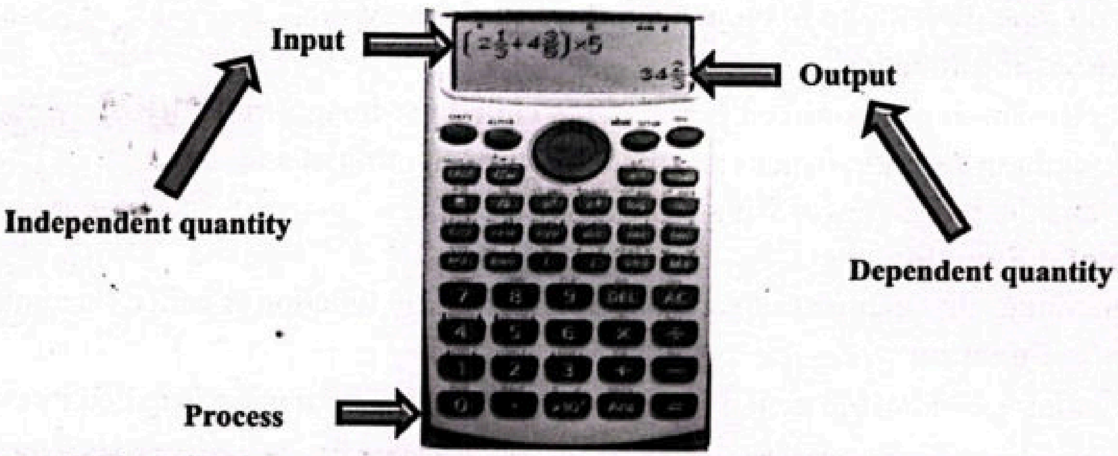
06

Functions And Graphs

After studying this unit students will be able to:

- Recognize function as a rule of correspondence and find domain and range of a function.
- Identify types of functions (into, onto, one-to-one, injective, surjective and bijective) by using Venn diagrams.
- Determine value of a function and perform operations on functions.
- Find inverse of a function.
- Formulate composite functions as: $fg(x) = f(g(x))$.
- Identify, sketch and interpret graphs of constant function, identity function, linear and non-liner function, absolute valued function and function of the type $y = x^n$ where n is a + ve or – ve integer and and $x > 0$
- Determine gradient of the curve through tangent line.
- Solve a system of one linear and one quadratic equation graphically.
- Solve real life problems related to functions.

Every machine needs some input to perform its functions. When we go shopping, we pay our bill at the counter. The cashier inputs the prices of all the items purchased into the calculating machine. The machine performs its function and processes out a bill. This bill is the output produced by the machine. It means every machine needs some input to give some output, i.e; every output is related to some input.



Our calculator is also a machine, we input some values in it and it gives us the result after performing some process.

Cartesian product of any two non-empty sets A and B denoted by $A \times B$ (read as A cross B) is a set containing all ordered pairs (x, y) such that $x \in A$ and $y \in B$. We can write it as:

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

Key Fact:

- If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$
- Number of subsets of $A \times B = 2^{n(A \times B)} = 2^{pq}$

For example, if $A = \{a, b\}$ and $B = \{2\}$, then

$$A \times B = \{(a, 2), (b, 2)\}$$

All the subsets of $A \times B$ are:

$R_1 = \{\ \}$, $R_2 = \{(a, 2)\}$, $R_3 = \{(b, 2)\}$ and $R_4 = \{(a, 2), (b, 2)\}$

Any subset of the cartesian product $A \times B$ is called a binary relation(or simply a relation) from set A to set B. Basically, a binary relation relates an element of one set to an element of the other set. All the above four subsets are binary relations from set A to set B.

Domain of any relation R denoted by dom(R) is a set containing first element of each ordered pair in the relation. $\text{Dom}(R_4) = \{a, b\}$

Range of a relation R denoted by ran(R) is a set containing second element of each ordered pair in the relation. $\text{Ran}(R_4) = \{2\}$

In R_4, 2 is image of a and b but a and b are preimages of 2

Function

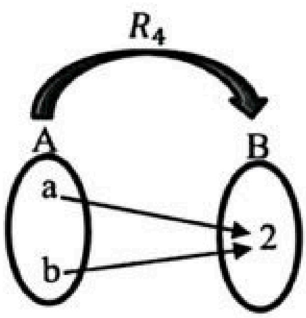
A function f from set A to set B is a relation, rule or mapping which maps each element of set A to a unique element of set B such that domain of f = A and there is no repetition in the first element of ordered pairs in f.

Mathematically, it is written as $f : A \rightarrow B$ and is read as f is a function from A to B.

In the above example in which $A = \{a, b\}$ and $B = \{2\}$,

Consider $R_4 = \{(a, 2), (b, 2)\}$,

which is shown in arrow diagram at right.



Domain of $R_4 = \{a, b\} = A$

There is no repetition in the first element of ordered pairs in R_4.

Therefore, R_4 is a function.

The first element of each ordered pair in R_4 is called pre-image while its corresponding second element is called image of the first element. In (a, 2), a is the pre-image of 2 and 2 is the image of a.

Check Point:

R_1, R_2 and R_3 are not functions. Justify.

Domain of a Function

A set containing first element of each ordered pair in the function is called its domain.

Range of a Function

A set containing second element of each ordered pair in the function is called its range.

Key Fact:

If f is a function from set A to set B, i.e., $f : A \rightarrow B$, then B is called co-domain of f and range of f is a subset of co-domain.

Example: If A = Set of first three natural numbers

B = Set of prime factors of 6

Is f = {(1, 2), (2, 3),(3, 2)} a function from A to B?

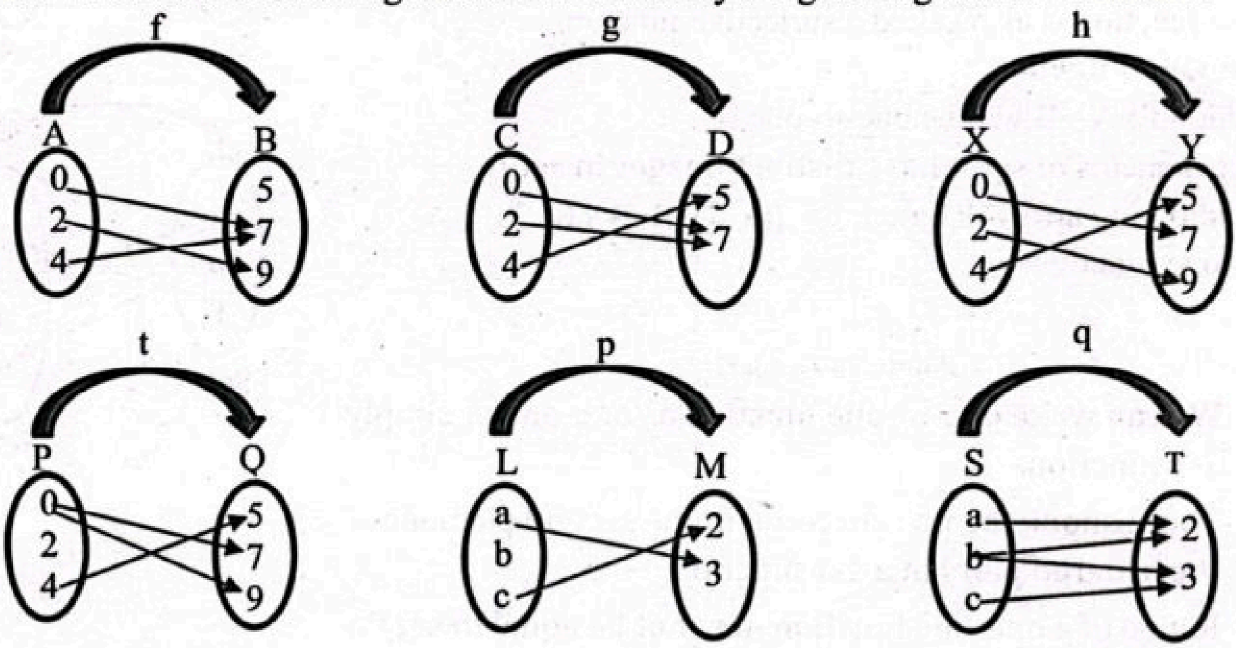
Solution: A = {1, 2, 3} , B = {2, 3}

Here f is a subset of A × B and dom f = {1, 2, 3} = A.

