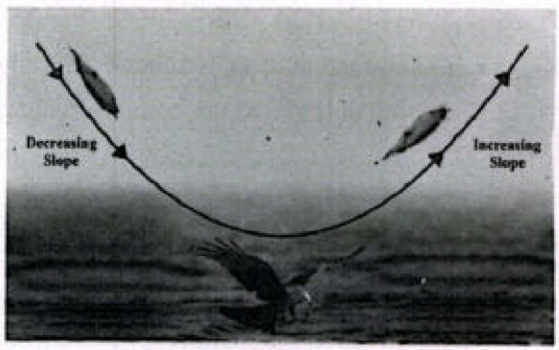
LIMIT, CONTINUITY AND DERIVATIVE

After studying this unit, students will be able to:

- Demonstrate and find the limit of a function.
- State and apply theorems on limit of sum, difference, product and quotient of functions to algebraic, exponential and trigonometric functions.
- Demonstrate and test continuity, discontinuity of a function at a point and in an interval.
- Apply concepts of transcendental functions, limit of a function and its continuity to real world problems.
- Calculate inflation over a period. Calculate depreciation with the help of straight-line method.
- Recognize the meaning of the tangent to a curve at a point.
- Calculate the gradient of a curve at a point. Identify the derivative as the limit of a difference quotient. Calculate the derivative of function. Estimate the derivative as rate of change of velocity, temperature and profit. Recognize the derivative function.
- State the connection between derivative and continuity.
- Find the derivative: function, square root, quadratic and logarithmic functions.
- Apply the differentiation rules to polynomials, rational and trigonometric functions.
- Apply the differentiation to state the increasing and decreasing function.
- Apply differentiation to real world problems.
- Find higher order derivatives of algebraic, implicit, parametric, trigonometric, inverse trigonometric functions. Describe the ability to approximate functions.
- Explain differentials to approximate the change in quantity. Calculate errors.
- Find extreme values by applying second derivative test. Explain and find critical point.
- Apply derivative and higher order derivative to real world problems.

The word calculus is a diminutive form of the Latin word calx, which means stone. In ancient civilization, small stones or pebbles were often used as a means for reckoning consequently, the word calculus can refer to any systematic method of computation. However, over the last. several hundred years, a definition of calculus means that the branch of mathematics concerned with the calculation and application of entities known as derivatives and integrals.



2.1 Limits of Functions

Two of the most fundamental concepts in the study of calculus are the notions of function and the limit of the function. In this first section, we shall be especially interested in determining whether the values $f(x)$ of a function f approach a fixed number L as x approaches a number 'a' using the symbol '$\rightarrow$' for the word **'approach'** we ask $f(x)\rightarrow L$ as $x\rightarrow a$.

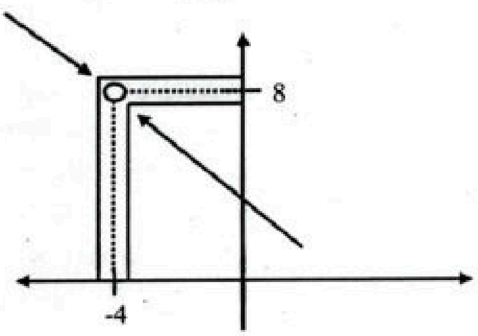
2.1.1 Limit of a Function as x Approaches to a Number

Consider a function: $$f(x)=\frac{16-x^2}{4+x}$$

Whose domain is set of all real numbers except -4. Although $f(-4)$ is not defined, nonetheless, $f(x)$ can be calculated for any value of x near -4. The table shows that, as x approaches to -4 from either the left or right, the functional values $f(x)$ approaches to 8. That is, when x is near -4, $f(x)$ is near 8. We say 8 is the limit of $f(x)$ as x approaches to -4. We can write as:

$$f(x)\rightarrow 8 \text{ as } x\rightarrow -4 \text{ or } \lim_{x\to -4}\frac{16-x^2}{4+x}=8$$

x	$f(x)$
-4.1	8.1
-4.01	8.01
-4.001	8.001
-3.9	7.9
-3.99	7.99
-3.999	7.999



For $x\neq -4$, f can be simplified by cancellation $f(x)=\dfrac{16-x^2}{4+x}=\dfrac{(4+x)(4-x)}{4+x}=4-x$.

The graph of f is essentially the graph of $y=4-x$ with the exception that the graph of f has a hole at the point that corresponds to $x=-4$. As x get closer and closer to -4, represented by the two arrowheads on the x-axis. The two arrowheads on the y-axis simultaneously get closer and closer to the number 8.

Intuitive Definition: If $f(x)$ can be made arbitrarily closer to a finite number by taking x sufficiently close to but different from a number a, from both the left and right side of a, then $\lim_{x\to a} f(x)=L$

$x\rightarrow a^-$ denote that x approaches a from the left and $x\rightarrow a^+$ denote that x approaches a from the right.

Thus, if both sides have the common value L,

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=L$$

We say that: $\lim_{x\to a} f(x)$ exist and write $\lim_{x\to a} f(x)=L$

Note: The existence of a limit of a function f at a does not depend on whether f is actually defined for a but only on whether f is defined for near a.

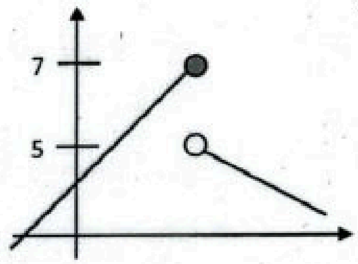
Example 1: Using the graph, check whether the limit of the function exists or not.

$$f(x) = \begin{cases} x+2 & x \le 5 \\ -x+10 & x > 5 \end{cases}$$

$\lim_{x\to 5^-} f(x) = x + 2 = 7$

$\lim_{x\to 5^+} f(x) = -x + 10 = -5 + 10 = 5$

$x \to 5^-$	$f(x)$
4.9	6.9
4.99	6.99
4.999	6.999



$x \to 5^+$	$f(x)$
5.1	4.9
5.01	4.99
5.001	4.999

Since $\lim_{x\to 5^-} f(x) \neq \lim_{x\to 5^+} f(x)$, we concluded that $\lim_{x\to 5} f(x)$ does not exist.

Example 2: Evaluate. a. $\lim_{x\to 0} \frac{\sin x}{x}$ b. $\lim_{x\to 0} \frac{1-\cos x}{x}$

Solution:

a.

$x \to 0^-$	$\frac{\sin x}{x}$
-0.1	0.998341
-0.01	0.9999833
-0.001	0.999998

a. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$x \to 0^+$	$\frac{\sin x}{x}$
0.1	0.998341
0.01	0.9999833
0.001	0.999998

b.

$x \to 0^-$	$\frac{1-\cos x}{x}$
-0.1	-0.0499583
-0.01	-0.0049999
-0.001	-0.0005001
-0.0001	-0.000510

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$x \to 0^+$	$\frac{1-\cos x}{x}$
0.1	0.0499583
0.01	0.0049999
0.001	0.0005001
0.0001	0.000510

Example 3: Evaluate: a. $\lim_{x\to 3} 15$ b. $\lim_{x\to 5} 10x$ c. $\lim_{x\to 5}(x^2 - 5x + 6)$

d. $\lim_{x\to -1} \frac{3x-1}{6x+2}$ e. $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$ f. $\lim_{x\to 2}(3x-2)^6$

Solution:

a. $\lim_{x\to 3} 15 = 15$

b. $\lim_{x\to 5} 10x = 10(5) = 50$

c. $\lim_{x\to 5}(x^2-5x+6) = 25-25+6 = 6$

d. $\lim_{x\to -1}\frac{3x-1}{6x+2} = \frac{3(-1)-1}{6(-1)+2} = \frac{-3-1}{-6+2} = 1$

e. $\lim_{x\to 1}\frac{x-1}{x^2+x-2}$, $\lim_{x\to 1} x^2+x-2=0$

By simplifying first we can apply theorem v,

$$= \lim_{x\to 1}\frac{x-1}{x^2+x-2} = \lim_{x\to 1}\frac{(x-1)}{(x-1)(x+2)}$$

$$= \lim_{x\to 1}\frac{1}{(x+2)} = \frac{1}{3}$$

f. $\lim_{x\to 2}(3x-2)^6 = (3(2)-2)^6 = (4)^6 = 4096$

Theorems on Limits:

i. If c is constant, then $\lim_{x\to a} c = c$.

ii. If c is constant, then

$$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$$

iii. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$= L_1 + L_2$

iv. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}, L_2 \neq 0$

v. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$

Example 4: Evaluate: a. $\lim_{x\to 5}\frac{4x+5}{x^2-25}$ b. $\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$

Solution:

a.

$\lim_{x\to 5}\frac{4x+5}{x^2-25}$, $\lim_{x\to 5} 4x+5 = 25$,

$\lim_{x\to 5} x^2-25 = 0$

$$\lim_{x\to 5}\frac{4x+5}{x^2-25} = \frac{25}{0}$$

We can't simplify to remove zero from the denominator, so limit $x \to 5$ doesn't exist.

b.

$\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$, $\lim_{x\to -8} 2x+10 = -6 \neq 0$

$= \lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10}$ (apply theorem iv)

$$= \lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10} = \frac{\lim_{x\to -8} x-\sqrt[3]{x}}{\lim_{x\to -8} 2x+10} = \frac{-8-(-8)^{1/3}}{2(-8)+10}$$

$$= \frac{-8-((-2)^3)^{1/3}}{-6} = \frac{-8+2}{-6} = 1$$

Exercise 2.1

1. Use a graph to find the given limit, if it exists.

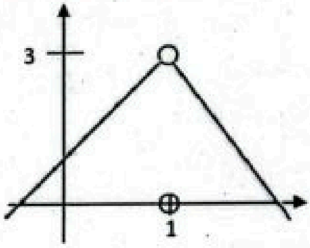
a. $\lim_{x\to 5}\sqrt{x-1}$ b. $\lim_{x\to 1}\frac{x^2-1}{x-1}$ c. $\lim_{x\to 0}\frac{x^2-3x}{x}$ d. $\lim_{x\to 0}\frac{|x|}{x}$

e. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x & x<2 \\ x+1 & x\geq 2 \end{cases}$ f. $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2 & x<0 \\ 2 & x=0 \\ \sqrt{x}-1 & x>0 \end{cases}$

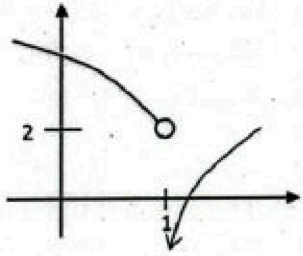
g. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

2. Use the given graph to find each limit ($x \to 1$), if it exists.

a.



b.



Evaluate the following.

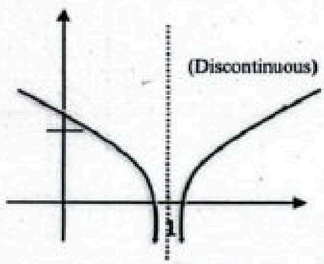
3. $\lim\limits_{x \to -2} \frac{x^3+8}{x+2}$
4. $\lim\limits_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
5. $\lim\limits_{x \to 7} \frac{x^2-21}{x+2}$
6. $\lim\limits_{x \to 0} \frac{x^2-6x}{x^2-7x+6}$
7. $\lim\limits_{y \to 1} \frac{y^3-1}{y-1}$
8. $\lim\limits_{x \to 3^+} \frac{(x+3)^2}{\sqrt{x-3}}$
9. $\lim\limits_{x \to 2} (x-4)^4(x^2-3)^{10}$
10. $\lim\limits_{x \to 0} \left(x - \frac{1}{x-2}\right)$
11. $\lim\limits_{x \to -3} \frac{2x+6}{4x^2-36}$
12. $\lim\limits_{x \to 0} \frac{\tan x}{x}$
13. $\lim\limits_{x \to 0} \frac{x}{\sin 3x}$

2.2 Continuity

In the case of limit, we have used the phrase "connect the points with smooth curve". The phrase provides the concept of graph that is a nice continuous curve that is, a curve with no gaps or breaks. Indeed, a continuous function is often described as one whose graph can be drawn without lifting pencil from paper.

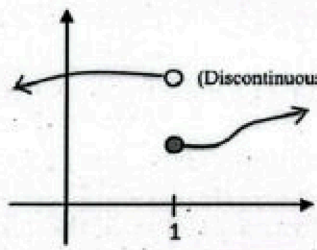
Before moving towards the precise definition of continuity, we demonstrate in figures some intuitive examples of functions that are not continuous or continuous at a number.

Fig (i)



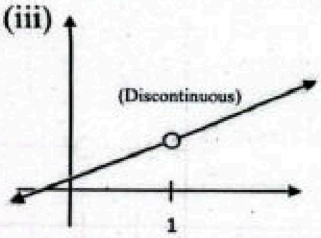
$\lim\limits_{x \to 1} f(x)$ does not exist and $f(1)$ is not defined.

Fig (ii)



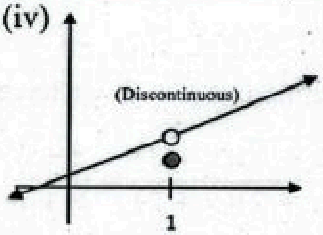
$\lim\limits_{x \to 1} f(x)$ does not exist and $f(1)$ is defined.

Fig (iii)



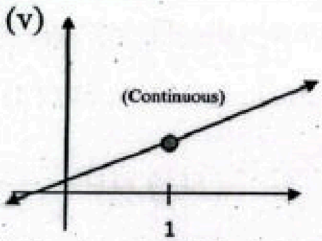
$\lim\limits_{x \to 1} f(x)$ exist but $f(1)$ is not defined.

Fig (iv)



$\lim\limits_{x \to 1} f(x)$ exist and $f(1)$ is defined but $\lim\limits_{x \to 1} f(x) \neq f(1)$

Fig (v)



$\lim\limits_{x \to 1} f(x)$ exist and $f(1)$ is defined and $\lim\limits_{x \to 1} f(x) = f(1)$

2.2.1 Continuity at a Number

Figures (i) - (v), at page 47, suggest the threefold conditions of continuity of a function at a number a (instead of 1 we consider a).

