08

Application of Trigonometry

After studying this unit students will be able to:

- Extend sine and cosine functions to angles between 90° and 180°.
- Solve problems using laws of sines, cosines and the area formula for any triangle.
- Solve simple trigonometric problems in three dimensions.
- Apply concepts of trigonometry to real life world problems such as video games, flight engineering, navigation and sound waves.

Burj Khalifa at Dubai is the tallest building in the world with tallest free-standing structure and highest number of stories. It is 828 metres (2,716.5 feet) high and consists of more than 160 stories. How do scientists know its true height? A common way to measure the height involves determining the angle of elevation, which is formed by the top of building and the ground at a point some distance away from the base of the building. This method is much more practical than climbing the building and dropping a very long tape measure or rope.

Trigonometric Ratios

Quadrantal Angles

In a coordinate plane the two axes divide the plane in four equivalent parts called quadrants. If terminal ray of an angle in standard position coincides with any axis, then it is called quadrantal angle. The measure of a quadrantal angle is multiple of 90°.

For example; 0°, 90°, 180°, 270°, 360°, 450°, ... are quadrantal angles.

In radian measure quadrantal angle is a multiple of $\frac{\pi}{2}$.

For example; $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \ldots$ are quadrantal angles.

Key Fact:

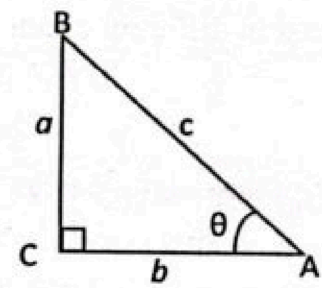
Two angles having same terminal ray in standard position, are called coterminal angles.

Trigonometric Ratios

The ratio between any two sides of a right-angled triangle is called trigonometric ratio.

In triangle ABC, $\angle C = 90°$ and $\angle A = \theta$.

With respect to acute angle θ, $BC = a$ is length of perpendicular, $AC = b$ is length of base and $AB = c$ is length of hypotenuse.



Various trigonometric ratios are defined as:

- The ratio of perpendicular and hypotenuse is sine of θ denoted by sin θ.

  $\therefore \sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a}{c}$

- The ratio of base and hypotenuse is cosine of θ denoted by cos θ.

  $\therefore \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{b}{c}$

- The ratio of perpendicular and base is tangent of θ denoted by tan θ.

  $\therefore \tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a}{b}$

- Furthermore, $\operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{c}{a} = \frac{1}{\sin\theta}$, $\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{c}{b} = \frac{1}{\cos\theta}$

  and $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{b}{a} = \frac{1}{\tan\theta}$

Trigonometric Ratios with the help of Unit Circle

A circle whose radius is 1 unit is called a unit circle.

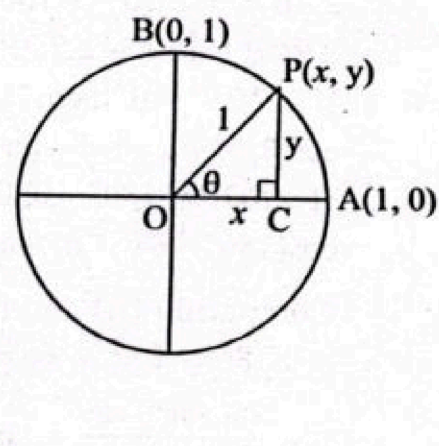
Consider a unit circle with centre O.

Take a point $P(x, y)$ on the circle. Join P with O.

Draw PC perpendicular to x-axis.

Let $OC = x$, $PC = y$ and $OP = 1$ (unit circle).

If $\angle COP = \theta$ where $0 < \theta < 180°$, then:



$\cos\theta = \frac{x}{1}$ and $\sin\theta = \frac{y}{1}$ ...... (i)

Other trigonometric ratios can be defined as:

$\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$

Now from (i), we have:

$x = \cos\theta$ and $y = \sin\theta$

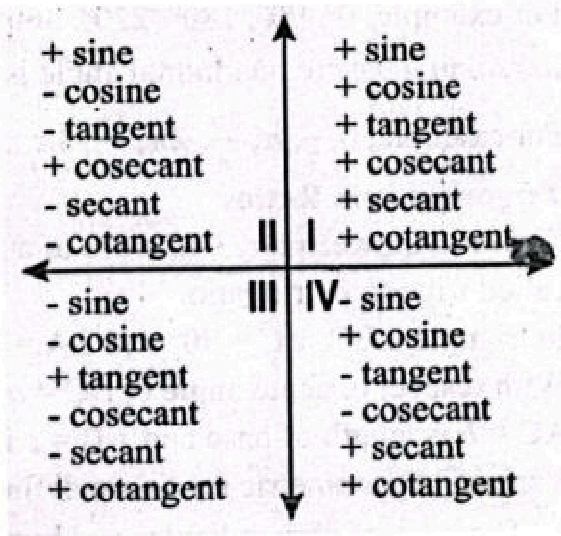
So, $P(x, y) = P(\cos\theta, \sin\theta)$.

$\therefore$ If P is a point on a unit circle with centre O and $\angle COP = \theta$, then coordinates of P are:

$$(\cos\theta, \sin\theta).$$

Signs of Trigonometric Ratios in Different Quadrants

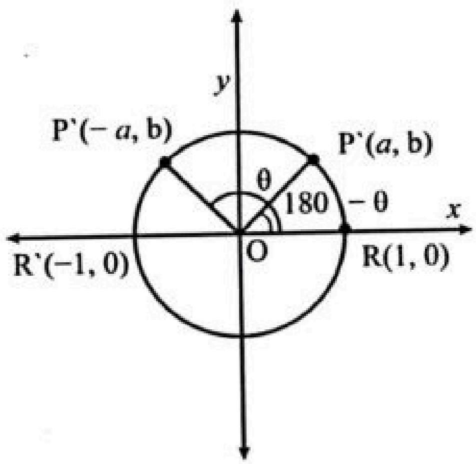
As $\cos\theta = x$ and $\sin\theta = y$ in the unit circle. Therefore, depending upon the signs of x and y in different quadrants, the signs of trigonometric ratios change. The adjoining figure shows signs of trigonometric ratios in different quadrants.



Trigonometric Ratios between 90° and 180°

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in second quadrant ($90^\circ < \theta < 180^\circ$)?

In the diagram, $R(1, 0)$ and $P(a, b)$ are points on the unit circle. Under a reflection in the y-axis, the image of $P(a, b)$ is $P'(-a, b)$ and the image of $R(1, 0)$ is $R'(-1, 0)$.



Since angle measure is preserved under a line reflection,

$\therefore \quad \angle ROP = \angle R'OP'$

Also $\angle ROP'$ and $\angle R'OP'$ are supplementary angles.

$\therefore \quad \angle ROP' + \angle R'OP' = 180^\circ \quad$ (i)

But, $\quad \angle R'OP' = \angle ROP$

Therefore, (i) becomes:

$$\angle ROP' + \angle ROP = 180^\circ$$

$$\angle ROP = 180^\circ - \angle ROP' = 180^\circ - \theta$$

Case 1: If $\angle ROP' = \theta$, then:

$\cos\theta = -a$ and $\sin\theta = b$ .... (ii) (using the properties of unit circle)

Case 2: If $\angle ROP = 180^\circ - \theta$, then:

$\cos(180^\circ - \theta) = a$ and $\sin(180^\circ - \theta) = b$ .... (iii)

Comparing (ii) and (iii), we get:

$\cos(180^\circ - \theta) = -\cos\theta$ and $\sin(180^\circ - \theta) = \sin\theta$

As, $\dfrac{\sin(180^\circ - \theta)}{\cos(180^\circ - \theta)} = \tan(180^\circ - \theta)$ and $\dfrac{\sin\theta}{-\cos\theta} = -\tan\theta$

$\therefore \tan(180^\circ - \theta) = -\tan\theta$

To find sine and cosine of angle lying between 90° and 180°, proceed as follows:

a. Measure the angle between the terminal side of the given angle and the horizontal axis. This is the reference angle.

b. Determine the values of the cosine and sine of the reference angle.

c. Give the cosine the same sign as the x-values in the quadrant of the original angle.

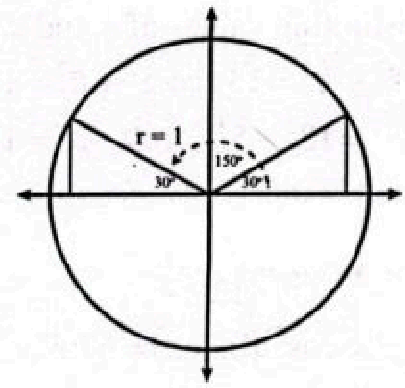
d. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example:

Find the exact values of sin 150°, cos 150°, and tan 150°.

Solution:

Using above relations, we have:



$$\sin 150^\circ = \sin(180^\circ - 150^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\cos 150^\circ = -\cos(180^\circ - 150^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 150^\circ = -\tan(180^\circ - 150^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

Key Fact:

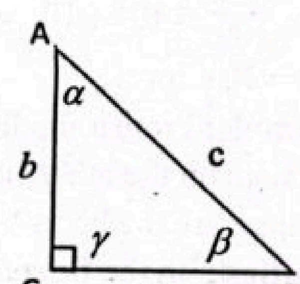
- Angles have cosines and sines with the same absolute value as their reference angles.
- The sign (positive or negative) can be determined from the quadrant of the angle.

Exercise 8.1

1. Find the reference angles of the following angles.
   (i) 125° (ii) 138° (iii) 111° (iv) 142°
2. If $\theta = 36^\circ$. Find equivalent angle α in the second quadrant for which $\sin\alpha = \sin\theta$.
3. If $\theta = 87^\circ$. Find equivalent angle β in the second quadrant for which $\cos\beta = -\cos\theta$.
4. Using a reference angle, find the exact values of sin 120°, cos 120° and tan 120°.
5. Using the reference angle, find $\cos\frac{3\pi}{4}$, $\sin\frac{3\pi}{4}$ and $\tan\frac{3\pi}{4}$.
6. If $\cos\theta = 0.559$, find the values of:
   (i) $\cos(180^\circ - \theta)$ (ii) $\sin(180^\circ - \theta)$ (iii) $\tan(180^\circ - \theta)$
   (iv) $\cot(180^\circ - \theta)$ (v) $\sec(180^\circ - \theta)$ (vi) $\mathrm{cosec}(180^\circ - \theta)$

Solution of Right Angled Triangles

We have studied about the solution of right triangles in the previous grade. We know that any triangle (in particular a right triangle) has six quantities, three sides and three angles. In the adjoining figure, a right triangle ABC contains three sides a, b, c and three angles α, β, γ. We can solve a right-angled triangle when its:



(i) Two sides are given,

(ii) One acute angle and one side are given.

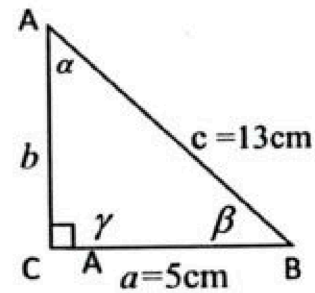
Let's solve some examples to discuss the two cases.

Case 1: When Measures of two Sides are Given

Example: Solve the right triangle ABC in which $\gamma=90^\circ, a=5cm, c=13cm.$

Solution: Given: A right triangle ABC in which $\gamma=90^\circ, a=5cm, c=13cm.$

To Find: b, α and β.



Using Pythagoras theorem, we have:

$$a^2+b^2=c^2$$

Substituting values of a and c, we get:

$$5^2+b^2=13^2 \Rightarrow 5^2+b^2=13^2$$

$$\Rightarrow b^2=13^2-5^2=169-25=144$$

$$\Rightarrow b=12cm$$

Now $\cos\beta=\dfrac{5}{13}=0.385$

$$\Rightarrow \beta=\cos^{-1}(0.385)=67.4^\circ$$

As $\alpha+\beta=90^\circ$

$$\Rightarrow \alpha=90^\circ-\beta=90^\circ-67.4^\circ=22.6^\circ$$

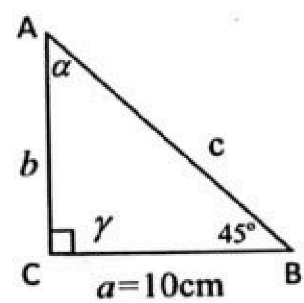
Case 2: When Measures of one Side and one Angle are Given

Example: Solve the right triangle ABC in which $\gamma=90^\circ$, $\beta=45^\circ$, $a=10cm.$

Solution: Given: A right triangle ABC in which $\gamma=90^\circ$, $\beta=45^\circ$, $a=10cm.$

To Find: b, c and α.

