VECTORS

UNIT 2

Student Learning Outcomes (SLOs)

The students will

- Represent a vector in 2-D as two perpendicular components.
- Describe the product of two vectors (dot and cross-product) along with their properties.

In our daily life, we used to deal with physical quantities, such as mass, time, length, velocity, acceleration, force, work and torque etc. Some physical quantities, known as scalars, can be completely understood by a number and a proper unit only. On the other hand, some physical quantities require direction along with a number and a unit; such quantities are known as vectors. Displacement, force, acceleration, torque and momentum etc., are the examples of vector quantities. Vectors are very useful in our daily life. For example, a signboard provides information about distances and directions of other places relative to the location of the signboard. Distance is a scalar quantity. Knowing the distance alone is not enough to get to the desired location. We must also know the direction from the signboard to the destination. The direction along with the distance, is a vector quantity called the displacement. Therefore, signboards give information about displacement from the signboard to the destination.

Graphically, a vector can be completely expressed by a line with an arrow head. Length of the line represents the magnitude according to the proper scale and the arrow head represents the direction. We have studied about the addition of vectors in grade 9; here we shall study about the components of vectors and about the products of vectors.

2.1 VECTOR RESOLUTION

Components are the effective parts of a vector in different directions. In 2-D, we take x-component and y-component. The component along x-axis is called the horizontal or the x-component and the component along the y-axis is called the vertical or the y-component. Both the x-component and y-component are perpendicular to each other and are called rectangular components.

The components of a vector which are mutually perpendicular are called rectangular components.

Consider a vector 'A' making an angle 'θ' with the horizontal, as shown in the Fig. 2.1. Perpendiculars are drawn from the head of vector A on the x-axis and y-axis. The projection OQ of vector A along x-axis is x-component of vector A and is represented by A_x. The projection OS of vector A along y-axis is y-component of vector A and is represented by A_y. So, the vector A in its component form can be written as:

$$\mathbf{A} = \mathbf{A_x} + \mathbf{A_y} \qquad \text{\_\_\_\_\_\_ (2.1)}$$

OR

$$\mathbf{A} = A_x\,\hat{i} + A_y\,\hat{j} \qquad \text{\_\_\_\_\_\_ (2.2)}$$

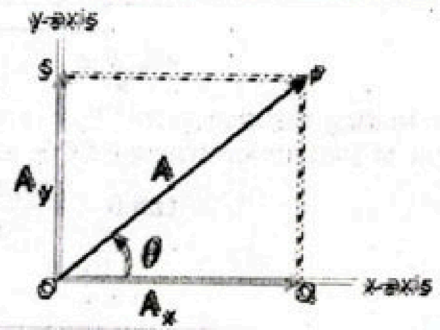

Figure 2.1: Rectangular components of a vector.

Here, $\hat{i}$ and $\hat{j}$ are representing directions along +x-axis and +y-axis, respectively, and are called unit vectors. Unit vectors specify the direction of any vector and has magnitude equal to one.

2.1.1 Finding Magnitude of Rectangular Components

In order to find the magnitude of rectangular components A_x and A_y of a vector **A**, we apply the trigonometric ratios. For this, consider the triangle OPQ, as shown in Fig. 2.1. As:

$$\sin\theta = \frac{perpendicular}{hypotenuse} = \frac{\overline{QP}}{\overline{OP}} = \frac{\overline{OS}}{\overline{OP}}$$

or $$\sin\theta = \frac{A_y}{A}$$

or $$A_y = A\sin\theta \qquad \text{\_\_\_\_\_ (2.3)}$$

and,

$$\cos\theta = \frac{base}{hypotenuse} = \frac{\overline{OQ}}{\overline{OP}}$$

or $$\cos\theta = \frac{A_x}{A}$$

or $$A_x = A\cos\theta \qquad \text{\_\_\_\_\_ (2.4)}$$

From equation (2.3) and (2.4), we can find the magnitude of perpendicular components A_x and A_y.