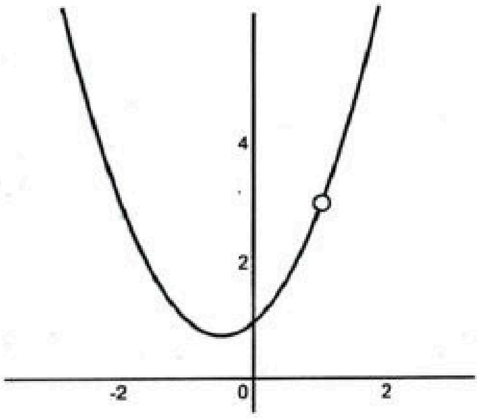
Definition: Continuity

A function is said to be continuous at a number a if

i. $f(a)$ is defined
ii. $\lim_{x\to a} f(x)$ exits, and
iii. $\lim_{x\to a} f(x) = f(a)$

Example 5: The rational function

$$f(x) = \frac{x^3 - 1}{x - 1}$$
$$= \frac{(x-1)(x^2+x+1)}{x-1}$$
$$= x^2 + x + 1, x \neq 1$$

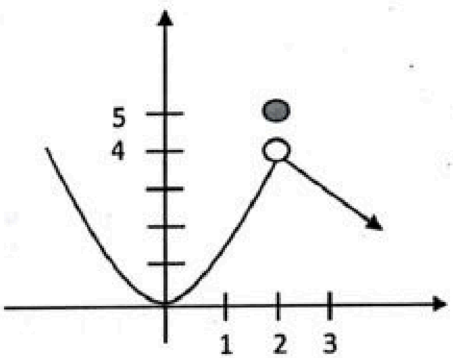


is discontinuous at 1 since $f(1)$ is not define.

From graph, we observe that $\lim_{x\to 1} f(x) = 3$. We can also state that f is continuous at any other number $x \neq 1$.

Example 6: Given figure shows the graph of the piecewise function defined

$$f(x) = \begin{cases} x^2 & x < 2 \\ 5 & x = 2 \\ -x + 6 & x > 2 \end{cases}$$



Now $f(2)$ is defined and is equal to 5. Next, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x + 6 = 4$$

This implies limit exists: $\lim_{x\to 2} f(x) = 4$.

Since $\lim_{x\to 2} f(x) \neq f(2) = 5$, therefore f is discontinuous at 2.

Example 7: Let $f(x) = \frac{x^2+x-6}{x^2-4}$, for $x \neq 2$. Show how to define $f(2)$ in order to make f continuous function at 2.

Solution: Although $f(2)$ is not defined, if $x \neq 2$, we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

The function $f(x) = \frac{x+3}{x+2}$ is equal to $f(x)$ for $x \neq 2$, but is also

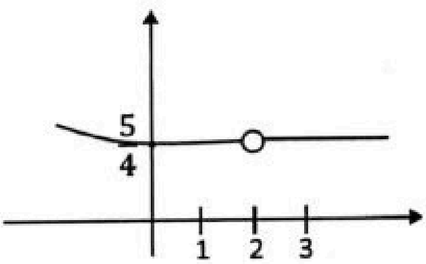


Fig (i)

continuous at $x = 2$ having the value of $\frac{5}{4}$. Thus f is the continuous extension of f to $x = 2$ and

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \lim_{x\to 2}\frac{x+3}{x+2} = \frac{5}{4}$$

The graph of f is shown in figure (i).

The graph of its continuous extension is shown in figure (ii).

$$f(x) = \frac{x+3}{x+2} = \begin{cases} \frac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \frac{5}{4}, & x = 2 \end{cases}$$

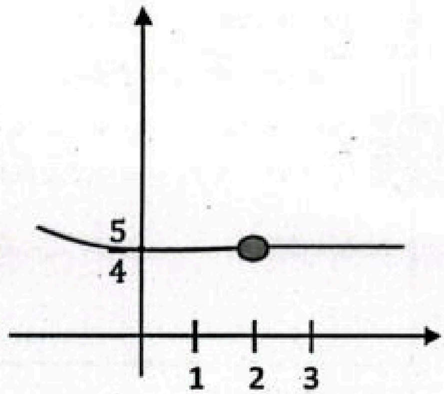


Fig (ii)

We can also observe that $x = 2$ is removable discontinuity for the $f(x) = \frac{x^2+x-6}{x^2-4}$.

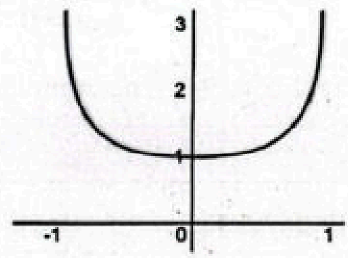
2.2.2 Continuity on an Interval

A function is said to be continuous on an open interval (a, b) if it is continuous at every number in the interval. A function f is continuous on a closed interval $[a, b]$ if it is continuous on $(a, b]$ and in addition, it is continuous on $[a, b)$

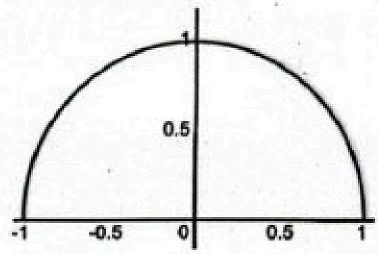
$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to b^-} f(x) = f(b)$$

Example 8:

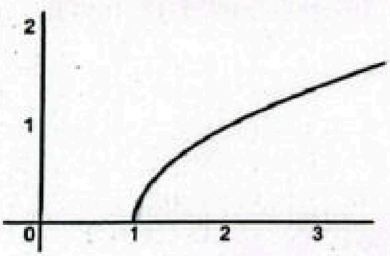
a. $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous on the open interval $(-1, 1)$ but is not continuous on the closed interval $[-1, 1]$, since neither $f(-1)$ nor $f(1)$ is defined.



b. $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$ we can observe from figure that $\lim_{x\to -1^+} f(x) = f(-1) = 0$ and $\lim_{x\to 1^-} f(x) = f(1) = 0$



c. $f(x) = \sqrt{x-1}$ is continuous on $[1, \infty)$ since $\lim_{x\to 1^+} f(x) = f(1) = 0$



Key Facts

Continuity of a Sum, Product and Quotient: If f and g are functions continuous at a number a, then cf(c a constant), $f+g, fg$ and $\frac{f}{g}, (g(a)\neq 0)$ are also continues at a.

Removable Discontinuity: If $\lim_{x\to a} f(x)$ exists but f is either not defined at a or $f(a)\neq \lim_{x\to a} f(x)$, then f is said to have a removable discontinuity at a. For example the function $\frac{x^2-1}{x-1}$ is not defined at 1 but $\lim_{x\to 1} f(x) = 2$. By definition $f(1)=2$, the new function

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 2, & x=1\end{cases}$$

is continuous at every number.

Exercise 2.2

In problems 1-8, determine the numbers (if any), at which the given function is discontinuous.

1. $f(x)=x^2-5x+6$
2. $f(x)=\frac{2x}{x^2+5}$
3. $f(x)=\frac{1}{x^2-9x+8}$
4. $f(x)=\frac{x^2-1}{x^4-1}$
5. $f(x)=\frac{x-1}{\sin 2x}$
6. $f(x)=\begin{cases}x, & x<0\\ x^2, & 0\leq x\leq 2\\ x, & x\geq 2\end{cases}$
7. $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ \frac{1}{2}, & x=0\end{cases}$
8. $f(x)=\begin{cases}\frac{x^2-36}{x-6}, & x\neq 6\\ 12, & x=6\end{cases}$

In problems 9-14, determine whether the given function is continuous in the indicated intervals.

9. $f(x)=x^2+1$	a. $[-1, 3]$	b. $[3, \infty)$
10. $f(x)=\frac{1}{x}$	a. $(-3, 3)$	b. $(0, 10]$
11. $f(x)=\frac{1}{\sqrt{x}}$	a. $[1, 4)$	b. $[1, 9]$
12. $f(x)=\sqrt{x^2-9}$	a. $[-3, 3]$	b. $[3, \infty)$
13. $f(x)=\frac{x}{x^3+8}$	a. $[-4, -3]$	b. $[-10, 10]$
14. $f(x)=\sin\frac{1}{x}$	a. $[\frac{1}{\pi}, 5)$	b. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

In problems 15-18, find the values of m and n so that the given function is continuous.

15. $f(x)=\begin{cases}mx, & x<4\\ x^2, & x\geq 4\end{cases}$
16. $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ m, & x=2\end{cases}$
17. $f(x)=\begin{cases}mx, & x<3\\ n, & x=3\\ -2x+9, & x>3\end{cases}$
18. $f(x)=\begin{cases}mx-n, & x<1\\ 5, & x=1\\ 2mx+n, & x>1\end{cases}$

19. Prove that the equation $\frac{x^2+1}{x+3}+\frac{x^4+1}{x-4}=0$ has a solution in the interval $(-3,\ 4)$.

20. Prove that $f(x)=\begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

is discontinuous at every real number. What does the graph of f look like?

2.3 Rate of Change of Functions

2.3.1 Tangent of a Graph

Suppose $y=f(x)$ is a continuous function. In the figure(i), the graph of f possesses a tangent line L at a point P, and then we would like to find its equation.

To do so we need: (i) the coordinates of P and

(ii) the slope m_{tan} of L.

The coordinates of P pose no difficulty since a point on a graph is obtained by specifying a value of x, say $x=a$ in domain of f. The coordinates of point of tangency are $(a, f(a))$.

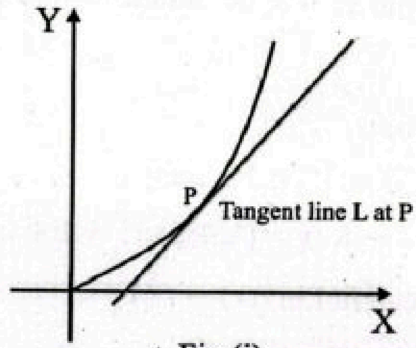


Fig (i)

As a means of approximating the slope m_{tan}, we find the slope of secant lines that pass through the fixed-point P and any other point Q on the graph.

If P has coordinates $(a, f(a))$ and if we let Q have coordinates $(a+\Delta x, f(a+\Delta x))$, then from fig (ii) the slope of the secant line through P and Q is

$$m_{sec}=\frac{\text{change in y-coordinate}}{\text{change in x-coordinate}}$$

$$=\frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}=\frac{\Delta y}{\Delta x}$$

Then, $m_{sec}=\frac{\Delta y}{\Delta x}$

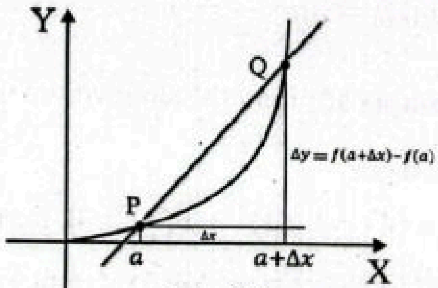


Fig (ii)

When the value of Δx is close to zero either positive or negative, we get points Q and Q' on the graph on each side of P, but close to the point P, we expect that the slopes m_{PQ} and $m_{PQ'}$ are very close to the slope of the tangent line L. See fig (iii)

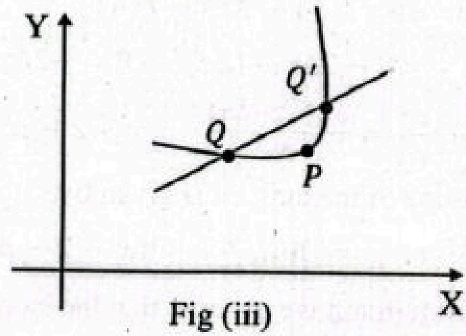


Fig (iii)

Definition: Tangent line

Let $y = f(x)$ be a continuous function. At a point (a, *f*(a)) the tangent line to the graph is the line that passes through the point with slope.

$$\text{Slope} = m_{tan} = \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

whenever the limit exists.

The slope of the tangent line at $(a, f(a))$ is also called the slope of the curve at the point. The tangent at $(a, f(a))$ is unique since a point and a slope determine a single line.

Example 9: Use definition to find the slope of the tangent line to the graph of $f(x) = x^2$ at $(1, f(1))$.

Solution:

i. $f(1) = 1^2 = 1$ for any $\Delta x \neq 0$

$f(1 + \Delta x) = (1 + \Delta x)^2 = 1 + 2\Delta x + (\Delta x)^2$

ii. $\Delta y = f(1 + \Delta x) - f(1)$

$= 1 + 2\Delta x + (\Delta x)^2 - 1 = 2\Delta x + (\Delta x)^2 = \Delta x(2 + \Delta x)$

iii. $\frac{\Delta y}{\Delta x} = \frac{\Delta x(2+\Delta x)}{\Delta x} = 2 + \Delta x$

Slope of the tangent is given by:

iv. $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2 + \Delta x = 2$

We summarize the definition into 4 steps:

- Evaluate f at a and $a + \Delta x$: $f(a)$ and $f(a + \Delta x)$
- Find Δy
- Divide Δy by Δx, $\Delta x \neq 0$

$$\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

- Compute $\lim_{\Delta x \to 0}$

$$m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Example 10: Find the slope of the tangent line to the graph $f(x) = -x^2 + 6x$ at $(4, f(4))$.

Solution:

i. $f(4) = -(4)^2 + 6(4) = 8$, for any $\Delta x \neq 0$

$f(4 + \Delta x) = -(4 + \Delta x)^2 + 6(4 + \Delta x) = 8 - 2\Delta x - (\Delta x)^2$

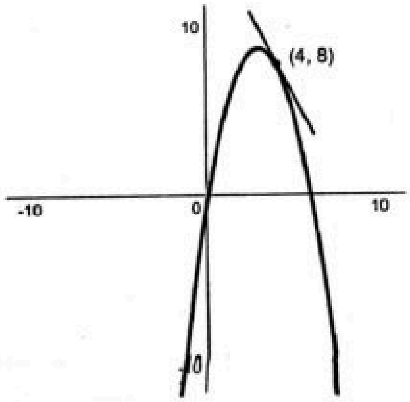
ii. $\Delta y = f(4 + \Delta x) - f(4)$

$= 8 - 2\Delta x - (\Delta x)^2 - 8 = -2\Delta x - (\Delta x)^2 = \Delta x(-2 - \Delta x)$

iii. $\frac{\Delta y}{\Delta x} = \frac{\Delta x(-2-\Delta x)}{\Delta x} = -2 - \Delta x$

Slope of the tangent is given by:

iv. $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} -2 - \Delta x = -2$



From graph we observe that the slope of line is -2 at (4, 8).