As $\alpha+\beta=90^\circ$



As $\alpha+\beta=90^\circ$

$$\Rightarrow \alpha=90^\circ-\beta=90^\circ-45^\circ=45^\circ$$

Now $\dfrac{b}{a}=\tan 45^\circ$

$$\Rightarrow b=a\times\tan 45^\circ=10\times 1=10cm$$

To find c, we use Pythagoras theorem as follows:

$$c^2=a^2+b^2$$

Substituting values of a and b, we get:

$$c^2=10^2+10^2=100+100=200$$

$$\Rightarrow c=10\sqrt{2}cm$$

Example: From a small plane Z the angle of depression of a boat is 25°. The angle of depression of a boat on the other side of the plane is 50°. The plane is flying at an altitude of 1500 m. How far apart are the boats?

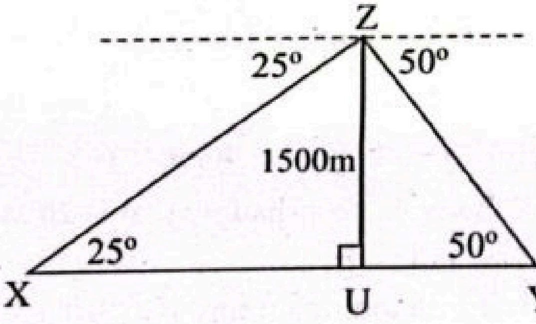
Solution: In the figure,

distance between the boats = XY = XU + UY

Using the properties of angles,

$\angle X = 25^\circ$ and $\angle Y = 50^\circ$

In right ΔXUZ,



$$\tan 25^\circ = \frac{ZU}{XU}$$

$$XU = \frac{ZU}{\tan 25^\circ} = \frac{1500}{0.466} = 3216.76m$$

In right ΔYUZ,

$$\tan 50^\circ = \frac{ZU}{YU}$$

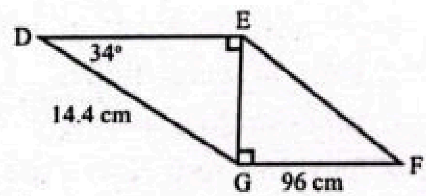
$$YU = \frac{ZU}{\tan 50^\circ} = \frac{1500}{1.192} = 1258.65m$$

$\therefore$ Distance between the boats = XY = XU + UY

= 3216.76 + 1258.65 = 4475.41 m

Example:

Find the measure of $\angle F$ in the adjoining figure.



Solution:

To find $\angle F$, we need measure of $\overline{GE}$.

In right ΔDEG,

$$\sin 34^\circ = \frac{GE}{GD}$$

$$GE = GD \times \sin 34^\circ = 14.4 \times 0.559 = 8.05cm$$

Now in right ΔEGF,

$$\sin(\angle F) = \frac{GE}{GD} = \frac{8.05}{9.6} = 0.839$$

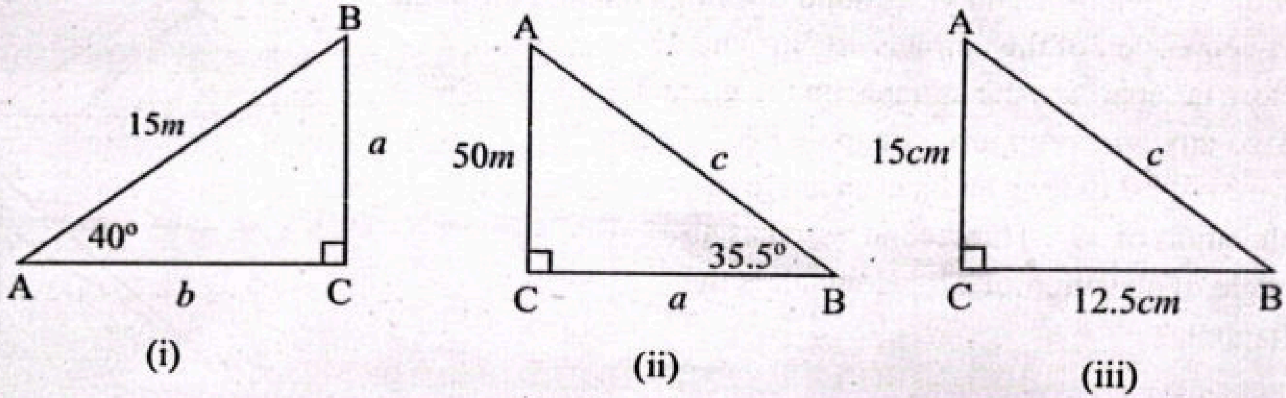
$$\angle F = \sin^{-1}(0.839) = 57^\circ 2'$$

Check Point:

Abbas is flying a kite to which the angle of elevation is 70°. The string of the kite is 65 meters long. How far is the kite above the ground?

Exercise 8.2

1. Solve the following right-angled triangles.

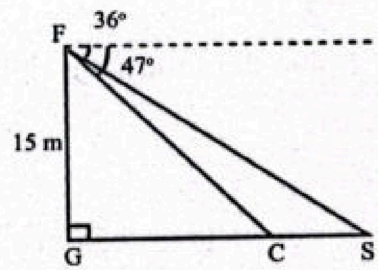


2. Solve right-angled triangles ABC in which $\gamma = 90^\circ$ and

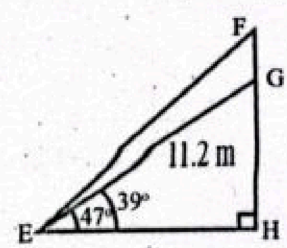
(i) $a = 12cm, \beta = 35^\circ$ (ii) $b = 30cm, \alpha = 25^\circ\ 35'$ (iii) $a = 50cm, b = 25cm$

(iv) $a = 30cm, c = 40cm$ (v) $a = 24cm, \alpha = 36^\circ\ 15'$ (vi) $b = 10cm, \alpha = 70.5^\circ$

3. A tower casts a shadow that is 20 *m* long when the angle of elevation of the sun is 65°. How tall is the tower?
4. Arif is standing on top of a cliff 105 *m* above a lake. The measurement of the angle of depression to a boat on the lake is 42°. How far is the boat from Arif?
5. A ladder that is 20 ft. long is leaning against the side of a building. If the angle formed between the ladder and the ground is 75°, how far is the bottom of the ladder from the base of the building?
6. Uzma is standing 50 meters from a hot air balloon that is preparing to take off. The angle of elevation to the top of the balloon is 28°. Find the height of the balloon.
7. A man is in a boat that is floating 175 feet from the base of a 200-foot cliff. What is the angle of depression between the cliff and the boat?
8. A flagpole casts a shadow 40 feet long when the measurement of the angle of elevation to the sun is 31°. How tall is the flagpole?
9. A straight waterslide is 175 feet above ground and is 200 feet long. What is the angle of depression to the bottom of the slide?
10. Zain wants to measure the height of a tree. He walks exactly 50 *m* from the base of the tree and looks up that the angle from the ground to the top of the tree is 33°. How tall is the tree?
11. From a 100*m* observation tower on the beach, a man sights a whale in difficulty. The angle of depression of the whale is 7°. How far is the whale from the shoreline?
12. Urooba is sittng on the ground midway between two trees, 100 *m* apart. The angles of elevation of the tops of the trees are 13° and 18°. How much taller is one tree than the other?
13. The angle of elevation of the top of a tree, T is 27°. From the same point on the ground, the angle of elevation of a hawk H, flying directly above the tree is 43°. The tree is 12.7m tall. How high is the hawk above the ground?
14. This diagram shows a falcon F, on a tree, with a squirrel S, and a chipmunk C, on the ground. From the falcon, the angles of depression of the animals are 36° and 47°. How far apart are the animals on the ground?

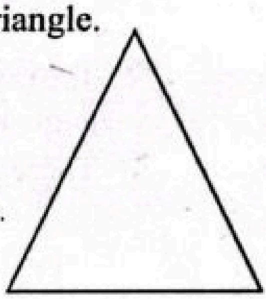


15. Two guy wires support a flagpole FH. The first wire is 11.2 m long and has an angle of elevation of 39°. The second wire has an angle of elevation of 47°. How tall is the flagpole?

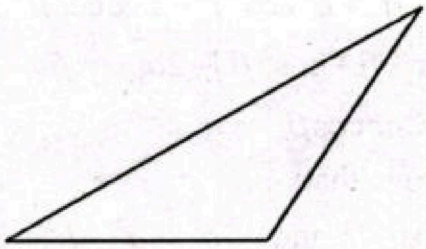


Oblique Triangles

An oblique triangle is any triangle that is not a right triangle. It could be an acute triangle or it could be an obtuse triangle.



Acute Triangle



Obtuse Triangle

The Pythagoras Theorem along with trigonometric ratios allow us to easily handle any given right triangle problem, but what if the triangle is not a right triangle i.e. the triangle is oblique?
To solve oblique triangles, we use certain laws known as "law of cosines", "law of sines" and "law of tangents".

Law of Cosines

When we know the measures of two sides and the included angle of a triangle (SAS), the size and shape of the triangle are determined. Therefore, we should be able to find the measure of the third side of the triangle. Sometimes three sides of oblique triangles are given (SSS) and we have to find the three angles of triangle. Such cases are easily handled by using law of cosines.

Statement:

If a, b, c are three sides and α, β, γ are three angles of an oblique triangle with usual notation, then law of cosines is stated as follows:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \text{ .... (1)}$$

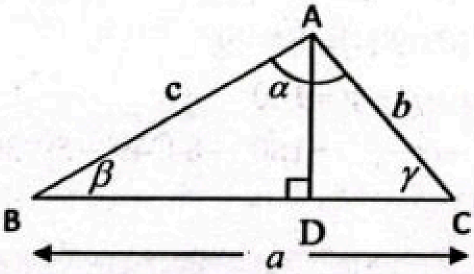
$$b^2 = a^2 + c^2 - 2ac\cos\beta \text{ .... (2)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \text{ .... (3)}$$

Proof: Consider an acute angled triangle ABC with usual notations as shown in the figure.

Draw $\overline{AD} \perp \overline{BC}$.

In right ΔABD,



$$\frac{BD}{AB} = \cos\beta$$

$$\Rightarrow BD = AB\cos\beta = c\cos\beta \text{.....}(i)$$

Again, $\dfrac{AD}{AB} = \sin\beta$

$$\Rightarrow AD = AB\sin\beta = c\sin\beta \text{.....}(ii)$$

Now, $CD = BC - BD = a - c\cos\beta$

In right ΔACD,

$(AC)^2 = (AD)^2 + (CD)^2$ .....(Pythagoras Theorem)

$$b^2 = (c\sin\beta)^2 + (a - c\cos\beta)^2$$
$$b^2 = c^2\sin^2\beta + a^2 + c^2\cos^2\beta - 2ac\cos\beta$$
$$b^2 = a^2 + c^2(\sin^2\beta + \cos^2\beta) - 2ac\cos\beta$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

Key Fact:

Law of cosines is applicable when:

- Two sides and included angle are given.
- Three sides are given.

Similarly, we can prove that:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$ and $c^2 = a^2 + b^2 - 2ab\cos\gamma$

From laws of cosines,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc},\ \cos\beta = \frac{a^2 + c^2 - b^2}{2ac},\ \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

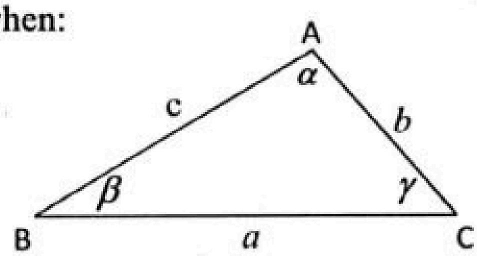
These laws are useful when three sides of oblique triangles are given and we have to find their angles.