Also, there is no repitition in the first element of the ordered pairs in f, therefore f is a function.

> **Thinking Corner!**
> Every function is a relation but its converse is not true.
> Do you agree?

Example: Which of the following relations indicated by the given figures are functions?



Solution: Here f, g and h are functions as the domain of each of them is equal to their corresponding first set and no first element is repeated in their respective mapping.

But in mapping t, dom(t) ≠ P(∵ 2 has no image in set Q) and there is repetition of 0 (∵ it is mapped to two elements 7 and 9 of Q), therefore t is not a function.

In mapping p, dom(p) ≠ L (∵ b has no image in set M), so p is not a function.

In mapping q, although dom(q) = S but b is repeated (∵ it is the pre-image of two elements 2 and 3 of T), therefore q is not a function.

Types of Functions

Range of a function is required for identifying the type of a function.

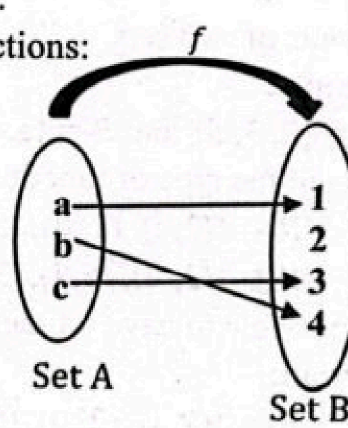
For any two non-empty sets A and B, these are the five types of functions:

Into Function

A function f : A→B will be into if there is atleast one element in set B which is not an image of any element of set A.

i.e, range(f) ⊂ B.

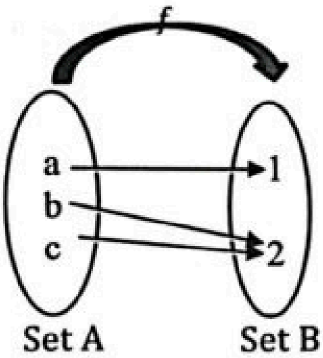
In the adjoining arrow diagram, f = {(a, 1), (b, 4), (c, 3)}is an into function.



> **Key Fact:**
> In into function, range f ≠ B.

Onto Function

A function f : A→B will be onto if very element of set B is an image of atleast one element of set A. i.e, range(f) = B.



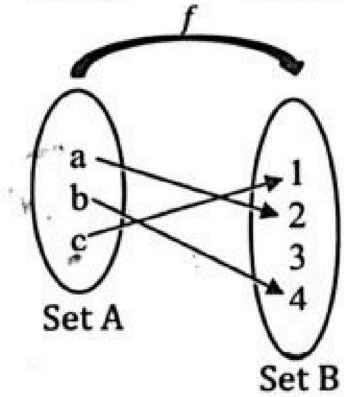
In the adjoining arrow diagram, f = {(a, 1), (b, 2), (c, 2)} is an onto function.

An onto function is also called a surjective function.

One-to-One Function

A function f : A→B will be one-to-one if distinct elements of set A have distinct images in set B.

In the adjoining arrow diagram, f = {(a, 2), (b, 4), (c, 1)} is a one-to-one function.



Points To Ponder!

- We can write one-to-one function as one-one or simply 1-1 function.
- There should be no repetition in the second element of the ordered pairs in a 1-1 function.
- Range of a one-one function may not be equal to set B.

Injective Function

A function which is into as well as 1-1 is called an injective function.

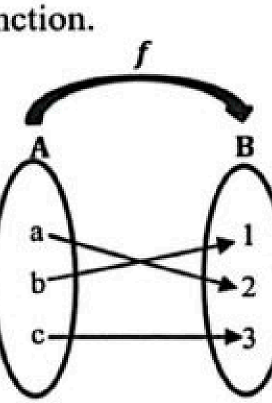
In the above arrow diagram, f = {(a, 2), (b, 4), (c, 1)} is an injective function.

Bijective Function

A function which is both onto and 1-1 is called a bijective function.

In the arrow diagram at right,

f = {(a, 2), (b, 1), (c, 3)} is a bijective function.



Bijective function shows one-one correspondence between the elements of two sets.

Example:

If A = {2, 3, 5} and B = {a, e, i, o, u}, which of the following relations from A to B are functions? Also tell the type of function.

(i) R_1= {(2, i), (2, o), (5, u), (3, e)} **(ii)** R_2= {(3, i), (2, o), (5, i)}

(iii) R_3= {(2, u), (5, i), (3, o)}

Is it possible to have a function from A onto B?

Solution:

(i) R_1= {(2, i), (2, o), (5, u), (3, e)}

$\text{Dom}(R_1) = \{2, 3, 5\} = A$, but there is repitition in the first element of first two ordered pairs, therefore R_1 is not a function.

(ii) $R_2 = \{(3, i), (2, o), (5, i)\}$

$\text{Dom}(R_2) = \{2, 3, 5\} = A$ and there is no repitition in the first element of the ordered pairs in R_2, therefore R_2 is a function. $\text{Range}(R_2) = \{i, o\} \subset B$ which implies that R_2 is an into function.

(iii) $R_3 = \{(2, u), (5, i), (3, o)\}$

$\text{Dom}(R_3) = \{2, 3, 5\} = A$ and there is no repitition in the first element of the ordered pairs in R_3, therefore R_3 is a function.

$\text{Range}(R_3) = \{i, o, u\} \subset B$ which implies that R_3 is an into function. Also, there is no repitition in the second element of the ordered pairs in R_3, so R_3 is one-one function.

Hence, R_3 is an injective function.

Also number of elements in set B is more than number of elements in set A, therefore A onto B function is not possible.

Evaluating a Function

A function is usually written as y = f(x), in which the values of y (dependent variable) depends upon the values of x (independent variable) and is read as 'y is a function of x' or simply y is equal to f of x'.

Evaluation of a function is a process of finding the value of dependent variable by substituting any specific value of the independent variable.

