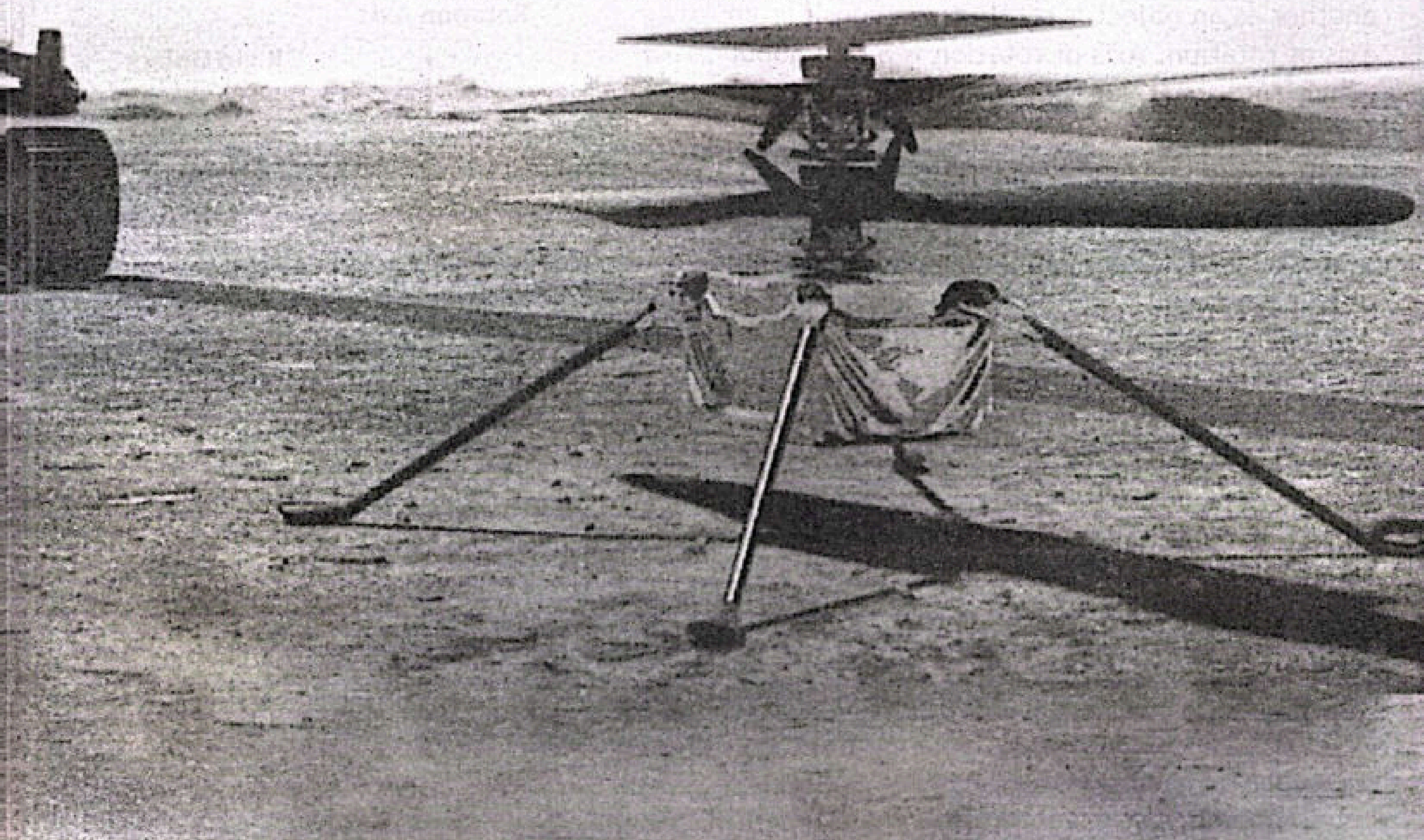
UNIT 4

ROTATIONAL AND CIRCULAR MOTION



Student Learning Outcomes (SLOs)

The students will:

- Express angles in radians.
- Define and calculate angular displacement, angular velocity and angular acceleration [This involves use of $S = r\,\theta$, $v = r\omega$, $\omega = 2\pi/T$, $a == r\omega^2$ and $a = v^2/r$ to solve problems].
- Use **equations of angular motion** to solve problems involving rotational motions.
- Analyse qualitatively motion in a curved path due to a perpendicular force.
- Define and calculate centripetal force [Use $F = mr\omega^2$, $F = mv^2/r$].
- Analyze situations involving circular motion in terms of centripetal force [e.g. situations in which centripetal acceleration is caused by a tension force, a frictional force, a gravitational force, or a *normal force*].
- Define and calculate moment of inertia of a body and angular momentum.
- State and apply the law of conservation of angular momentum. Illustrate the applications of conservation of angular momentum in real life. [Such as by flywheels to store rotational energy, by gyroscopes in navigation systems, by ice skaters to adjust their angular velocity].
- Justify how a centrifuge is used to separate materials using centripetal force.
- Derive and apply the relation between torque, moment of inertia and angular acceleration.
- Explain why the objects in orbiting satellites appear to be weightless.
- Describe how artificial gravity is created to counter weightlessness.

ROTATIONAL AND CIRCULAR MOTION

Rotational motion is the turning or spinning motion of an object about an axis that passes through it. For rotational motion of rigid objects, which are non-deformable and the particles forming it stay in fixed positions relative to one another as an object is rotated, we consider an axis of rotation. Axis of rotation is a line about which rotation takes place. This line remains fixed during rotational motion, while the other points of the body move in circles about it.

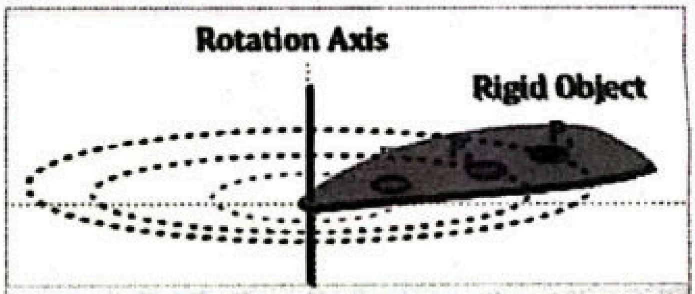


Figure 4.1: Axis of rotation.

The axis of rotation may be a pivot, hinge or any other support. Every point in a rotating rigid object moves in a circle (shown dashed in Fig. 4.1 for points P_1, P_2 and P_3) with the center on the axis of rotation. A straight line drawn from the axis to any point in the object sweeps out the same angle in the same time interval.

4.1 ROTATIONAL KINEMATICS

Rotational kinematics deals with motion of objects along a circular path without any reference to forces or torques.

4.1.1 Angular Position (θ)

The angle through which position vector of a moving object is displaced with respect to some chosen reference direction is called angular position 'θ'.

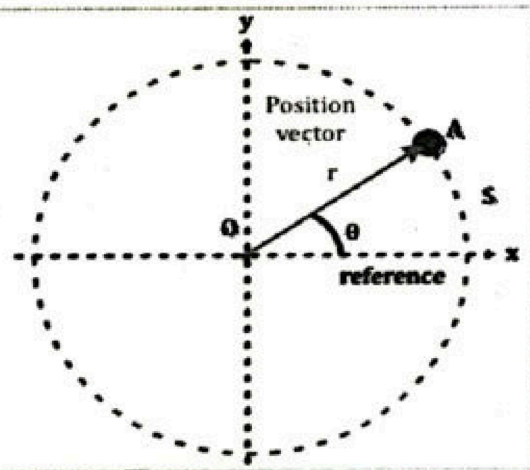


Figure 4.2: Angular Position.

Let an object 'A' is rotated through arc length 'S' from a certain reference axis, along a circle of radius 'r' as shown in Fig. 4.2. The angular position of the rigid object is the angle 'θ' between this radial line (represented by position vector 'r') and the fixed reference line in space (often chosen as the + x axis). Mathematically,

$$\theta = \frac{S}{r} \qquad \text{\_\_\_\_\_\_ (4.1)}$$

This resembles the way we identify the position of an object in translational motion as the distance x between the object and the reference position, which is the origin (x = 0).

4.1.2 Angular Displacement (Δθ)

The change in angular position with respect to chosen reference direction is termed as angular displacement.

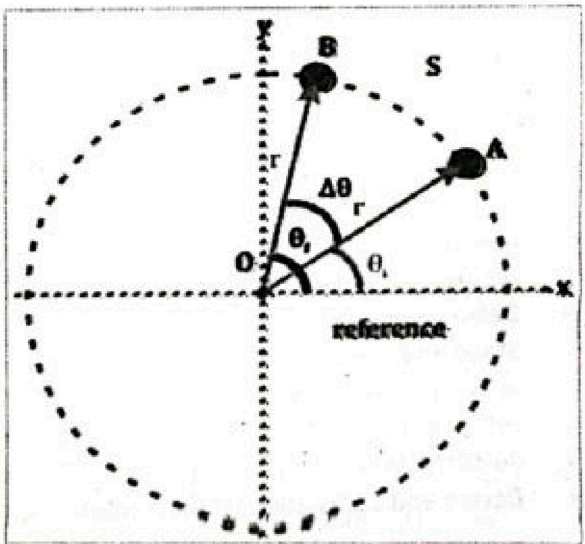


Figure 4.3: Angular Displacement.

As a particle on a rigid object travels from position A to position B in a time interval t, as in Fig. 4.3, the reference line fixed to the object sweeps out an angle, given by:

$$\Delta\theta = \theta_f - \theta_i \qquad \text{\_\_\_\_\_\_ (4.2)}$$

Conventionally, positive angular displacements represent anti-clockwise motion and negative represents clockwise motion.

Units of Angular Displacement: The SI unit of angular displacement is radian. Other units are degrees and revolutions.

Relation between radian and degree:

In one complete rotation, there are 360°.