Check Point:

Prove law of cosines by taking an obtuse triangle.

Example: Use the law of cosines to solve the triangle ABC when:

$a = 12cm,\ b = 7cm,\ \gamma = 59°30'$



Solution: Using law of cosines, we have:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$c^2 = 12^2 + 7^2 - 2\times 12\times 7\times \cos 59°30'$$
$$c^2 = 144 + 49 - 168\times 0.506 = 108$$
$$c = \sqrt{108} = 10.4cm$$

Now to find angle α, we use following formula.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7^2 + 10.4^2 - 12^2}{2\times 7\times 10.4}$$
$$= \frac{13.16}{145.6} = 0.0904$$

$\alpha = \cos^{-1}(0.0904) = 84°49'$

Finally, $\alpha + \beta + \gamma = 180°$

$\beta = 180° - \alpha - \gamma \ = 180° - 84°49' - 59°30'$

$\beta = 35°41'$

Key Fact:

If in the law of cosines,

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

angle α is 90°, then

$a^2 = b^2 + c^2 - 2bc\cos 90°$

$a^2 = b^2 + c^2 - 2bc\times 0 \Rightarrow a^2 = b^2 + c^2$

which becomes Pythagoras Theorem. Thus Pythagoras Theorem is a special case of "law of cosines".

Check Point:

Two forces of magnitude 20N and 30N are inclined at angle of 105° with each other. Find the magnitude of resultant force. Can you find the angle between force of magnitude 20N and resultant force?

Law of Sines

Law of cosines is not applicable when two angles and one side (AAS or ASA) or one angle and two sides (angle not included (SSA)) of an oblique triangle are given. In order to handle such situation, we use another law known as "law of sines".

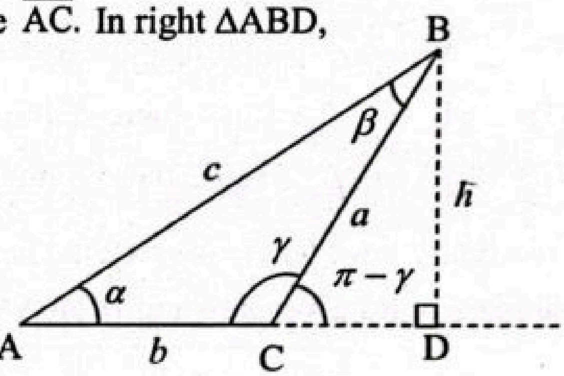
Statement:

If a, b, c are three sides and α, β, γ are three angles of an oblique triangle with usual notation, then law of sines is stated as follows:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof: Consider an obtuse triangle ABC as shown in the figure below.

Let h be the height of triangle ABC with respect to base $\overline{AC}$. In right ΔABD,



$$\frac{BD}{AB} = \sin\alpha$$

$$\Rightarrow BD = AB\sin\alpha = c\sin\alpha$$

$$\Rightarrow h = c\sin\alpha \ldots\ldots (i)$$

In right ΔCBD,

$$\frac{BD}{CB} = \sin(\pi - \gamma)$$

$$\Rightarrow BD = CB\sin(\pi - \gamma) \qquad [\text{as } \sin(\pi - \gamma) = \sin\gamma]$$

$$\Rightarrow h = a\sin\gamma \ldots\ldots (ii)$$

From (i) and (ii), we get:

$$a\sin\gamma = c\sin\alpha$$

Or $\quad \dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

Check Point:

Prove law of sines by taking an acute triangle.

Similarly, by drawing perpendiculars on other two sides of ΔABC, we get:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \text{ and } \frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

Combining above results, we obtain the following law known as "law of sines".

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad \text{or} \quad \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

To Determine the Number of Solutions in ΔABC

Task: Given measure of sides a, b and angle α. Use the law of sines to solve the ΔABC for $\sin\beta$.

Case-1:

(i) When $\sin\beta > 1$, there is no triangle, so no solution.

(ii) When $\sin\beta = 1$, there is one right triangle (one solution) if α is acute and no triangle (no solution) if α is obtuse.

Case-2:

When α is acute and $\sin\beta < 1$, there are two possible values of β, acute or obtuse.

$0 < \beta < 90^\circ$ and $\beta' = 180^\circ - \beta$

(i) If $\alpha + \beta' < 180^\circ$, there are two possible triangles ΔABC and $\Delta AB'C$.

(ii) If $\alpha + \beta' \geq 180^\circ$, there is only one ΔABC.

Case-3:

When α is obtuse and $\sin\beta < 1$, then β must be acute.

(i) If $\alpha + \beta < 180^\circ$, there is only one ΔABC.

(ii) If $\alpha + \beta' \geq 180^\circ$, there is no triangle, so no solution.

Alternatively, if we let $h = b\sin\alpha$, the height of triangle, we can summarize the number of possible triangles given a, b and α in ΔABC.

Angle is α	Acute $a < h$	Acute $a = h$	Acute $h < a < b$	Acute $a > b$	Obtuse $a < h$	Obtuse $a > b$
Possible Triangles	none	one right triangle	two triangles	one triangle	none	one triangle

Example:

Use the law of sines to solve the triangle ABC when:
$a = 25cm,\ \alpha = 66^\circ 51',\ \gamma = 44^\circ 12'$

Check Point

Solve the triangle ABC by using both law of sines and law of cosines when:
$a = 25cm, \alpha = 66^\circ 51',\ \gamma = 44^\circ 12'$

Solution:

Given: $a = 25cm,\ \alpha = 66^\circ 51',\ \gamma = 44^\circ 12'$

To Find: b, c and β

$$\alpha + \beta + \gamma = 180^\circ$$

$$66^\circ 51' + \beta + 44^\circ 12' = 180^\circ$$

$$\beta = 180^\circ - 66^\circ 51' - 44^\circ 12' = 68^\circ 57'$$

Using law of sines, we have:

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \Rightarrow \frac{25}{\sin 66^\circ 51'} = \frac{c}{\sin 44^\circ 12'}$$

$$c = \frac{25 \times \sin 44^\circ 12'}{\sin 66^\circ 51'} = 18.96cm$$

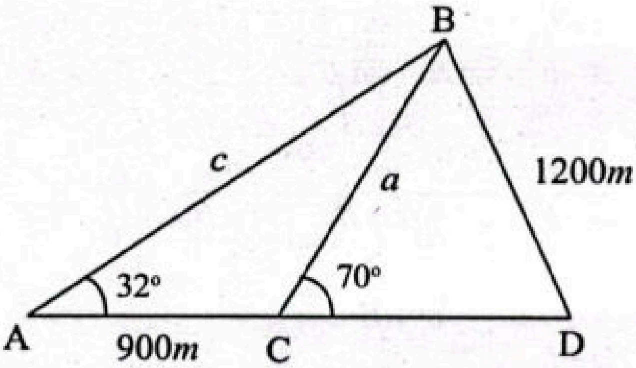
Again, by law of sines,

$$\frac{b}{\sin\beta}=\frac{a}{\sin\alpha} \Rightarrow \frac{b}{\sin 68^\circ 57'}=\frac{25}{\sin 66^\circ 51'}$$

$$\Rightarrow b=\frac{25}{\sin 66^\circ 51'}\times \sin 68^\circ 57' = 25.37\,cm$$

Example:

Find the values of unknowns in the adjoining figure.
Figure is not drawn according to scale.



Solution:

In ΔABC,

$$\angle ACB = 180^\circ - 70^\circ = 110^\circ$$

and $\angle ABC = 180^\circ - 32^\circ - 110^\circ = 38^\circ$

Using law of sines for triangle ABC,

$$\frac{a}{\sin 32^\circ}=\frac{900}{\sin 38^\circ}$$

$$\Rightarrow a=\frac{900}{\sin 38^\circ}\times \sin 32^\circ = 774.66m$$

Again for the ΔABC,

$$\frac{c}{\sin 110^\circ}=\frac{774.66}{\sin 32^\circ}$$

$$c=\frac{774.66}{\sin 32^\circ}\times \sin 110^\circ = 1373.69m$$

Now in ΔBCD,

$$\frac{a}{\sin\angle D}=\frac{1200}{\sin 70^\circ} \Rightarrow \frac{774.66}{\sin\angle D}=\frac{1200}{\sin 70^\circ}$$

$$\sin\angle D=\frac{774.66\times \sin 70^\circ}{1200}=0.607$$

$$\angle D=\sin^{-1}(0.607)=37^\circ 21'$$

Law of Tangents

Law of tangents is also used to solve oblique triangles. This law handles the same situations for which the law of cosines is used.

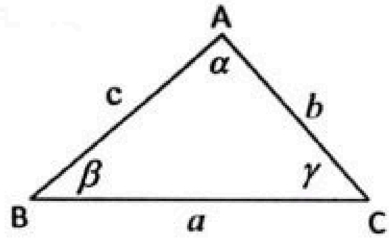
Statement:

If a, b, c are three sides and α, β, γ are three angles of an oblique triangle with usual notation, then law of tangents is stated as follows:

$$\frac{a+b}{a-b}=\frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)},\quad \frac{b+c}{b-c}=\frac{\tan\left(\frac{\beta+\gamma}{2}\right)}{\tan\left(\frac{\beta-\gamma}{2}\right)},\quad \frac{c+a}{c-a}=\frac{\tan\left(\frac{\gamma+\alpha}{2}\right)}{\tan\left(\frac{\gamma-\alpha}{2}\right)}$$

Proof:

For any oblique triangle ABC, the law of sines is given as follows:



$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$\frac{a}{b}=\frac{\sin\alpha}{\sin\beta}$$

Using componendo and dividendo property, we get:

$$\frac{a+b}{a-b}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

$$\frac{a+b}{a-b}=\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\frac{a+b}{a-b}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$$

Check Point:

Prove that

$$\frac{a+c}{a-c}=\frac{\tan\left(\frac{\alpha+\gamma}{2}\right)}{\tan\left(\frac{\alpha-\gamma}{2}\right)}$$

Similarly, we can prove that:

$$\frac{b+c}{b-c}=\frac{\tan\left(\frac{\beta+\gamma}{2}\right)}{\tan\left(\frac{\beta-\gamma}{2}\right)},\quad \frac{c+a}{c-a}=\frac{\tan\left(\frac{\gamma+\alpha}{2}\right)}{\tan\left(\frac{\gamma-\alpha}{2}\right)}$$

Example:

Use the law of tangents to solve the triangle ABC when $a=500mm$, $b=600mm$, $\gamma=50°20'$

Solution:

Given: $a=500cm$, $b=600cm$, $\gamma=50°20'$

To Find: c,α and β

As in any triangle,

$$\alpha+\beta+\gamma=180° \Rightarrow \beta+\alpha=180°-\gamma$$

$$\Rightarrow \beta+\alpha=180°-50°20'=129°40' \text{ ......}(i)$$

Now using law of tangents for $b>a$,

$$\frac{b+a}{b-a}=\frac{\tan\frac{\beta+\alpha}{2}}{\tan\frac{\beta-\alpha}{2}}$$

Substituting the values of b, c and $\beta+\alpha$, we get:

Key Fact:

It is better to use formula

$$\frac{a+c}{a-c}=\frac{\tan\left(\frac{\alpha+\gamma}{2}\right)}{\tan\left(\frac{\alpha-\gamma}{2}\right)}$$

rather than the formula

$$\frac{c+a}{c-a}=\frac{\tan\left(\frac{\gamma+\alpha}{2}\right)}{\tan\left(\frac{\gamma-\alpha}{2}\right)}$$

when $a>c$ in the given situation.

$$\frac{600+500}{600-500}=\frac{\tan\frac{129°40'}{2}}{\tan\frac{\beta-\alpha}{2}} \Rightarrow \frac{1100}{100}=\frac{\tan 64°50'}{\tan\frac{\beta-\alpha}{2}}$$

$$\tan\frac{\beta-\alpha}{2}=\frac{\tan 64°50'}{11} \Rightarrow \tan\frac{\beta-\alpha}{2}=0.193$$

$$\frac{\beta-\alpha}{2}=\tan^{-1}(0.193)=10°57'$$

$$\beta-\alpha=21°54' \;.......(ii)$$

Solving (i) and (ii), we get:

$$\beta=75°47',\ \alpha=53°53'$$

To find c, we use law of sines (we can also use law of cosines).

$$\frac{c}{\sin\gamma}=\frac{b}{\sin\beta} \Rightarrow c=\frac{b\times\sin\gamma}{\sin\beta}$$

$$c=\frac{600\times\sin 50°20'}{\sin 75°47'}=476.45mm$$

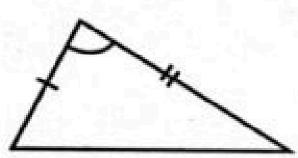
Check Point:

Solve the triangle ABC by using both law of tangents and law of sines when:
$b=50cm,\ c=40,\ \alpha=80°30'$

When Law of Sines and Law of Cosines are Used

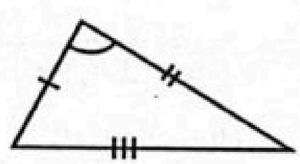
The Law of Cosines and the Law of Sines can be used to find the remaining three measures of any triangle when we know the measure of a side and the measures of any two other components (two sides, two angles, one side one angle) of the triangle.