2.1.2 Finding a Vector from its Components

If the rectangular components A_x and A_y of a vector **A** are given, then its magnitude 'A' and direction 'θ' can be found. Again, we consider the triangle OPQ, as shown in Fig. 2.1. By applying Pythagoras theorem, we get:

$$(OP)^2 = (OQ)^2 + (QP)^2 \qquad \text{\_\_\_\_\_ (2.5)}$$

As QP = OS, so equation (2.5) can be written as:

$$(OP)^2 = (OQ)^2 + (OS)^2 \qquad \text{\_\_\_\_\_ (2.6)}$$

Putting OP = A, OQ = A_x and OS = A_y, in equation (2.6), we get:

$$A^2 = (A_x)^2 + (A_y)^2$$

or $$A = \sqrt{(A_x)^2 + (A_y)^2} \qquad \text{\_\_\_\_\_ (2.7)}$$

For Your Information

A vector which specifies the location of a point P (a, b) with respect to origin is called position vector, and can be given as: $r = a\hat{i} + b\hat{j}$ and its magnitude is given as: $r = \sqrt{a^2 + b^2}$.

For finding the direction 'θ' of a vector, we apply trigonometric ratio of tangent on triangle OPQ, as:

$$\tan\theta = \frac{perpendicular}{base}$$

or $$\tan\theta = \frac{A_y}{A_x}$$

or $$\theta = \tan^{-1}\frac{A_y}{A_x} \qquad \text{\_\_\_\_\_ (2.8)}$$

By using equations (2.7) and (2.8) , we can find the magnitude and direction of a vector.

Example 2.1: A peddler is pushing a trolley on a horizontal road with a force of 50 N making an angle 30° with the road. Find the horizontal and vertical components of the force.

Given: Magnitude of force = F = 50 N

Angle of the force with horizontal = θ = 30°

To Find: F_x = ? F_y = ?

Solution: To find x-component of the force F, we use the equation:

$$F_x = F\cos\theta$$

Putting values, we get:

$$F_x = 50\cos 30^\circ$$

or, $F_x = 43.3\ \text{N}$

To find y-component of the force F, we use the equation:

$$F_y = F\sin\theta$$

Putting values, we get:

$$F_y = 50\sin 30^\circ$$

or, $F_y = 25\ \text{N}$

Assignment 2.1

Fatima is pulling her trolley bag while climbing up the ramp at her school gate. Find the force with which she is pulling her bag, if the x-component and y-component of her force are 12 N and 5 N, respectively.

2.2 PRODUCT OF VECTORS

When two vector quantities are multiplied, then the product may be a scalar quantity or a vector quantity. The product obtained depends upon the nature of the given vectors. Let us discuss the two types of products in the following sections.

2.2.1 Scalar Product or Dot Product:

When the product of two vector quantities gives a scalar quantity then the product is called scalar product.

The scalar product of two vectors is represented by putting a dot (·) between the symbols of the two vectors; therefore, it is also known as dot product.

Let us take two vectors **A** and **B** that are making an angle θ with each other, as shown in the Fig. 2.2. The dot product between these two vectors can be denoted by **A.B** and is defined as:

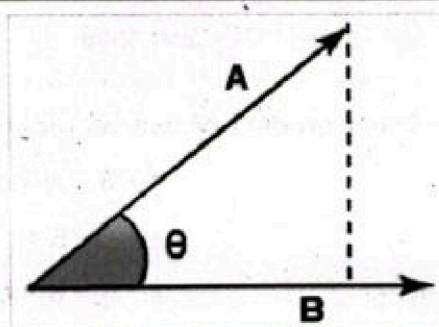

Figure 2.2: Two vectors **A** and **B** that are making an angle θ with each other.

$$\mathbf{A}\cdot\mathbf{B} = A\,B\cos\theta \quad \text{\_\_\_\_\_\_ (2.9)}$$

Here, A and B are the magnitudes of the vectors **A** and **B**. Thus, the scalar product of two vectors is obtained by multiplying their magnitudes with the cosine of the angle between them.