Example: Find the value of $f(x) = -2x^3 + x^2$ at:

(i) $f(-1)$ **(ii)** $f(t)$ **(iii)** $f(\frac{-1}{2})$ **(iv)** $f(x+2)$

Solution: **(i)** Substituting $x = -1$ in $f(x) = -2x^3 + x^2$, we have

$$f(-1) = -2(-1)^3 + (-1)^2 = -2(-1) + 1 = 3$$

(ii) Replacing x by t in f(x), we have

$$f(t) = -2(t)^3 + (t)^2 = -2t^3 + t^2$$

(iii) Substituting $x = \frac{-1}{2}$ in f(x), we have

$$f(\tfrac{-1}{2}) = -2\left(\frac{-1}{2}\right)^3 + \left(\frac{-1}{2}\right)^2 = -2\left(\frac{-1}{8}\right) + \frac{1}{4} = \frac{1}{2}$$

(iv) Replacing x by $x + 2$ in f(x), we have

$$f(x+2) = 2(x+2)^3 + (x+2)^2 = -2x^3 - 16 - 12x^2 - 24x + x^2 + 4x + 4$$
$$= -2(x^3 + 8 + 6x^2 + 12x) + (x^2 + 4x + 4)$$
$$= -2x^3 - 11x^2 - 20x - 12$$

Example: The value of a photocopy machine t years after its purchase is given by
$V(t) = 9500 - 860t$ dollars., Find:

(i) V(5) and tell what does V(5) mean?

(ii) Find t when $V(t) = 6920$ and explain what it represents.

Solution:

(i) $V(t) = 9500 - 860t$

$V(5) = 9500 - 860 \times 5 = 5200$ dollars

It means the value of machine after 5 years will be 5200 dollars.

(ii) When $V(t) = 6920$, then $6920 = 9500 - 860t$

$\Rightarrow 860t = 9500 - 6920 = 2580$

$\Rightarrow t = \frac{2580}{860} = 3$

Hence, t = 3, which tells us that the value of photocopy machine will be 6920 dollars after 3 years.

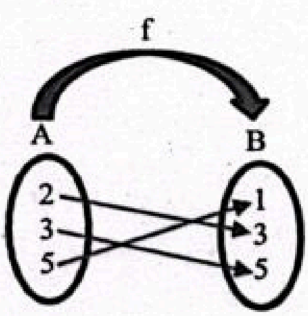
Exercise 6.1

1. If $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5\}$, which of the following relations are functions from A to B?

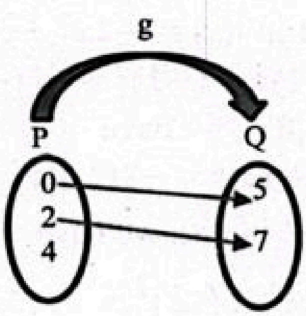
   (i) $R_1 = \{(2, 3), (6, 5), (8, 3), (4, 1)\}$
   (ii) $R_2 = \{(2, 3), (6, 1), (8, 3), (6, 5)\}$
   (iii) $R_3 = \{(2, 3), (6, 5), (8, 3)\}$
   (iv) $R_4 = \{(2, 3), (6, 3), (8, 3), (4, 1)\}$

2. Identify the functions from the following diagrams. Also write their type.

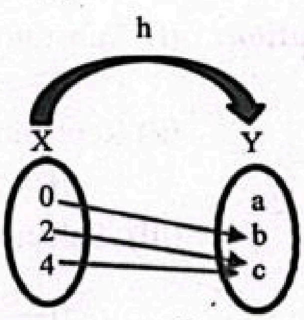
(i)



(ii)



(iii)



3. If $f(x) = x^2 - \frac{1}{2}x + 3$, evaluate the following.

   (i) f(2) (ii) f(−1) (iii) $f(\frac{2}{3})$ (iv) $f(t + 1)$

4. If X = Set of prime factors of 6 and Y = Set of first three non-negative integers, check whether the function defined by $f : X \rightarrow Y = \{(x, y) \mid x - y = 1\}$ is injective or bijective.

5. A function f is defined by $f(x) = t - ax$, find the values of a and t if 0 and 2 are the images of 1 and -1 respectively.
6. A function f is defined by $f(x) = mx + c$, is f bijective? Justify your answer.

Operations on Functions

Operations on functions are the ways of combining functions to create new functions. Some common operations on any two functions $f(x)$ and $g(x)$ are:

(i) Addition of functions: $(f+g)(x) = f(x) + g(x)$.

For example, if $f(x) = x^2$ and $g(x) = 2x$, then

$(f+g)(x) = f(x) + g(x) = x^2 + 2x$

(ii) Subtraction of functions: $(f-g)(x) = f(x) - g(x)$.

For example, if $f(x) = 2x^2 + 3x$ and $g(x) = x^2 - 2x$, then,

$(f-g)(x) = f(x) - g(x) = (2x^2 + 3x) - (x^2 - 2x)$

$= 2x^2 + 3x - x^2 + 2x = x^2 + 5x$

(iii) Multiplication of functions: $(f \times g)(x) = f(x) \times g(x)$.

For example, if $f(x) = x^2$ and $g(x) = -2x$, then $(f \times g)(x) = f(x) \times g(x)$

$= x^2 \times (-2x) = -2x^3$

(iv) Division of functions: $(f \div g)(x) = f(x) \div g(x)$, where $g(x) \neq 0$

For example, if $f(x) = x^2$ and $g(x) = 2x$, then $(f \div g)(x) = f(x) \div g(x)$

$= x^2 \div (2x) = \frac{x^2}{2x} = \frac{x}{2}$

(v) Composition of functions: The composition of $f(x)$ and $g(x)$ is a function denoted by fog and is defined as:

$$(fog)(x) = f(g(x))$$

For example, if $f(x) = x^2$ and $g(x) = 2x - 1$, then:

$$(fog)(x) = f(g(x)) = f(2x-1) = (2x-1)^2 = 4x^2 - 4x + 1$$

In other words, the output of inner function $g(x)$ will become the input of the outer function $f(x)$.

In this example, we first applied the function g to x, which resulted in $2x - 1$, then applied the function f to the result which gave us $(2x-1)^2$.

Furthermore, for finding $(gof)(x)$, $f(x)$ will be the input of the outer function $g(x)$.

So, $(gof)(x) = g(f(x)) = g(x^2) = 2x^2 - 1$

Key Fact:

- $(fog)(x)$ and $(gof)(x)$ can simply be written as $fg(x)$ and $gf(x)$ respectively.
- $f(f(x)) = f^2(x)$ and $f(f^2(x)) = f^3(x)$

Example: Aahil wants to have a new bike which costs 24000 rupees but he has only a saving of 12000 rupees. If his parents have agreed to give him additional 1000 rupees per week for doing chores. However, he also spends 500 rupees per week on snacks. How much will Aahil have as his saving after 10 weeks. Also, tell in how many weeks he will be able to save the money for getting a new bike?

Solution: Aahil's saving function: $S(x) = 12000 + 1000x$, where x represents number of weeks

His spending function: $T(x) = 500x$

Total money with Aahil after x weeks $= S(x) - T(x)$

$$= 12000 + 1000x - 500x = 12000 + 500x$$

After 10 weeks, saving with Aahil $= 12000 + 500(10) = 17000$ rupees

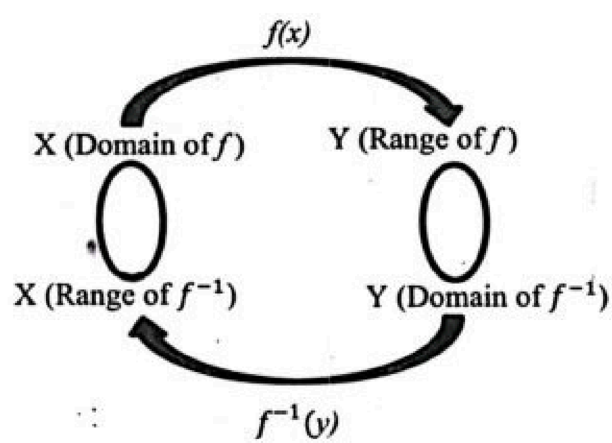
Now, $24000 = 12000 + 500x \Rightarrow 24000 - 12000 = 500x \Rightarrow x = \frac{12000}{500} = 24$

Hence, he can save required amount in 24 weeks.