1. **Given:** Two sides and the included angle.



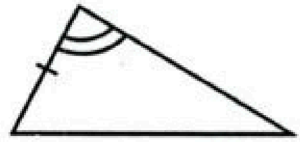
- Use the Law of Cosines to find the measure of the third side.
- Use the Law of Sines or the Law of Cosines to find the measure of another angle.
- Use the sum of the angles of a triangle to find the measure of the third angle.

2. **Given:** Three sides.



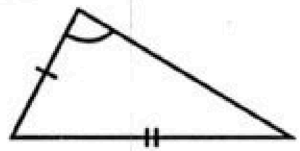
- Use the Law of Cosines to find the measure of an angle.
- Use the Law of Sines or the Law of Cosines to find the measure of another angle.
- Use the sum of the angles of a triangle to find the measure of the third angle.

3. **Given:** Two angles and a side.



- Use the sum of the angles of a triangle to find the measure of the third angle.
- Use the Law of Sines to find the remaining sides.

4. **Given:** Two sides and an angle opposite to one of them



- Use the Law of Sines to find the possible measure(s) of another angle.
- Determine if there are two, one, or no possible triangles.
- If there is a triangle, use the sum of the angles of a triangle to find the measure(s) of the third angle.
- Use the Law of Sines or the Law of Cosines to find the measure(s) of the third side.

Half Angle Formulae

Sometimes it is convenient to use half angle formulae for the solution of oblique triangles when measures of three sides of a triangle are given. In this section, we will study half angles formulae for sines, cosines and tangents. Let us derive them one by one.

(a) Cosines of Half Angle

Statement: In any triangle ABC with usual notations,

$$\cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}},\quad \cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}},\quad \cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}$$

where $S=\dfrac{a+b+c}{2}$.

Proof: We know that:

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$2\cos^2\frac{\alpha}{2}-1=\frac{b^2+c^2-a^2}{2bc} \Rightarrow 2\cos^2\frac{\alpha}{2}=\frac{b^2+c^2-a^2}{2bc}+1$$

$$\Rightarrow 2\cos^2\frac{\alpha}{2}=\frac{b^2+c^2+2bc-a^2}{2bc}=\frac{(b+c)^2-(a)^2}{2bc}$$

$$\Rightarrow 2\cos^2\frac{\alpha}{2}=\frac{(b+c+a)(b+c-a)}{2bc}$$

As $a+b+c=2S \Rightarrow b+c-a=2S-2a=2(S-a)$

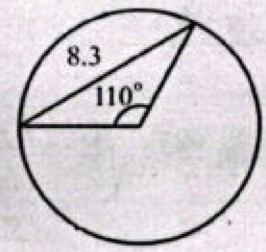
$$\therefore\ 2\cos^2\frac{\alpha}{2}=\frac{2S\times 2(S-a)}{2bc}$$

$$\cos^2\frac{\alpha}{2}=\frac{S(S-a)}{bc}$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}}$$

Check Point:

Find the diameter of the circle shown in the figure below.



Similarly, it can be proved that:

$$\cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}}\quad \text{and}\quad \cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}$$

(b) Sines of Half Angle

Statement: In any triangle ABC with usual notations,

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}},\quad \sin\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{ac}},\quad \sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$$

where $S=\dfrac{a+b+c}{2}$.

Proof: We know that

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$1-2\sin^2\frac{\alpha}{2}=\frac{b^2+c^2-a^2}{2bc}\Rightarrow 2\sin^2\frac{\alpha}{2}=1-\frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow 2\sin^2\frac{\alpha}{2}=\frac{(a)^2-(b-c)^2}{2bc}=\frac{(a-b+c)(a+b-c)}{2bc}$$

As $a+b+c=2S \Rightarrow a+c-b=2S-2b=2(S-b)$ and $a+b-c=2S-2c=2(S-c)$

$$\therefore\ 2\sin^2\frac{\alpha}{2}=\frac{2(S-b)\times 2(S-c)}{2bc}=\frac{2(S-b)(S-c)}{bc}$$

$$\sin^2\frac{\alpha}{2}=\frac{(S-b)(S-c)}{bc}$$

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}}$$

Check Point:

Prove that

$$\cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}\ \text{and}\ \sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$$

Similarly, it can be proved that:

$$\sin\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{ac}}\quad \text{and}\quad \sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$$

(c) Tangents of Half Angle

Statement: In any triangle ABC with usual notations,

$$\tan\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}},\quad \tan\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{S(S-b)}},\quad \tan\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

where $S=\dfrac{a+b+c}{2}$.

Proof: $\tan\dfrac{\alpha}{2}=\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\dfrac{\sqrt{\frac{(S-b)(S-c)}{bc}}}{\sqrt{\frac{S(S-a)}{bc}}}=\sqrt{\dfrac{(S-b)(S-c)}{S(S-a)}}$

$$\therefore \tan\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

Similarly, it can be proved that:

$$\tan\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{S(S-b)}},\quad \tan\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}.$$

Check Point:

Solve the triangle ABC when $a=5cm$, $b=6cm$, $c=7cm$ using half angle formulae for sines and tangents. Which one method do you think easy?

Example:

Use half angle formulae to solve the triangle ABC when $a=5cm$, $b=6cm$, $c=7$cm.

Solution:

Given: $a=5cm$, $b=6cm$, $c=7$cm.

To Find: α, β and γ

We use half angle formula for cosines here.

Now $S=\dfrac{a+b+c}{2}=\dfrac{5+6+7}{2}=9$

$$\therefore \cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}}=\sqrt{\frac{9(9-5)}{6\times 7}}=\sqrt{\frac{36}{42}}=0.926$$

$$\frac{\alpha}{2}=\cos^{-1}(0.926)=22.18^\circ \Rightarrow \alpha=44.36^\circ$$

Now $\cos\dfrac{\beta}{2}=\sqrt{\dfrac{S(S-b)}{ac}}=\sqrt{\dfrac{9(9-6)}{5\times 7}}$

$$=\sqrt{\frac{27}{35}}=0.878$$

$$\frac{\beta}{2}=\cos^{-1}(0.878)=28.58^\circ \Rightarrow \beta=57.20^\circ$$

To find third angle, we have:

$$\alpha+\beta+\gamma=180^\circ \Rightarrow \gamma=180^\circ-\alpha-\beta$$

$$\Rightarrow \gamma=180^\circ-44.36^\circ-57.20^\circ=78.44^\circ$$

Check Point:

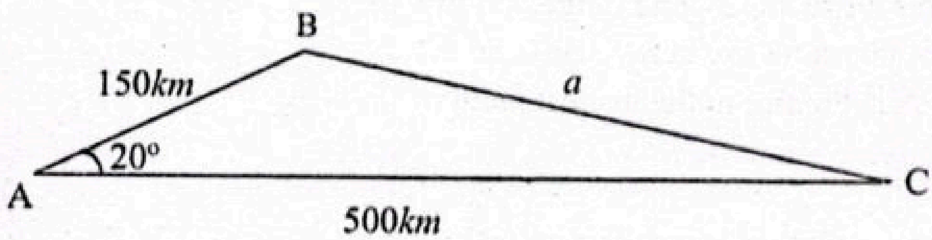
The base of an isosceles triangle measures 14.5 *cm* and the vertex angle measures 110°.

a. Find the measure of one of the congruent sides of the triangle to the nearest hundredth.

b. Find the perimeter of the triangle.

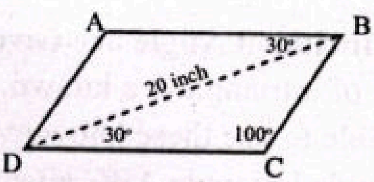
Exercise 8.3

1. Use the Law of Cosines to find the remaining side(s) and angle(s) in triangle ABC.
   (i) $a=7, b=12, \gamma=58.5^\circ$ (ii) $b=20, c=15, \alpha=36^\circ$ (iii) $a=150, c=150, \beta=15^\circ$
   (iv) $\beta=45^\circ$, $a=30, c=42$ (v) $a=7, b=10, c=12$ (vi) $a=3, b=3, c=5$
   (vii) $a=30, b=45, c=50$ (viii) $a=6, b=6, c=6$ (ix) $a=5, b=12, c=13$
2. Use the half angles formulae to solve parts (v) to (ix) in Q.1 where possible.
3. Use the Law of Sines to solve triangle ABC in the following (where possible).
   (i) $a=10, \alpha=40^\circ, \beta=60^\circ$ (ii) $b=20, \alpha=50^\circ, \beta=70^\circ$
   (iii) $c=12, \alpha=45^\circ, \beta=75^\circ$ (iv) $b=18, \beta=40^\circ35', \gamma=120^\circ$
   (v) $a=14.6, \alpha=25^\circ10', \beta=85.5^\circ$ (vi) $c=52, \alpha=42.3^\circ, \gamma=85^\circ14'$
4. Solve parts (i) to (iv) of Q.1 by using the Law of Tangents.
5. Solve by using appropriate law.
   (i) $a=10, b=8, \beta=80^\circ$ (ii) $b=20, c=14, \beta=70^\circ$
   (iii) $c=12, b=10, \gamma=64^\circ$ (iv) $b=18, \alpha=55^\circ5', a=37$
   (v) $a=14.6, b=10.6, c=17.2$ (vi) $c=88, \beta=23.2^\circ, \gamma=73^\circ14'$
6. A pilot is flying from city A to city C, 500 *km* apart. He starts his flight 20° off course and flies on this course for 150 *km* and is above city B. How far is he from city C?

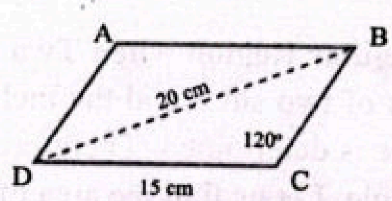


7. Two sides of a triangular plot have lengths 400m and 600m. The measurement of angle between the sides is 45°. Find the perimeter and area of the plot.
8. The sides of a triangle are 6.5cm, 8.2cm and 5.8cm. Find the measurement of smallest and largest angles.
9. The sides of a parallelogram are 50cm and 70cm. Find the length of each diagonal if the larger angle measures 110 °.
10. For parallelogram ABCD, find :

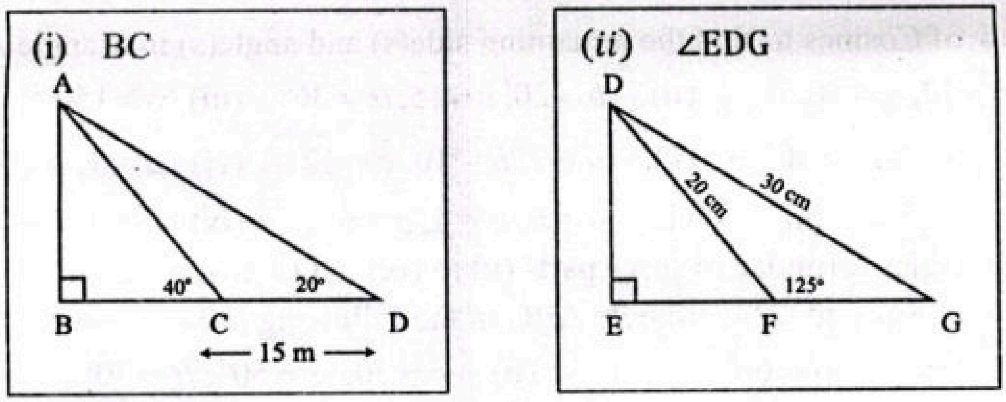
(i) BC



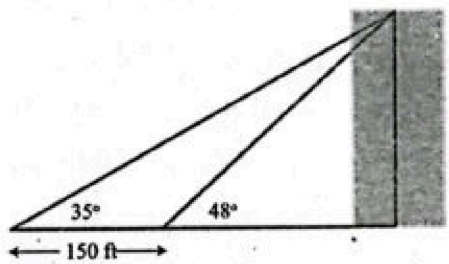
(ii) ∠BDC



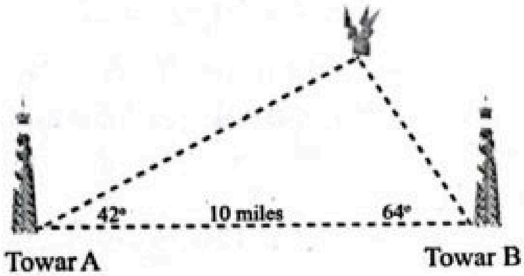
11. For the figure below find:



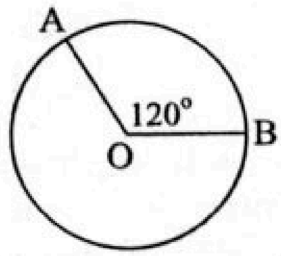
12. Find the height of the building in the figure below.