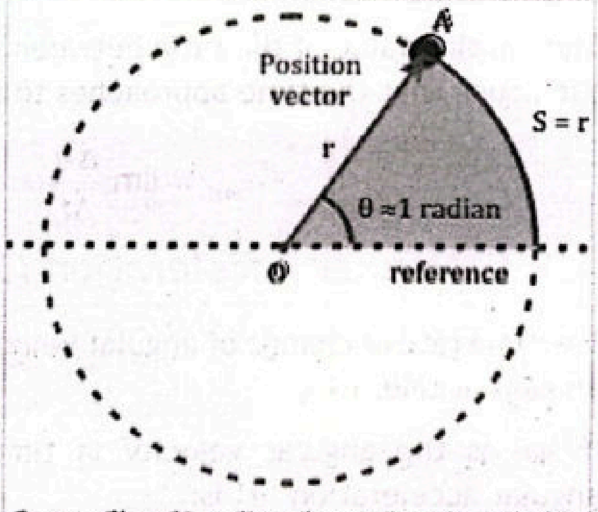
Number of degrees in one revolution = 360°

To find the number of radians in one revolution, we put S as circumference of circle, which is $2\pi r$, in equation 4.1, we get:

number of radians in one revolution = 2π rad

As for one complete revolution the number of radians must be equal to the number of degrees, therefore:

$$2\pi\, rad = 360^0 \text{ or } 1 rad = \frac{360^0}{2\pi} = \frac{360^0}{2\times 3.14} = 57.3^0$$

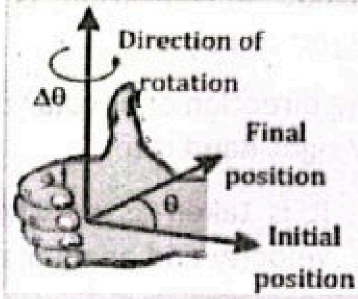


One radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle.

Figure 4.4: Radian Measurement.

Direction of Angular Displacement: Angular displacement is a vector quantity, having both magnitude and direction. The right hand rule is used to specify the direction.

Holding the axis of rotation in right hand with fingers curling in the direction of rotation; the thumb gives the direction of angular displacement.



Right Hand Rule

4.1.3. Angular Velocity (ω)

The time rate of change of angular displacement of a body is called angular velocity.

If '$\Delta\theta$' is the small angular displacement in time 'Δt', then angular velocity 'ω' is:

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad \text{\_\_\_\_\_\_ (4.3)}$$

Units of Angular Velocity: The SI unit of angular velocity is radian per second (rad s^{-1}).

Other units are deg/s or rev/s or rev /min (rpm). The direction of angular velocity is same as that of angular displacement.

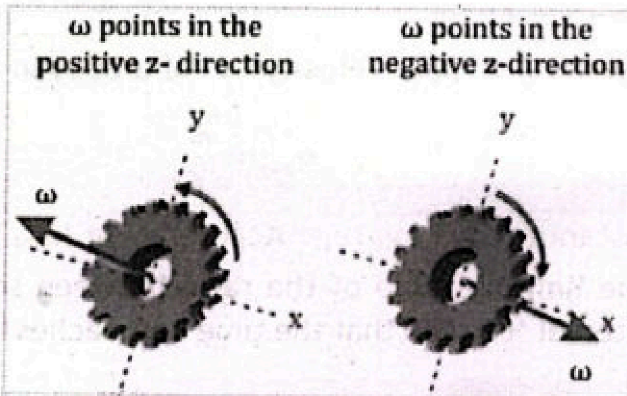


Figure 4.5: Angular velocity.

Average Angular Velocity (ω_{av}):

The total angular displacement 'θ' of a body during time 't' is called average angular velocity.

$$\omega_{av} = \frac{\theta}{t} \qquad \text{\_\_\_\_\_ (4.4)}$$

Instantaneous Average Velocity (ω_{inst}):

The limiting value of the ratio between small angular displacement '$\Delta\theta$' and small time interval 'Δt', such that the time approaches to zero, is called instantaneous angular velocity.

$$\omega_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \qquad \text{\_\_\_\_\_ (4.5)}$$

4.1.4. Angular Acceleration (α)

The time rate of change of angular velocity is called angular acceleration.

If 'ω' is the angular velocity in time 't', then angular acceleration 'α' is:

$$\alpha = \frac{\Delta\omega}{\Delta t} \qquad \text{\_\_\_\_\_ (4.6)}$$

Units of Angular Acceleration: The SI unit of angular acceleration is rad/s^2. Other units are deg/s^2 or rev/s^2.

The direction of angular acceleration is determined by right hand rule.

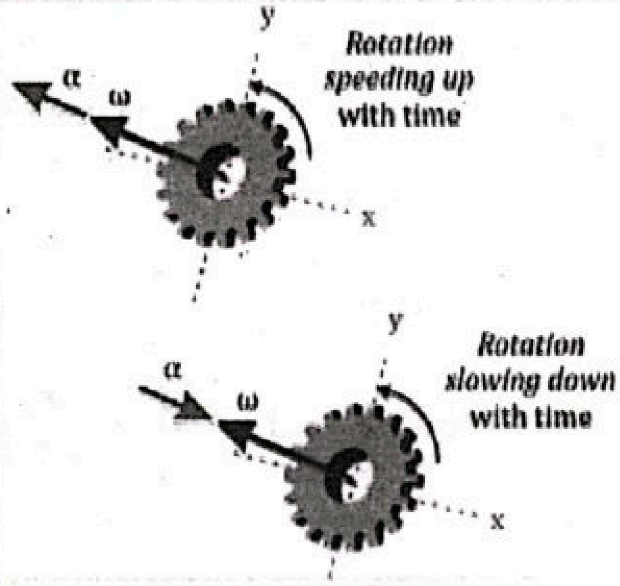


Figure 4.6: Angular acceleration.

- It is taken as positive when angular velocity of a body increases. In such case angular velocity and angular acceleration have same direction.
- It is taken as negative when angular velocity of a body decreases. In such case angular velocity and angular acceleration are anti-parallel.

Average Angular Acceleration (α_{av})

The total angular velocity 'ω' of a body in time 't' is called average angular acceleration.

$$\alpha_{avg} = \frac{\omega}{t} \qquad \text{\_\_\_\_\_ (4.7)}$$

Instantaneous Average Acceleration (α_{inst})

The limiting value of the ratio between small change in angular velocity 'ω' and small time interval 't', such that the time approaches to zero, is called instantaneous angular acceleration.

$$\alpha_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \qquad \text{\_\_\_\_\_ (4.8)}$$

4.1.5 Relationship between Linear and Angular Kinematic Quantities

Linear kinematic quantities like displacement, velocity and acceleration can be related with their rotational analogue.

A. Relation between Linear and Angular Displacement: Consider the Fig. 4.7, in which a particle that moves in circle of radius 'r' with center at 'O'. Let the particle moves from point A to B, and there is another point C such that $\angle AOC = 1$ radian, therefore Arc AC must be equal to radius 'r'. By using simple geometry, we can write:

$$\frac{\text{Arc AB}}{\text{Arc AC}} = \frac{\angle AOB}{\angle AOC}$$

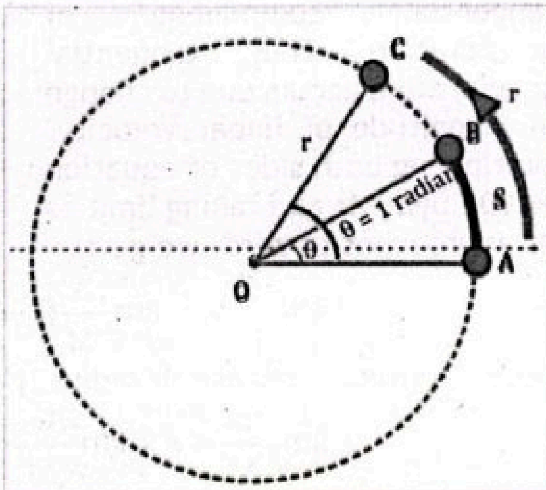


Figure 4.7: Linear and angular acceleration.

Here, Arc AB is linear displacement 'S' and $\angle AOC$ is the angular displacement 'θ'. As Arc AC = r and $\angle AOC$ = 1 radian, so the above equation becomes:

$$\frac{S}{r} = \frac{\theta}{1rad} \qquad \text{or} \qquad \theta = \frac{S}{r}$$

For angular displacement θ in radians, we can write:

$$S = r\theta \quad \text{\_\_\_\_\_\_ (4.9)}$$

B. Relation between Linear and Angular Velocity:
Multiplying both sides of equation (4.9) by $\Delta/\Delta t$ and taking limit Δt approaches to zero, we get:

$$\lim_{\Delta t \to 0} \frac{\Delta}{\Delta t} S = \lim_{\Delta t \to 0} \frac{\Delta}{\Delta t}(r\theta)$$

Since there is no change in radius 'r' with respect to time, therefore:

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = r \times \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \quad \text{\_\_\_\_\_\_ (4.10)}$$

Now by definitions of linear and angular velocities:

$$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} \quad \text{\_\_\_\_\_ (4.11)} \qquad \text{and} \qquad \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \quad \text{\_\_\_\_\_\_ (4.12)}$$

Putting values from equation (4.11) and equation (4.12) in equation (4.10), we get:

$$v = r\omega \quad \text{\_\_\_\_\_\_ (4.13)}$$

The points A and B move closer together as Δt approaches to zero. And the direction of linear velocity is along the tangent to the circle. Therefore, this velocity is also called as tangential velocity.

C. Relation between Linear and Angular Acceleration: In angular motion, the linear acceleration has two components, tangential component and the radial component, as shown in the Fig. 4.8.

In vector form: $\mathbf{a} = \mathbf{a_t} + \mathbf{a_R}$

In magnitude $a = \sqrt{a_T^2 + a_R^2}$

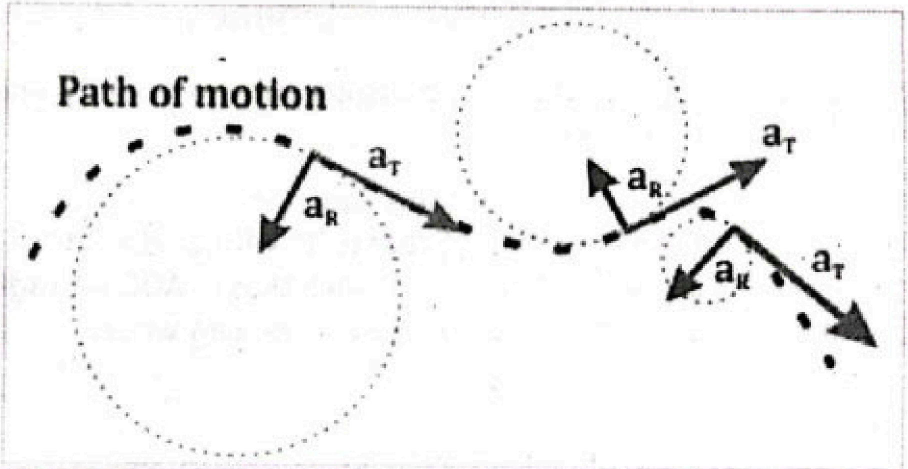


Figure 4.8: Acceleration components.