Equation (2.9) can also be written as:

$$\mathbf{A}\cdot\mathbf{B} = (A\cos\theta)\,B \quad \text{\_\_\_\_\_\_ (2.10)}$$

It is shown in the Fig. 2.3 that (A cos θ) is the component of vector **A** in the direction of vector **B**. So, equation (2.10) can also be written as:

$$\mathbf{A}\cdot\mathbf{B} = (\text{component of vector } \mathbf{A} \text{ along } \mathbf{B})\,B \quad \text{\_\_\_\_\_\_ (2.11)}$$

Similarly, it can be shown that:

$\mathbf{A} \cdot \mathbf{B}$ = A (component of vector **B** along **A**) ______ (2.12)

Hence, from equation (2.11) and (2.12), we can also define scalar product as:

The product of magnitude of either vector with the component of other vector in the direction of first vector.

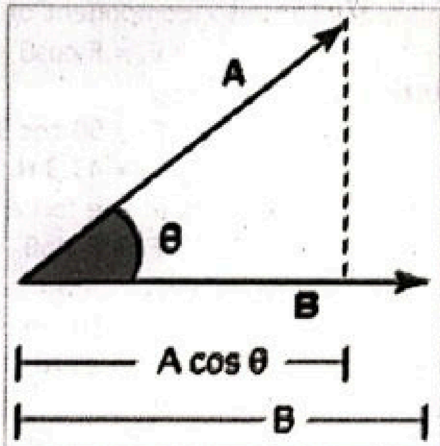

Figure 2.3: component of vector **A** in the direction of vector **B**.

Examples of Scalar Product:

- Work: Work (W) is an example of scalar product. Work is a scalar quantity which is the scalar product of displacement **d** and force **F**. Mathematically, it can be written as:

  $W = \mathbf{F} \cdot \mathbf{d}$

  or $W = F\, d \cos\theta$

- Power: Power (P) is also an example of scalar product. It is equal to the scalar product of force **F** and velocity **v**. Mathematically; it can be written as:

  $P = \mathbf{F} \cdot \mathbf{v}$

  or $W = F\, v \cos\theta$

- Electric Flux: Electric flux (Φ) is a scalar quantity which is the scalar product of electric field intensity **E** and vector area **A**. Mathematically, it can be written as:

  $\Phi = \mathbf{E} \cdot \mathbf{A}$

  or $W = E\, A \cos\theta$

Properties of Scalar Product:

1) Scalar product of two vectors obeys the commutative property, i.e.,

$$\mathbf{A} \cdot \mathbf{B} = A\, B \cos\theta$$

$$= B\, A \cos\theta = \mathbf{B} \cdot \mathbf{A}$$

Hence, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

2) Scalar product of two orthogonal (perpendicular) vectors is equal to zero, i.e.,

$$\mathbf{A} \cdot \mathbf{B} = A\, B \cos 90^\circ = 0$$

Similarly, for mutually perpendicular unit vectors, we can show:

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$$

Note that the dot product of a vector with the null vector is also zero.

3) Scalar product of two parallel vectors has maximum value, and is equal to the product of their magnitudes only. Hence:

$$\mathbf{A} \cdot \mathbf{B} = A\, B \cos 0^\circ = AB$$

Scalar product of a vector with itself is equal to the square of its magnitude, i.e.,

$$\mathbf{A} \cdot \mathbf{A} = A\, A \cos 0^\circ = A^2$$

Similarly, we can show that the dot product of a unit vector with itself is unity, i.e.,

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

4) Scalar product of two antiparallel vectors is negative, i.e.,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos 180^\circ = -AB$$

5) Scalar product of two vectors in terms of their rectangular components can be given as:

$$\mathbf{A}\cdot\mathbf{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

OR $$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \qquad \text{\_\_\_\_\_ (2.13)}$$

As, $$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB}$$

So, in components form: $$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} \qquad \text{\_\_\_\_\_ (2.14)}$$

Example 2.2: Find the power delivered by the engine for attaining velocity (3, 4) m s^{-1}, while it exerts a force (8, -2) N.

Given: Velocity = $\mathbf{v} = (3\hat{i} + 4\hat{j})$ m s^{-1} Force = $\mathbf{F} = (8\hat{i} - 2\hat{j})$ N

To Find: Power Delivered = P = ?

Solution: As we know that power is the scalar product of force and velocity hence:

$$P = \mathbf{F}\cdot\mathbf{v}$$

$$P = (3\hat{i} + 4\hat{j})\cdot(8\hat{i} - 2\hat{j})$$

$$P = 24 - 8 = 16\text{ W}$$

Assignment 2.2

If vectors A= 5î + ĵ and B = 2î + 4ĵ, then find:
(i) projection of A on B. (ii) projection of B on A.

