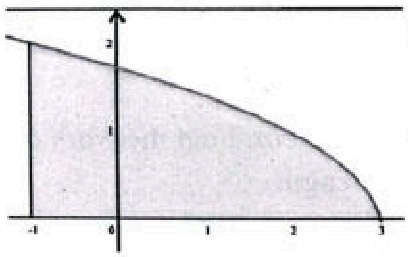
$$W = \int_0^{\frac{1}{5}} 260x\, dx = |130x^2|_0^{\frac{1}{5}} = \frac{26}{5} = 5.2\,J$$

Exercise 3.8

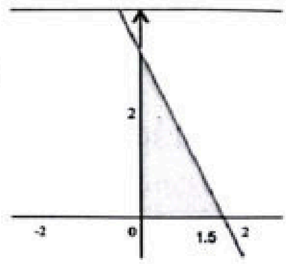
1. Find the area of region bounded by the curve $y = x^2$, the x-axis, lines $x = 1$ and $x = 3$.
2. Find the area under the curve $y = \sqrt{6x+4}$ (above x-axis) from $x = 0$ to $x = 2$.
3. Find the area of region bounded by the curve $y^2 = 4x$ and line $x = 3$.
4. In the figure, a sketch of the function $y = \frac{1}{2}(0.2x^2 + x)$ is shown. Find:
   (i) the area of region A.
   (ii) the area of region B.
   (iii) area of the region from $x = 1$ to $x = 4$.
   (iv) area of the region from $x = -1$ to $x = -4$.



5. Find the area bunded by the graph:
   (i) $y = 1 + \cos x$; $[0, 3\pi]$ (ii) $y = -1 + \sin x$; $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$
6. Find the area of the region bounded by the graphs of $y = x, y = -2x$ and $x = 3$.
7. Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$ and bounded on the sides by the lines $x = 0$ and $x = 2$.
8. Find the area bounded by the curve $y = x^3 + 1$, the x-axis and the line $x = 1$.
9. Find the area of the region enclosed by $x = y^2$ and $y = x - 2$ integrating with respect to y.
10. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis.
11. Find the volume of the sloid that results when the shaded region is revolved about the indicated axis.
   (i) $y = \sqrt{3 - x}$ about x-axis



   (ii) $y = 3 - 2x$ about y-axis



12. An object moves in a straight line according to the position function given below. If f is measured in centimetres, find the distance travelled by the object in the indicated time interval:
   (i) $S(t) = t^2 - 2t$; $[0, 5]$ (ii) $S(t) = t^3 - 3t^2 - 9t$; $[0, 4]$
   (iii) $S(t) = 6 \sin \pi t$; $[1, 3]$
13. It takes a force of 50 N to stretch a spring of 0.5m. Find the work done in stretching the spring 0.6m beyond its natural length.
14. A force $F = \frac{3}{2}x\ lb$ is needed to stretch a 10 inch spring an additional x inch. Find the work done in stretching the spring 16 inch.
15. Find the consumer and producer surpluses, when:
   (i) $S(x) = 24,\quad D(x) = 100 - 2x$
   (ii) $S(x) = x^2 - 4,\quad D(x) = -x + 8$
   (iii) $S(x) = 2x^2 + 3x,\quad D(x) = 36 - x^2$
16. Find the total revenue obtained in 4 years if the rate of increase in dollars per year is:
$$f(t) = 200(t - 5)^2$$
17. Find the total revenue obtained in 8 years if the rate of increase in dollars per year is:
$$f(t) = 600\sqrt{1 + 3t}$$
18. Find the area bounded by the curve $f(x) = x^3 - 2x^2 + 1$ and the x-axis in the first quadrant bonded by the line $x = 1.5$.

Review Exercise

1. Select the correct option in the following.
   (i) If f is integrable, then it is:
   (a) discontinuous (b) unbounded (c) continuous (d) linear
   (ii) If $f'(x) = 3x^2 + 2x$, then $f(x)$ is:
   (a) $6x + 2 + c$ (b) $x^3 + x^2 + c$ (c) $3x^3 + 2x^2 + c$ (d) $1.5x^3 + x^2 + c$
   (iii) $\int \frac{d}{dx}(x^2)dx$ is equal to:
   (a) $x^2 + c$ (b) $2x + c$ (c) $\frac{x^3}{3} + c$ (d) $2x + c$
   (iv) $\int \sin 2x dx$ is:
   (a) $\frac{\cos 2x}{2} + c$ (b) $2\cos 2x + c$ (c) $-\frac{\sin 2x}{2} + c$ (d) $-\frac{\cos 2x}{2} + c$
   (v) $\int_3^7 dx$ is:
   (a) 3 (b) 4 (c) 5 (d) 6
   (vi) $\int_{\frac{\pi}{6}}^{\pi} \cos x dx$ is:
   (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$
   (vii) $\frac{d}{dx}\int_{-2}^{x} t^3 dt$ is equal to:
   (a) t^4 (b) t^3 (c) x^3 (d) $x^3 - 16$
   (viii) What is relation between $\int_1^2 x dx$ and $\int_1^2 t dt$?
   (a) $\int_1^2 x dx < \int_1^2 t dt$ (b) $\int_1^2 x dx > \int_1^2 t dt$
   (c) $\int_1^2 x dx \neq \int_1^2 t dt$ (d) $\int_1^2 x dx = \int_1^2 t dt$
   (ix) Area under the graph of $f(x) = 4$; [2, 5] is:
   (a) 2 (b) 4 (c) 5 (d) 12
   (x) $\int \sqrt{x} dx$ is:
   (a) $x^{\frac{3}{2}} + c$ (b) $\frac{2}{3}x^{\frac{3}{2}} + c$ (c) $\frac{3}{2}x^{\frac{3}{2}} + c$ (d) $x^{\frac{1}{2}} + c$
2. Evaluate:
   (i) $\int \frac{4x+2}{x^2+x+1} dx$ (ii) $\int x(x^2+1)^4 dx$ (iii) $\int \cos^2 3x\, dx$
   (iv) $\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)} dx$ (v) $\int \sin^{-1} x dx$ (vi) $\int 2x \sin 3x\, dx$
   (vii) $\int x^2 e^x dx$ (viii) $\int_0^{\frac{\pi}{4}} (\sin 2x - 5\cos 4x) dx$ (ix) $\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}} dx$
3. Use the substitution $u = 2x + 1$ to evaluate $\int_0^1 \frac{x^2}{\sqrt{2x+1}} dx$.
4. A model rocket is launched upward from ground level with an initial speed of 60m/s.
   (a) How long does it take for the rocket to reach its highest point?
   (b) How high does the rocket go?
5. Suppose that a parachute moves with a velocity $V(t) = \cos \pi t$ m/s along a coordinate line. Assuming that the parachute has the coordinate $S = 4m$ at time $t = 0$ sec, find its position.