Tangential Component:
The component of angular acceleration which is parallel to linear instantaneous velocity is tangential component of acceleration. Thus, tangential acceleration occurs due to change in magnitude of linear velocity. Multiplying both sides of equation (4.10) by $\Delta/\Delta t$ and taking limit as Δt approaches to zero, we get:

$$\lim_{\Delta t\to 0}\frac{\Delta}{\Delta t}v = \lim_{\Delta t\to 0}\frac{\Delta}{\Delta t}(r\omega)$$

Since there is no change in radius 'r' with respect to time, therefore:

$$\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = r\times\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} \quad \text{\_\_\_\_ (4.14)}$$

Now by definitions of linear and angular accelerations:

$$a = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = a_T \quad \text{\_\_\_\_ (4.15)}$$

and
$$\alpha = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} \quad \text{\_\_\_\_ (4.16)}$$

Putting values from equations (4.15) and (4.16) in equation (4.14), we get:

$$a_T = r\alpha \quad \text{\_\_\_\_ (4.17)}$$

This enables us to write all the kinematic equations in rotational form, as shown in the Table 4.1. Kinematics for rotational motion is similar to translational kinematics.

Table 4.1: Kinematic Equations for Rotational Motion

Equations for Linear Motion	Equations for Angular Motion
$S = v\,t$	$\theta = \omega\,t$
$v_f = v_i + at$	$\omega_f = \omega_i + at$
$2\,a\,S = v_f^2 - v_i^2$	$2\,\alpha\,\theta = \omega_f^2 - \omega_i^2$
$S = v_i\,t + ½\,at^2$	$\theta = \omega_i\,t + ½\,\alpha t^2$

Radial Component: The component of acceleration in angular motion which is along radius of the circular path is radial component of acceleration. This acceleration arises due to change in direction of linear instantaneous velocity. For an object moving in a circular path with constant speed, there is only the radial acceleration, also called centripetal acceleration.

Example 4.1: In a workshop, a bicycle tyre of radius 33.1 cm is rolled across the level floor with an initial velocity of 6.80 m s^{-1}. Assuming constant angular acceleration, the tyre comes to rest at a distance of 74.8 m. Determine (a) initial angular velocity of the tyre; (b) the total number of revolutions it made before coming to rest; (c) the angular acceleration of the tyre; and (d) the time it took before coming to rest.

Given: Initial velocity 'v_i' = 6.80 m s^{-1} Final angular velocity 'ω_f' = 0.00 rad s^{-1}
Radius 'r' = 33.1 cm = 0.331 m   Distance 'S' = 74.8 m

To Find: (a) Angular velocity 'ωi' = ?   (b) Number of revolutions 'N' = ?
(c) Angular acceleration 'α' = ?   (d) Time 't' = ?

Solution: (a) The relation between linear and angular velocity is $v = r\omega$ or $\omega = \frac{v}{r}$

Putting values: $\omega = \frac{6.80}{0.331}$ therefore, $\omega = 20.54\,rad\,s^{-1}$

(b) When the tyre completes one revolution, it moves a distance equal to the circumference of the tyre ($2\pi r$), as long as there is no slipping or sliding. The number of revolutions will be the total distance divided by distance covered during each revolution ($2\pi r$). $N = \frac{S}{2\pi r}$

Putting values: $N = \frac{74.8}{2\times 3.14\times 0.331}$ therefore, $N = 35.9\,rev$

(c) In one revolution there are 2π radians, the total angular displacement θ will be $35.9 \times 2\pi$ radians = 225.6 radians (θ = 225.6 radians). To find angular acceleration we would use the equation independent of time (3rd equation) i.e.

$$2\alpha\theta = \omega_f^2 - \omega_i^2 \quad \text{or} \quad \alpha = \frac{\omega_f^2 - \omega_i^2}{2\theta}$$

Putting values: $\alpha = \frac{(0)^2 - (20.54)^2}{2\times 225.6}$ or $\alpha = -0.94\,rad\,s^{-2}$

(d) To find 't', we can use any of the equation involving time, however the simpler equation $\omega_f = \omega_i + \alpha t$, by rearranging this equation for time, we get:

$$t = \frac{\omega_f - \omega_i}{\alpha}$$

putting values: $t = \frac{0 - 20.54}{-0.94}$ therefore, $t = 21.9\,s$

So, the tyre will take about 22 seconds before coming to rest.

Assignment 4.1

The front wheel of a tractor travels 700 revolutions while the rear wheel 280 in a time interval of 40 seconds. Find their angular velocities.

4.2 CENTRIPETAL ACCELERATION AND CENTRIPETAL FORCE

Consider a particle is moving in a circular path of radius r with constant speed, this means that direction of velocity is changing. This change in velocity of the particle produces acceleration which is directed towards the center of the circle, this type of acceleration is called centripetal acceleration.

Consider the Fig. 4.9 (a) in which a particle follows a circular path. The particle is at point A at time t_i with velocity v_i. It reaches at point B at a later time t_f with velocity v_f. For uniform circular motion v_i and v_f differ only in direction; their magnitudes are same, i.e. $|v_i| = |v_f| = |v|$

In Fig. 4.9 (b) velocity vectors have been redrawn tail to tail. The vector Δv joins the heads of two vectors, representing vector addition,

$$v_f = v_i + \Delta v$$

The angle '$\Delta\theta$' between the two position vector 'r_i' and 'r_f' is the same as the angle '$\Delta\theta$' between the two velocity vectors 'v_i' and 'v_f'. This is because the velocity vector is perpendicular to the position vector, thus the two angles must be same. This allows us to write a relation for the lengths of the sides of the two triangles.

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

Where $|r_i| = |r_f| = |r|$ and $|v_i| = |v_f| = |v|$

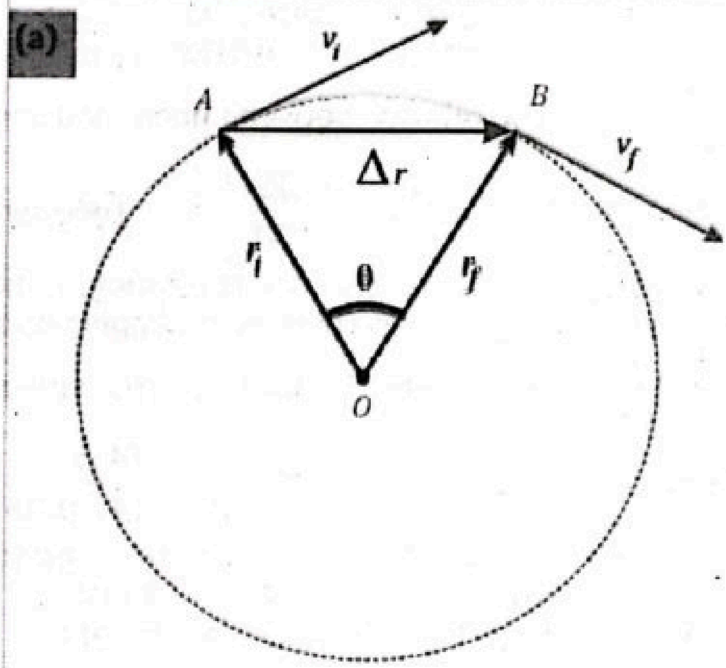
or $$\Delta v = v\frac{\Delta r}{r}$$

Dividing both sides by Δt, we get:

$$a_{av} = \frac{v}{r}\frac{\Delta r}{\Delta t}$$

Now imagine the points 'A' and 'B' in the figure are extremely close together. As 'A' and 'B' approach each other, 'Δt' approaches to zero. The acceleration at this stage will now be instantaneous acceleration.

$$a = \frac{v}{r}\lim_{\Delta t \to 0}\frac{\Delta r}{\Delta t}$$



As the particle moves from point A to B its velocity vector changes from v_i to v_f.

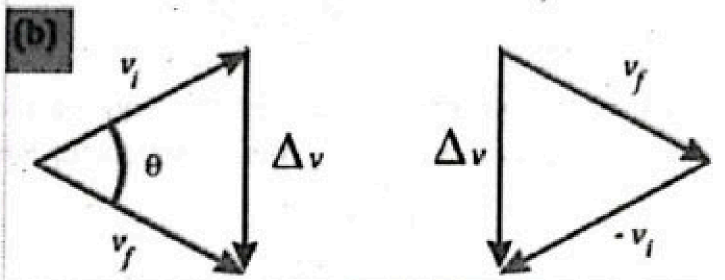


Figure 4.9: Centripetal Acceleration.

Since, $v = \lim_{\Delta t \to 0}\frac{\Delta r}{\Delta t}$. Therefore, $a = \frac{v}{r} \times v$

This acceleration is referred to as centripetal acceleration a_c.

$$a_c = \frac{v^2}{r} \qquad \text{or} \qquad a_c = \left(\frac{v^2}{r}\right)\hat{r}$$

As centripetal acceleration a_c is directed towards the center of the circle, the radial vector r is directed outwards from the center of the circle, thus a negative sign can be added to the equation.

$$a_c = -\left(\frac{v^2}{r}\right)\hat{r} \qquad \text{\_\_\_\_\_\_ (4.18)}$$

as v = r ω, therefore, $$a_c = -\left(\frac{r^2\omega^2}{r}\right)\hat{r}$$

Hence, $$a_c = -r\omega^2\hat{r} \qquad \text{\_\_\_\_\_\_ (4.19)}$$

According to Newton's second law, an object that is accelerating must have a net force acting on it. for example, to open a door, force must be applied to produce tangential acceleration thereby creating torque. Similarly, for an object to move in a circle, a force must be applied to it to keep it moving in that circle. Thereby giving it necessary radial (centripetal) acceleration.

4.2.1 Centripetal Force

The net force that causes the particle to undergo centripetal acceleration is called centripetal force F_c.

When Newton's second law is applied to a particle moving in a uniform circular motion, we can write: $F_c = m\,a_c$ ______ (4.20)

Putting equation (4.18) or equation (4.19) in equation (4.20), we can write centripetal force F_c as: $F_c = -\left(\frac{mv^2}{r}\right)\hat{r}$

and $F_c = -mr\omega^2\hat{r}$ ______ (4.21)

The direction of the centripetal force is always directed towards the center of the circle. Centripetal force is not a new force, but any net force that makes an object move towards the center of the circle can be termed as centripetal force. For example, to swing a ball in a circle at the end of a string, the tension in the string act as centripetal force. For a moon revolving around the Earth, or planets revolving around the Sun, gravity act as centripetal force. In other situation, it can be a normal force, or even an electric force (as in CD players and computer hard disks).