2.2.2 Vector Product or Cross Product:

When the product of two vector quantities gives a vector quantity then the product is called vector product.

vector product of two vectors is represented by putting a cross (x) between the symbols of the two vectors, therefore it is also known as cross product.

Let us take two vectors **A** and **B** that are making an angle θ with each other, as shown in the Fig. 2.4. The cross product between these two vectors can be denoted by **A×B** and is defined as:

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{n} \qquad \text{\_\_\_\_\_ (2.15)}$$

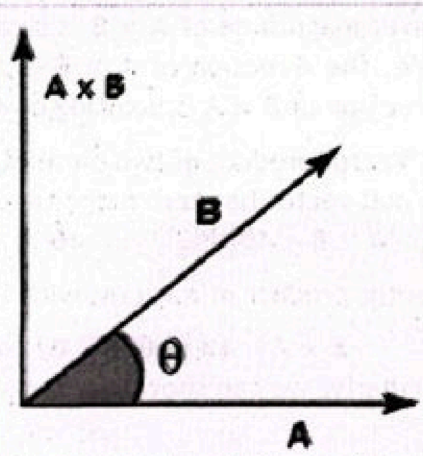

Figure 2.4: Cross product of two vectors **A** and **B** is shown by A×B.

Here, A and B are the magnitudes of the vectors **A** and **B**. Thus, the vector product of two vectors is obtained by multiplying their magnitudes with the sine of the angle between them.

In equation (2.15), $\hat{n}$ represents the direction of the vector product, which is always perpendicular to the plane containing **A** and **B**. The direction of cross product ($\hat{n}$) can be found by right hand rule of vector product, as shown in Fig. 2.5. It can be stated as:

Rotate the fingers of your right hand in the direction from first vector to the second vector through smaller angle of the two possible angles with erect thumb. The direction of the product will be along the direction of erect thumb.

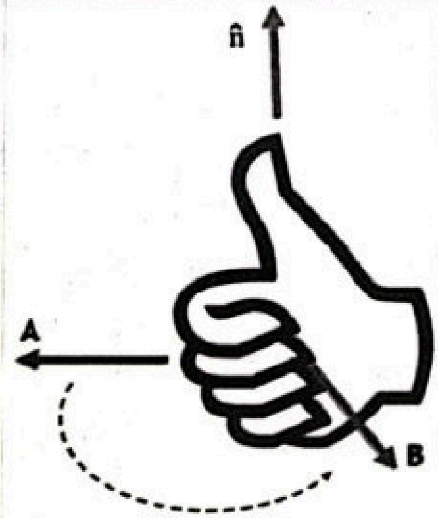

Figure 2.5: Right hand rule.

Examples of Vector Product:

- Torque (τ) is an example of vector product. It is the cross product of force (**F**) and position vector (**r**).

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = r F \sin\theta \, \hat{n}$$

- Angular momentum (**L**) is also an example of vector product. It is the cross product of linear momentum **P** and position vector **r**, i.e.,

$$\mathbf{L} = \mathbf{r} \times \mathbf{P} = r p \sin\theta \, \hat{n}$$

Properties of Vector Product:

1) Vector product is not commutative but anti-commutative. It means that by changing the order of vectors, the direction of vector product gets reversed, as shown in Fig. 2.6. i.e.,

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \qquad \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

Here, magnitude of $\mathbf{A} \times \mathbf{B}$ is same as $\mathbf{B} \times \mathbf{A}$. According to right hand rule, the direction of $\mathbf{A} \times \mathbf{B}$ is pointing upward given by $\hat{n}$ and the direction of $\mathbf{B} \times \mathbf{A}$ is pointing downward given by $-\hat{n}$.

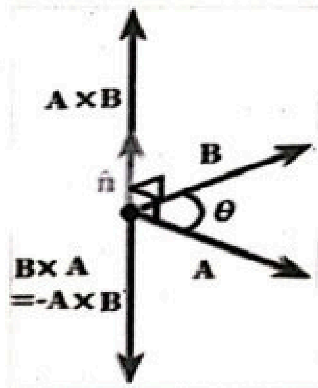

Figure 2.6: By changing the order of vectors, the direction of vector product gets reversed.