Key Facts

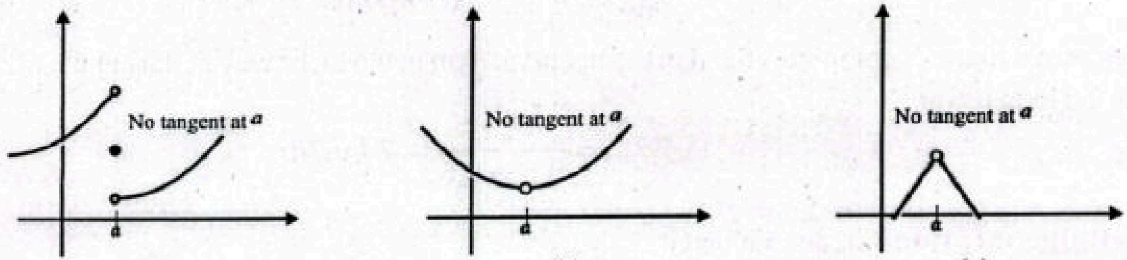
A Tangent May Not Exist: The graph of a function f will not have a tangent line at a point whenever.

i. f is discontinuous at $x = a$, or

ii. The graph of f has corner at $(a, f(a))$.

Moreover, the graph f may not have a tangent line at a point where

iii. The graph has a sharp peak.

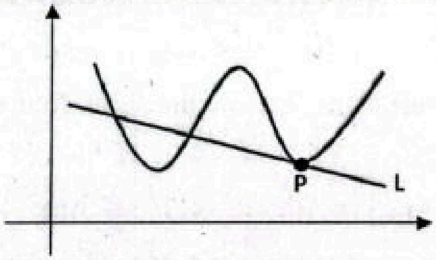


(a) (b) (c)

2.5.2 Rate of Change

The slope $\frac{\Delta y}{\Delta x}$ of a secant through $(a, f(a))$ is also called the average rate of change of f at a. The slope $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is said to be the instantaneous rate of change of the functions at a, if $m_{tan} = \frac{1}{10}$ at a point $(a, f(a))$, we would not expect the values of f to change drastically for x values near a.

Remark: The line L is tangent at P but intersects the graph of f at three points. but is not tangent to the graph.



2.4 Instantaneous Velocity

Almost everyone has an intuitive notion of speed or velocity as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 miles in one hour, the average velocity of the bus must have been 60 mil/hr. Of course, it is difficult to maintain the rate of 60 mil/hr for the entire trip because the bus slow down for towns and speeds up when it passes cars. In other words, the velocity changes with time. If a bus company's schedule demands that the bus travel the 60 miles from one town to another in one hour, the driver knows instinctively that he must compensate for velocities or speeds below 60 mil/hr by travelling at speeds greater than this at other points in journey. Knowing that the average velocity is 60 mil/hr doesn't, however, answer the questions, what is the velocity of the bus at a particular instant?

$$\text{Average velocity:} \qquad V_{ave} = \frac{\text{distance travelled}}{\text{time of travel}}$$

Consider a runner who finishes a 10 km race in an elapsed time of 1 hour and 15 min (1.25 hr). The runner's average velocity or average speed for the race was

$$V_{ave} = \frac{10}{1.25} = 8km/hr$$

But suppose we now wish to determine velocity at the instant the runner is one half hour into the race. If the distance run in the time interval from 0 hr to 0.5 hr is measured to be 5 km, then

$$V_{ave} = \frac{5}{0.5} = 10\ km/hr$$

Suppose if a runner's completes 5 km in 0.5 hr and 5.7 km in 0.6 hr, however, during the time interval from 0.5 hr to 0.6 hr

$$V_{ave} = \frac{5.7-5}{0.6-0.5} = 7\ km/hr$$

Definition: Instantaneous Velocity

Let $s = f(t)$ be a function that gives the position of an object moving in a straight line. The instantaneous velocity at time t_1 is

$$V(t_1) = \lim_{\Delta t\to 0}\frac{f(t_1+\Delta t)-f(t_1)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

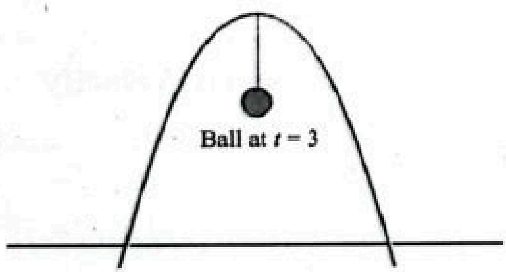
whenever the limit exists.

Example 11: The height s above ground of a ball dropped from the top of the tower is given by $s = -4.9t^2 + 192$ where s is measured in meters and t in seconds. Find the instantaneous velocity of the falling ball at $t_1 = 3\ sec.$

Solution: We use the same four step procedure:

Step 1: $f(3) = -4.9(3)^2 + 192 = 147.9$ for any $\Delta t \neq 0$

$$f(3+\Delta t) = -4.9(3+\Delta t)^2 + 192$$
$$= -4.9(\Delta t)^2 - 29.4\Delta t + 147.9$$



Step 2: $\Delta s = f(3+\Delta t) - f(3)$

$$= [-4.9(\Delta t)^2 - 29.4\Delta t + 147.9] - 147.9$$
$$= \Delta t[-4.9\Delta t - 29.4]$$

Step 3: $\frac{\Delta s}{\Delta t} = \frac{\Delta t(-4.9\Delta t - 29.4)}{\Delta t}$

$$= -4.9\Delta t - 29.4$$

Step 4: $v(3) = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}(-4.9\Delta t - 29.4)$

$$= -29.4\ m/sec$$

The minus sign is significant because the ball is moving opposite to the positive or upward direction. The number $f(3) = 147.9\ m$ is the height of the ball above the ground at 3 seconds.

Exercise 2.3

In problem 1-6, find the slope of the tangent line to the graph of the given function at the indicated point.

1. $f(x) = 2x - 1$; $(x,\ f(x)) = (4,\ 7)$
2. $f(x) = -\frac{1}{2}x + 3$; $(a,\ f(a))$
3. $f(x) = x^2 + 4$; $(-1,\ 5)$
4. $f(x) = x^2 - 5x + 4$; $(2,\ -2)$
5. $f(x) = x^3$; $(1,\ f(1))$
6. $f(x) = \frac{1}{x}$; $\left(\frac{1}{3},\ f\left(\frac{1}{3}\right)\right)$

In problem 7-8, find the average rate of change of the given function in the indicated interval.

7. $f(x) = x^3 + 2x^2 - 4x$; $[-1,\ 2]$
8. $f(x) = cosx$; $[-\pi,\ \pi]$

In problem 9-10, find the instantaneous velocity of the particle at the indicated time.

9. $f(t) = -4t^2 + 10t + 6$; $t = 3$
10. $f(t) = t^2 + \frac{1}{5t+1}$; $t = 0$
11. The height above ground of a ball dropped from an initial altitude of 122.5 m is given by $s(t) = 122.5 - 4.9t^2$, where s is measured in meters and t in seconds.
    i. What is the instantaneous velocity at $= \frac{1}{2}$?
    ii. At what time does the ball hit the ground?
    iii. What is the impact velocity?
12. The height of a projectile shot from ground level is given by $s(t) = -16t^2 + 256t$, where s is measured in feet and t in seconds.
    i. Determine the height of the projectile at $t = 2,\ t = 6,\ t = 9$ and $t = 10$.
    ii. What is the average velocity of the projectile between $t = 2$ and $t = 5$.
    iii. Show that the average velocity between $t = 7$ and $t = 9$ is zero, also interpret.
    iv. At what time does the projectile hit the ground?
    v. Determine the instantaneous velocity at time $t = 8$.
    vi. What is the maximum height that the projectile attains?

2.5 The Derivative Functions

In this section we will discuss the concept of a "derivative" which is the primary mathematics tool that is used to calculate and study rates of change.

We have studied a slope of tangent line: $\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

For any x, if the limit exists, then it can be interpreted either on the slope of a tangent line to the curve $y = f(x)$ as $x = x_0$ or as the instantaneous rate of change of y with respect to $x = x_0$. This limit is so important that it has special notations.

$$f'(x_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x)}{\Delta x}$$

You can think of f'(read "f prime").

Definition: The Derivative Functions

The function f' defined by the formula: $f'(x_0) = \lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x)}{\Delta x}$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit exists.

Example 12: Find the derivative of $f(x) = x^2$, by definition.

Solution: We have: $f'(x) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-(x)^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+\Delta x^2-x^2}{\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x} = \lim_{\Delta x\to 0} 2x+\Delta x = 2x$$

Example 13: Find the derivative of

$$y = f(x) = -x^2 + 4x + 1$$

Solution: $\Delta y = f(x+\Delta x) - f(x)$

$$= \Delta x[-2x - \Delta x + 4]$$

Therefore $f'(x) = y' = \frac{\Delta y}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{\Delta x[-2x-\Delta x+4]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}[-2x-\Delta x+4] = -2x+4$$

Key Point: Notation

Many ways to denote the derivative of a function

$y = f(x)$

- y' "y prime"
- $\frac{dy}{dx} = \frac{df}{dx} = \frac{df(x)}{dx} = D_x f = y'$

We also read $\frac{dy}{dx}$ as "the derivative of y with respect of x" and $\frac{df}{dx}$ and $\left(\frac{d}{dx}\right)f(x)$ as "the derivative of f with respect of x".

- y' and f' (used by Newton).
- $\frac{d}{dx}$ (used by Leibniz).

Input

- Function $y = f(x)$. operator $\frac{d}{dx}$

Output

- Derivative $y' = \frac{df}{dx}$
- Process is also called differentiation.

Example 14:

a. Find the derivative of $y = f(x) = \sqrt{x}$, by definition.

b. Find the slope of the tangent at $x = 9$.

Solution:

a. $f(x) = \sqrt{x}$, $f(x + \Delta x) = \sqrt{x + \Delta x}$

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} - \sqrt{x}} \text{(rationalise)} = \frac{1}{2\sqrt{x}}$$

b. The slope of the tangent at $x = 9$ is

$$\frac{dy}{dx}\Big|_{x=9} = \frac{1}{2\sqrt{x}}\Big|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

2.6 Rules of Differentiation

2.6.1 Power and Sum Rules

The definition of derivative has the obvious drawback of being rather clumsy and tiresome to apply. For example, to find the derivative of function like $f(x) = 5x^{100} + x^{\frac{7}{5}}$ is a time taking job. Here, we will develop some important theorems that will enable us to calculate derivatives more efficiently.

Theorem 2.1: Power Rule

If n is a positive integer, then: $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

Proof:

Let $f(x) = x^n, n$ a positive integer. By binomial theorem we can write:

$$f(x + \Delta x) = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-1}(\Delta x)^2 + \cdots + (\Delta x)^n$$

Thus: $\frac{d}{dx}[x^n] = f'(x) = \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-1}(\Delta x)^2 \cdots + (\Delta x)^n] - x^n}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(\Delta x) + \cdots + (\Delta x)^{n-1}]}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(\Delta x) + \cdots + (\Delta x)^{n-1}\right)$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

A power rule simply states that differentiate x^n: $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

Example 15: Find: $\frac{d}{dx}[x^4] = 4x^3$, $\frac{d}{dx}[x^7] = 7x^6$, $\frac{d}{dx}[x^{50}] = 50x^{49}$, $\frac{d}{dx}[x^{200}] = 200x^{199}$,

$$\frac{d}{dx}[x^{31}] = 31x^{30}$$

We can apply this formula for all real numbers like:

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[x^{\frac{4}{5}}\right] = \frac{4}{5}x^{\frac{4}{5}-1} = \frac{4}{5x^{\frac{1}{5}}}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left[\frac{1}{x^{50}}\right] = \frac{d}{dx}x^{-50} = -\frac{50}{x^{51}}$$

$$\frac{d}{dx}\left[10x^{\frac{1}{3}}\right] = 10\frac{d}{dx}x^{\frac{1}{3}} = \frac{10x^{\frac{-2}{3}}}{3} = \frac{10}{3x^{\frac{2}{3}}}$$

Derivative of constant function:

$$\frac{d}{dx}[c] = \frac{d}{dx}[cx^0] = c\frac{d}{dx}[x^0] = c.0x^{0-1} = 0$$

$\frac{d}{dx}[c] = 0$ like $\frac{d}{dx}[10] = 0$

Theorem:

If n is any real number

$$\frac{d}{dx}x^n = nx^{n-1}$$

Theorem:

If c is any real number

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Sum and Difference Rule:

If f and g are differentiable function, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

In words, the derivative of a sum equals to the sum of the derivatives and the derivative of difference is equal to the difference of the derivatives.