Frictional force and normal force as centripetal force:

When a car travels without skidding around an un-banked curve, the static frictional force between the tyres and the road provides the necessary centripetal force. The reliance on friction can be eliminated completely for a given speed, if the roads are banked at an angle relative to the horizontal while making a turn (Fig. 4.10).

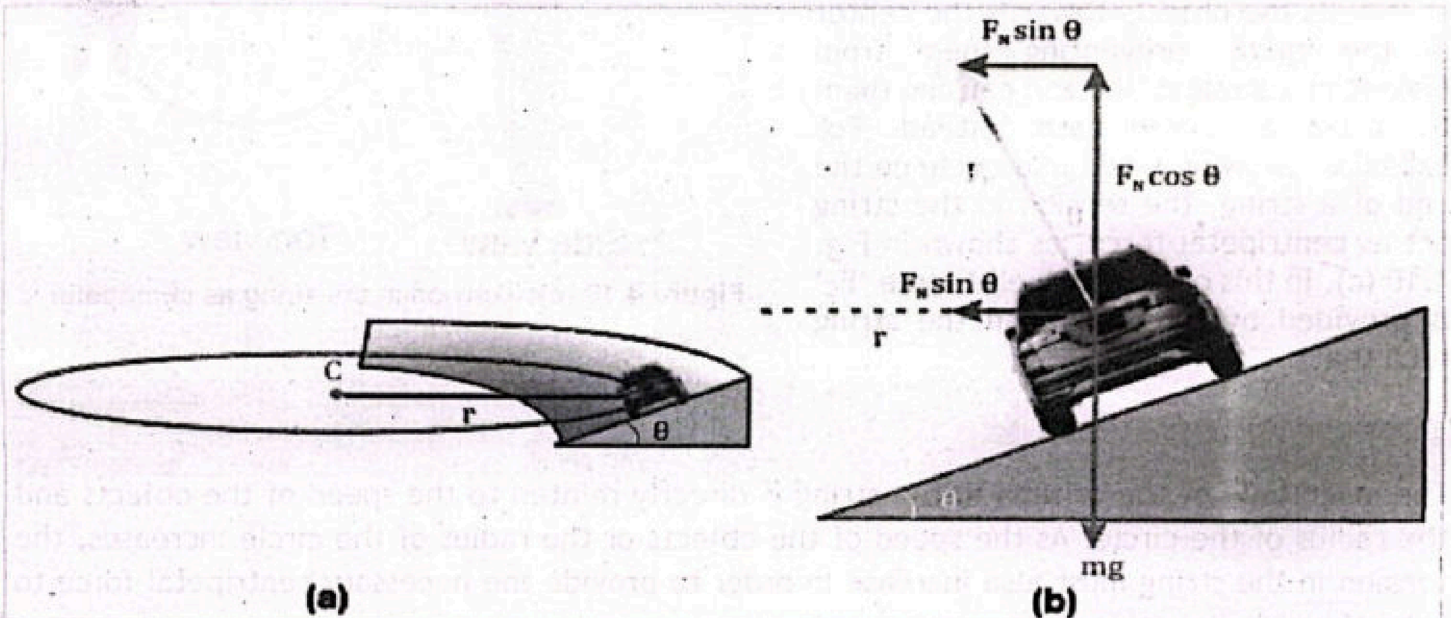


Figure 4.10: Friction and normal force as centripetal force.

Because the roadbed makes an angle with respect to the horizontal, the normal force has a component '$F_N \sin\theta$' that points toward the center 'C' of the circle and provides the centripetal force.

In the Fig. 4.10 (a) part shows a car going around a friction-free banked curve. The radius of the curve is 'r', where 'r' is measured parallel to the horizontal. Part (b) of the figure shows the normal force 'F_N' that the road applies on the car, the normal force is perpendicular to the road. Because the roadbed makes an angle 'θ' with respect to the horizontal, the normal force has a component '$F_N \sin\theta$' that points toward the center C of the circle and provides the centripetal force.

$$F_c = F_N \sin\theta = \frac{mv^2}{r} \quad \text{\_\_\_\_\_\_ (4.22)}$$

Since the car does not accelerate along the component of normal force '$F_N \cos\theta$', this component only balances the weight 'mg' of the car. Therefore, '$F_N \cos\theta = mg$'.

Dividing equation (4.22) by this equation, we get: $$\frac{F_N \sin\theta}{F_N \cos\theta} = \frac{mv^2/r}{mg}$$

or $$\tan\theta = \frac{v^2}{rg} \quad \text{\_\_\_\_\_\_ (4.23)}$$

This equation indicates that for a given speed v, the centripetal force needed for a turn of radius 'r' at an angle 'θ' is independent of the mass of the vehicle. higher speeds and smaller radii require more steeply banked curves—that is, larger values of 'θ'. At a speed that is too low for a given 'θ', a car would slide down a frictionless banked curve; at a speed that is too higher, a car would slide off the curve.

Tension force as centripetal force: When objects are connected by a string or rope and moving in a circle, the tension in the string acts as the centripetal force. The tension in the string is responsible to provide the necessary centripetal force, as it pulls the objects towards the center of the circle, preventing them from moving in a straight line and causing them to follow a curved path instead. For example, to swing a ball in a circle on the end of a string, the tension in the string act as centripetal force, as shown in Fig. 4.10 (c). In this case centripetal force 'F_c' is provided by tension 'T' in the string such that:

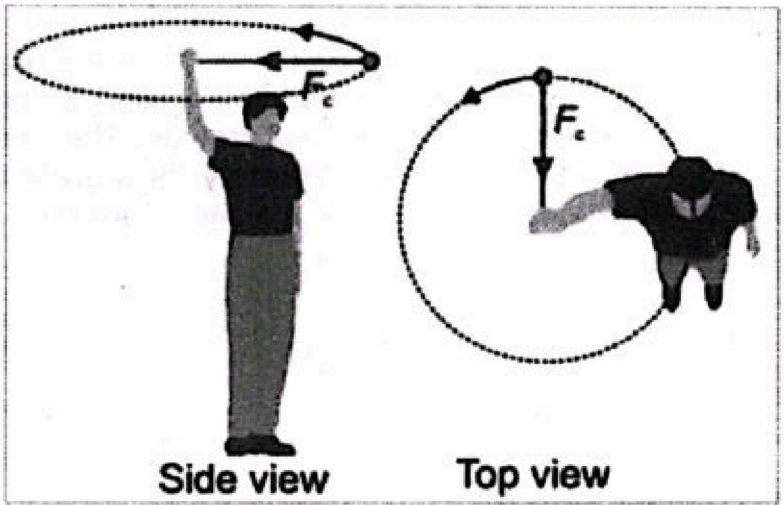


Figure 4.10 (c): Tension in the string as centripetal force.

$$T = \frac{mv^2}{r}$$

The magnitude of the tension in the string is directly related to the speed of the objects and the radius of the circle. As the speed of the objects or the radius of the circle increases, the tension in the string must also increase in order to provide the necessary centripetal force to keep the objects moving in a circular path.

Gravitational force as centripetal force: The force of gravity keeps planets in orbit around the sun and satellites in orbit around Earth, serving as the centripetal force. Without the force of gravity acting as the centripetal force, planets and satellites would not be able to maintain their orbits and would instead drift off into space. This force is essential for maintaining the balance between the inward force of gravity and the outward force of the object's inertia. This is why the sun's gravitational pull keep planets in orbit around it, and Earth's gravitational pull is able to keep satellites in orbit around it, as shown in Fig. 4.10 (d). In this case, the gravitational force 'F_g' is responsible for providing the centripetal acceleration required for the circular motion. Mathematically:

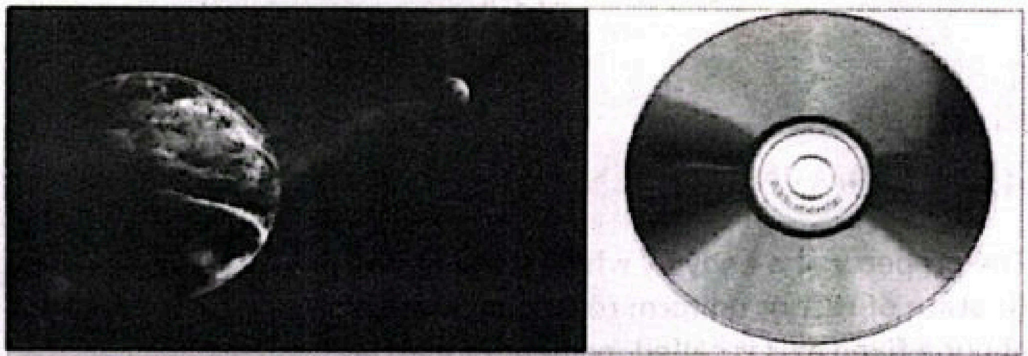


Figure 4.10 (d): Examples of centripetal force.

$$F_g = \frac{mv^2}{r}$$

Centrifuge: A centrifuge is a device that separates substances suspended in a liquid mixture by spinning a sample of liquid mixture very quickly around an axle. Any small denser particles found in the liquid travel in a straight line inside the test tube, obeying Newton's first law. The liquid in the test tube applies a centripetal force on these particles to keep them moving in a circle. After running the centrifuge at high speed for a period of time, the particles become clumped together at the bottom of the test tube, which can be collected and the sample is analyzed, as shown in Fig. 4.11.

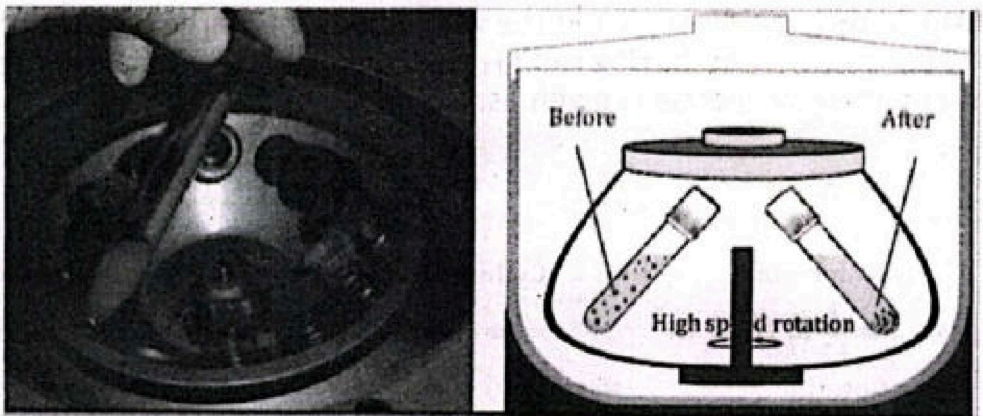


Figure 4.11: Centrifuge

The same centrifugation principle can be applied in the following commonly used devices.