2) Vector product of two parallel or anti-parallel vectors is null vector (a null vector has zero magnitude and arbitrary direction). i.e.,

$$\mathbf{A} \times \mathbf{B} = AB \sin 0^\circ \, \hat{n} = 0 \quad \text{and} \quad \mathbf{A} \times \mathbf{B} = AB \sin 180^\circ \, \hat{n} = 0$$

Vector product of a vector with itself is equal to null vector. i.e.,

$$\mathbf{A} \times \mathbf{A} = AA \sin 0^\circ \, \hat{n} = 0$$

Similarly, we can show that the vector product of a unit vector with itself is null vector. i.e.,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

3) Vector product of two perpendicular vectors has a maximum value, and is equal to the product of their magnitudes only. Hence:

$$\mathbf{A} \times \mathbf{B} = A B \sin 90^0 \, \hat{n} = A B \, \hat{n}$$

Similarly, for mutually perpendicular unit vectors, we can show:

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

4) Vector product of two vectors in terms of their rectangular components can be given as:

$$\mathbf{A} \times \mathbf{B} = (A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}) \times (B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k})$$

By using the determinant and solving, we get:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{\_\_\_\_\_\_ (2.16)}$$

OR $$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\,\hat{\imath} + (A_zB_x - A_xB_z)\,\hat{\jmath} + (A_xB_y - A_yB_x)\,\hat{k} \quad \text{\_\_\_\_\_\_ (2.17)}$$

Physical Significance of Vector Product:

Vector product gives us the area of a plane. For example, if two vectors **A** and **B** represent adjacent sides of a plane, as shown in the Fig. 2.7, then the magnitude of **A** × **B** gives us the magnitude of that area (the area may be rectangle or parallelogram).

Area of parallelogram = |**A** × **B**|

The unit vector $\hat{n}$ gives the direction of that area, which is normal to the plane and can be found by right hand rule, as shown in Fig. 2.5.

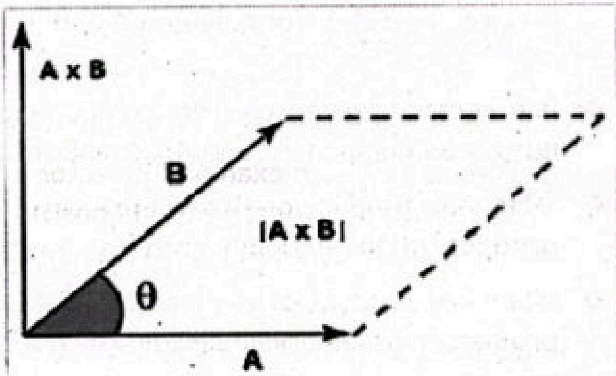

Figure 2.7: Magnitude of **A x B** gives area of parallelogram.

Example 2.3: Find the area of the parallelogram whose adjacent sides are given by the vectors: A = ($\hat{\imath}$ + 6 $\hat{\jmath}$ + 2 $\hat{k}$) m and B = (7 $\hat{\imath}$ + $\hat{\jmath}$ + 5 $\hat{k}$) m.

Given: **A** = ($\hat{\imath}$ + 6 $\hat{\jmath}$ + 2 $\hat{k}$) m   **B** = (7 $\hat{\imath}$ + $\hat{\jmath}$ + 5 $\hat{k}$) m

To Find: **A** × **B** = ?

Solution: As we know that vector product gives us the area of parallelogram. So,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 6 & 2 \\ 7 & 1 & 5 \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} = \hat{\imath}\begin{vmatrix} 6 & 2 \\ 1 & 5 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 2 \\ 7 & 5 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 6 \\ 7 & 1 \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} = 28\,\hat{\imath} + 9\,\hat{\jmath} - 41\,\hat{k}\ \ m^2$$

Assignment 2.3

Two vectors A and B having magnitudes 3.2 unit and 5.2 unit respectively, making an angle of 60° with each other. What is the magnitude of their cross products?