Example 16:

i. $\frac{d}{dx}[2x^6 + x^{-9}]$

$= \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}]$

$= 2.\frac{d}{dx}[x^6] + (-9)x^{-9-1}$

$= 2(6)x^5 - 9x^{-10}$

$= 12x^5 - \frac{9}{x^{10}}$

ii. $\frac{d}{dx}\left[4x^5 - \frac{1}{2}x^4 + 9x^3 + 7\right]$

$= 4\frac{d}{dx}[x^5] - \frac{1}{2}\frac{d}{dx}[x^4] + 9\frac{d}{dx}[x^3] + \frac{d}{dx}[7]$

$= 4(5)x^4 - \frac{1}{2}(4)x^3 + 9(3)x^2 + 0$

$= 20x^4 - 2x^3 + 27x^2$

iii. $\frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right]$

$= 5\frac{d}{dx}[x^{-1}] - \frac{1}{5}\frac{d}{dx}[x]$

$= -5x^{-2} - \frac{1}{5}x^{1-1} = \frac{-5}{x^2} - \frac{1}{5}$

Example 17: Find the derivative of the following w.r.t. x.

a. $y = (x+1)^2$ b. $y = (x+1)(x-2)$ c. $y = \frac{x^3+x^2}{3x}$

Solution:

a. $y = (x+1)^2 = x^2 + 2x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}[x^2 + 2x + 1] = \frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + \frac{d}{dx}(1) = 2x + 2 + 0 = 2x + 2$$

b. $y = (x+1)(x-2) = x^2 - x - 2$

$$\frac{dy}{dx} = \frac{d}{dx}x^2 - \frac{d}{dx}x - \frac{d}{dx}2$$

$$= 2x - 1 - 0 = 2x - 1$$

c. $y = \frac{x^3+x^2}{3x} = \frac{x^3}{3x} + \frac{x^2}{3x}$

$$y = \frac{x^2}{3} + \frac{x}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}\frac{d}{dx}x^2 + \frac{1}{3}\frac{d}{dx}x = \frac{1}{3}(2x) + \frac{1}{3} = \frac{2}{3}x + \frac{1}{3}$$

Note: In the different contents of science, engineering and business functions are often expressed in variable other than x and y. Correspondingly, we must adapt the derivative notation to new symbols, for example:

Function	Derivative	Function	Derivative
$V(t) = 4t$	$V'(t) = \frac{dV}{dt} = 4$	$H(z) = \frac{1}{4}z^6$	$H'(z) = \frac{dH}{dz} = \frac{3}{2}z^5$
$A(r) = \pi r^2$	$A'(r) = \frac{dA}{dr} = 2\pi r$	$r(\theta) = 4\theta^3 - 3\theta$	$r'(\theta) = \frac{dr}{d\theta} = 12\theta^2 - 3$

Exercise 2.4

1. Find the derivative of the functions w. r. t. x.

   a. $y = x^9$ b. $f(x) = 4x^{\frac{1}{3}}$ c. $f(x) = 9$ d. $f(x) = 6x^3 + 3x^2 - 10$

2. Determine $f'(x)$.

   a. $f(x) = \sqrt{5}$ b. $f(x) = \sqrt{5}x$ c. $f(x) = 5\sqrt{x}$ d. $f(x) = \sqrt{5x}$

3. Determine $f'(x)$.

   a. $f(x) = x^2(x^3 + 5)$ b. $f(x) = (x+9)(x-9)$ c. $f(x) = (x^2 + x^3)^3$

   d. $f(x) = -3x^{-8} + 2\sqrt{x}$ e. $f(x) = ax^3 + bx^2 + cx + d$, ($a, b, c$ and d are constants)

   f. $f(x) = x^{24} + 2x^{\frac{1}{3}} + 3x^8 + 9x^4$

4. Find $\frac{dy}{dx}$.

a. $y=\frac{x+2x^{\frac{3}{2}}}{\sqrt{x}}$ b. $y=(x^3-5)(2x+3)$ c. $y=(4x^2-3)(7x^2+x)$

5. Find slope of tangent at $x=1$.

a. $y=x^2+3x$ b. $y=x^4-x^2$

2.7 The Product and Quotient Rules

We will develop techniques for differentiating products and quotients. If functions whose derivative are known.

2.7.1 Derivative of a Product.

You might be considered conjecture that the derivative of a product of two functions is the product of their derivatives. However, simple examples will show this not possible.

Consider:

$f(x)=x^2$ and $g(x)=x^3$

The product of their derivative is:

$f'(x)g'(x)=(2x)(3x^2)=6x^3$

But their product is:

$y=f(x)g(x)=x^5$ and $\frac{dy}{dx}=y'=5x^4\neq 6x^3$

Thus, the derivative of the product is not equal to the product of their derivative.

Theorem: Product Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)$$

The first function times the derivative of the second function plus the second function times the derivative of first function.

Example 18: Find $\frac{dy}{dx}$ if $y=(4x^2-1)(7x^3+x)$.

Solution: We can use two methods to find $\frac{dy}{dx}$. We can either use the product rule or we can multiply out the factors in y and then differentiate. We provide both methods.

Method I: The Product Rule

$$\frac{dy}{dx}=\frac{d}{dx}[(4x^2-1)(7x^3+x)]$$

$$\frac{dy}{dx}=\overbrace{(4x^2-1)}^{\text{First}}\overbrace{\frac{d}{dx}(7x^3+x)}^{\text{Derivative of second}}+\overbrace{(7x^3+x)}^{\text{Second}}\overbrace{\frac{d}{dx}(4x^2-1)}^{\text{Derivative of first}}$$

$$\frac{dy}{dx}=(4x^2-1)(21x^2+1)+(7x^3+x)(8x)$$

$$\frac{dy}{dx}=140x^4-9x^2-1$$

Method II: Multiplying First

$$y=(4x^2-1)(7x^3+x)=28x^5-3x^3-x$$

$$\frac{dy}{dx}=\frac{d}{dx}[28x^5-3x^3-x]=140x^4-9x^2-1$$

Both derivatives are same.

Example 19: Find $\frac{dy}{dx}$ if $y = [(1 + x^3)\sqrt{x}]$

Solution: Apply the product rule $\frac{dy}{dx} = \frac{d}{dx}[(1 + x^3)\sqrt{x}]$

$$= (1 + x^3)\frac{d}{dx}\sqrt{x} + \sqrt{x}\frac{d}{dx}(1 + x^3) = (1 + x^3)\frac{1}{2}x^{\frac{1}{2}-1} + \sqrt{x}(3x^2)$$

$$= \frac{(1+x^3)}{2\sqrt{x}} + 3x^{\frac{5}{2}} = \frac{1+x^3+6x^3}{2\sqrt{x}} = \frac{7x^3 + 1}{2\sqrt{x}}$$

2.7.2 Derivative of a Quotient

Just as the derivative of a product is not generally the product of derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship/method is given by the following.

Theorem: Quotient Rule

If f and g are differentiable functions and $g(x) \neq 0$, then,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The denominator times the derivative of numerator minus the numerator times the derivative of denominator all divided by the denominator square.

Example 20: Differentiate $y = \frac{3x^2-1}{2x^3+5x^2+7}$ w.r.t. x.

Solution: Apply the quotient rule:

$$\frac{dy}{dx} = \frac{\overbrace{(2x^3 + 5x^2 + 7)}^{\text{Denominator}}\frac{d}{dx}\overbrace{[3x^2 - 1]}^{\text{Derivative of numerator}} - \overbrace{(3x^2 - 1)}^{\text{Numerator}}\frac{d}{dx}\overbrace{[2x^3 + 5x^2 + 7]}^{\text{Derivative of Denominator}}}{(2x^3 + 5x^2 + 7)^2}$$

$$= \frac{(2x^3 + 5x^2 + 7)(6x) - (3x^2 - 1)(6x^2 + 10x)}{(2x^3 + 5x^2 + 7)^2}$$

$$= \frac{-6x^4 + 6x^2 + 52x}{(2x^3 + 5x^2 + 7)^2}$$

2.8 The Connection Between Derivatives and Continuity

- If a function is differentiable at a point, it is automatically continuous at that point.
- But the reverse is not always true. A function can be continuous at a point and still not be differentiable (like a sharp corner or cusp, for example | x | is continuous but not differentiable).

Exercise 2.5

Find $\frac{dy}{dx}$ if

Summary of Differentiation Rules:

- $\frac{d}{dx}[c] = 0$, $\frac{d}{dx}[cf] = cf'$, $\frac{d}{dx}[f \pm g] = f' \pm g'$
- $\frac{d}{dx}[f.g] = fg' + gf'$
- $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf'-fg'}{g^2}$

1. $y = \frac{1}{x}$
2. $y = (x^2 - 7)(x^2 + 4x + 2)$
3. $y = (7x + 1)(x^4 - x^3 - 9x)$
4. $y = \frac{3x+4}{x^2+1}$
5. $y = \frac{x-2}{x^4+x+1}$
6. $y = \frac{3x^2+5}{3x-1}$
7. $y = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27)$
8. $y = \frac{2-3x}{7-x}$
9. $y = \frac{x^2-10x+2}{x^3-x}$
10. $y = \frac{x^4+2x^3-1}{x^2}$
11. $y = \frac{10}{(x^3-10)^9}$
12. $y = \frac{(x^2+1)^2}{3x-2}$
13. $y = \frac{(x+1)^2}{(x-1)^2}$

Find the slope of the tangent line to the curve at the point whose abscissa is given.

14. $y = \frac{4x-1}{x}$, $x = -1$
15. $y = \frac{54}{x^2+1}$, $x = 2$
16. $y = \frac{2x+5}{x+2}$, $x = 1$
17. $y = (2\sqrt{x} + 1)(x^3 - 6)$, $x = 0$

2.9 Derivations of Trigonometric Functions

The main objective of this section is to obtain formulas for the derivatives of six basic trigonometric functions. We will assume in this section that the variable x in the trigonometric functions $\sin x, \cos x, \tan x, \cot x, \sec x$ and $\operatorname{cosec} x$ is measured in radians. We also need the limits in results and restated as follows:

$$\lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x} = 1 \text{ and } \lim_{\Delta x \to 0} \frac{1 - \cos\Delta x}{\Delta x} = 0$$

We start the problem of differentiating $f(x) = \sin x$. Using the definitions of derivative

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\sin x\left[\frac{\cos\Delta x - 1}{\Delta x}\right] + \cos x\left[\frac{\sin\Delta x}{\Delta x}\right]\right]$$

$$= \sin x \lim_{\Delta x \to 0}\frac{\cos\Delta x - 1}{\Delta x} + \cos x \lim_{\Delta x \to 0}\frac{\sin\Delta x}{\Delta x}$$

$$= \sin x(0) + \cos x(1)$$

$\sin x$ and $\cos x$ independent of Δx

Thus, we have $\boxed{\frac{d}{dx}\sin x = \cos x}$

In a similar manner it can be shown that $\boxed{\frac{d}{dx}\cos x = -\sin x}$

Example 21: Find $\frac{dy}{dx}$ if $y = x\sin x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}[x\sin x]$

$= x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x$ use product rule

$= x\cos x + \sin x(1) = x\cos x + \sin x$

Example 22: Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\sin x}{1+\cos x}\right]$

$$= \frac{1 + \cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}[1 + \cos x]}{(1 + \cos x)^2}$$

$$= \frac{(1 + \cos x)\cos x - \sin x(0 - \sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}$$

The other Trigonometric Functions:

Let $y = \tan x$

$$\frac{dy}{dx} = \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{(\cos x)\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$

Similarly, $\boxed{\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x}$

For $y = \sec x$

$$\frac{dy}{dx} = \frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$

$$= \frac{\cos x\frac{d}{dx}(1) - (1)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x(0) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \sec x\tan x$$

$\boxed{\frac{d}{dx}\sec x = \sec x\tan x}$

Similarly, $\boxed{\frac{d}{dx}\mathrm{cosec}\,x = \mathrm{cosec}\,x\cot x}$

Example 23: Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{x-\cot x}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\cos x}{x-\cot x}\right]$

$$= \frac{(x - \cot x)\frac{d}{dx}\cos x - \cos x\frac{d}{dx}(x - \cot x)}{(x - \cot x)^2}$$

$$= \frac{(x - \cot x)(-\sin x) - \cos x(1-(-\mathrm{cosec}^2 x))}{(x - \cot x)^2}$$

$$= \frac{-x\sin x + \cot x\sin x - \cos x - \cos x\,\mathrm{cosec}^2 x}{(x - \cot x)^2}$$

$$= \frac{-x\sin x + \cos x - \cos x - \cos x\,\mathrm{cosec}^2 x}{(x - \cot x)^2}$$

$$= \frac{-x\sin x - \cos x\,\mathrm{cosec}^2 x}{(x - \cot x)^2}$$

Example 24: Find $\frac{dy}{dx}$ if $y = \sin x(2 + \sec x)$

Solution: $\frac{dy}{dx} = \frac{d}{dx}[\sin x(2 + \sec x)]$

$$= \sin x\frac{d}{dx}(2 + \sec x) + (2 + \sec x)\frac{d}{dx}(\sin x) = \sin x(0 + \sec x\tan x)) + (2 + \sec x)(\cos x)$$

$$= \sin x\sec x\tan x + 2\cos x + \sec x\cos x = \sin x\frac{1}{\cos x}\tan x + 2\cos x + \sec x\cos x$$

$$= \tan^2 x + 2\cos x + 1 = \tan^2 x + 1 + 2\cos x$$

$$= \sec^2 x + 2\cos x \qquad (1 + \tan^2 x = \sec^2 x)$$

2.10 Derivatives of Inverse Trigonometric Functions

The derivative of an inverse trigonometric function can be obtained. Research reveals that the inverse tangent and inverse cotangent are differentiable for all x. However the remaining four inverse trigonometric functions are not differentiable at either $x = -1$ or $x = 1$

Inverse sine function:

For $-1 < x < 1$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \sin^{-1} x$ if and only if $x = \sin y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\sin y$$

$$1 = \cos y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

Similarly, $\boxed{\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1}$

Inverse tangent function:

For $-\alpha < x < \alpha$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \tan^{-1} x$ if and only if $x = \tan y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\tan y$$

$$1 = \sec^2 y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \text{ for } x \in R}$

Similarly, $\boxed{\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}, \text{ for } x \in R}$

Inverse secant function:

For $|x| > 1$ and $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$

$y = \sec^{-1} x$ if and only if $x = \sec y$

Differentiate w.r.t x

$$\frac{dx}{dx} = \frac{d}{dx}\sec y$$

$$1 = \sec y\tan y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y\tan y}$$

$\therefore 1 + \tan^2 y = \sec^2 y$

$\tan^2 y = \sec^2 y - 1$

$\tan y = \sqrt{\sec^2 y - 1}$

$$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\boxed{\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \text{ for } |x| > 1}$$

Similarly, $\boxed{\frac{d}{dx}\mathrm{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}, \text{ for } |x| > 1}$

Example 25:

Differentiate $y = \sin^{-1}5x$ w.r.t. x.

Solution: $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}5x$

$$= \frac{1}{\sqrt{1-(5x)^2}} \cdot \frac{d}{dx}5x$$

$$\frac{d}{dx}\sin^{-1}5x = \frac{5}{\sqrt{1-25x^2}}$$

Example 26: Differentiate $y = \tan^{-1}12x$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}12x$

$$= \frac{1}{1+(12x)^2} \cdot \frac{d}{dx}12x$$

$$\frac{d}{dx}\tan^{-1}12x = \frac{12}{1+144x^2}$$

Example 27: Differentiate $y = \sec^{-1}x^2$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\sec^{-1}x^2$

$$= \frac{1}{x^2\sqrt{(x^2)^2-1}} \cdot \frac{d}{dx}x^2$$

$$= \frac{1}{x^2\sqrt{(x^2)^2-1}} \cdot (2x)$$

$$\frac{d}{dx}\sec tx^2 = \frac{2x}{x^2\sqrt{x^4-1}}$$

Exercise 2.6

Find the derivative of the given functions w.r.t. x.