Inverse of a Function

The inverse of any function $f(x)$ is a function denoted by $f^{-1}(x)$ which reverses the effect of $f(x)$ and it undoes what $f(x)$ does. If $f: X \to Y$, then $f^{-1}: Y \to X$, therefore inverse of only that function exists which is bijective(both one-one and onto).

Thus, if $y = f(x)\ \forall\ x \in X$, then $x = f^{-1}(y)\ \forall\ y \in Y$. It can be illustrated in the following diagram as :



Key Fact:
Mathematically,
$f(f^{-1}(x)) = f^{-1}(f(x)) = x$
and $(f^{-1})^{-1} = f$

It is clear from above diagram that domain of f = range of f^{-1} and range of f = domain of f^{-1}.

Finding the Inverse of an Algebraic Function

If $f(x)$ is any algebraic function, then its inverse $f^{-1}(x)$ involves the following steps:

I. Write $y = f(x)$

II. Find the value of x in terms of y from the equation of step I, means write x in terms of y.

III. Replace x by $f^{-1}(y)$ in the resulting equation of step II.

IV. Replace each y by x in the resulting equation of step III to get $f^{-1}(x)$.

The result obtained in step IV can be verified by showing $f^{-1}(f(x)) = x$.

Example: If $f: R \to R$ is defined by $f(x) = 3x + \frac{1}{2}$, find $f^{-1}(x)$.

Solution: Writing $f(x) = y$, we have,

$$y = 3x + \frac{1}{2}$$

$$3x = y - \frac{1}{2} = \frac{2y-1}{2} \quad \Rightarrow \quad x = \frac{2y-1}{6} \qquad \text{(Isolating } x\text{)}$$

So, $f^{-1}(y) = \frac{2y-1}{6}$

Hence, $f^{-1}(x) = \frac{2x-1}{6}$ (Replacing each y by x)

Exercise 6.2

1. If $f(x) = 4(x-1)$ and $g(x) = x^2 - 2x + 1$, find
   (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(f \times g)(x)$ (iv) $(f \div g)(x)$
2. If $f(x) = 4x$ and $g(x) = x + 1$, find,
   (i) $(fog)(x)$ (ii) $(gof)(x)$ (iii) $(fof)(x)$ (iv) $(gog)(x)$
3. Find $(fog)(x)$ and $(gof)(x)$, if
   (i) $f(x) = 3-2x$ and $g(x) = x + 1$
   (ii) $f(x) = \frac{2}{x}$ and $g(x) = \frac{2x}{x-1}$
   (iii) $f(x) = 3x$ and $g(x) = \frac{2}{\sqrt{x-1}}$
   (iv) $f(x) = x^2 - 1$ and $g(x) = \sqrt{x-1}$
4. If $f(x) = x^2$ and $g(x) = 2x + 1$, find the value of x if:
   (i) $(fog)(x) = (gof)(x)$ (ii) $f(x) = g(x)$ (iii) $(gof)(x) = 9$
5. Find the inverse of the following functions.
   (i) $f(x) = 2x - 1$ (ii) $g(x) = \frac{2}{x-3}$, where $x \neq 3$
   (iii) $f(x) = \sqrt{x+5}$, where $x \geq -5$ (iv) $g(x) = (x-3)^2$, where $x \geq 3$
6. If $f(x) = \frac{3}{x-5}$ and $g(x) = x + \frac{3}{2}$, find the value of $f^{-1}(1)$ and $g^{-1}\left(\frac{1}{2}\right)$.

Graphs of Some Algebraic Functions

Polynomial Functions

A function $f: R \to R$ defined as $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + +a_{n-1}x^{n-1} + a_nx^n$, where $a_0, a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are all real numbers and n (called the degree of polynomial) is a non-negative integer is called a polynomial function.

For example, $f(x) = 2x^3 - x^2 + x$ is a polynomial function of degree 3 having leading coefficient 2.

Key Fact:

- The highest exponent of the variable involved in a polynomial is called its degree.
- The coefficient of highest degree term in a polynomial is called the leading coefficient.

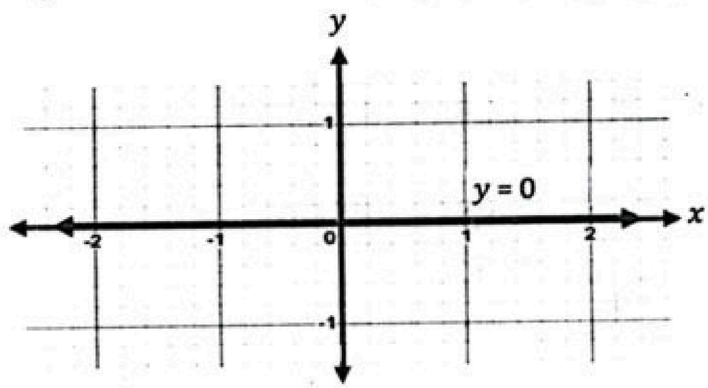
(i) Zero Polynomial Function

A polynomial function of no degree is called a zero polynomial function. This function is of the form:

$y = f(x) = 0$ in which $a_0, a_1, a_2, a_3, \dots, a_{n-1}, a_n$ are all zero.

The graph of $y = 0$ is a straight line that lies on the x-axis indicating that the value of y is always zero regardless of the value of x. It passes through origin and extends infinitely in both positive and negative x-directions.

The graph of $y = 0$ is shown in the given coordinate plane.



Do You Know?

A vertical line intersects the graph of a function only at one point. If it intersects the graph at more than a point, then that graph will not be the graph of a function. This is called vertical line test.

(ii) Constant Function

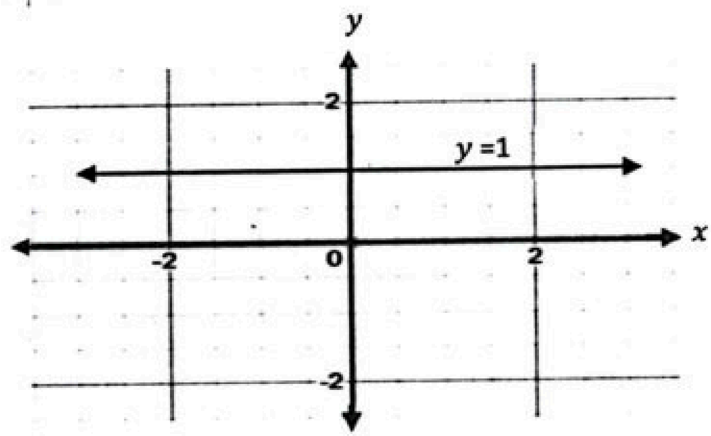
A polynomial function of degree zero is called a constant function.

This function is of the form,

$$y = f(x) = a = ax^0,$$

where a is any non-zero constant.

For example, $y = 3$, $y = \frac{-1}{2}$ etc.



The graph of $y = 1$ (shown in the given graph) is a straight line parallel to x-axis.