Cream Separator is a centrifugal device that separates milk into cream and skimmed milk.

Washing Machine Dryer consists of a long cylinder with small holes on its walls. Wet clothes are placed in this cylinder, and then rotated rapidly to dry it.

Example 4.2: The centripetal force on a car of mass 856 kg moving along a curve is 7250 N. If its speed is 12.0 m s^{-1}, what is the radius of the curve?

Given: Mass 'm' = 856 kg   Centripetal force 'F_c' = – 7250 N.   Speed 'v' = 12.0 m s^{-1},

Solution: Radius 'r' = ?

Solution: The centripetal force is given as:

$$F_c = -\frac{mv^2}{r} \quad \text{or} \quad r = -\frac{mv^2}{F_c}$$

Putting values: $r = -\dfrac{856 \times (12)^2}{-7250}$ or $r = 17\,m$

Assignment 4.2

A car, with its centre of gravity 0.4 m above the Earth's surface, is passing through a curve whose speed limit is 15 m s^{-1}. Find radius of the curve.

4.3 MOMENT OF INERTIA

The property of a body by which it maintains its state of rest or uniform rotational motion about a fixed axis is called moment of inertia (or rotational inertia).

The moment of inertia (or rotational inertia) is the rotational equivalent of mass. Objects with larger mass have a larger inertia, meaning that they are harder to accelerate linearly. Similarly, an object with a larger moment of inertia is harder to angularly accelerate. The moment of inertia is given by:

$$I = mr^2 \quad (4.24)$$

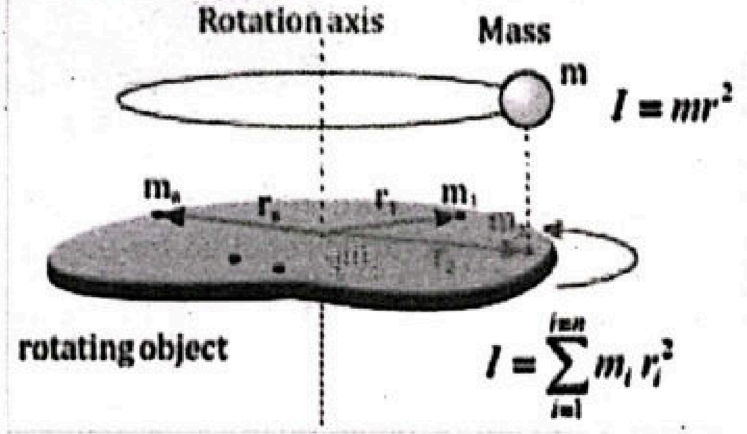


Figure 4.12: Moment of inertia.

If the body is rigid, we divide the whole body into large number of small portions having masses $m_1, m_2, m_3, \ldots, m_n$ having radii $r_1, r_2, r_3, \ldots, r_n$ from its axis of rotation, as shown in Fig. 4.12, and moment of inertia is given as:

$$I = \sum_{i=1}^{i=n} m_i r_i^2 \quad (4.25)$$

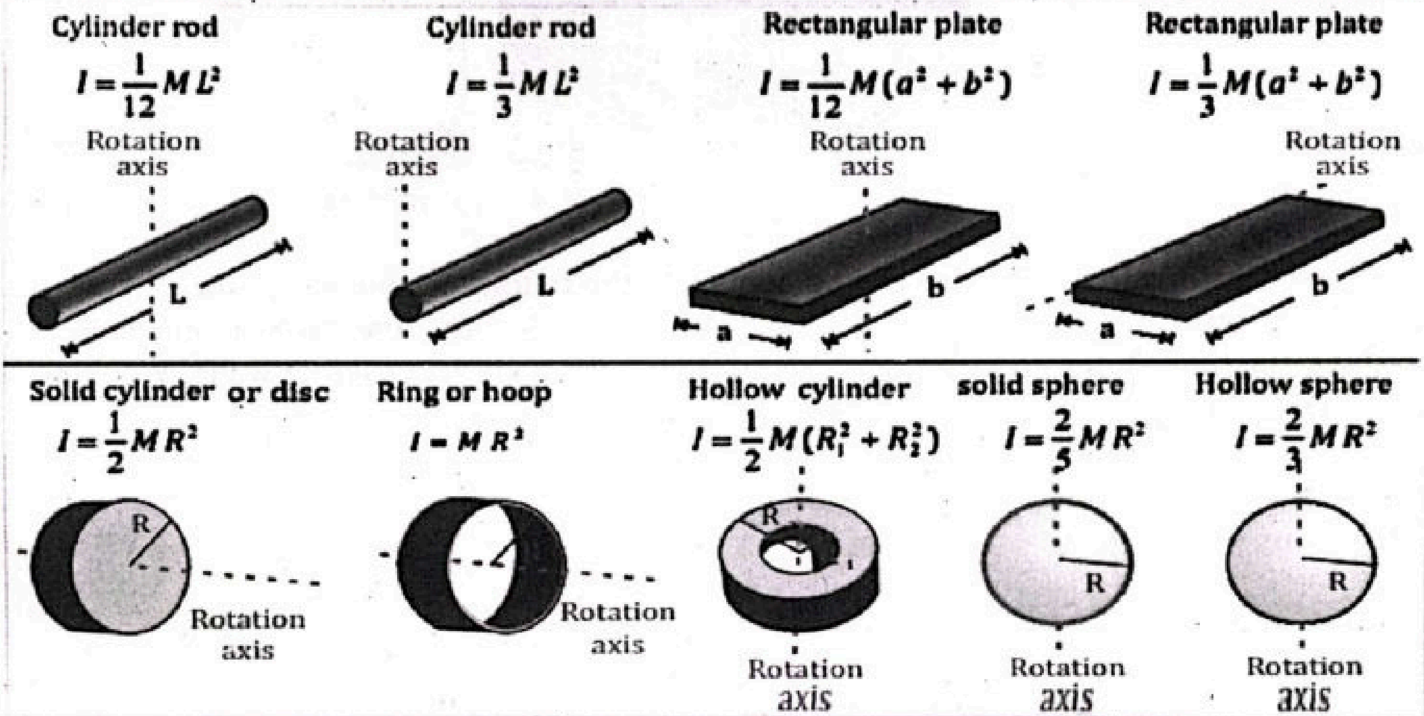


Figure 4.13: Moment of inertia for uniform objects.

Fig. 4.13 shows the calculated moments of inertia for various objects of uniform composition, each with mass 'M'.

4.4 ANGULAR MOMENTUM

The angular momentum 'L' of an object is defined as:

The cross product of position vector 'r' with respect to axis of rotation and linear momentum 'p' of an object. $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

The SI unit of angular momentum is kg m^2 s^{-1}, and dimensions are [ML^2T^{-1}].

4.4.1. For a Point Mass

Consider a mass 'm' rotating at distance 'r' from the axis of rotation, as shown in Fig. 4.14. By definition of angular momentum:

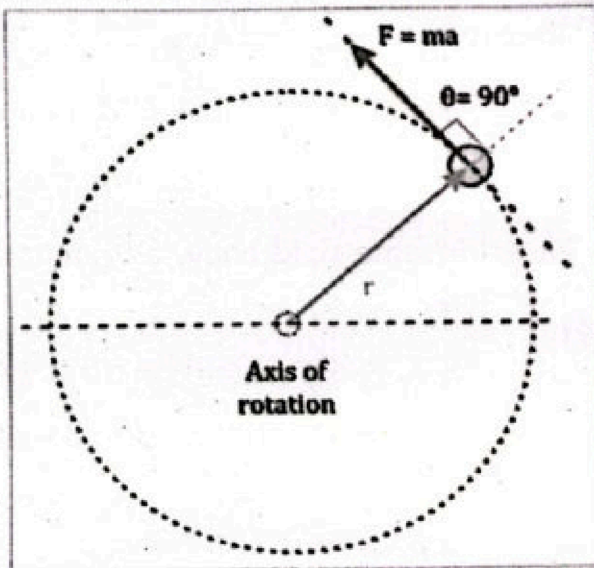


Figure 4.14: Mass 'm' rotating at distance 'r' from axis of rotation.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad \text{or} \quad \mathbf{L} = r\,p \sin\theta\ \hat{n}$$

Since $\theta = 90^\circ$ and $\sin 90^\circ = 1$, therefore, magnitude of angular momentum is given by:

$$L = rp \qquad \text{\_\_\_\_\_\_ (4.26)}$$

From the definition of linear momentum:

$$p = mv \qquad \text{\_\_\_\_\_\_ (4.27)}$$

The relation between linear and angular velocity is:

$$v = r\omega \qquad \text{\_\_\_\_\_\_ (4.28)}$$

Putting equation (4.28) in equation (4.27), we get:

$$p = mr\omega \qquad \text{\_\_\_\_\_\_ (4.29)}$$

Now putting equation (4.29) in equation (4.26), we get: $p = r(mr\omega)$

Hence $L = mr^2\omega = I\omega$ ______ (4.30)

From Eq. (4.30), the angular momentum of an object can also be defined as the product of moment of inertia and its angular velocity, just like linear momentum is defined as product of mass and velocity.

4.4.2. For a Rigid Body

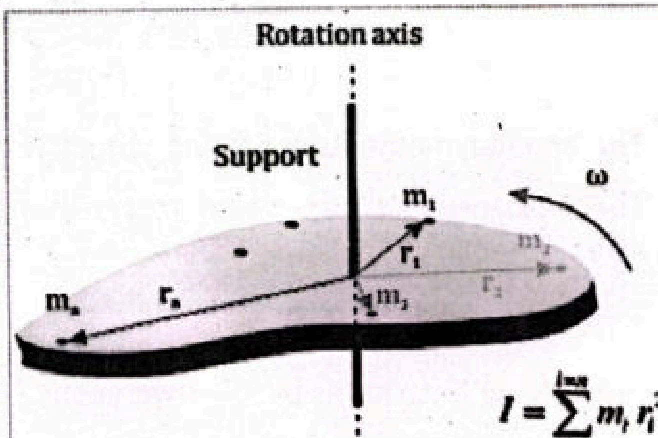


Figure 4.15: Rigid body rotating about axis of rotation.