1. $y = x^2 - \cos x$
2. $y = 4x^3 + x + \sin x$
3. $y = 3\cos x - 5\cot x$
4. $y = \sin x\cos x$
5. $y = (x^2 + \sin x)\sec x$
6. $y = \frac{5-\cos x}{5+\sin x}$
7. $y = \frac{\sec x}{1+\tan x}$
8. $y = \frac{\sin x}{x^2+\sin x}$
9. $y = \frac{\cot x}{x+1}$
10. $y = (1+\cos x)(x - \sin x)$

Find the derivative of the given functions w.r.t. x.

11. $y = \sin^{-1}(5x-1)$
12. $y = 4\cot^{-1}\frac{x}{2}$
13. $y = \frac{\sin^{-1}x}{\sin x}$
14. $y = \frac{\sec^{-1}x}{x}$
15. $y = x\sin^{-1}x + x\cos^{-1}x$
16. $y = \frac{1}{\tan^{-1}x^2}$

2.11 Product Rule

In this section, we will derive a formula that expresses the derivative of a composition *fog* in terms of the derivative of *f* and *g*. This formula will enable us to differentiate complicated functions.

Suppose we wish to differentiate:

$$y = (x^5+1)^2 \quad \ldots\ldots \quad \text{(i)}$$

We can write $y = (x^5+1)(x^5+1)$

$$\frac{dy}{dx} = (x^5+1)\frac{d}{dx}(x^5+1) + (x^5+1)\frac{d}{dx}(x^5+1)$$

$$= (x^5+1)(5x^4) + (x^5+1)(5x^4)$$

$$= 2(x^5+1)(5x^4) \ldots\ldots \quad \text{(ii)}$$

2.11.1 Power Rule for Functions

From (i), $y = (x^5+1)^2$

$$\frac{dy}{dx} = 2(x^5+1)^{2-1}\frac{d}{dx}(x^5+1)$$

$$= 2(x^5+1)(5x^4) \ldots\ldots \quad \text{(iii)}$$

From (ii) and (iii), both expressions are same.

Theorem: Power Rule for Functions

If n is an integer and g is a differentiable function then,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Example 28: Differentiate w.r.t. x.

a. $y = (2x^3+4x+1)^4$

b. $y = \frac{1}{(7x^5-x^4+2)^{10}}$

Solution:

a. $\frac{dy}{dx} = \frac{d}{dx}(2x^3+4x+1)^4$

$$= 4(2x^3+4x+1)^{4-1}\frac{d}{dx}(2x^3+4x+1)$$

$$= 4(2x^3+4x+1)^3(6x^2+4)$$

b. $y = (7x^5-x^4+2)^{-10}$

$$\frac{dy}{dx} = \frac{d}{dx}(7x^5-x^4+2)^{-10}$$

$$= -10(7x^5-x^4+2)^{-10-1}\frac{d}{dx}(7x^5-x^4+2)$$

$$= -10(7x^5-x^4+2)^{-11}(35x^4-4x^3)$$

Example 29: Differentiate $y = \frac{(x^2-1)^3}{(5x+1)^8}$ w.r.t. x.

Solution: $\frac{dy}{dx} = \frac{d}{dx}\frac{(x^2-1)^3}{(5x+1)^8}$

$$= \frac{(5x+1)^8\frac{d}{dx}(x^2-1)^3 - (x^2-1)^3\frac{d}{dx}(5x+1)^8}{[(5x+1)^8]^2}$$

$$= \frac{(5x+1)^8 3(x^2-1)^2(2x) - (x^2-1)^3 8(5x+1)^7(5)}{(5x+1)^{16}}$$

$$= \frac{6x(5x+1)^8(x^2-1)^2 - 40(x^2-1)^3(5x+1)^7}{(5x+1)^{16}}$$

$$= \frac{(x^2-1)^2(5x+1)^7[6x(5x+1) - 40(x^2-1)]}{(5x+1)^{16}}$$

$$= \frac{(x^2-1)^2[-10x^2+6x+40]}{(5x+1)^9}$$

2.11.2 Chain Rule: A power of a function can be written as a composite function. If $f(x) = x^n$ and $u = g(x)$, then $f(x) = f(g(x)) = [g(x)]^n$ is a special case of the chain rule for differentiating composite function.

> **Theorem: Chain Rule**
> If $y = f(x)$ is a differentiable formula of u and $u = g(x)$ is a differentiable function, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f'(g(x)).g'(x)$

Example 30: Differentiate w.r.t. x.

a. $y = \tan^2 x$

b. $y = (9x^3 + 1)^2 \sin 5x$

Solution:

a. $y = \tan^2 x$

$$\frac{dy}{dx} = 2\tan^{2-1} x \frac{d}{dx}\tan x$$

$$= 2\tan x \sec^2 x$$

b. $y = (9x^3 + 1)^2 \sin 5x$

$$\frac{dy}{dx} = (9x^3 + 1)^2 \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}(9x^3 + 1)^2$$

$$= (9x^3 + 1)^2 \cos 5x(5) + \sin 5x.\,2(9x^3 + 1)27x^2$$

$$= (9x^3 + 1)[45x^3 \cos 5x + 5\cos 5x + 54x^2 \sin 5x]$$

2.12 Implicit Differentiation

2.12.1 Explicit and Implicit Functions

A function in which the dependent variable is expressed solely in terms of the independent variable x, namely $y=f(x)$ is said to be an explicit function, for example, $y = \frac{1}{4}x^3 - 1$ is an explicit function, whereas an equivalent equation $3y - x^3 - 4 = 0$ is said to define the function implicitly or y is an implicit of x.

2.12.2 Explicit Differentiation

To illustrate this, let us consider the simple equation:

$$xy = 1 \quad \ldots\ldots \quad \text{(i)}$$

One way to find $\frac{dy}{dx}$ is to rewrite this equation as:

$$y = \frac{1}{x}$$

From which it follows that: $\frac{dy}{dx} = -\frac{1}{x^2} \quad \cdots \quad$ (ii)

Another way to obtain this derivative is to differentiate both sides of (i) before solving for y in terms of x.

From (i) $\quad \frac{d}{dx}(xy) = \frac{d}{dx}1$

$$x\frac{d(y)}{dx}+y\frac{d(x)}{dx}=0$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

If we take, $y=\frac{1}{x}$, we get

$$\frac{dy}{dx}=-\frac{1}{x^2}$$

This method of obtaining derivatives is called implicit differentiation.

Example 31: Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2+\sin y=x^2$

Solution: $\frac{d}{dx}[5y^2+\sin y]=\frac{d}{dx}[x^2]$

$$5\frac{d}{dx}y^2+\frac{d}{dx}\sin y=2x$$

$$5\left(2y\frac{dy}{dx}\right)+\cos y\frac{dy}{dx}=2x$$

$$(10y+\cos y)\frac{dy}{dx}=2x$$

Solving for $\frac{dy}{dx}$ we obtain: $\frac{dy}{dx}=\frac{2x}{10y+\cos y}$

2.13 Derivative of Exponential Functions

The derivative of exponential is: $\boxed{\frac{d}{dx}e^x=e^x}$ like $\frac{d}{dx}e^{3x}=e^{3x}.3$

Example 32: Differentiate $y=x^2e^{5x}$ w.r.t. x.

Solution: $\frac{dy}{dx}=\frac{d}{dx}[x^2e^{5x}]$

$$=x^2\frac{d}{dx}e^{5x}+e^{5x}\frac{d}{dx}x^2 \quad =x^2e^{5x}.5+e^{5x}.2x \quad =5x^2e^{5x}+2xe^{5x} \quad =xe^{5x}(5x+2)$$

2.14 Derivative of Logarithmic Functions

We find the derivative of common logarithmic which is continuous functions.

$\boxed{\frac{d}{dx}\ln x=\frac{1}{x}}$ like $\frac{d}{dx}\ln(x^3+1)=\frac{1}{x^3+1}\frac{d}{dx}(x^3+1)=\frac{3x^2}{x^3+1}$.

Example 33: Differentiate $\ln(4x^3+2x^2+9)$ w.r.t. x.

Solution: $y=\ln(4x^3+2x^2+9)$

$$\frac{dy}{dx}=\frac{1}{4x^3+2x^2+9}\frac{d}{dx}(4x^3+2x^2+9)=\frac{1}{4x^3+2x^2+9}(12x^2+4x)=\frac{4x(3x+1)}{4x^3+2x^2+9}$$

Derivative of $y=a^x$: $\frac{d}{dx}a^x=a^x.\frac{1}{\ln a}$

We will apply the chain rule to find the derivative of parametric equations.

Example 34: Differentiate $y=4^{3x^2+5}$ w.r.t. x.

Solution: Taking $\ln$ both sides

$$\ln y=\ln 4^{3x^2+5}$$

$$\ln y=(3x^2+5).\ln 4$$

$$\frac{1}{y}\frac{dy}{dx}=\ln 4.\frac{d}{dx}(3x^2+5),\ \frac{dy}{dx}=y\ln 4\,(6x)=\ln 4\left(4^{3x^2+5}\right)6x=6\ln 4\left(4^{3x^2+5}\right)x$$

Example 35: Find $\frac{dy}{dx}$ if $x = \tan t,\ y = 4t^3 + 1$

Solution: $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{dy}{dt} = \frac{d}{dt}(4t^3 + 1) = 12t^2$$

$$\frac{dx}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t$$

$$\frac{dy}{dx} = \frac{12t^2}{\sec^2 t}$$

Example 36: Find $\frac{dy}{dx}$ if $x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$

Solution: $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$= \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$= \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}$$

$$= \frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}$$

$$= \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{(t^2-1)}{2t}$$

2.15 Differentials

We have already discussed the derivative of finding slope of a tangent line to the graph of a function $y = f(x)$.

$$m_{\text{sec}} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

For small values of Δx ,

$$m_{sec} \cong m_{tan} \text{ or } \frac{\Delta y}{\Delta x} = m_{tan} = f'(x)$$

We have: $\frac{\Delta y}{\Delta x} = f'(x)$

$$\Delta y = f'(x)\Delta x$$

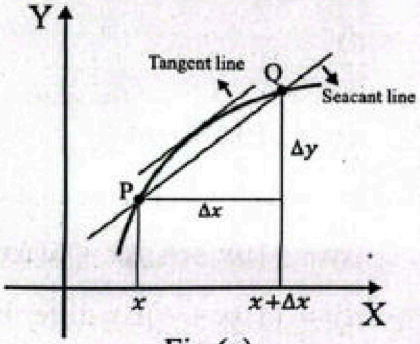


Fig (a)

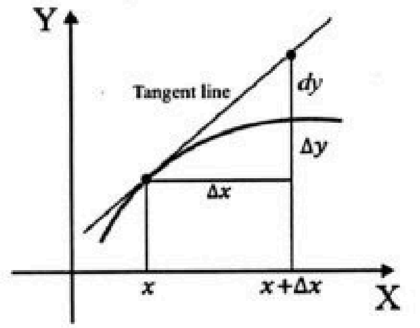
Fig (a): Geometric representation of the derivative at a point on the curve.

Definition: The increment Δx is called the differential of the independent variable x and is denoted by dx. i.e.

The function $f'(x)\Delta x$ is called differential of the dependent variable y and is denoted by dy. i.e. $dy = f'(x)\Delta x = f'(x)dx$

Since the slope of a tangent to graph is

$$m_{\tan} = \frac{rise}{run} = f'(x) = \frac{f'(x)\Delta x}{\Delta x}, \Delta x \neq 0$$



It follows that the rise of the tangent line can be interrupted in dy

$\Delta y \cong dy$

Example 37: a) Find Δy and dy for $y = 5x^2 + 4x + 1$

b) Compare the values of Δy and dy for $x = 6, \Delta x = dx = 0.02$

Solution:

a) $\Delta y = f(x + \Delta x) - f(x)$

$= [5(x+\Delta x)^2 + 4(x+\Delta x) + 1] - [5x^2 + 4x + 1]$

$= 10x\Delta x + 4\Delta x + 5(\Delta x)^2$

$$\frac{\Delta y}{\Delta x} = \frac{10x\Delta x + 4\Delta x + 5(\Delta x)^2}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{(10x + 4 + 5\Delta x)\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = 10x + 4$$

$$\frac{dy}{dx} = 10x + 4$$

$$dy = (10x + 4)dx$$

Since $dx = \Delta x$. We observe that

$\Delta y = (10x + 4)\Delta x + 5(\Delta x)^2$ and

$dy = (10x + 4)\Delta x$ differ by the amount $5(\Delta x)^2$.

b) When $x = 6, \Delta x = 0.02$

$\Delta y = 10(6)(0.02) + 4(0.02) + 5(0.02)^2$

$= 1.282$

Whereas $dy = (10(6) + 4)(0.02) = 1.28$

$\Delta y \cong dy$

$1.282 \cong 1.28$

The difference in answers is, of course

$5(0.02)^2 = 0.002$

2.16 Approximations

When $\Delta x = 0$, differentials give a means of "predicting" the value of $f(x + \Delta x)$ by knowing the value of the function and its derivative at x. From fig if x is changes by an amount Δx, then the corresponding change in the function is

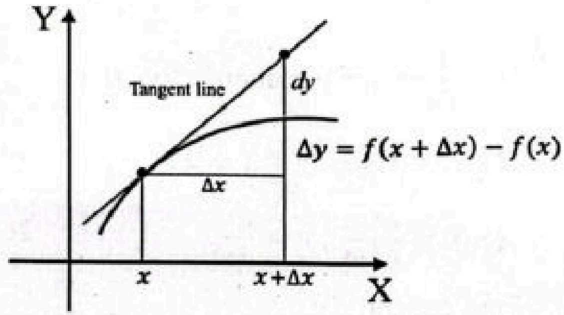
$\Delta y = f(x + \Delta x) - f(x)$ and so

$f(x + \Delta x) = f(x) + \Delta y$

For small change in x, take $\Delta y = dy$

$f(x + \Delta x) = f(x) + dy$

$= f(x) + f'(x)dx$



Example 38: Find the approximated value of $\sqrt{25.4}$.