(iii) Linear Function

A polynomial function of degree one is called a linear function. This function is of the form, $y = f(x) = ax + b$, where $a, b \in R$ and $a \neq 0$. For example, $y = 2x + 3$, $y = 2x$ etc.

The graph of a linear equation is a straight line. For sketching the graph of a linear function, we first find two points which satisfy the given equation. It is convenient to find x-intercept and y-intercept for getting two points.

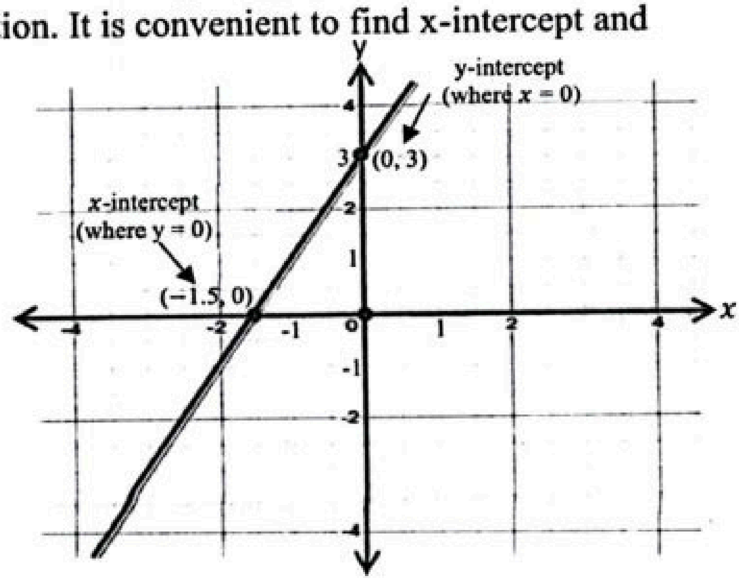
To calculate x-intercept, substitute $y = 0$ in $y = 2x + 3$ and to calculate y-intercept, substitute $x = 0$ in the equation.

***x*-intercept:** Put $y = 0$, $0 = 2x + 3$

$$\Rightarrow x = -\frac{3}{2} = -1.5$$

So, the point where the graph intersects x-axis is $(-1.5, 0)$.

***y*-intercept:** Put $x = 0$, $y = 3$



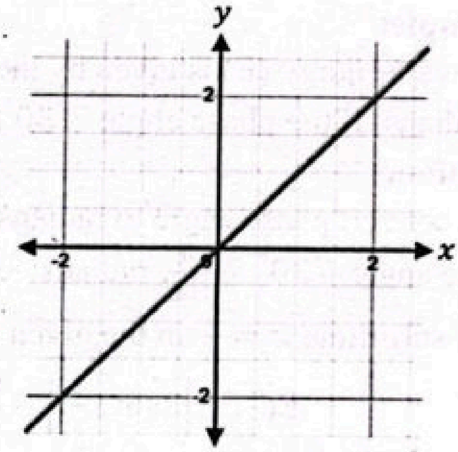
The point where the graph intersects y-axis is $(0, 3)$.
From the graph, it is clear that this line does not pass through origin. We can find the coordinates of any point on the graph just by taking its distance from the axis.

(iv) Identity Function

An identity function can be represented as:
$y = f(x) = x$, means it has the same value as the input.

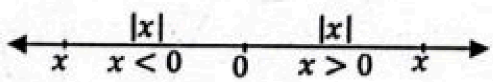
For example, $f(2) = 2$, $f(-3) = -3$ etc.

It is a linear function and its graph is a straight line bisecting first and third qudrants passing through origin.

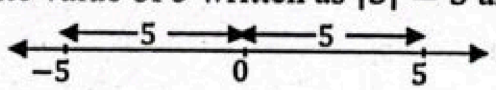


(v) Absolute Valued Function

The absolute value of a real number is its distance from 0 on the number line.



For example, the absolute value of 5 written as $|5| = 5$ and of -5 written as $|-5| = 5$.

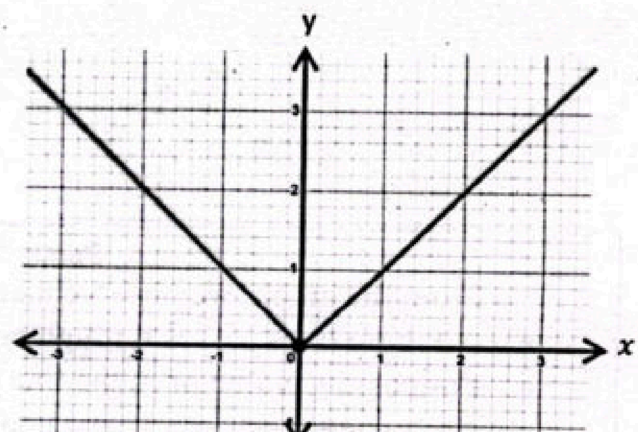


The absolute value of a real number x denoted by $|x|$ is defined as:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The function defined by $f(x) = |x|$, is called the absolute valued function or the modulus function.

Its graph is V-shaped with vertex at origin.
It bisects first quadrant for $x > 0$ and bisects second quadrant for $x < 0$.
Its domain is set of all real numbers but range is set of non-negative real numbers.



Absolute valued functions are widely used in many real life problems including distance, displacement, magnitude and to calculate energy wave etc.

Example:

The profit (in rupees) of a company is given by the function $P(x) = |2x - 100|$, where x is the number of units sold. If 120 units are sold, what is the profit?

Solution:

Profit function, $P(x) = |2x - 100|$

Substituting $x = 120$, $P(120) = |2 \times 120 - 100| = 140$

Thus, the company makes a profit of 140 rupees.

Example:

A wave's energy (E) is given by the function $E(x) = |3\sin(x)|$, where x is the wave's phase angle in radians. If the phase angle is 30°, what is the wave's energy ?

Solution:

First convert phase angle to radians as under:

Phase angle $= 30^\circ \times \frac{\pi}{180^\circ}$ radians $= \frac{\pi}{6}$ radians

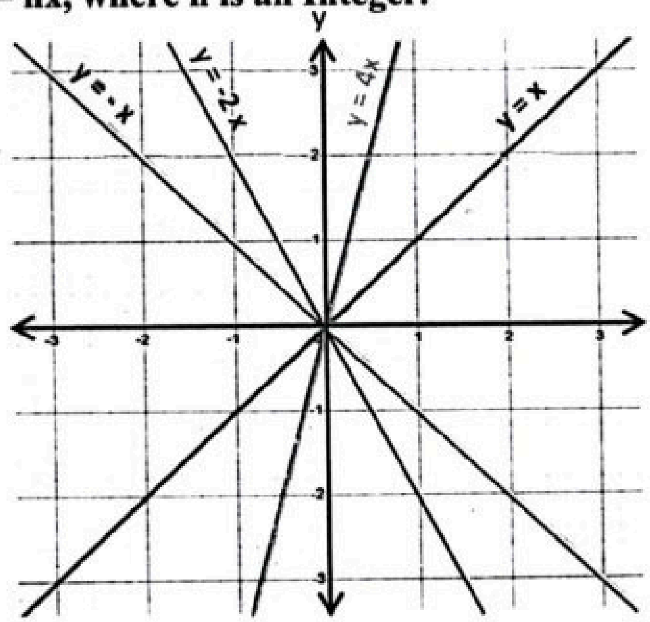
Now substitute $x = \frac{\pi}{6}$ in the given function,

$$E\left(\frac{\pi}{6}\right) = \left|3\sin\left(\frac{\pi}{6}\right)\right| = \left|3 \times \frac{1}{2}\right| = \frac{3}{2}$$

Hence, the wave's energy is $\frac{3}{2}$ units.