13. Fire towers A and B are located 10 miles apart at the same level of ground. Rangers at fire tower A spots a fire at 42°, and rangers at fire tower B spot the same fire at 64°. How far from tower A is the fire to the nearest tenth of a mile?



14. Circle O has a radius of 15cm. The angle between radii $\overline{OA}$ and $\overline{OB}$ is 120°. Find the length of chord $\overline{AB}$.



15. Two lighthouses are 12 miles apart along a straight shore. A ship is 15 miles from one lighthouse and 20 miles from the other. Find, to the nearest degree, the measure of the angle between the lines of sight from the ship to each lighthouse.

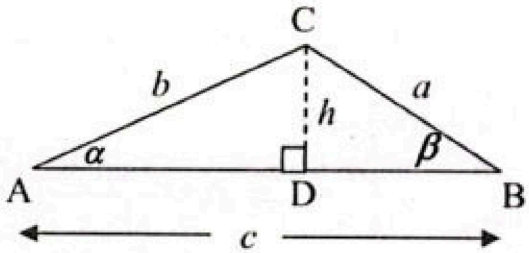
Area of Triangular Region

(a) Area of Triangular Region when Two Sides and Included Angle are Given

When the measures of two sides and the included angle of a triangle are known, the size and shape of the triangle is determined. Therefore, it is possible to use these known values to find the area of the triangle. Let us find the area of an acute angled triangle ABC given below.

We know that:

Area (Δ) of triangle ABC $= \frac{1}{2} \times$ base$\times$ height



Area (Δ) of triangle ABC $= \frac{1}{2} \times \text{AB} \times \text{CD} = \frac{1}{2} \times c \times h$ .... (i)

Now in right angled triangle ACD, AB $= c$

and $\frac{h}{b} = \sin\alpha \Rightarrow h = b\sin\alpha$

Therefore from (i),

$$\Delta = \frac{1}{2} \times c \times b\sin\alpha = \frac{1}{2}bc\sin\alpha$$

Again, in right angled triangle BCD

$$\frac{h}{a} = \sin\beta \Rightarrow h = a\sin\beta$$

Therefore from (i),

$$\Delta = \frac{1}{2} \times c \times a\sin\beta = \frac{1}{2}ac\sin\beta$$

Check Point:

1. Rashid found the area of parallelogram $ABCD$ by using $(AB)(BC)(\sin\angle B)$.
Shehzad found the area of parallelogram $ABCD$ by using $(AB)(BC)(\sin\angle A)$. Explain why Rashid and Shehzad both got the correct answer.

Similarly, we can prove that:

$$\Delta = \frac{1}{2}ab\sin\gamma$$

Hence $\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma$

Example:

Find the area of triangle DEF if DE $= 10$, EF $= 8$ and $\angle$E $= 30°$.

Solution:

$$\text{Area of triangle DEF} = \frac{1}{2} \times \text{DE} \times \text{EF} \times \sin(\angle \text{E})$$

$$= \frac{1}{2} \times 10 \times 8 \times \sin 30°$$

$$= 40 \times 0.5 = 20 \textit{ sq. units}$$

Challenge:

Take an obtuse angled triangle and prove that $\Delta = \frac{1}{2}ab\sin\gamma$

(b) Area of Triangular Region when Two Angles and One Side are Given

If a is measure of side of a triangle ABC and any two angles let's say β and γ are given then:

$$\text{Area of triangle ABC} = \Delta = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$$

Let us prove this formula.

We have proved in the previous section that:

Check Point:

Urwa said that the area of rhombus $PQRS$ is $(PQ)^2(\sin P)$. Do you agree with Urwa? Explain why or why not.

Area of triangle ABC $=\Delta=\frac{1}{2}ab\sin\gamma$ ......(*i*)

But in the current situation, *b* is not given. We can replace *b* by using law of sines as follows.

$$\frac{b}{\sin\beta}=\frac{a}{\sin\alpha} \Rightarrow b=\frac{a\sin\beta}{\sin\alpha}$$

Substituting the value of *b* in (i), we get:

$$\Delta=\frac{1}{2}a\left(\frac{a\sin\beta}{\sin\alpha}\right)\sin\gamma=\frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$$

In the same way, we can prove that:

$$\Delta=\frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta} \quad \text{and} \quad \Delta=\frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$$

Hence, $\Delta=\frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$, $\Delta=\frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$, $\Delta=\frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$

Check Point:

Prove that

(i) $\Delta=\frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$

(ii) $\Delta=\frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$

Example:

Calculate the cost of grass cutting @ Rs.10 per sq. unit of a triangular plot ABC if $a=16.5,\ \alpha=40.5^\circ,\ \beta=65^\circ$.

Solution: To find the cost of grass cutting, we need area of plot and to find area, we need the third angle γ.

$$\gamma=180^\circ-\alpha-\beta$$

$$\Rightarrow \gamma=180^\circ-40.5^\circ-65^\circ=74.5^\circ$$

Now, area of triangle ABC $=\frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$

$$=\frac{1}{2}\times(16.5)^2\times\frac{\sin 65^\circ\times\sin 74.5^\circ}{\sin 40.5^\circ}$$

$$=183.05\ sq.\ units$$

Therefore, the cost of grass cutting @ Rs.10 per sq. unit = Rs. 10×183.05 = Rs.1830.50

(c) Area of Triangular Region when Three Sides are Given

If *a, b, c* are measure of sides of a triangle ABC then:

Area of triangle ABC $=\Delta=\sqrt{S(S-a)(S-b)(S-c)}$ where $S=\frac{a+b+c}{2}$

Let us prove this formula.

We have proved in the previous section that:

Area of triangle ABC $=\Delta=\frac{1}{2}bc\sin\alpha$

Using half angle formula, we have:

$$\Delta=\frac{1}{2}bc\times2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=bc\times\sqrt{\frac{(S-b)(S-c)}{bc}}\times\sqrt{\frac{S(S-a)}{bc}}$$

Check Point:

Two sides of a triangle are 20cm long each. Find the angle between both sides if area of triangle is 200 sq. unit.

$$\Delta = bc\times\sqrt{\frac{(S-b)(S-c)}{bc}}\times\sqrt{\frac{S(S-a)}{bc}}=\sqrt{S(S-a)(S-b)(S-c)}$$

This formula is known as hero's formula or Heron's formula.

Example:

Find area of a triangle with dimensions 16, 18, 20.

Solution: Let $a=16, b=18, c=20.$

Area of triangle = $\sqrt{S(S-a)(S-b)(S-c)}$

Where $S=\frac{16+18+20}{2}=\frac{54}{2}=27$

Now, area of triangle = $\sqrt{27(27-16)(27-18)(27-20)}=\sqrt{18711}$

$=136.79 sq.unit$

Exercise 8.4

1. Find the area of triangular region ABC in the following.

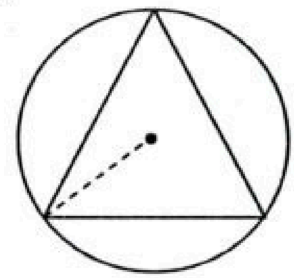
(i) $a=8, b=14, \gamma=68.7^\circ$	(ii) $b=30, c=25, \alpha=46^\circ$
(iii) $a=20, c=15, \beta=25^\circ$	(iv) $\beta=46^\circ 50'$, $a=43, c=52$
(v) $b=4.5, c=2.5, \alpha=65.2^\circ$	(vi) $a=2, b=2, c=5$
(vii) $a=12, \alpha=44^\circ, \beta=60^\circ$	(viii) $b=30, \alpha=40^\circ, \beta=80^\circ$
(ix) $c=10, \alpha=75^\circ, \beta=45^\circ$	(x) $b=21, \beta=30^\circ 15', \gamma=110^\circ$
(xi) $a=18.4, \alpha=65^\circ 10', \beta=95.5^\circ$	(xii) $c=25, \alpha=52.7^\circ, \gamma=79^\circ 24'$

(xiii) 18, 21, 32 (xiv) 20, 26, 37 (xv) $\frac{1}{2},\frac{1}{3},\frac{1}{4}$

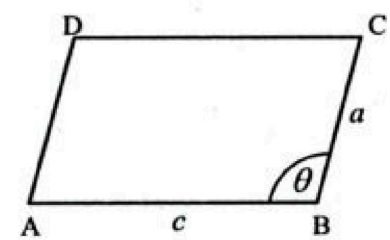
(xvi) 12.4, 13.7, 20.2 (xvii) 1.6, 2.6, 4.1

2. The adjacent sides of parallelogram ABCD measure 12 and 15. The measure of one angle of the parallelogram is 135°. Find the area of the parallelogram.
3. Three streets intersect in pairs enclosing a small triangular park. The measures of the distances between the intersections are 30 *m*, 34 *m*, and 27 *m*. Find the area of the park.
4. A field is bordered by two pairs of parallel roads so that the shape of the field is a parallelogram. The lengths of two adjacent sides of the field are 2 *km* and 3 *km*, and the length of the shorter diagonal of the field is 3 *km*.
   a. Find the cosine of the acute angle of the parallelogram.
   b. Find the exact value of the sine of the acute angle of the parallelogram.
   c. Find the exact value of the area of the field.
   d. Find the area of the field to the nearest integer.
5. The roof of a shed consists of four congruent isosceles triangles. The length of each equal side of one triangular section is 22.0 feet and the measure of the vertex angle of each triangle is 75°. Find the area of one triangular section of the roof.

6. A garden is in the shape of an isosceles trapezoid. The lengths of the parallel sides of the garden are 30 feet and 20 feet, and the length of each of the other two sides is 10 feet. If a base angle of the trapezoid measures 60°, find the exact area of the garden.
7. In triangle ABC, $\angle B = 30^\circ$ and in triangle DEF, $\angle E = 150^\circ$. Show that if AB = DE and BC = EF, the areas of the two triangles are equal.
8. Area of a triangular garden is $150m^2$. If two corner angles of a side measure 40° and 65°, find the length of that side. Also find the third angle of garden.
9. An equilateral triangle is inscribed in a circle of radius $6cm$. Find the area of triangle.



10. Laiba wants to draw triangle ABC with AB = 15 cm, BC= 8 cm, and an area of 40 cm^2.
    a. What must be the sine of $\angle B$? b. Find, the measure of $\angle B$.
11. Let ABCD be a parallelogram with $AB = c$, $BC = a$, and $\angle B = \theta$.
    a. Write a formula for the area of parallelogram in terms of c, a, and θ.
    b. For what value of θ does parallelogram have the greatest area?



Circles Connected with Triangles

(a) Circumcircle

A circle passing through the vertices of any triangle (polygon) is called circumcircle. The centre of this circle is called the circumcentre and its radius is called the circumradius. The radius of circumcircle is denoted by R.