Solution: First, identify the function $f(x) = \sqrt{x}$

We wish to calculate the approximated value of $f(x + \Delta x) = \sqrt{x + \Delta x}$ when $x = 25$ and $\Delta x = 0.4$; $(25.4 = 25 + 0.4)$

Now, $dy = \frac{1}{2}x^{\frac{-1}{2}}dx = \frac{1}{2\sqrt{x}}\Delta x$

We have; $f(x + \Delta x) = f(x) + dy$

$$= f(x) + \frac{1}{2\sqrt{x}}\Delta x = \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x = \sqrt{25} + \frac{1}{2\sqrt{25}}(0.4) = 5.04$$

Exercise 2.7

Find the derivative of functions w.r.t. variable involved.

1. $y = \left(x - \frac{1}{x^2}\right)^5$
2. $f(x) = \left(\frac{x^2-1}{x^2+1}\right)^2$
3. $y = (3x - 1)^4(-2x + 9)^5$
4. $f(\theta) = (2\theta + 1)^3\tan^2\theta$
5. $y = \sin2x\cos3x$
6. $f(x) = (\sec4x + \tan2x)^5$
7. $h(t) = \frac{t+\sin4t}{10+\cos3t}$
8. $f(x) = \tan\left(\cos\frac{x}{2}\right)$

Use implicit differentiation to find $\frac{dy}{dx}$.

9. $4x^2 + y^2 = 8$
10. $x + xy - y^2 - 20 = 0$
11. $y^4 - y^2 = 10x - 3$
12. $x^3y^2 = 2x^2 + y^2$
13. $xy = \sin x + y$
14. $x + y = \cos xy$
15. $x\sin y - y\cos x = 1$
16. $\sin y = y\cos2x$

Find $\frac{dy}{dx}$.

17. $y = x^3e^{5x}$
18. $y = e^{4x}(1 + \ln x)$
19. $y = \frac{e^{2x}}{e^{-2x}+1}$
20. $y = \ln(e^x + e^{-x})$
21. $y = \ln\left(x + \sqrt{x^2 + 1}\right)$
22. $y = e^{-3x}\cos x$

Find $\frac{dy}{dx}$ of the parametric functions.

23. $x = t + \frac{1}{t}, y = t + 1$
24. $x = t^2 + \frac{1}{t^2}, y = t - \frac{1}{t}$
25. $x = \frac{\theta^2 - 1}{\theta^2 + 1}, y = \frac{\theta - 1}{\theta + 1}$
26. $x = \sin 2\theta, y = \cos 4\theta$

Find Δy and dy.

27. $y = x^2 + 1$
28. $y = \sin x$

Use the concept of the differential to find the approximated value of the given expressions.

29. $(1.8)^5$
30. $\sqrt{37}$
31. $\sin 31°$
32. $\tan\left(\frac{\pi}{4} + 0.1\right)$

2.17 Higher Order Derivatives

2.17.1 The Second Derivative

The derivative $f'(x)$ is a function derived from a function $y = f(x)$. By differentiating the first derivative $f'(x)$, we obtain another function called the second derivative, which is denoted by $f''(x)$. In terms of the operation symbol $\frac{d}{dx}$ we define the second derivative with respect to x as the function obtained by differentiating $y = f(x)$ twice is successive.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The second derivative is commonly denoted by

$$f''(x), \quad y'', \quad \frac{d^2y}{dx^2}, \quad D^2y$$

Normally, we shall use one of the first three symbols.

Example 39: Find the second derivative of $y = x^3 - 2x^2$ w.r.t. x.

Solution: The first derivative is: $\frac{dy}{dx} = 3x^2 - 4x$

The second derivative follows from differentiating the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x) = 6x - 4$$

Example 40: Find the second derivative:

a. $\sin 3x$ b. $(x^3+1)^4$ c. e^{2x}

Solution:

a. The first derivative is: $y' = \frac{dy}{dx} = \frac{d}{dx}(\sin 3x) = 3\cos 3x$

The second derivative is: $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(3\cos 3x) = -9\sin 3x$

b. The first derivative is:

$$y' = \frac{dy}{dx} = \frac{d}{dx}(x^3+1)^4 = 4(x^3+1)^3\frac{d}{dx}x^3 = 12x^2(x^3+1)^3$$

To find the second derivative, we will use product and power rule

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}[12x^2(x^3+1)^3] = 12\left[x^2\frac{d}{dx}(x^3+1)^3 + (x^3+1)^3\frac{d}{dx}x^2\right]$$

$$= 12[x^2 3(x^3+1)^2 3(x^2) + (x^3+1)^3(2x)] = 12x(x^3+1)^2[11x^3+2]$$

c. The first derivative is: $y' = \frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = 2e^{2x}$

The second derivative is: $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(2e^{2x}) = 4e^{2x}$

2.18 Higher Derivatives

Assuming all derivatives exist, we can differentiate a function $y = f(x)$ as many times as we want. The third derivative is the derivative of the second derivative. The fourth derivative is the derivative of the third derivative and so on. We denote the third and fourth derivative, by $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$, respectively and define them by:

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

In general, if n is a positive integer, then the nth derivative is denoted by:

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

Other notations for the first n derivatives are:

$$f'(x), f''(x), f'''(x), f^{(4)}(x), \dots\dots\dots f^{(n)}(x)$$

$$y',\ y'',\quad y''',\quad y^{(4)}, \dots\dots\dots\ y^{(n)}$$

$$D_xy,\quad D^2{}_xy,\quad D^3{}_xy,\quad D^4{}_xy, \dots\dots\dots D^n{}_xy$$

Example 41: Find the first five derivatives of $f(x) = 2x^4 - 6x^3 + 7x^2 + 5x - 10$ w.r.t. x.

Solution: We have

$f'(x) = 8x^3 - 18x^2 + 14x + 5$

$f''(x) = 24x^2 - 36x + 14$

$f'''(x) = 48x - 36$

$f^{(4)}(x) = 48$

$f^{(5)}(x) = 0$

Example 42: Find the third derivatives of $y = \frac{1}{x^3}$

Solution: We have $y = \frac{1}{x^3} = x^{-3}$

$\frac{dy}{dx} = -3x^{-4}$

$\frac{d^2y}{dx^2} = (-3)(-4)x^{-5} = 12x^{-5}$

$\frac{d^3y}{dx^3} = (12)(-5)x^{-6} = -60x^{-6} = \frac{-60}{x^6}$

Exercise 2.8

Find the second derivative of the functions w.r.t. the variable involved.

1. $y = -x^3 + 6x + 9$
2. $f(x) = 30x^2 - x^3$
3. $f(x) = (-5x + 9)^2$
4. $y = 2x^6 + 5x^3 - 6x^2$
5. $y = 20x-3$
6. $y = \frac{2}{x^4}$
7. $f(x) = x^2(3x - 4)^3$
8. $f(x) = (x^2 + 5x - 1)^4$
9. $f(x) = \cos 10x$
10. $f(x) = \tan\frac{x}{2}$
11. $f(\theta) = \sin^2 5\theta$
12. $f(\theta) = \frac{1}{3+2\cos\theta}$
13. $f(x) = e^{2x}(x^2 + 1)$
14. $f(x) = (x^2 + 1)\ln(x^2 + 1)$

Find the indicated derivative.

15. $y = 4x^7 + x^6 - x^4;\ \frac{d^4y}{dx^4}$
16. $y = \frac{2}{x};\ \frac{d^5y}{dx^5}$
17. $f(x) = \cos\pi x;\ f'''(x)$
18. $f(x) = \frac{1}{\sec(2x+1)};\ f^{(4)}(x)$
19. Let $f(x) = x^3 + 2x$
    a. Find $f'(x)$ and $f''(x)$
    b. In general; $f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$

    provided limit exists. Use $f''(x)$ obtained in part (a) and use definition to find $f''(x)$.
20. Show that $\frac{d^2}{dx^2}(fg) = f''g + 2f'g' + fg''$

    $\frac{d^3}{dx^3}(fg) = f'''g + 3f''g' + 3f'g'' + fg'''$

2.19 Extrema of Functions

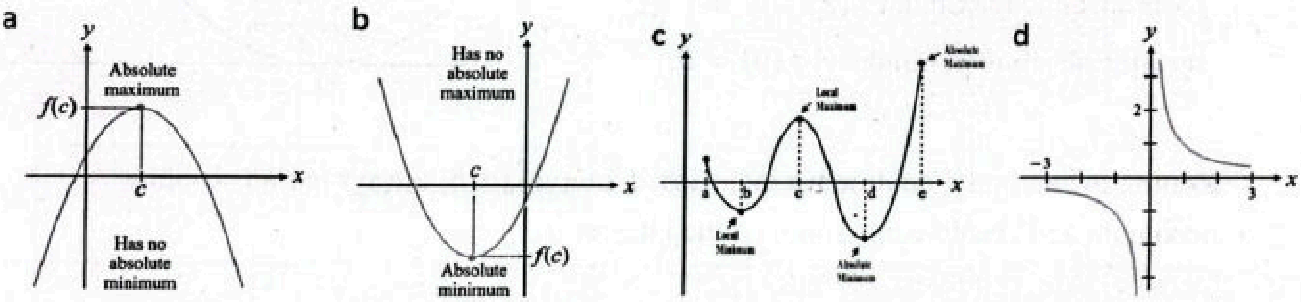
Suppose a function f is defined on an interval I. The maximum and minimum values of f on I (if exist) are said to be extrema of the functions. We have two kinds of extrema.

Definition: Absolute Extrema

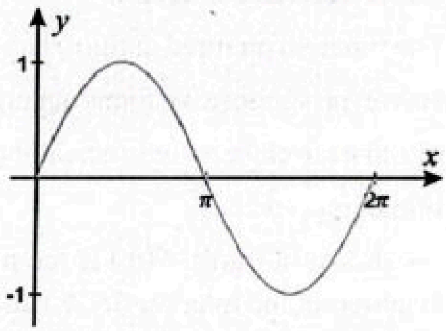
i. A number $f(c)$ is an **Absolute Maximum** of a function f if $f(x) \leq f(c)$ for every x in the domain of f.

ii. A number $f(c)$ is an **Absolute Minimum** of a function f if $f(x) \geq f(c)$ for every x in the domain of f.

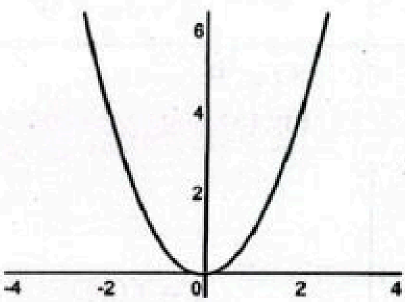
Absolute extrema are called global extrema. Figure shows several possibilities:



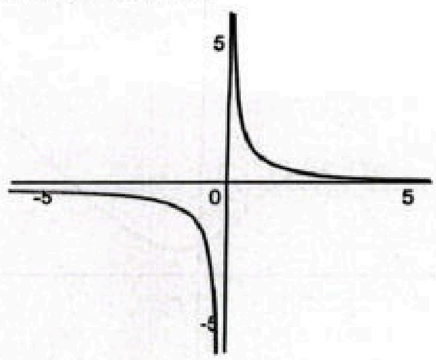
Example 43:



i. For $f(x) = \sin x$, $f\left(\frac{\pi}{2}\right) = 1$ is its absolute maximum and $f\left(\frac{3\pi}{2}\right) = -1$ is its absolute minimum.



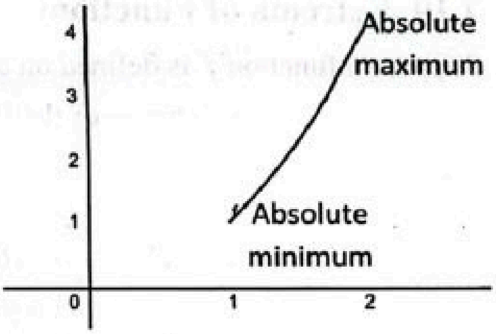
ii. The function $f(x) = x^2$ has the absolute minimum $f(0) = 0$ but has no absolute maximum.



iii. $f(x) = \frac{1}{x}$ has neither an absolute maximum nor an absolute minimum.

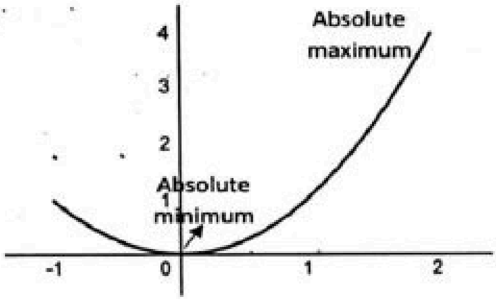
Example 44:

i. $f(x) = x^2$ defined only on the closed interval at $[1, 2]$ has the absolute maximum $f(2) = 4$ and the absolute minimum $f(1) = 1$



ii. On the other hand, if $f(x) = x^2$ is defined on the interval $(1, 2)$. f has no absolute extrema.

iii. $f(x) = x^2$ is defined on the interval $[-1, 2]$. f has absolute maximum $f(2) = 2$ and now the absolute minimum is $f(0) = 0$.



Result: A function f continuous on a closed interval $[a, b]$ always has an absolute maximum and absolute minimum on the interval.

2.19.1 Relative Extrema

The function pictured in fig(a) has no absolute extrema. However, suppose we focus our attention on values of x that are close to, or in a neighborhood of the numbers c_1 and c_2.