Consider a rigid body and divide it into large number of small masses '$m_1, m_2, m_3, \ldots, m_n$' having distances '$r_1, r_2, r_3, \ldots\ldots\ldots r_n$' from the axis of rotation, as shown in the Fig. 4.15. The net angular momentum will be sum of all the individual angular momenta:

$$L_{net} = L_1 + L_2 + L_3 + L_4 + - - - + L_n \quad \text{\_\_\_ (4.31)}$$

The angular momentum about point 1 will be:

$$L_1 = m_1 r_1^2 \omega_1 \quad \text{\_\_\_\_\_ (4.32)}$$

Similarly, the angular momentum about point 2 will be:

$$L_2 = m_2 r_2^2 \omega_2 \quad \text{\_\_\_\_\_ (4.33)}$$

and
$$L_3 = m_3 r_3^2 \omega_3 \quad \text{\_\_\_\_\_ (4.34)}$$

Similarly,
$$L_n = m_n r_n^2 \omega_n \quad \text{\_\_\_\_\_ (4.35)}$$

Putting equations (4.32), (4.33), (4.34) and (4.35) in equation (4.31), we get:

$$L_{net} = m_1 r_1^2 \omega_1 + m_2 r_2^2 \omega_2 + m_3 r_3^2 \omega_3 + - - - - - + m_n r_n^2 \omega_n \quad \text{\_\_\_\_\_ (4.36)}$$

Since for same rigid body, all points on the body rotate with the same angular velocity 'ω', therefore,

$$\omega_1 = \omega_2 = \omega_3 = - - - - = \omega_n = \omega$$

Therefore, equation (4.36) can be written as:

$$L_{net} = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega + - - - - - + m_n r_n^2 \omega$$

or
$$L_{net} = \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + - - - - - + m_n r_n^2 \right) \omega$$

The term in parenthesis in above equation is moment of inertia of a rigid body, so,

$$L_{net} = \left(\sum_{i=1}^{i=n} m_i r_i^2 \right) \omega = I \omega \quad \text{\_\_\_\_\_ (4.37)}$$

4.4.3. Relation between Torque and Angular Momentum

The angular momentum L of an object is defined as:

The cross product of position vector r with respect to axis of rotation and linear momentum P of an object.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Multiplying both sides by $\frac{\Delta}{\Delta t}$, we get: $\frac{\Delta}{\Delta t}\mathbf{L} = \frac{\Delta}{\Delta t}(\mathbf{r} \times \mathbf{p})$

or
$$\frac{\Delta \mathbf{L}}{\Delta t} = \mathbf{r} \times \frac{\Delta \mathbf{p}}{\Delta t} \quad \text{\_\_\_\_\_ (4.38)}$$

According to Newton's second law of motion in terms of momentum:

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t} \qquad \text{\_\_\_\_\_\_ (4.39)}$$

Putting equation (4.39) in equation (4.38), we get:

$$\frac{\Delta \mathbf{L}}{\Delta t} = \mathbf{r} \times \mathbf{F} \qquad \text{\_\_\_\_\_\_ (4.40)}$$

By the definition of torque: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ ______ (4.41)

Therefore, $\dfrac{\Delta \mathbf{L}}{\Delta t} = \boldsymbol{\tau}$ ______ (4.42)

4.4.4. Conservation of Angular Momentum

In the absence of any external torque, the angular momentum of a system remains constant.

i.e., $\dfrac{\Delta \mathbf{L}}{\Delta t} = 0$

Therefore, $\Delta \mathbf{L} = 0$

Or $\mathbf{L}_f - \mathbf{L}_i = 0$

Hence, $\mathbf{L}_f = \mathbf{L}_i$

or $I_f\,\omega_f = I_i\,\omega_i$ ______ (4.43)

Equation (4.43) implies that

The final angular momentum should be equal to initial angular momentum.

A spinning ice skater is an interesting example of conservation of angular momentum. When the skater's arms are extended, the rotational inertia 'I' is relatively large and the angular velocity 'ω' is relatively small, as shown in Fig. 4.16. Often at the end of the spin, the skater pulls his arms close to his body resulting in a much faster spin (larger angular velocity) because of a much smaller rotational inertia. When a rotating body contracts, its angular velocity increases; and when a rotating body expands, its angular velocity decreases. This phenomenon is the

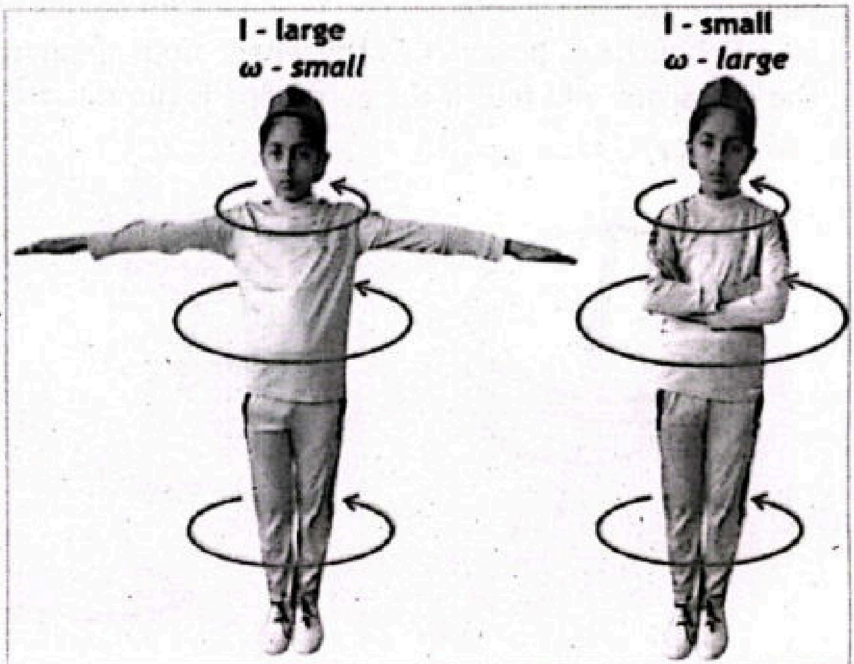


Figure 4.16: Spinning ice skater.

result of the conservation of angular momentum. As:

$$L_f = L_i \quad \text{or} \quad I_f\,\omega_f = I_i\,\omega_i$$

And moment of inertia is given by: $I = mr^2$

Therefore, $m_f r_f^2 \omega_f = m_i r_i^2 \omega_i$

As $r_f < r_i$ and $m_f = m_i$ therefore his rotational speed will increase to compensate for the decrease in rotational inertia.

Similarly, gymnasts and divers generate their spins (torque) from a solid base or a diving board after which the angular momentum remains unchanged, as shown in Fig. 4.17. The usual somersaults and twists result by making variations in their rotational inertia.

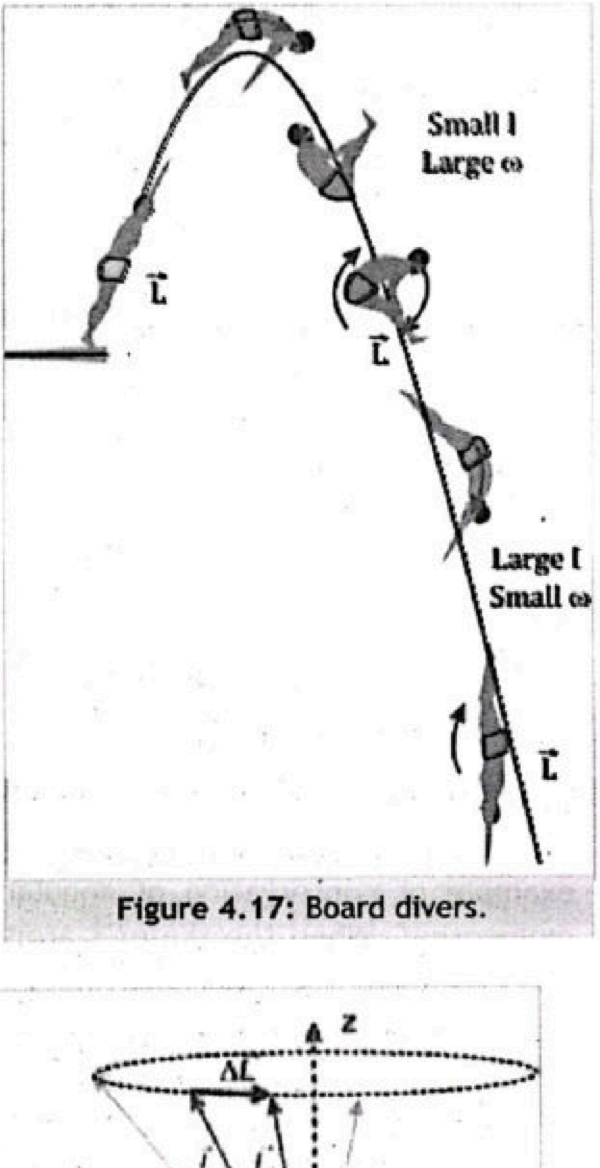


Figure 4.17: Board divers.

A **gyroscope** is a device that utilizes the principle of angular momentum to maintain its orientation relative to the Earth's axis or resist changes in its orientation. A very unusual and fascinating type of motion you probably have observed is that of a gyroscope, which utilize the principle of angular momentum.

Gyroscope usually consists of a wheel mounted on an axle which can rotate freely and is secured in a metal frame, as shown in Fig. 4.18 (a). When the wheel is made to spin the gyroscope can be balanced mounted on a flat surface, however as the wheel stops spinning the gyroscope will fall. If the gyroscope is tilted it also

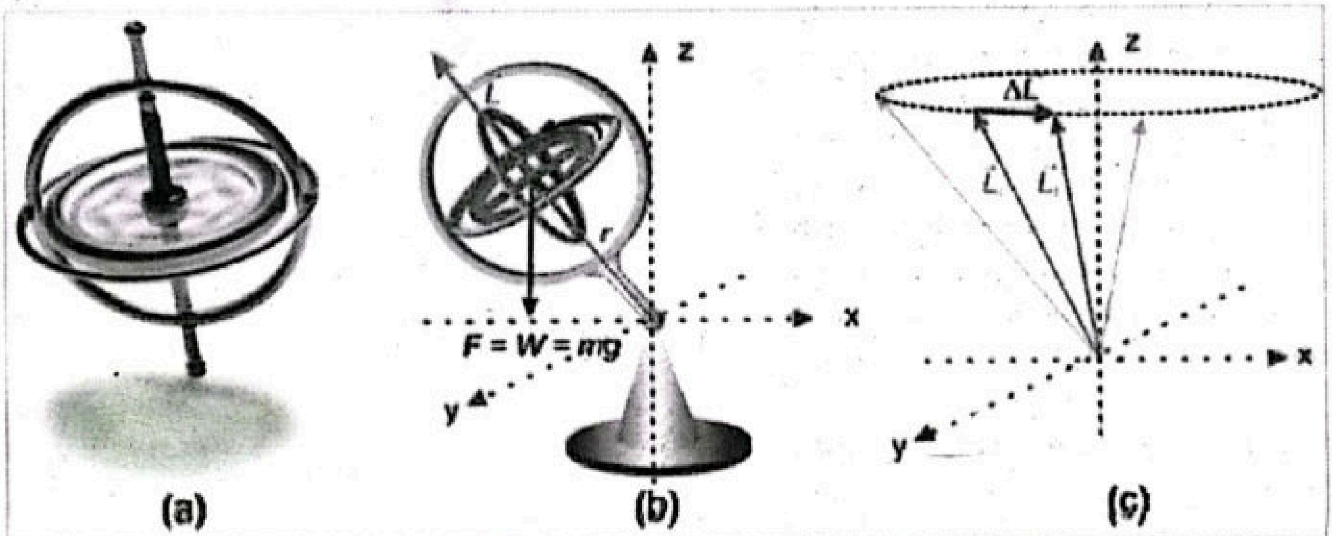


Figure 4.18: Spinning Gyroscope.

keep levitated without falling, but will start precession about gravitational force axis. This is the gravity-defying part of a gyroscope, as shown in the Fig. 4.18 (b).