SUMMARY

- **Vectors** are those quantities, which require direction along with their magnitude for complete description.
- A vector with unit magnitude is called **unit vector**, it is required for the information of direction of any vector.

VECTORS

- A vector with zero magnitude and arbitrary direction is called **null vector.**
- A vector which specifies the location of a point P (a, b) with respect to origin is called **position vector**, and can be given as: $r = a\,\hat{\imath} + b\,\hat{\jmath}$ and its magnitude is given as: $r = \sqrt{a^2 + b^2}$.
- The components of a vector which are mutually perpendicular are called **rectangular components** of that vector.
- A vector can be found, if its rectangular components are given by relations: $A = \sqrt{(A_x)^2 + (A_y)^2}$ for magnitude and $\theta = tan^{-1}\frac{A_y}{A_x}$ for direction.
- If a vector is given then its components can be found by the relation, $A_x = A\cos\theta$ for horizontal component and $A_y = A\sin\theta$ for vertical component.
- When the product of two vectors gives us scalar quantity, such product is called **scalar product,** mathematically given as: $A \cdot B = A\,B\cos\theta$.
- When the product of two vectors gives us vector quantity, such product is called **vector product,** mathematically given as: $A \times B = A\,B\sin\theta\,\hat{n}$.

EXERCISE

Multiple Choice Questions

Encircle the correct option.

1) The number of perpendicular components of a force (in 2-D) are:

A. 1 B. 2 C. 3 D. 4

2) $\hat{\jmath} \times \hat{\imath} =$ _____

A. 0 B. 1 C. $\hat{k}$ D. $-\hat{k}$

3) A force of 10 N is making an angle of 30° with the horizontal. Its x-component will be:

A. 4 N B. 5 N C. 7 N D. 8.7 N

4) If two forces of magnitude 3 N and 4 N are acting at right angle to each other than their resultant force will be:

A. 7 N B. 5 N C. 1 N D. Null vector

5) Angle between two vectors A and B can be easily determined by:

A. dot product B. cross product C. head to tail rule D. right hand rule

6) For which angle the equation $|A \cdot B| = |A \times B|$ is correct?

A. 30° B. 45° C. 60° D. 90°

Short Questions

Give short answers of the following questions.

2.1 If the cross product of two vectors vanishes, what will you say about their orientation?

2.2 Find the dot product of unit vectors with each other at (a) 0° and (b) 90°.

2.3 Show that scalar product obeys commutative property.

2.4 Solve by using the properties of dot and cross product: (a) $\hat{\imath} \cdot (\hat{\jmath} \times \hat{k})$ (b) $\hat{\jmath} \times (\hat{\jmath} \times \hat{k})$?

2.5 If both the dot product and the cross product of two vectors are zero. What would you conclude about the individual vectors?

2.6 What are rectangular components of a vector? How they can be found?

2.7 Give two examples for each of the scalar and vector product.

2.8 Show that: $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$.

2.9 What units are associated with the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$?

Comprehensive Questions

Answer the following questions in detail.

2.1 Explain the resolution of a vector in 2-D as two rectangular components.

2.2 Describe the dot product of two vectors along with their properties.

2.3 Describe the cross product of two vectors along with their properties.

2.4 What geometric interpretation does the cross product have? Explain with the help of a diagram.

Numerical Problems

2.1 If the magnitude of cross product between two vectors is $\sqrt{3}$ times the dot product, find angle between them. **(Ans: 60°)**

2.2 A force is acting on a body making an angle of 30° with the horizontal. The horizontal component of the force is 20 N. Find the force. **(Ans: 23.1 N)**

2.3 A vector **F** having magnitude 5.5 N makes 10° with x-axis and a vector **r** with magnitude 4.3 m makes 80° with x-axis. What is the magnitude of their dot and cross products? **(Ans: 8.1 N m and 22.2 N m)**

2.4 The magnitude of dot and cross product of two vectors are $6\sqrt{3}$ and 6, respectively. Find the angle between the vectors. **(Ans: θ = 30°)**

2.5 A force of 200 N is making an angle of 30° with the x-axis. Find the horizontal and vertical components of this force. **(Ans: 173.2 N and 100 N)**