As shown in fig(b), $f(c_1)$ is the maximum value of the function in the interval (a_1, b_1) and $f(c_2)$ is a minimum value in the interval (a_2, b_2). These local or relative, extrema are defined as follows:

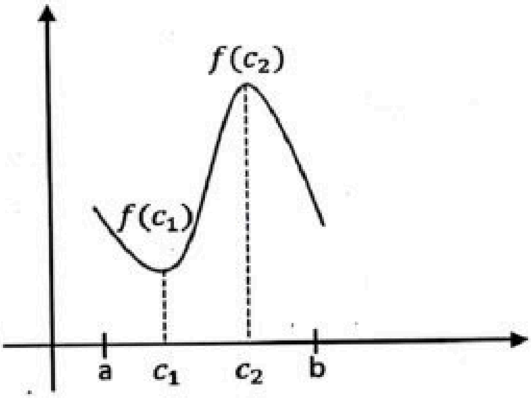
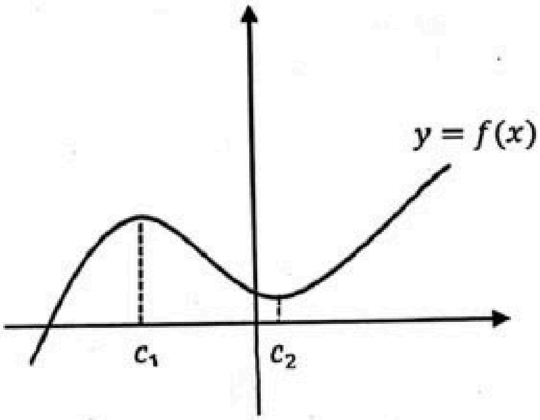


Fig (a)



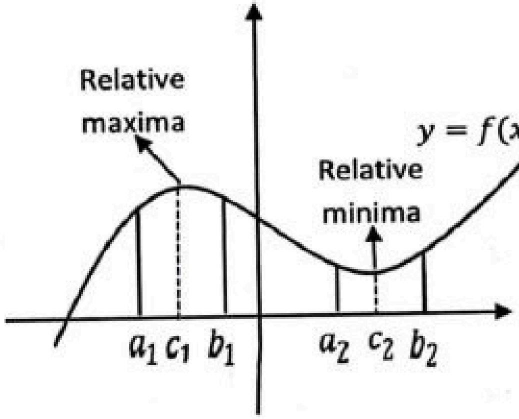


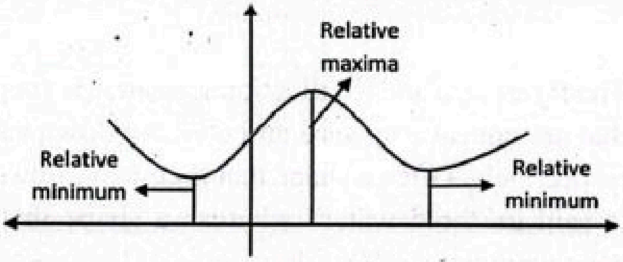
Fig (b)

Definition: Relative Extrema

i. A number $f(c_1)$ is a **Relative Maximum** of a function f if $f(x) \leq f(c_1)$ but every x in some open interval that contains c_1.

ii. A number $f(c_1)$ is a **Relative Minimum** of a function f if $f(x) \geq f(c_1)$ for every x in some open interval that contains c_1.

Result: From fig , we suggest if c is a value at which a function f has a relative extremum, then either $f'(c) = 0$ or $f'(c)$ does not exist.



Critical values: A critical value of a function f is a number in c in its domain for which $f'(c) = 0$ or $f'(c)$ does not exist.

Example 45: Find the critical values of

a. $f(x) = x^3 - 15x + 6$ c. $f(x) = \frac{x^2}{x-1}$ b. $f(x) = (x+4)^{\frac{2}{3}}$

Solution:

a. $f(x) = x^3 - 15x + 6$

$f'(x) = 3x^2 - 15$

$f'(x) = 3(x+\sqrt{5})(x-\sqrt{5})$

The critical values are those number for which $f'(x) = 0$, namely $-\sqrt{5}$ and $\sqrt{5}$.

b. $f(x) = (x+4)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x+4)^{\frac{-1}{3}}$

$f'(x) = \frac{2}{3(x+4)^{\frac{1}{3}}}$

We observe that $f'(x)$ doesnot exist, when $x = -4$ since -4 is in the domain of f. We conclude it is a critical value.

c. $f(x) = \frac{x^2}{x-1}$

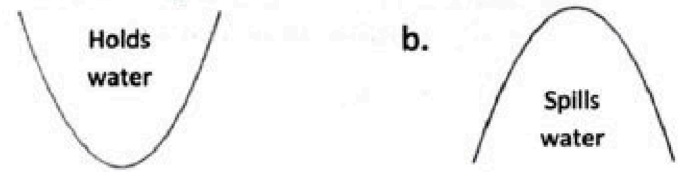
$f'(x) = \frac{x(x-2)}{(x-1)^2}$; by quotient rule

Now $f'(x) = 0$ when $x = 0$ and $x = 2$, whereas $f'(x)$ doesn't exist when $x = 1$. However, inspection of f reveals $x = 1$ is not in its domain and so the any critical values are 0 and 2.

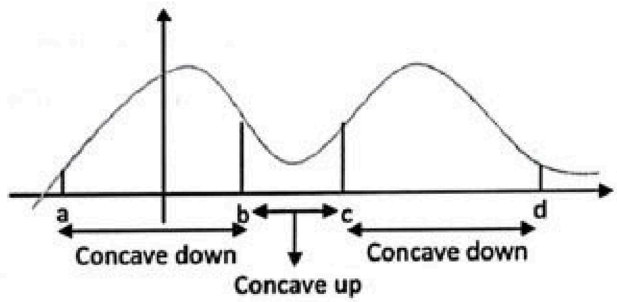
2.20 Second Derivative Test for Relative Extrema

Concavity: We know about the concavity:

a. b.



The figures (a) and (b) illustrates geometric shapes that are concave upward and concave downward, respectively. Often a shape that is concave upward is said to "hold water" whereas a shape that is concave downward "spills water".



The graph in the fig(c) is concave upward on the interval (b, c) and concave downward on (a, b) and (c, d).

Concavity and The Second Derivative Test

Definition: Test for concavity

Let f be a function for which f''exists on (a, b).

If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward on (a, b).

If $f''(x) < 0$ for all x in (a, b),then the graph of f is concave downward on (a, b).

Example 46: Determine the interval on which the graph of $f(x) = -x^3 + \frac{9}{2}x^2$ is concave upward and the intervals for which the graph is concave downward.

Solution: $f(x) = -x^3 + \frac{9}{2}x^2$

$f'(x) = -3x^2 + 9x$

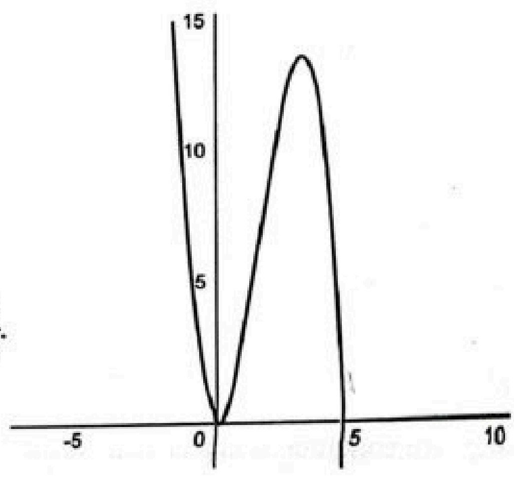
$f''(x) = -6x + 9 = 6\left(-x + \frac{3}{2}\right)$

We observe that $f''(x) < 0$ when $6\left(-x + \frac{3}{2}\right) > 0$ or

$x < \frac{3}{2}$ and that $f''(x) < 0$ when $6\left(-x + \frac{3}{2}\right) < 0$ or $x > \frac{3}{2}$.

If follows that the graph of f is concave upward on

$(-\infty, \frac{3}{2})$ and concave downward on $\left(-\frac{3}{2}, \infty\right)$.



2.21 Point of Inflection

In the example 47 function changes concavity at the point that corresponds to $x = \frac{3}{2}$. As x increases through $\frac{3}{2}$, the graph of f changes from concave upward to concave downward at the point $\left(\frac{3}{2}, \frac{27}{4}\right)$ a point on the graph of a function where the concavity changes from upward or downward or reverse is called a point of inflection.

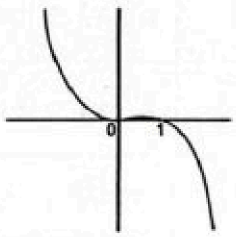
Definition: Point of Inflection

Let f be a continuous at c, a point $(c, f(c))$ is point of inflection if there exists an open interval (a, b) that contains c such that the graph of f is either:

i. Concave upward on (a, c) and concave downward on (c, b) or

ii. Concave downward on (a, c) and concave upward on (c, b).

Example 47: Find points of inflection of $f(x) = -x^3 + x^2$



Solution:

$$f'(x) = -3x^2 + 2x \text{ and } f''(x) = -6x + 2$$

Since $f''(x) = 0$ at $\frac{1}{3}$, the point $\left(\frac{1}{3}, \frac{2}{27}\right)$ is the only possible point of inflection. We have

$$f''(x) = 6\left(-x + \frac{1}{3}\right) > 0 \text{ for } x < \frac{1}{3}$$

$$f''(x) = 6\left(-x + \frac{1}{3}\right) < 0 \text{ for } x > \frac{1}{3}$$

Implies that the graph of f is concave upward on $\left(-\infty, \frac{1}{3}\right)$ and concave downward on $\left(\frac{1}{3}, \infty\right)$.

Thus, $\left(\frac{1}{3}, f\left(\frac{1}{3}\right)\right)$ or $\left(\frac{1}{3}, \frac{2}{27}\right)$ is a point of inflection.

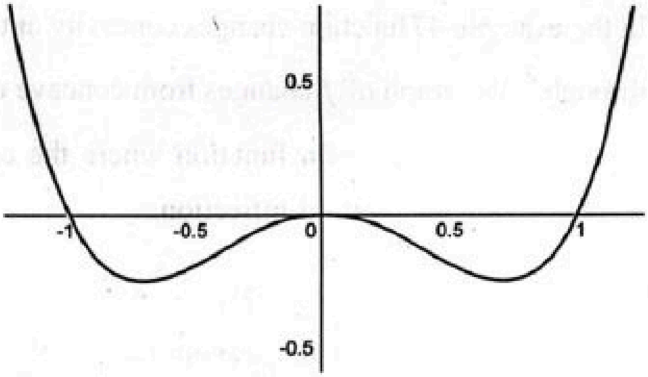
Definition: Second Derivative Test for Relative Extrema

Let f be function for which f'' exists on an interval (a, b) that contains the critical number c.

i. If $f''(c) > 0$, then $f(c)$ is a relative minimum.

ii. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

Example 48: Find the critical point and also relative extrema by second derivative test for $f(x) = x^4 - x^2$.



Solution: $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$

$$f''(x) = 12x^2 - 2$$

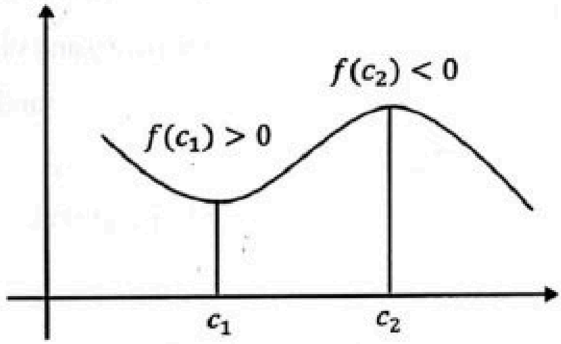
For critical values take $f'(x) = 0$

$$2x(2x^2 - 1) = 0$$

$$x = 0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$$

The second derivative test is summarized as:

x	Sign of $f''(x)$	$f(x)$	Conclusions
0	$f''(x) = -ve$	0	Rel. max
$\frac{\sqrt{2}}{2}$	$f''(x) = +ve$	$\frac{-1}{4}$	Rel. min
$\frac{-\sqrt{2}}{2}$	$f''(x) = +ve$	$\frac{-1}{4}$	Rel. min



Exercise 2.9

1. Find the critical values of the function.

   i. $f(x) = 2x^2 - 6x + 8$ ii. $f(x) = x^3 + x - 2$

   iii. $f(x) = \frac{x}{x^2+2}$ iv. $f(x) = \cos 4x$

   v. $f(x) = (4x - 3)^{\frac{1}{3}}$ vi. $f(x) = x^2(x + 1)^3$

2. Find the absolute extrema of the function on the indicated interval.

   i. $f(x) = -x^2 + 6x : [1, 4]$ ii. $f(x) = (x - 1)^2 : [2, 5]$

   iii. $f(x) = x^{\frac{2}{3}} : [-1, 8]$ iv. $f(x) = x^3 - 6x^2 + 2 : [-3, 2]$

   v. $f(x) = 1 + 5\sin 3x : \left[0, \frac{\pi}{2}\right]$ vi. $f(x) = 2\cos 2x - 4\cos x : [0, 2\pi]$

3. Use the second derivative to determine the intervals on which the function is concave upward and concave downward.

   i. $f(x) = -x^2 + 7x$ ii. $f(x) = -x^3 + 6x^2 + x - 1$

iii. $f(x) = (x+5)^5$

iv. $f(x) = x(x-4)^3$

v. $f(x) = x^{\frac{1}{2}} + 2x$

vi. $f(x) = x + \frac{9}{x}$

4. Use the second derivative to locate all points of inflection.

i. $f(x) = x^4 - x^3 + 2x^2 + x - 1$

ii. $f(x) = x^{\frac{5}{3}} + 4x$

iii. $f(x) = \sin x$

iv. $f(x) = \cos x$

v. $f(x) = x - \sin x$

vi. $f(x) = \tan x$

5. Use second derivative test to find the relative extrema of the function.

i. $f(x) = -(-2x-5)^2$

ii. $f(x) = x^3 + 3x^2 + 3x + 1$

iii. $f(x) = 6x^5 - 10x^2$

iv. $f(x) = x^2 + \frac{1}{x^2}$

v. $f(x) = \cos 3x, [0, 2\pi]$

vi. $f(x) = \cos x + \sin x, [0, 2\pi]$

6. Determine whether the give function has a relative extremum at the indicated points.

i. $f(x) = \cos x \sin x,\ x = \frac{\pi}{4}$

ii. $f(x) = x \sin x,\ x = 0$

iii. $f(x) = \tan^2 x,\ x = \pi$

iv. $f(x) = (1+\sin x)^3,\ x = \frac{\pi}{8}$

2.22 Applications of Derivatives

Many real world phenomenon involve changing quantities like the speed of the rocket, the inflation of currency, the number in a bacteria in a culture, the stoke intensity of an earth quake, the voltage of an electrical signal and so forth. In this section we will develop the concept of limits, continuity, derivative and extrema of function for use in real world problems. Another important application of the derivative is to find solution of the optimization problems. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task and if cost is the main consideration we might be interested in finding the least expensive way to perform a task. Mathematically, optimization problem can be reduced to finding the largest or smallest value of a function on some interval and determining where the largest and smallest values occurs. Using derivatives, we will develop the mathematical tools necessary for solving such problems.