This unusual behaviour can be explained by the vector nature of angular momentum, the change in direction of gyroscope will require a torque. The torque is provided by gravitational force as its weight towards the ground. The angular momentum will start to follow the torque, as shown in the Fig. 4.18 (c), the change in angular momentum 'ΔL' is:

$$\Delta L = \tau \times \Delta t$$

Where the torque has the same direction as 'ΔL' and 'Δt' is the duration of time. The same effects can also be observed even if it is lifted by string looped around its lower end.

A **flywheel** (as shown in Fig 4.19) is a mechanical component that stores energy by spinning a heavy disc or wheel about an axle. When torque is applied, the rotational speed increases, storing kinetic energy that can be used for different tasks. The concept of a flywheel is based on the principle of conservation of energy, where the energy input is stored in the form of rotational motion. This stored energy can then be released when needed, such as during power outages or to provide additional power for machinery.

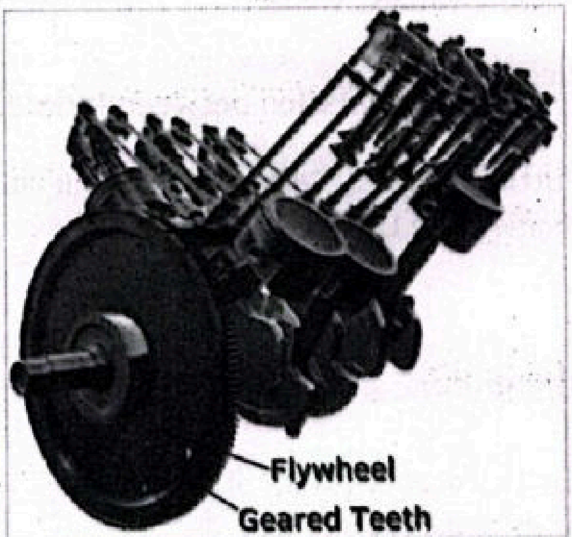


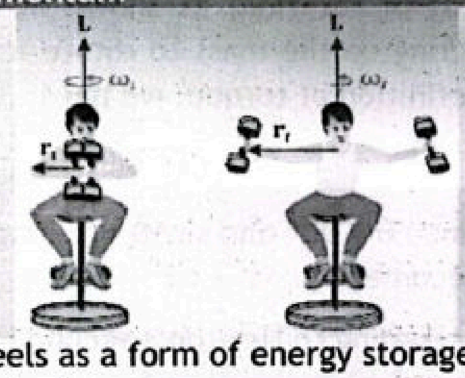
Figure 4.19: Flywheel.

Flywheels are commonly used in various applications, such as in engines, industrial machinery, and energy storage systems. In vehicles, flywheels can help smooth out the power delivery and improve fuel efficiency. In industrial settings, they can provide backup power and help regulate the speed of machinery.

The design of a flywheel involves careful consideration of the material, size, and shape of the disc or wheel, as well as the bearings and axle to minimize friction and maximize energy storage. Additionally, the speed and torque at which the flywheel operates must be carefully controlled to ensure safe and efficient operation.

Activity - Conservation of Angular Momentum

Hold pair of dumbbells in your hand and find a turntable to rotate at full speed by holding dumbbells close to your body. As soon as you extend your arms your rotation speed (angular velocity) will decrease. Again, upon drawing your hands nearer towards your chest the angular velocity will increase.
Can you explain why does this happens?



In recent years, there has been growing interest in using flywheels as a form of energy storage for renewable energy sources, such as wind and solar power. By storing excess energy generated during peak production times, flywheels can help balance the supply and demand of electricity on the grid.

Example 4.3: What is the angular momentum of a 3.6 kg uniform cylindrical grinding wheel of radius 31 cm rotating at 1150 rpm? (b) How much torque is required to stop it in 7.8 s?

Given: Mass 'm' = 3.6 kg Radius 'R' = 31 cm = 0.31 m

Initial angular velocity 'ω_i' = 1150 rpm = 120.4 rad s^{-1} Time duration 'Δt' = 7.8 s

To Find: (a) Angular momentum $L = ?$ (b) Torque $\tau = ?$

Solution: (a) The angular momentum is given as: $L = I\omega$

Since, moment of inertia for disk is $I = \frac{1}{2}mR^2$, therefore $L = \frac{1}{2}mR^2\omega$

Putting values, we get: $L = \frac{1}{2} \times 3.6 \times (0.31)^2 \times 120.4 = 20.83$ J s

(b) From the relation between torque and angular momentum: $\tau = \frac{L_f - L_i}{\Delta t}$

Putting values, where initial angular momentum L_i is 20.83 kg m^2 s^{-1} and final angular momentum L_f is zero (0 kg m^2 s^{-1}).

$$\tau = \frac{0 - 20.83}{7.8}$$

Therefore, $\tau = -2.67\, kg\, m^2\, s^{-2} = -2.67\, Nm$

Assignment 4.3

Earth rotates about its own axis. What will be its angular momentum when its average angular speed around its axis is 7.29×10^{-5} rad s^{-1}?

4.5 TORQUE AND ANGULAR ACCELERATION

Relationship exists between torque and angular acceleration, just like force and acceleration as in Newton's second law of motion.

4.5.1 For a Point Mass

Consider a mass 'm' rotating at a distance 'r' from the axis of rotation, as shown in Fig. 4.20. The force 'F' acting on the mass to rotate it is a tangential force. By definition of torque, we have:

$$\tau = r F \sin\theta\, \hat{n} \quad \text{\_\_\_\_\_\_} \quad (4.44)$$

Since $\theta = 90°$ and $\sin 90° = 1$, magnitude of equation 1 will become: $\tau = rF$ ______ (4.45)

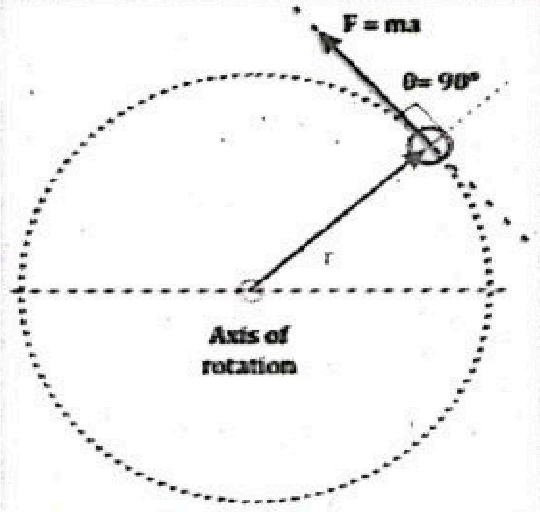


Figure 4.20: Mass 'm' rotating at distance 'r' from axis of rotation.

According to Newton's second Law, $F = ma$ ______ (4.46)

The relation between tangential and angular acceleration is given by:

$$a = r\alpha \quad \text{\_\_\_\_\_\_} \quad (4.47)$$

Putting equation (4.47) in equation (4.46), we get:

$$F = mr\alpha \quad \_\_\_\_\_ \quad (4.48)$$

Putting equation (4.48) in equation (4.45), we get:

$$\tau = r(mr\alpha) \quad \text{or} \quad \tau = mr^2\alpha \_\_\_\_\_ \quad (4.49)$$

Since, the term mr^2 is moment of inertia, therefore,

$$\tau = I\alpha \quad \_\_\_\_\_ \quad (4.50)$$

Equations (4.50) states that torque is moment of inertia times angular acceleration. This statement is similar to Newton's second law of motion F = ma, which gives force as equal to inertia (mass) times acceleration.

4.6 WEIGHTLESSNESS IN SATELLITES

Weightlessness occurs when the feeling of weight is completely or almost completely absent, meaning there is zero apparent weight. This occurs during free-fall, when the force of gravity is balanced out by the inertial force from orbital flight, like centrifugal force.

In a weightless environment, objects and individuals float freely, as there is no force pulling them towards the ground. This phenomenon is commonly experienced by astronauts in space, where they appear to be float inside their spacecraft with objects around them.

The term zero gravity is often used incorrectly to describe weightlessness, as astronauts in space stations are not in gravity free environment high above the Earth, 250 miles out in space, where most space stations orbit, the gravitational field is still quite strong there roughly 95% of its at on surface of the Earth. Weightlessness can be achieved in two ways. One that to travel millions of miles from gravitational force of large object, where the gravitational force reduces to nearly zero. Or the second and much more practical is to create weightless environment through act of free fall. The space stations are in constant free fall, having the right speed and at right altitude. Inside the space station the astronaut is also falling free, so they appear to float, as shown in Fig. 4.21, and physicists call it weightlessness.

Figure 4.21: Weightlessness in satellites.

Weightlessness can be enjoyed in amusement parks momentarily; it can also be simulated on Earth through techniques such as parabolic flights or neutral buoyancy tanks, allowing researchers to study the effects of microgravity on the human body and various materials. Living in a space station is not easy, besides the dangers of space travel and time spent away from family in isolation, astronauts feel many health issues related to microgravity. Their bones and muscles get weakened, cardiovascular system is affected and immune system is compromised.

Apart from all these health challanges some everyday activities become near-impossible. Their basic necessities like eating, sleeping, and showering habits are modified. They even can't cry; they face difficulty in digesting food and even in urination and excretion. Rotational simulated gravity has been proposed as a solution in human spaceflight to the adverse health effects caused by prolonged weightlessness.