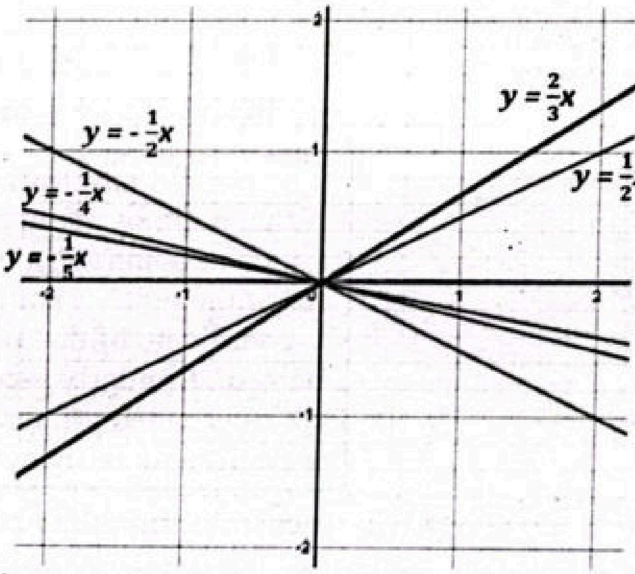
(vi) Graphs of the functions of the form y = nx, where n is an Integer.

For the graph of the function $y = nx$;
n is a + ve or – ve integer and and $x > 0$, we take different integral values of n.
The graph for $n = -2, -1, 1, 4$ is drawn at right.



> **Points To Ponder!**
>
> From the graph, we observe that:
> - All these lines passes through origin.
> - For positive value of n, the line passes through first and third quadrant.
> - For negative value of n, the line passes through second and fourth quadrant.

Similarly, we can sketch the linear graphs for fractional values of n as drawn here.



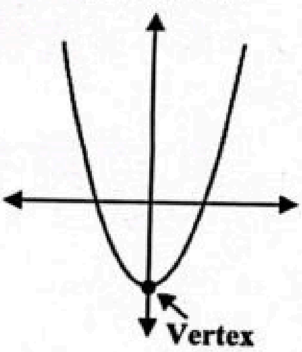
Identifying, Sketching and Interpreting Graphs of Some Non-Linear Functions

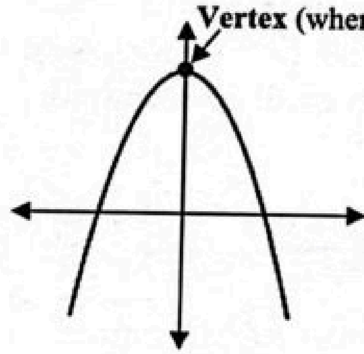
(i) Quadratic Function

A polynomial function of degree two is called a quadratic function. This function is of the form, $f(x) = ax^2 + bx + c \quad ; a, b, c \in R$ and $a \neq 0$.

For example, $y = x^2 - 2x - 3$, $y = -4x^2 + 1$ are quadratic functions.

The graph of a quadratic function is a U-shaped curve called parabola that opens either upward (if $a > 0$) or downward (if $a < 0$).





The vertical line passing through the vertex of parabola is called 'axis of symmetry' as the graph is symmetrical about y-axis.

In order to sketch the graph of such quadratic functions, follow the following steps.

Step-1: Identify the shape of the graph whether it opens upward or downward.

Step-2: Find x-intercept and y-intercept.

Step-3: Find the vertex of the parabola.

For finding vertex of the parabola, we proceed as follows:

$$y = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

(Isolate the constant term and take coefficient of x^2 as common from first two terms)

$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$ (Adding and subtracting square of half of the coefficient of x to complete the square)

$= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c$

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ --------→ (i)

Enlighten Yourself!
If the graph of a function intersects x-axis at point$(a, 0)$, then x-coordinate 'a' of the point is called x-intercept and y-coordinate of that point will always be zero. Similarly y-coordinate 'b' of $(0, b)$ is called y-intercept in which x-coordinate is always zero.

Comparing (i) with

$y = a(x - h)^2 + k$ --------→ (ii) (general form of the curve)

We have,

$h = \frac{-b}{2a}$ and $k = c - \frac{b^2}{4a}$

So, the graph of the curve, y $= ax^2 + bx + c$ is a parabola with vertex at (h, k). The point (h, k) is the turning point of the graph. The curve is symmetrical about the line $x = h$.

To sketch the graph of parabola, we draw its vertex , x- and y-intercepts and some other points to get a smooth curve.

Example:

Sketch the graph of y $= x^2 - 2x - 3$.

Solution:

$y = x^2 - 2x - 3$ ----------→ (i)

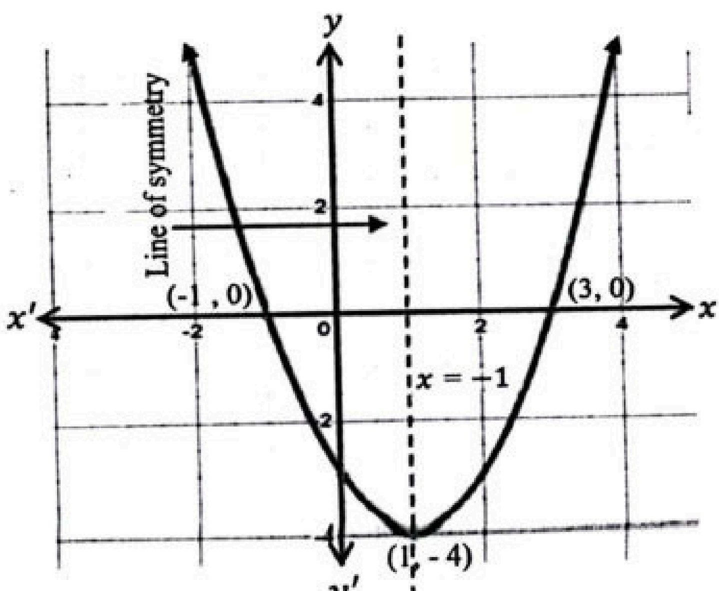
Comparing (i) with y $= ax^2 + bx + c$, we have $a = 1, b = -2$ and $c = -3$

Here, $a = 1 > 0$, so the curve will open upward.

$h = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$ and

$k = c - \frac{b^2}{4a} = -3 - \frac{(-2)^2}{4(1)} = -4$

So, the **vertex** of parabola is $(h, k) = (1, -4)$.



For **y-intercept**, put $x = 0$, y $= -3$

For **x -intercept**, put y $= 0$.