Example 49: A side of a cube is measured to be $30cm$ with the possible error of $\pm 0.02cm$. What is the approximate maximum possible error in the volume of the cube?

Solution: The volume of a cube is $V = x^3$, where x the length of one side. If Δx represents the error in the length of one side, then the corresponding error in the volume is:

$\Delta V = (x + \Delta x)^3 - x^3$

We use differential: $dv = 3x^2 dx = 3x^2 \Delta x$

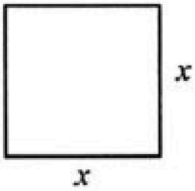
as an approximate to ΔV. Thus, for $x = 30$ and $\Delta x = \pm 0.02$, the approximate maximum error is:

$dv = 3(30)^2(\pm 0.02) = \pm 54cm^3$

Example 50: A square is expanding with time. What is the rate at which the area increases related to the rate at which a side increases?

Solution: At any time the area A of a square is a function of length of one side of x:

$$A = x^2$$



Thus, the related rates are derived from the time derivative.

$$\frac{dA}{dt} = 2x\frac{dx}{dt} \text{ (diff w.r.t "t")}$$

is the same as: $\frac{dA}{dt} = 2x\frac{dx}{dt}$

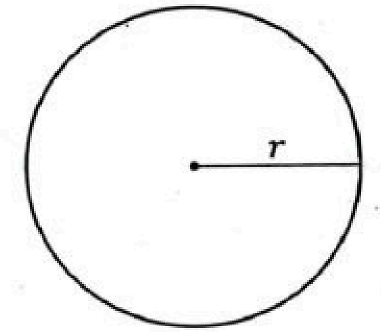
Example 51: Air is being pumped into a spherical baloon at a rate of 20 cubic feet/min. At what rate is the radius changing when the radius is 3ft?

Solution: As shown in fig, we denote the radius of the baloon by r and its volume by V. As per statement, air is being pumped at the rate $20ft^3/min$, means we have: $\frac{dV}{dt} = 20ft^3/min$

In addition, we require $\frac{dr}{dt}\Big|_{r=3}$

We know the relation between V and r is $V = \frac{4}{3}\pi r^3$

Diff w.r.t "t"



$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

But $\frac{dV}{dt} = 20$, therefore $20 = 4\pi r^2\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{5}{\pi r^2}$$

Thus, $\frac{dr}{dt}\Big|_{r=3} = \frac{5}{9\pi}\frac{ft}{min} = 0.18\,ft/min$

Example 52: Find two non-negative numbers whose sum is 15 such that the product of one with the square of other is a maximum.

Solution: Let x and y denote the two non-negative numbers (i.e. $x \geq 0$ and $y \geq 0$). It is given that: $x + y = 15$ ....(i)

Let p denote the product: $p = x.y^2$ (Product = one number. square of the other)

We can use $y = 15 - x$ to express p in terms of x: $p(x) = x(15 - x)^2$

The function $p(x)$ defined any for $0 \leq x \leq 15$.

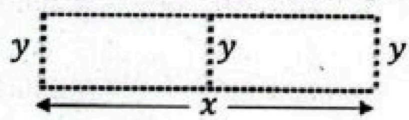
If $x > 15$, then $y = 15 - x$ would be negative.

$p'(x) = x.2(15 - x)(-1) + (15 - x)^2 = (15 - x)(15 - 3x)$

Thus, any critical value is $x = 5$.

Testing the end points of the interval reveal $p(0) = p(15) = 0$, is the minimum value of the product. Hence, $p(5) = 5(10)^2 = 500$ must be the maximum value. The two non-negative numbers are 5 and 10.

Example 53: A rectangular plot of land that contain $1500\ m^2$ will be fenced and divided into equal portions by any additional fence parallel to two sides. Find the dimensions of the land that require the least amount of fencing.



Solution: Let us introduce variable x and y so that $xy = 1500$. Then the function we wish to minimize is the sum of the lengths of the five portions of the fence.

$L = 2x + 3y$

But $y = \dfrac{1500}{x}$, we have

$L(x) = 2x + \dfrac{4500}{x}$

$L'(x) = 2 - \dfrac{4500}{x^2}$

For critical value, $L'(x) = 0$

$x^2 = 2250$

$x = 15\sqrt{10}$

For 2nd derivative: $L''(x) = \dfrac{13500}{x^3}$

When $x = 15\sqrt{10}$

$L''(15\sqrt{10}) > 0$

Hence $x = 15\sqrt{10}m$, is required minimum amount of fencing.

So,

$L(15\sqrt{10}) = 2(15\sqrt{10}) + \dfrac{4500}{15\sqrt{10}} = 15\sqrt{10}$

$xy = 1500$

$y = \dfrac{1500}{x} = \dfrac{1500}{15\sqrt{10}}$

$y = 10\sqrt{10}m$

Dimension of land:

$xy = 15\sqrt{10} \times 10\sqrt{10}$

Price Growth Model: The price level at time P, considering inflation can be modeled as: $P(t)=P_0e^{rt}$, where, $P(t)$= Price at time t, P_0 = Initial price, continuous annual inflation rate,

t= Time(in years)

To find the rate of change of price with respect to time, take the derivative of $P(t)$ with respect to t. $\frac{d}{dt}P(t)=P_0re^{rt}$, The $\frac{dP(t)}{dt}$, represents the instantaneous rate of change of the price level or how fast prices are increasing at a time t.

Example 54: The price of a product is modeled as: $P(t)=200e^{0.03t}$, where t is the time in years and P (t) is the price at a time t. Find the rate at which the price is increasing after:

a. 0 years b. 5 years c. 10 ears

Solutions: The price inflation is $P(t)=200e^{0.03t}$, the derivative gives the rate of price increase:

$$\frac{dP(t)}{dt}=200(0.03)e^{0.03t}=6e^{0.03t}$$

a. At t = 0, $\frac{dP(0)}{dt}=200(0.03)e^{0.03(0)}=6e^{0}=6\,\text{units/year}$

b. At t = 5, $\frac{dP(5)}{dt}=200(0.03)e^{0.03(5)}=6e^{0.15}=6.92\,\text{units/year}$

c. At t = 10, $\frac{dP(10)}{dt}=200(0.03)e^{0.03(10)}=6e^{0.3}=8.10\,\text{units/year}$

Using Straight Lines: Derivatives help analyze a line relationship in real life scenarios. Straight lines appear in situations, where variables change at a constant rate and derivatives calculate the rate or optimize related process.

Example 55: Economics, Marginal Cost and Revenue: A company's Revenue R(x) from selling x units is given by: $R(x)=50x$ The total cost $C(x)$ for producing x units is: $C(x)=30x+200$

a. Find the marginal revenue and marginal cost.
b. Determine the break-even point (units sold where revenue equals cost).
c. Interpret the meaning of the straight-line equations and slopes.

Solutions: a. Marginal Revenue and Marginal Cost:

- Marginal Revenue $(R'(x))$: $R'(x)=\frac{d}{dx}(50x)=50$, revenue increases by 50/units.
- Marginal Cost $(C'(x))$: $C'(x)=\frac{d}{dx}(30x+200)=30$, cost increase by 30/units.

b. Break-Even Point: At break even, revenue equal costs:

$R(x) = C(x)$

$50x = 30x + 200$, which gives, $x = 10$.

The company breaks even when 10 units are sold.

c. Interpretation of Slopes:

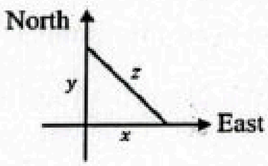
- The slopes of $R(x)$ is 50, showing revenue grows faster than cost.
- The slopes of $C(x)$ is 30, representing slower cost growth.

Exercise 2.10

1. According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$?
2. $f(x) = \begin{cases} kx+1, & x \leq 3 \\ 2-kx, & x > 3 \end{cases}$ is continuous at 3. What is K?
3. The volume v of a sphere of radius r is $v = \left(\frac{4\pi}{3}\right)r^3$. Find the surface area s of the sphere if s is the instantaneous rate of change of the volume with respect to the radius.
4. The height S above ground of a projectile time t is given by

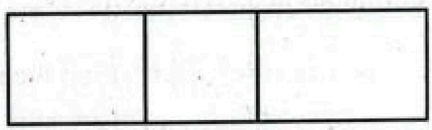
   $S(t) = \frac{1}{2}gt^2 + v_0t + s_0$.

   Where g, v_0 and s_0 are constants. Find the instantaneous rate of change of S with respect to t at $t = 4$.
5. The side of a square is measured to be 10cm with a possible error of ± 0.3 cm. Use differentials to find an approximation to the maximum error in the area. Find the approximate relative error and the approximate error.
6. A woman jogging at a constant rate of 10km/hr crosses a point A heading north. Ten minutes later a man jogging at a constant rate of 9km/hr crosses the same point heading east. How fast is the difference between the joggers hanging 20 minutes after the man crosses A?



7. A plate in the shape of an equilateral triangle is expanding with time. A side increases at a constant rate of 2cm/hr. At what rate is the area increasing when side in 8cm?

8. A rectangle expands with time. The diagonal of the rectangle increases at a rate of 1in/hr and length increases at a rate of $\frac{1}{4}$ min/hr. How fast is its width increasing when the width is 6 in and length is 8 in?
9. The side of a cube increases at a rate of 5cm/hr. At what rate does the diagonal of the cube increases?
10. A particle moves on a graph of $y^2 = x + 1$ so that $\frac{dx}{dt} = 4x + 4$. What is $\frac{dx}{dt}$ when $x = 8$?
11. At 8:00 am ship S_1 is 20km due north of S_2. Ship S_1 sails south at a rate of 9km/hr and S_2 sails west at a rate of 12 km/hr at 9:20 am. At what rate is the distance between the two ships changing?
12. Find two non-negative numbers whose sum is 60 and whose product is a maximum?
13. If the total fence to be used is 8000 m, find the dimensions of the enclosed land in figure that has the greatest area.



14. An open rectangular box is to be constructed with a square base and a volume of 32,000 cm^3. Find the dimensions of box that require the least amount of material.
15. A company determines that for the production of x units of a commodity its remove and cost functions are, respectively, $R(x) = -3x^2 + 970x$ and $G(x) = 2x^2 + 500$. Find the maximum profit and minimum average cost.
16. If the inflation rate is continuously compounded 4% per year and the price of a commodity is $50 today.
    a. Derive the function for the price of the commodity over time.
    b. Find the price after 8 years.
    c. Find the instantaneous rate of price at t=8 years.
17. A company models its operational cost as: $C(t) = 500e^{0.04t} - 100t$, where t is the time in years.
    a. Find the rate of change of cost at any time t.
    b. Determining the rate of increase in cost is minimal.
18. The price of commodity $P(t)$ is given by: $P(t) = 150(1 + 0.05t)^2$, where t is measured in years and $P(t)$ is the price level.
    a. Find the instantaneous rate of change of prices at t= 3years.
    b. Calculate the inflation rate at t=3 years.

19. A ship sails in a straight line. Its distance $d(t)$ (in nautical miles) from port is modeled by: $d(t) = 15t$ where t is time in hours.

a. Find the speed of the ship.
b. Calculate the distance after 3 hours.
c. Explain the meaning of the slope.

20: A cyclist is traveling along a straight path, and the distance traveled s(t) (in meters) is given by: $s(t) = 5t^2 + 3t$, where, t is the time in seconds.

a. Find the speed at any time t.
b. Determine the speed at t=4 seconds.
c. Interpret the significance of the slope in this context.

Review Exercise

1. Tick the correct options.

i. If $\lim_{x \to a} f(x) = 3$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$:

a. 3 b. 0 c. Exist d. Doesn't exist

ii. If $f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x > 0 \end{cases}$, then $\lim_{x \to 0^-} f(x) = 0$, is:

a. 1 b. −1 c. 0 d. 2

iii. If f and g are continuous at 2, then $\frac{f}{g}$ is continuous at:

a. 0 b. 1 c. 2 d. 3

iv. The function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, is continuous at:

a. 0 b. 1 c. −1 d. 0.1

v. If f is differentiable for every value of x, then f is:

a. Discontinuous b. Continuous
c. Finite d. Infinite

vi. If k is a constant and n is positive integer, then $\frac{d}{dx} k^n$ is:

a. nk^{n-1} b. k^{n-1} c. $\ln n . k^n$ d. 0

vii. If $f(2) = 2, g(x) = x^2$, then $\frac{d}{dx}[\frac{3g(x)}{f(2)}]$ is:

a. $2x$ b. $3x$ c. $\frac{3}{2}x$ d. $\frac{3}{2}x^2$

viii. If $y = f(x)$ is a polynomial function of degree 2, then $\frac{d^3}{dx^3} f(x)$ is:

a. 0 b. 1 c. -1 d. 2

ix. If f is differentiable for every value of x, then f is continuous for:

a. Some value of x b. $[0, \infty]$

c. Every value of x d. $[0, -\infty]$

x. If $f(t) = 2t^3$ is absolute minimum at:

a. 3 b. 0 c. -1 d. 1

2. Evaluate. $\lim_{x \to 0} \frac{x^3}{\sin^2 3x}$

3. Find the derivative. $y = \frac{1+\sin x}{x\cos x}$

4. If $f'(0) = -1$ and $g'(0) = 6$, what is $\frac{d^2}{dx^2}[xf(x) + xg(x)]$ at $x = 0$?

5. Find $\frac{d^2y}{dx^2}$, when $x^3 + y^3 = 27$

6. Use differential to find an approximate of $\sqrt{65}$.

7. An oil storage tank in the form of circular cylinder has a height of 5 m. The radius is measured to be 8 m with a possible error of ± 0.25 m. Use differentials to estimate the maximum error in the volume. Find the approximate relative error and the approximate percentage error.

8. A 15 ft ladder is leaning against a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 ft/min. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 ft from the wall?

9. Find the absolute extrema of $f(x) = x^3 - 3x^2 - 24x + 2$,

a. $[-3, 1]$ b. $[-3, 8]$

10. Graph the function: $f(x) = x + \frac{1}{x}$