Even though things and people may feel weightless in a weightless setting, they still possess mass and inertia, causing them to maintain their straight-line motion unless an outside force intervenes.

4.7 ARTIFICIAL GRAVITY

The gravity produced artificially in the satellites to counteract the effect of weightlessness is called artificial gravity.

It can be generated by rotating a space-station around its own axis, as shown in Fig. 4.22. The surface of the rotating space station exerts a force on objects with in contact with it and thereby provides the centripetal force that keeps the object moving on a circular path. In space stations, the astronauts feel weightless and cannot work effectively. In order to overcome this difficulty artificial gravity can be provided by rotating it about its own axis.

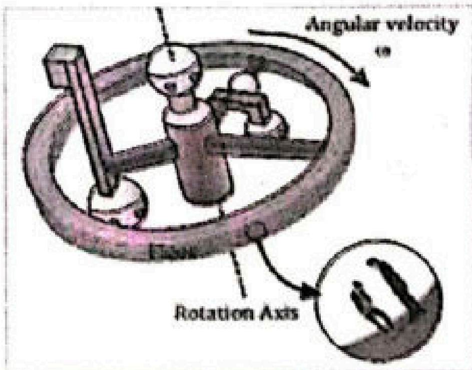


Figure 4.22: Artificial gravity.

To describe artificial gravity, consider a circular tube shaped part of the space station in which artificial gravity will be provided to the occupants of the space station. Let it have the radius 'R' and rotate with velocity 'v' as shown in the Fig. 4.21. The centripetal acceleration experienced at any point on the outer rim is:

$$a_c = \frac{v^2}{R} \qquad \text{\_\_\_\_\_\_ (4.52)}$$

Linear Velocity: From equation (4.52): $v^2 = a_c R$

Therefore, $v = \sqrt{a_c R}$

To provide the same force as the force of gravity this centripetal acceleration, and hence centripetal force, must be equal to the acceleration due to gravity i.e. $a_c = g$

Hence $$v = \sqrt{gR} \qquad \text{\_\_\_\_\_\_ (4.53)}$$

Angular Velocity of Satellites: The relation between linear and angular velocity is

$$v = \omega R \qquad \text{\_\_\_\_\_\_ (4.54)}$$

By comparing equations (4.53) and (4.54), we get:

$$\omega R = \sqrt{gR} \qquad \text{or} \qquad \omega = \frac{\sqrt{gR}}{R} = \sqrt{\frac{gR}{R^2}}$$

Therefore, $\omega = \sqrt{\frac{g}{R}}$ ______ (4.55)

Time Period of Satellites: Time period is the time required for the satellite to complete one rotation, i.e., $T = \frac{2\pi R}{v}$

Since $v = \omega R$, therefore: $T = \frac{2\pi R}{\omega R}$

or $T = \frac{2\pi}{\omega}$ ______ (4.56)

Putting the values from equation (4.55) in equation (4.56), we get:

$$T = 2\pi\sqrt{\frac{R}{g}} \quad \text{\_\_\_\_\_\_ (4.57)}$$

Frequency of Satellites: Since frequency is the reciprocal of time period, i.e., $f = \frac{1}{T}$, so, from equation (4.57) we get:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{R}} \quad \text{\_\_\_\_\_\_ (4.57)}$$

EXAMPLE 4.4: An 80.0 kg astronaut stands on the rim of rotating ring-shaped space station providing him sufficient artificial gravity g = 9.8 m s^{-2}. If the radius of the space station is 1.5 km. Calculate his (a) angular velocity, (b) time period and (c) frequency of rotation.

Given: Mass of astronaut = m = 80.0 kg   Radius of space ship = R = 1.50 km = 1500 m

To Find: (a) Angular velocity 'ω' =?   (b) Time period 'T' = ?   (c) Frequency 'f' = ?

SOLUTION: (a) The angular velocity for artificial satellite is:

$$\omega = \sqrt{\frac{g}{R}}$$

Putting values, we get: $\omega = \sqrt{\frac{9.8}{1500}}$ or $\omega = 0.08\,rad\,s^{-1}$

(b) The time period for artificial satellite is:

$$T = 2\pi\sqrt{\frac{R}{g}}$$

Putting values, we get: $T = 2 \times 3.14\sqrt{\frac{1500}{9.8}}$ or $T = 77.73\,s$

(c) Since frequency is the reciprocal of time period $f = \frac{1}{T}$

Putting values, we get: $f = \frac{1}{77.73}$ or $f = 0.013\,Hz$

Assignment 4.4

A space ship, having cylindrical shape, is rotated at a speed of 20 rpm about its axis in order to provide artificial gravity to its inhabitants. If the spaceship has a diameter of 8 m, find the artificial gravity it provides.

SUMMARY

- **Angular velocity:** The rate at which an object changes the angle while moving on a circular path.
- **Tangential acceleration:** The acceleration in a direction tangent to the circle at the point of interest in circular motion.
- **Angular acceleration:** The rate of change of angular velocity with respect to time.
- **Centripetal acceleration:** The acceleration of an object moving in a circle, directed toward the center.
- **Centripetal force:** Any net force causing uniform circular motion.
- **Moment of inertia:** Mass times the square of perpendicular distance from the rotation axis; for a point mass, it is $I = mr^2$ and, because any object can be built up from a collection of point masses, this relationship is the basis for all other moments of inertia.
- **Torque:** The turning effectiveness of a force and is defined as product of moment of inertia and angular acceleration ($\tau = I\alpha$).
- **Angular momentum:** The product of moment of inertia and angular velocity ($L = I\omega$).
- Angular momentum is conserved, i.e., the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system.

EXERCISE

Multiple Choice Questions

Encircle the correct option.

1) The term "centrifugation" means separation

A. through spinning
B. of components at higher temperature
C. through evaporation
D. of components at lower temperature

2) A car turns around a curve at 30 km h^{-1}. If it turns at double the speed, the tendency to overturn is:

A. doubled
B. quadrupled
C. halved
D. unchanged

3) The moment of inertia of a spinning body about a certain axis, doesn't depend on:

A. distribution of mass around the axis
B. orientation of the axis
C. mass of the body
D. angular velocity of the body

4) The change in angular momentum of a rod, when a torque of 2.5 N m is acted upon it for 2 s, is:

A. 1.25 J s
B. 2.5 J s
C. 5 J s
D. 0 J s

5) If size (length) of the wings of a fan is increased, its rotational speed, for the same voltage and current, will:

A. increase B. decrease C. remain constant D. may increase or decrease

6) In a body, angular acceleration is produced by:

A. net force B. power C. pressure D. net torque

7) An astronaut feels weightless inside the International Space Station. It is because the International Space Station is:

A. outside the gravitational field of Earth B. freely falling

C. at rest D. in motion

8) How many radians account for circumference of a circle?

A. 1 rad B. 2 rad C. π rad D. 2π rad

Short Questions

Give short answers of the following questions.

4.1 What is the value of angular acceleration of the minute hand of your wrist watch?

4.2 Is there a real force that removes water from wet clothes in a washing machine? Explain how the water is removed.

4.3 Determine the relation between (a) linear and angular displacement. (b) linear and angular velocity. (c) linear and angular acceleration.

4.4 Is centripetal force a fundamental force or a force provided by any of the fundamental forces? Can any combination of the fundamental forces provide centripetal force?

4.5 There are generally double tyres in heavy vehicles on one side of an axle. Will its moment of inertia be different from that of a single tyre?

4.6 Why is it best to have the blades rotate in opposite directions for a helicopter having two sets of lifting blades?

4.7 If diameter of Earth becomes half and there is no change in its mass, what affect will be there on the rotational speed of Earth around its own axis?

4.8 Why does in circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction?

4.9 Why does usually the value of artificial gravity is smaller than $9.8\ m\ s^{-2}$?

4.10 Why is a gyroscope used in aeroplanes?

4.11 How does the rotation of a flywheel helps to even out the power delivery from the engine?

4.12 A wall clock's arms show time as 09:15. Express the angle between the arms in radians.

Comprehensive Questions

Answer the following questions in detail.

4.1 What is centripetal force? Explain. Write down at least two applications where centripetal force plays its role.

4.2 What is moment of inertia? Derive its relation for rigid body.

4.3 Derive the expression for angular momentum of a body. Also deduce the relation between angular momentum and torque.

4.4 Explain conservation of angular momentum using practical life examples.

4.5 Derive the relation between torque and angular acceleration.

4.6 Justify how a centrifuge is used to separate materials using centripetal force.
4.7 Explain why the objects in orbiting satellites appear to be weightless.
4.8 Describe how artificial gravity be produced in a satellite to counter weightlessness.
4.9 Analyse motion in curved path due to perpendicular forces.

Numerical Problems

4.1 What will be the angular velocity of fly wheel of an engine if it completes 3000 revolutions in a minute? **(Ans. 314 rad s^{-1})**

4.2 A car is passing through a turn that is in the form of an arc of a circle of radius 14.5 m. What will be the maximum speed limit (the speed at which the car can cross the bridge without losing contact with the road) if the centre of gravity of the car is 0.5 m from the ground? **(Ans. 12.1 m s^{-1})**

4.3 A PT teacher rotates his stick at the axis that passes through its centre. If mass of the stick is 200 g and its length is 0.8 m, find its moment of inertia. **(Ans. 0.01 kg m^2)**

4.4 A football of mass 450 g rotates with an angular speed of 10 rev s^{-1}. If its radius is 11 cm, compute its angular momentum. **(Ans. 0.137 J s)**

4.5 A merryman in a circus is standing with his arms extended on a turn table rotating with angular velocity 10 rad/s. He brings his arms closer to his body so that his moment of inertia is reduced to one third of the initial value. Find his new angular velocity. **(Ans. 30 rad s^{-1})**

4.6 A boy exerts a force of 200 N at the edge of the 30.0 kg merry-go-round, which has a 2.0 m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when the boy, having 20.0 kg weight, sits 1.5 m away from the center. (ignore friction). **(Ans. 6.67 rad s^{-2}, 3.8 rad s^{-2})**

4.7 A wheel-shape space station provides an artificial gravity of 5.00 m s^{-2} to its inhabitants. If it has a diameter of 100 m, find its angular speed in rpm. **(Ans. 3 rpm)**

4.8 The minute hand of a watch is 2 cm long. If it travels 9.4 cm of length, how many radians will it travel? **(Ans. 4.7 rad)**

4.9 How many revolutions does the fidget spinner make after being flicked with an initial angular velocity of 10 revolutions per second and coming to rest in 5 seconds? *(ignore air resistance)* **(Ans. 25 revolutions)**