(b) Incircle

A circle drawn inside and touching the sides of a triangle (polygon) is called incircle or inscribed circle. It is the largest circle contained in the triangle. The centre of this circle is called the incentre usually denoted by I and its radius is called the inradius denoted by r.

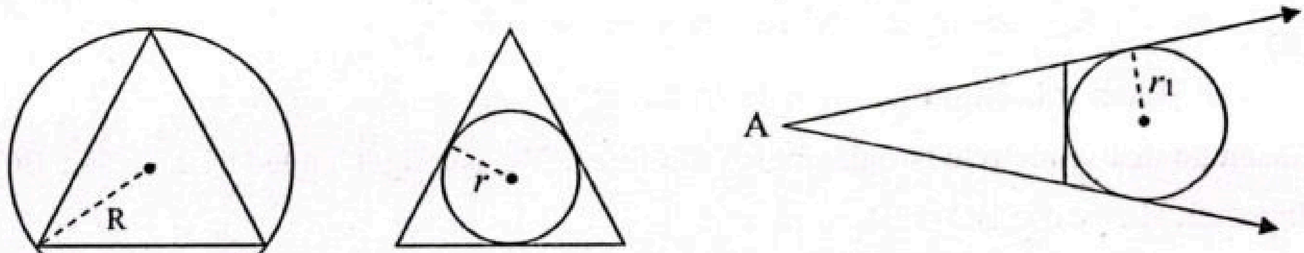
(c) Escribed Circle

A circle touching one of the sides of a triangle externally and the extensions of its two other sides internally, is called an escribed circle (e-circle or ex-circle). The centre of this circle is called the e-centre and its radius is called the e-radius.

If a circle touches extended arms of angle A (circle is drawn opposite to vertex A) of a triangle ABC, then its centre and radius are respectively denoted by I_1 and r_1. Similarly,

(i) Centre and radius of e-circle opposite to vertex B are denoted by I_2 and r_2.

(ii) Centre and radius of e-circle opposite to vertex C are denoted by I_3 and r_3.



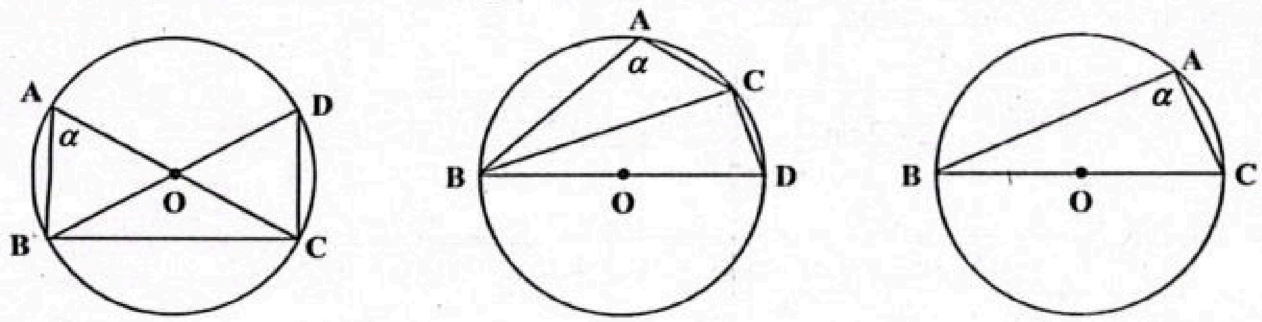
Circumcircle Incircle Escribed circle

Key Fact:

(i) Circumcentre is the point of intersection of right bisectors of sides of a triangle.
(ii) Incentre is the point of intersection of angle bisectors of a triangle.
(iii) E-centre is the point of intersection of one internal bisector of angle and external bisectors of remaining angles of a triangle.

Circum-Radius of Circle In-terms of Side and Opposite Angle of Triangle

Let O be the circumcentre of triangle ABC. Draw a diameter BD. Also join C to D.



(i) (ii) (iii)

Case-1: When α is Acute

In figure (i), α and $\angle BDC$ are angles in the same segment associared with chord BC.

$\therefore \ \angle BDC = \alpha$

Now in right angled triangle BCD,

$$\frac{BC}{BD} = \sin(\angle BDC) = \sin\alpha$$

$$\frac{a}{2R} = \sin\alpha \ \Rightarrow \ R = \frac{a}{2\sin\alpha} \text{ where } BD = 2R \text{ and } BC = a.$$

Case-2: When α is Obtuse

In figure (ii), $\alpha + \angle BDC = 180^\circ$ because ABDC is a cyclic quadrilateral.

$\therefore \ \angle BDC = \pi - \alpha$

Now in right angled triangle BCD,

$$\frac{BC}{BD}=\sin(\angle BDC)=\sin(\pi-\alpha) \text{ where } \sin(\pi-\alpha)=\sin\alpha. \quad \frac{a}{2R}=\sin\alpha \Rightarrow R=\frac{a}{2\sin\alpha}$$

Case-3: **When α is Right**

As, angle in a semicircle is right angle. Therfore, ABC is a right angled triangle and $\overline{BC}$ is a diameter. Hence $a = BC = 2R$

$$\therefore \quad \frac{a}{2R}=1=\sin\frac{\pi}{2}=\sin\alpha$$

$$\frac{a}{2R}=\sin\alpha \Rightarrow R=\frac{a}{2\sin\alpha}$$

Similarly, we can prove that:

$$R=\frac{b}{2\sin\beta} \text{ and } R=\frac{c}{2\sin\gamma}$$

Combining all results, we get:

$$R=\frac{a}{2\sin\alpha}=\frac{b}{2\sin\beta}=\frac{c}{2\sin\gamma}$$

Circum-Radius of Circle In-terms of Sides of Triangle

We have proved in the previous section, that:

$$R=\frac{a}{2\sin\alpha}=\frac{a}{2\times 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{a}{4\times\frac{\sqrt{(S-b)(S-c)}}{bc}\frac{\sqrt{S(S-a)}}{bc}}$$

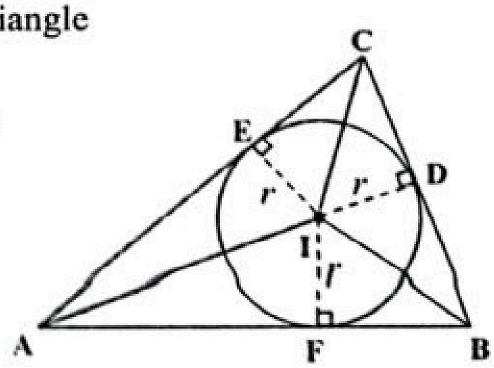
$$R=\frac{a}{4\times\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{S(S-a)}{bc}}}=\frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}}$$

$$R=\frac{abc}{4\Delta}$$

Where $\Delta=\sqrt{S(S-a)(S-b)(S-c)}$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$.

In-Radius of a Circle in a Triangle

Consider a triangle ABC. Draw the three angle bisectors of triangle which intersect at point I which is in-centre of circle.



By drawing angle bisectors, the given triangle ABC has been divided into three triangles IAB, IBC and IAC.

Draw altitude ID of triangle IBC, altitude IE of triangle IAC and altitude IF of triangle IAB. Note that

ID = IE = IF = radius of in-circle = r

Now if AB = c, BC = a, AC = b and Δ is the area of triangle ABC, then from the figure:

area of triangle ABC = area of triangle IBC + area of triangle IAC + area of triangle IAB

$$\Delta = \frac{1}{2}\times BC\times ID+\frac{1}{2}\times AC\times IE+\frac{1}{2}\times AB\times IF$$

$$\Delta=\frac{1}{2}\times a\times r+\frac{1}{2}\times b\times r+ \frac{1}{2}\times c\times r=\frac{1}{2}\times r\times (a+b+c)$$

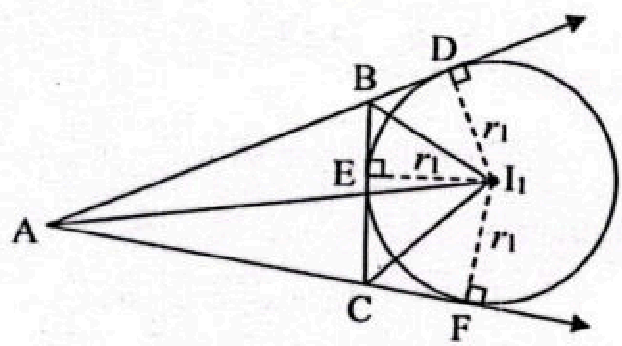
$$\Delta=r\times\frac{a+b+c}{2}=rS$$

$$r=\frac{\Delta}{S}$$

Key Facts

The two geometrical figures, circle and triangle are made up of distinct conception. The triangle is made with straight lines, whereas circle is made by curve lines (arcs). The beauty of circum- circle and in-circle of a triangle is a combination of a triangle and a circle. Engineers use this combination in making parts of machines.

E-Radius of a Circle of a Triangle



Consider a triangle ABC. Draw internal bisector of $\angle A$ and external bisectors of $\angle B$ and $\angle C$ of triangle intersecting at I_1 which is the centre of ex-circle.

By drawing angle bisectors, the given triangle ABC has been divided into three triangles I_1AB, I_1BC and I_1AC.

Draw altitude I_1D of triangle I_1AB, altitude I_1E of triangle I_1BC and altitude I_1F of triangle I_1AC. Note that

$I_1D = I_1E = I_1F$ = radius of e-circle = r_1

Now if AB = c, BC = a, AC = b and Δ is the area of triangle ABC, then from the figure:

area of triangle ABC = area of triangle I_1AB + area of triangle I_1AC + area of triangle I_1BC

$$\Delta = \frac{1}{2}\times AB\times I_1D+\frac{1}{2}\times AC\times I_1F-\frac{1}{2}\times BC\times I_1E$$

$$\Delta=\frac{1}{2}\times c\times r_1+\frac{1}{2}\times b\times r_1- \frac{1}{2}\times a\times r_1=\frac{1}{2}\times r_1\times (c+b-a)$$

$$\Delta=r_1\times\frac{2S-a-a}{2}=r_1\times\frac{2S-2a}{2}=r_1\times\frac{2(S-a)}{2}$$

$$r_1=\frac{\Delta}{S-a}$$

Similarly, we can prove that:

$$r_2=\frac{\Delta}{S-b} \quad \text{and} \quad r_3=\frac{\Delta}{S-c}$$

Example:

Find r, R and r_2 when $a = 8, b = 9, c = 11$.

Solution: **Given:** $a = 8, b = 9, c = 11$

To find: r, R and r_2

First of all, we find S and Δ.

$$S = \frac{8+9+11}{2} = \frac{28}{2} = 14$$

and $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

$$= \sqrt{14(14-8)(14-9)(14-11)} = \sqrt{1260} = 35.5$$

Now, $r = \frac{\Delta}{S} = \frac{35.5}{14} = 2.54$

$$R = \frac{abc}{4\Delta} = \frac{792}{142} = 5.58$$

$$r_2 = \frac{\Delta}{S-b} = \frac{35.5}{14-9} = \frac{35.5}{5} = 7.1$$

Key Fact:

We know that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Which is law of sines.

Check Point:

Find r_1, r_2 and r_3 when $a = 5.4$, $b = 7.6$ and $c = 10.2$.