$0 = x^2 - 2x - 3$

$0 = x^2 - 3x + x - 3$

$0 = x(x - 3) + 1(x - 3)$

$0 = (x - 3)(x + 1)$

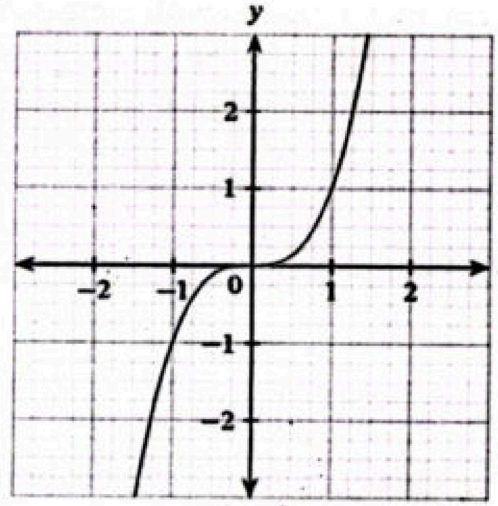
This implies, either $x - 3 = 0$ or $x + 1 = 0$

$\Rightarrow x = 3$ or $x = -1$

So, the graph will intersect x-axis at $(3, 0)$ and $(-1, 0)$. The parabola is symmetrical about the vertical line $x = -1$ through the turning point.

(ii) Cubic Function

From bottom left to top right



$y = x^3$

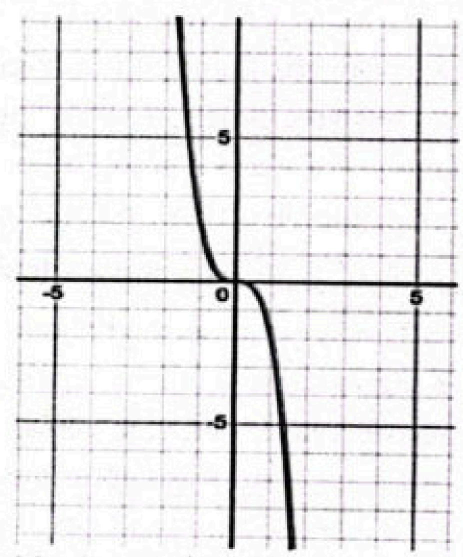
A polynomial function of degree three is called a cubic function. This function is of the form:
$f(x) = ax^3 + bx^2 + cx + d \quad ; a, b, c, d \in R$ and $a \neq 0$.

We will discuss here the graph of the functions $y = ax^3$.

To sketch the graph of $y = x^3$(when $a = 1$) , we need the shape, x and y intercepts and turning point.

Here leading coefficient is 1 which is positive which shows that the graph will concave up in the first quadrant and concave down in the third quadrant. For each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. At $x = 0$, y = 0, so this graph intersects both the axis at origin and is symmetric about origin as shown in the figure. Also origin is the turning point of this graph. The domain and range of this function is the set of all real numbers.

From top left to bottom right



Example:
Sketch the graph of $y = -2x^3$.

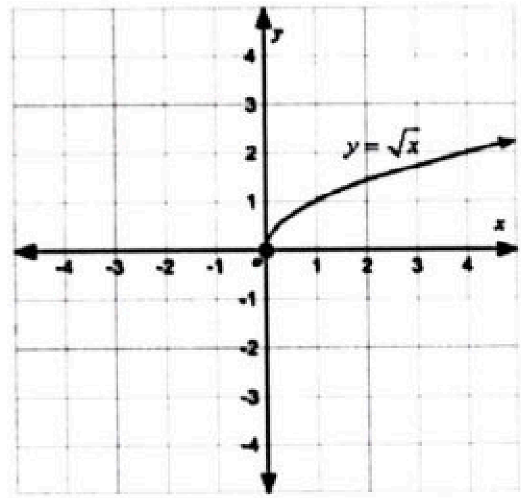
Solution: $y = -2x^3$ --------→ (i)

Here, the leading coefficient is -2 (negative), which shows that the graph will concave up in the second quadrant and concave down in the fourth quadrant.

At $x = 0$, y = 0, so the graph passes through origin which is the turning point of the graph.For each point $(-x, y)$ on the graph, the point $(x, -y)$ is also on the graph, so this graph is symmetric about origin. The domain and range of this function is the set of all real numbers.

(iii) Square Root Function

The function defined by $f(x) = \sqrt{x}$, where $x \geq 0$ is called a square root function.



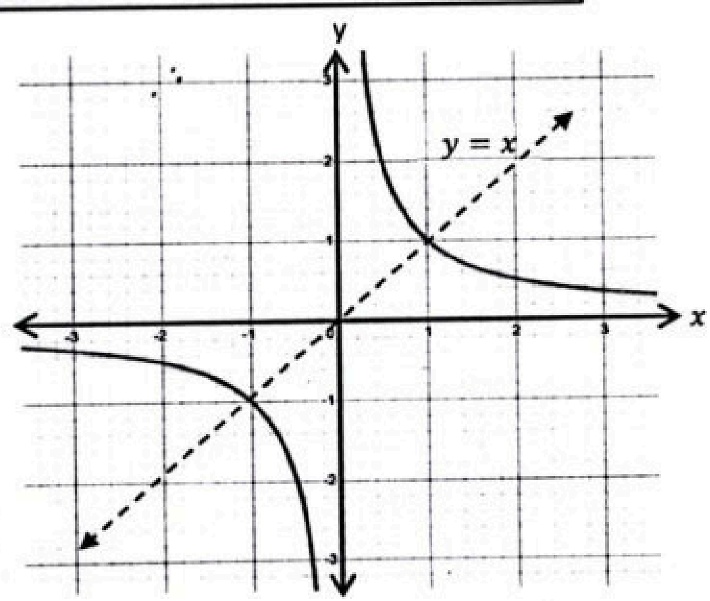
The graph of this function will remain in first quadrant and increases with increase in the the value of x as shown in the given graph.
The domain and range of this function is the set of non-negative real numbers.

(iv) Reciprocal Function

The reciprocal function is of the form $f(x) = \frac{k}{x}$,
where $x \neq 0$ and k(a constant)$\neq 0$. We will discuss here simple example of reciprocal function $f(x) = \frac{1}{x}$. The graph of reciprocal function shows how the function's output decreases as the input increases illustrating inverse relationships.

Enlighten Yourself!

- The reciprocal of a very large number is a very small number.
- The reciprocal of a very small number is a very large number.
- The reciprocal of zero is undefined.



First we draw the graph of the original function $y = x$ whose reciprocal is to be drawn.
The graph of $y = x$ is a straight line passing through origin bisecting first and third quadrants.
The graph of $y = \frac{1}{x}$ is a curve which is undefined at $x = 0$ and is symmetrical about the line $y = x$.
As we increase the value of x , the value of the function decreases and vice versa.
This graph never touches coordinate axis.
$y \to \infty$ as $x \to 0$, but it will never touch y-axis and
$x \to \infty$ as $y \to 0$, but it will never touch x-axis.
The domain of this curve is all real numbers except zero.

Have You Noticed?

- The graph of $f(x) = \frac{1}{x}$ has two branches.
- The graph of $y = \frac{1}{x}$ exists in the first and third quadrants only.
- As $x \to \infty$, $f(x) \to 0$ from above.
  As $x \to -\infty$, $f(x) \to 0$ from below.
  As $x \to 0$ from the right, $f(x) \to \infty$.
  As $x \to 0$ from the left, $f(x) \to -\infty$.