Example:

Prove that $r_2 = 4R\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}\sin\frac{\beta}{2}$

Solution:

$$\text{R.H.S} = 4R\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}\sin\frac{\beta}{2}$$

$$= 4\times\frac{abc}{4\Delta}\times\sqrt{\frac{S(S-a)}{bc}}\times\sqrt{\frac{S(S-c)}{ab}}\times\sqrt{\frac{(S-a)(S-c)}{ac}}$$

$$= \frac{abc}{\Delta}\times\sqrt{\frac{S^2(S-a)^2(S-c)^2}{a^2b^2c^2}} = \frac{abc}{\Delta}\times\frac{S(S-a)(S-c)}{abc}$$

$$= \frac{1}{\Delta}\times\frac{S(S-a)(S-b)(S-c)}{S-b} = \frac{\Delta^2}{\Delta(S-b)}$$

$$= \frac{\Delta}{S-b} = r_2 = \text{L.H.S}$$

Check Point:

Show that

$$4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} = r_3$$

Exercise 8.5

1. Find r, R, r_1, r_2 and r_3 when
   - (i) $a = 10, b = 13, c = 17$
   - (ii) $a = 22, b = 24, c = 30$
   - (iii) $a = 3, b = 4, c = 5$
   - (iv) $a = 50, b = 60, c = 70$
2. If ABC is an equilateral triangle, prove that:
   - (i) $r : R : r_1 = 1 : 2 : 3$
   - (ii) $r_1 : r_2 : r_3 = 1 : 1 : 1$

3. Show that:

(i) $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{2S}{abc}=\frac{1}{2rR}$ (ii) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{S}{\Delta}$

(iii) $r_1r_2+r_2r_3+r_1r_3=S^2$ (iv) $r_1+r_2+r_3-r=4R$

(v) $\sqrt{r r_1 r_2 r_3}=\Delta$ (vi) $r_1 r_2 r_3 = rS^2$

4. Show that: $(\sin\alpha+\sin\beta+\sin\gamma)=\frac{S}{R}$

5. Prove that: $(r_1+r_2)\tan\frac{\gamma}{2}=(r_3-r)\cot\frac{\gamma}{2}=c$

6. Show that:

(i) $4rR\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}=\Delta$

(ii) $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=S\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(iii) $r=a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}=b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\sec\frac{\beta}{2}=c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$

7. Prove that:

(i) $4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=r_1$ (ii) $4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}=r_2$

8. Prove that:

(i) $r_1=S\tan\frac{\alpha}{2}$ (ii) $r_2=S\tan\frac{\beta}{2}$ (iii) $r_3=S\tan\frac{\gamma}{2}$

9. Prove that:

(i) $r=(S-a)\tan\frac{\alpha}{2}=(S-b)\tan\frac{\beta}{2}=(S-a)\tan\frac{\gamma}{2}$

10. Find the radius of in-circle and circum-circle of a triangle having sides: 7*cm*, 12*cm* and 15*cm*.

11. The measures of sides of a triangle are 5*cm*, 9*cm*, 10*cm*. Find the circumference of ex-circle:

(i) opposite to smaller side (ii) opposite to larger side

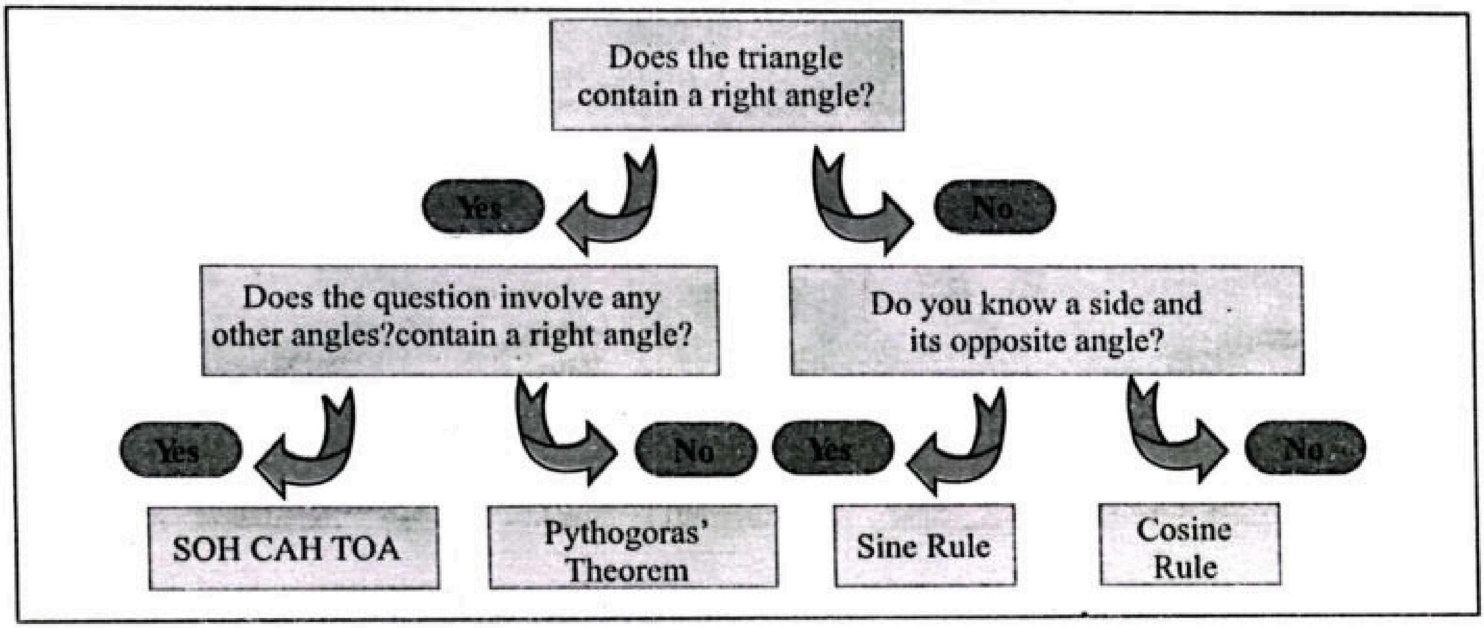
12. Find the area and circumference of in-circle of a triangle having sides: 2.5*cm*, 7.3 *cm* and 6.2cm.

Trigonometric Problems in Three Dimensions

3-D Trigonometry

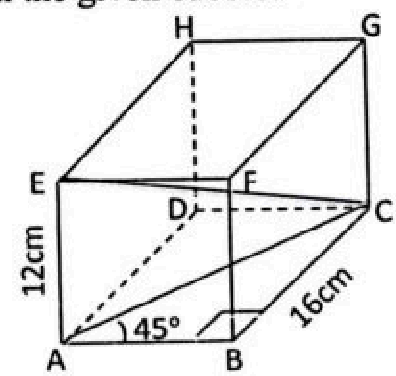
3-D trigonometry is an application of the trigonometric skills developed for 3-dimensional shapes in order to find unknown sides and angles.

In this topic, we will learn about 3-D trigonometry. We will use combined knowledge of Pythagoras' theorem, trigonometric ratios, the law of sines (sine rule) and the law of cosines (cosine rule) to find missing angles and sides of triangles in 3-dimensional shapes.
The flowchart below can help to determine which path we need to use:

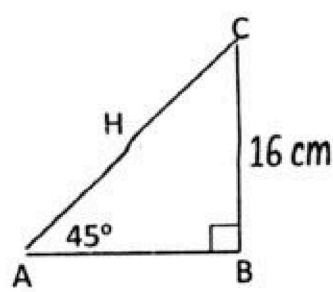


In order to find a missing angle or side within a 3-dimensional shape:
Step 1: Find the missing side or angle of a triangle.
Step 2: Sketch and label the second triangle using information from Step 1.
Step 3: Calculate the missing side or angle of the final triangle.
Let us solve some examples to understand the method.

Example: Find the length of $\overline{EC}$ in the given cuboid.



Step 1: Finding the length of missing side $\overline{AC}$ of right triangle ABC.
$\overline{AC}$ is hypotenuse of right triangle ABC. We can find its length by using simple trigonometric ratios.
In right triangle ABC:

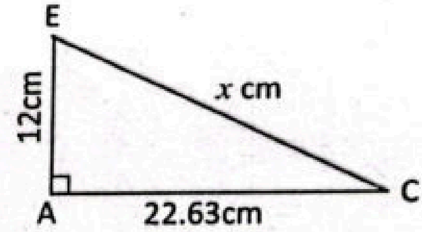


$$\sin\theta = \frac{16}{H}$$

$$\Rightarrow H = \frac{16}{\sin\theta} = \frac{16}{\sin 45^\circ}$$

$$= \frac{16}{0.707} = 22.63$$

Step 2: Sketching and labeling the second right triangle EAC using information from step 1.

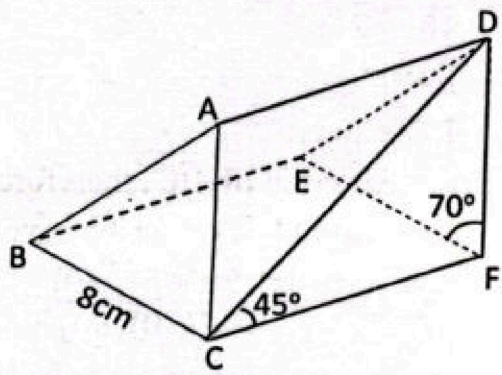


Step 3: EAC is a right-angled triangle, so we can use Pythagoras' Theorem to find the length of $\overline{EC}$.

Let EC = x cm

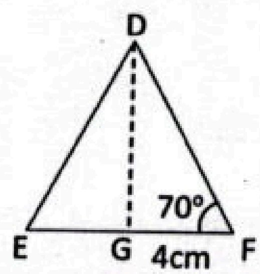
$x^2 = (12)^2 + (22.63)^2$
$x^2 = 144 + 512.12 = 656.12$
Taking square root on both sides, we get:
$x = \sqrt{656.12} = 25.61$
Therefore, EC = 25.61 cm

Example: ABCDEF is an isosceles triangular prism.
BC is 8cm, $\angle DFE = 70^\circ$ and angle FCD = 45°. Find CD.
Solution:
To find CD, we are required DF.
Therefore, first consider ΔDEF to find DF.



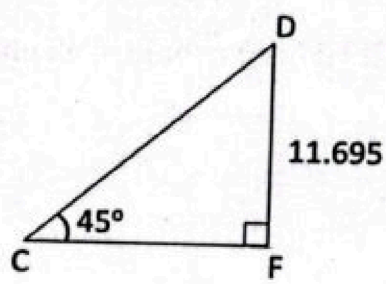
In right ΔDGF:

$$\cos 70^\circ = \frac{4}{DF}$$
$$\Rightarrow DF = \frac{4}{\cos 70^\circ}$$
$$= 11.695 \text{ cm}$$

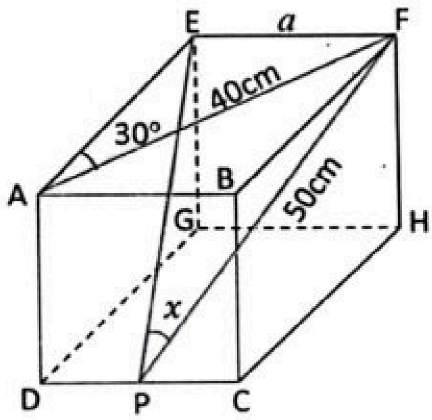


Now in right ΔCFD:

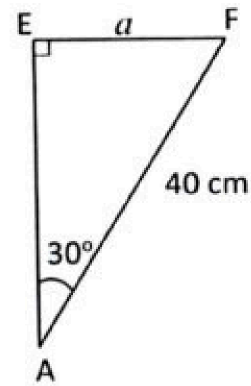
$$\sin 45^\circ = \frac{11.695}{CD}$$
$$\Rightarrow CD = \frac{11.695}{\sin 45^\circ}$$
$$= 16.54 \text{ cm}$$



Example: Find $\angle x$ in the following figure where P is mid-point of $\overline{CD}$.



Solution: To find the value of $\angle x$, we need EF = a which is length of a side of ΔAEF.



In right ΔAEF:

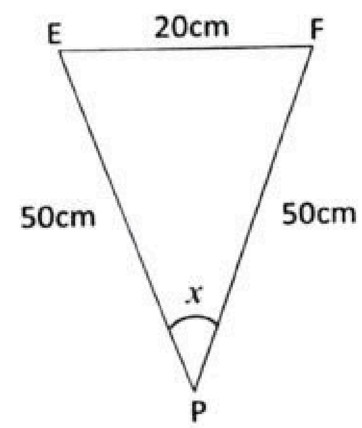
$$\frac{EF}{AF} = \sin 30^\circ$$

$$\Rightarrow \quad \frac{a}{40} = \frac{1}{2}$$

$$\Rightarrow \quad a = 20$$

$$\therefore \quad EF = 20 \text{ cm}$$

As P is a midpoint of $\overline{DC}$, therefore ΔPEF is an isosceles triangle.
Therefore, using law of cosines, we have:



$$\text{Cos } x = \frac{50^2 + 50^2 - 20^2}{2(50)(50)}$$

$$\text{Cos } x = \frac{2500 + 2500 - 400}{5000}$$

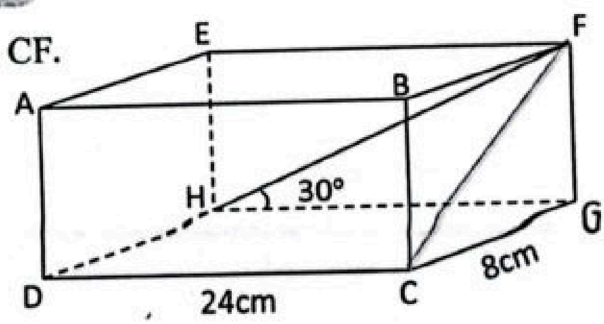
$$\text{Cos } x = \frac{4600}{5000} = 0.92$$

$$\Rightarrow \quad x = \text{Cos}^{-1}(0.92) = 23.1^\circ$$

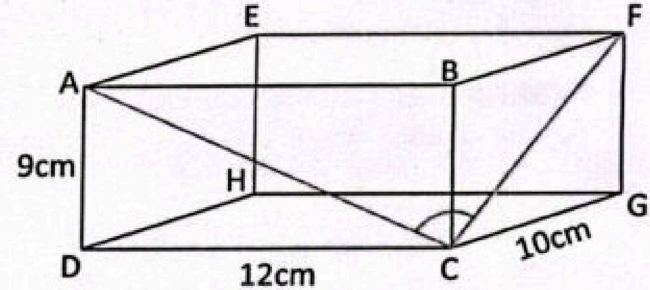
$$\therefore \quad \angle EPF = 23.1^\circ$$

Exercise 8.6

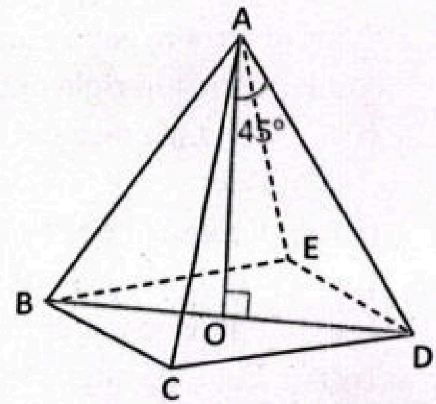
1. In the figure, ABCDEFGH is a cuboid. Calculate CF.



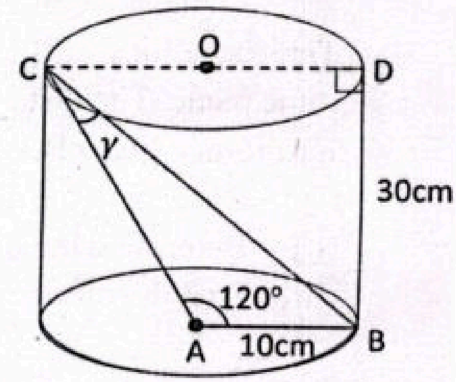
2. Find the angle ACF in the given cuboid in the following figure.



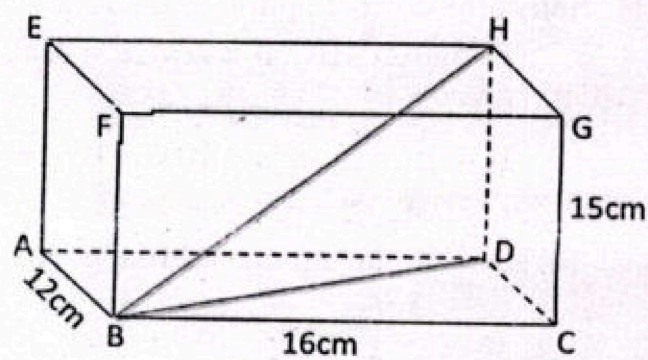
3. In the figure, perimeter of square based pyramid is 36 cm. Find the length of diagonal of the base and height of the pyramid.



4. The diagram shows a cylinder. Find the value of γ in triangle ABC.

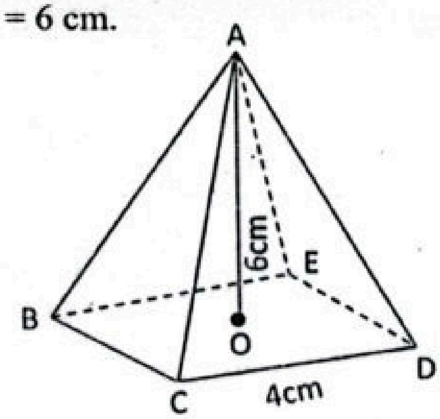


5. In the figure, three workers are standing at positions B, D and H of a container of cuboid shape.



(i) Find the distance between workers at B and D.
(ii) Find the distance between workers at B and H.
(iii) Find $\angle DBH$.

6. In the figure, O is the centre of the square based pyramid ABCDE. Calculate the angle between side AB and plane BCDE if OA = 6 cm.



7. Floor of a room is 9 m long and 6 m wide. The angle of elevation from the bottom left corner to the top right corner of the room is 49°.
   (i) Find the distance from one corner of the floor to the opposite corner of the floor.
   (ii) Find the height of the room.
   (iii) Find the angle of elevation from the bottom corner of the 9 m long wall to the opposite top corner of the wall.
   (iv) Find the angle of depression from the top corner of the 6 m long wall to the opposite bottom corner of the wall.

8. Three satellites A, B, and C, used for GPS navigation are orbiting the Earth in the same plane. The distance between satellites A and B is 15 km. If D is a house on Earth, such that $\angle BAD = 70°$ and ACBD is a rhombus then:
   (i) Determine the distance between satellite A and satellite C.
   (ii) Determine the distance between satellites B and C.
   (iii) Find the distance from satellite C to the house D.

MISCELLANEOUS EXERCISE-8

1. Select the correct option in the following.

   i. In right ΔABC, $a = 2cm$, $c = 4cm$, what is α ?
   (a) 30° (b) 45° (c) 60° (d) 120°

   ii. If in a triangle, $a = 10, b = 15, \alpha = 32°$ then $\beta = ...$
   (a) 42.5° (b) 46.5° (c) 52.7° (d) 62.8°

   iii. Area of an equilateral triangle having side a is:
   (a) $\frac{\sqrt{3}}{8}a$ (b) $\frac{\sqrt{3}}{4}a^2$ (c) $\frac{\sqrt{3}}{16}a^2$ (d) $\frac{\sqrt{3}}{16}a^2$

   iv. If a, a and b are length sides of an isosceles triangle, then $S = ...$
   (a) $\frac{a}{2} + b$ (b) $a + \frac{a+b}{2}$ (c) $a - \frac{a+b}{2}$ (d) $a + \frac{b}{2}$

v. Area of a triangle ABC with $a=20, b=30, \gamma=90^\circ$ is:

(a) 0 (b) 30 (c) 300 (d) 600

vi. For an equilateral triangle, $r : r_1 : R = \ldots$

(a) $1:2:3$ (b) $3:2:1$ (c) $1:1:2$ (d) $1:3:2$

vii. Radius of circum-circle for sides 6, 8, 10 is:

(a) 6 (b) 5 (c) 4 (d) 2

viii. If $a=5$, $b=10$, $c=20$ are sides of a triangle ABC, then angle α is:

(a) not possible (b) acute (c) obtuse (d) 0°

ix. Radius of circum-circle $R = \ldots$

(a) $\frac{a}{2}\sec\frac{\alpha}{2}$ (b) $\frac{b}{2}\sec\frac{\beta}{2}$ (c) $\frac{c}{2}\csc\frac{\gamma}{2}$ (d) $\frac{c}{2}\cos\frac{\gamma}{2}$

x. If in a triangle ABC, $a=b=c$, then $\tan\frac{\alpha}{2} = \ldots$

(a) $\sqrt{\frac{S-a}{a}}$ (b) $\sqrt{\frac{S-b}{b}}$ (c) $\sqrt{\frac{S-c}{c}}$ (d) all (a), (b) & (c)

xi. Shadow of a man 5.6ft tall is making an angle of elevation of 45° with the Sun. What is the length of shadow?

(a) 2.8ft (b) 5.6ft (c) 8.4ft (d) 11.2ft

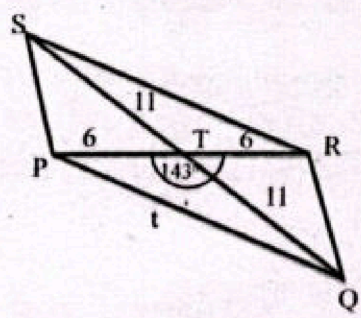
xii. $R(\sin\alpha + \sin\beta + \sin\gamma) = \ldots$

(a) S (b) S^2 (c) $\frac{1}{S}$ (d) $\frac{1}{S^2}$

2. Solve the following oblique triangles.

(i) $a=8, \alpha=15^\circ, \beta=20^\circ$
(ii) $a=34, b=41, \alpha=115^\circ$
(iii) $b=17.3, c=14.5, \beta=98.2^\circ$
(iv) $c=55.4, \alpha=115^\circ, \beta=45^\circ$
(v) $a=34, c=48, \beta=108^\circ$
(vi) $a=44, b=33, c=55$

3. The diagonals of a parallelogram measure 12 centimeters and 22 centimeters and intersect at an angle of 143 degrees. Find the length of the longer sides of the parallelogram.



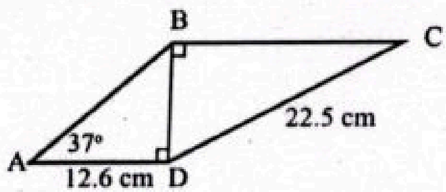
4. Usman and Abubakar follow a triangular path when they take a walk. They walk from home for 1.5*km* along a straight road, turn at an angle of 100°, walk for another 0.95*km* and then return home.
   a. Find the length of the last portion of their walk.
   b. Find the total distance that they walk.
5. A kite is in the shape of a quadrilateral with two pair of congruent adjacent sides. The lengths of two sides are 20.0 inches and the lengths of the other two sides are 35.0 inches. The two shorter sides meet at an angle of 115°.
   a. Find the length of the diagonal between the points at which the unequal sides meet.
   b. Using the answer to part **a**, find the measure of the angle at which the two longer sides meet.
6. Three streets intersect in pairs enclosing a small triangular park. The measures of the distances between the intersections are 55 *m*, 63 *m*, and 77 *m*. Find the area of the park.
7. The base of an isosceles triangle measures 14.5 centimeters and the vertex angle measures 110°.
   a. Find the measure of one of the congruent sides of the triangle.
   b. Find the perimeter of the triangle.
8. Three streets intersect in pairs enclosing a small park. Two of the angles at which the streets intersect measure 80° and 60°. The length of the longest side of the park is 90 *m*. Find the lengths of the other two sides of the park.
9. Aamir wants to draw a parallelogram with the measure of one side 12 *cm*, the measure of one diagonal 10 *cm* and the measure of one angle 120°. Is this possible? Explain why or why not. (Hint: Use law of sines.)
10. The angle of depression from an observer in an apartment complex to a pipe on the building next door is 55°. From a point five stories below the original observer, the angle of inclination to the pipe is 20°. Find the distance from each observer to the pipe and the distance from the pipe to the apartment complex. It is given that one story of a building is 9 feet high.
11. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater?
12. Side of an equilateral triangle is 6cm long. Find the circumference of circum-circle and ex-circle.
13. Use the Law of Cosines to prove that if the angle between two congruent sides of a triangle measures 60°, the triangle is equilateral.
14. Prove that: $\dfrac{1}{r^2}+\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}=\dfrac{a^2+b^2+c^2}{\Delta^2}$

15. Prove that: (i) $r_1 = a\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sec\frac{\alpha}{2}$ (ii) $r_2 = b\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}\sec\frac{\beta}{2}$

(iii) $r_3 = c\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sec\frac{\gamma}{2}$

16. A mountain climber is on top of a mountain that is 680 *m* high. The angles of depression of two points on opposite sides of the mountain are 48° and 32°. How long would a tunnel be that runs between the two points?

17. Find the measure of $\angle C$ in the adjoining figure.



18. Sides of a square prism are 12cm, 5cm and 5cm long as shown below. Find the measure of angle *a*.