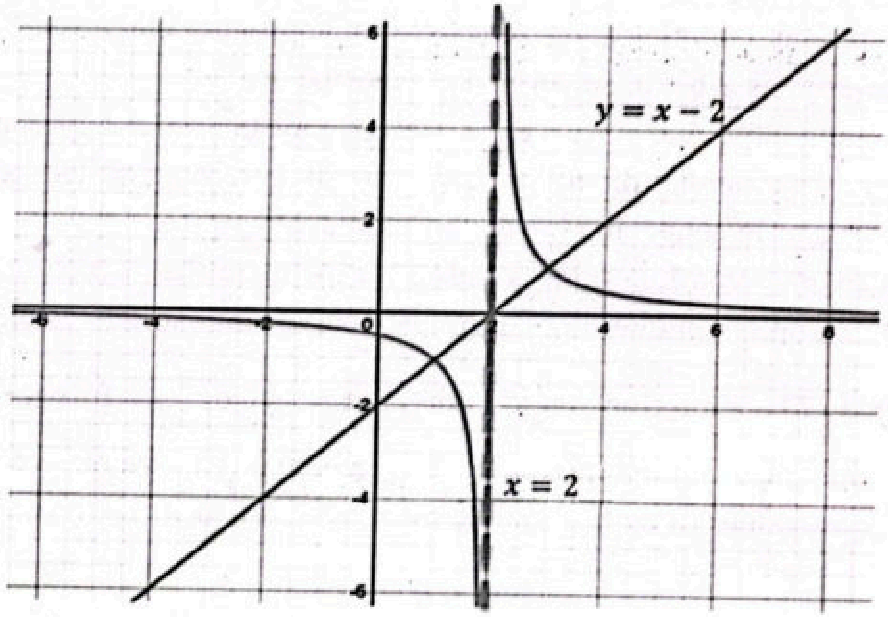
Enlighten Yourself!

The symbol ∞ was first used by John Wallis in 1655 which means unbouded.

Example:

Sketch the graph of $y = \frac{1}{x-2}$.

Solution



y-intercept: $y = \frac{-1}{2} = -0.5$

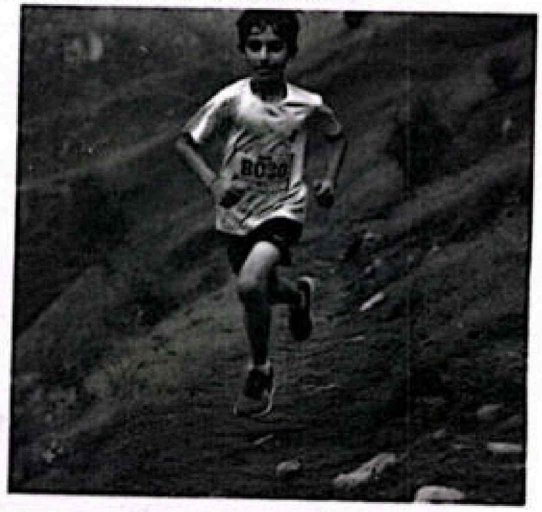
As $x \to \infty$, $y \to 0$

As $x \to -\infty$, $y \to 0$

Example:

Saim can jog 4 miles downhill in the same time which he takes to jog 2 miles uphill. He takes downhill 3mph faster than he jogs uphill. Find his jogging rate uphill and downhill.

Solution:



Let his uphill rate $= x$

Then, his downhill rate $= x + 3$

Uphill time $= \frac{2}{x}$

Downhill time $= \frac{4}{x+3}$

According to the given condition,

downhill time = uphill time

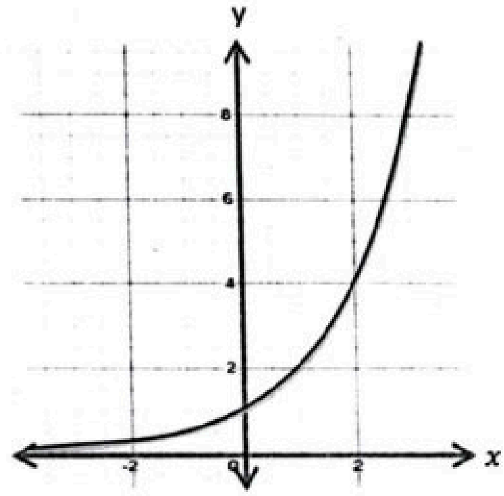
$$\Rightarrow \frac{4}{x+3} = \frac{2}{x}$$

$$\Rightarrow 4x = 2(x+3)$$

$$\Rightarrow 4x = 2x + 6 \Rightarrow 4x - 2x = 6 \Rightarrow x = 3$$

Thus, his uphill rate = 3mph and his downhill rate $= 3 + 3 =$ 6mph

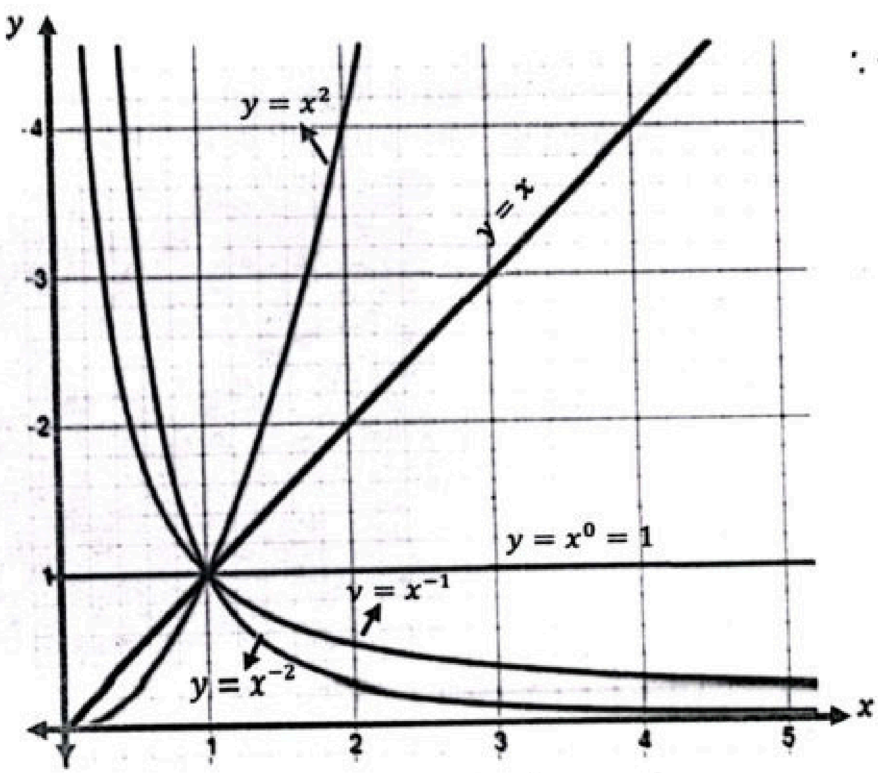
(v) Exponential Function



The most simple exponential function is of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
For example, $y = 2^x$ is an exponential function.
For sketching its graph, we find:
y-intercept: $y = 1$
Asymptote: Horizontal asymptote, $y = 0$
As x becomes large and negative, the graph of $y = 2^x$ becomes closer to x -axis but never quite reaches it.
We can say that as x approaches $-\infty$, y approaches 0.
Similarly, as x becomes large and positive, y also becomes large and positive, so we can say that as x approaches ∞, y also approaches ∞. The simple exponential functions have smooth increasing or decreasing graphs. All exponential graphs are similar in shape and have a horizontal asymptote.
The exponential functions are used for the quantities which increase or decrease exponentially. For example, population of animals, people and bacteria usually grow in an exponential way.

(vi) The Function $y = x^n$, where n is +ve or −ve Integer and $x > 0$

For the graph of the function $y = x^n$, where n is a + ve or – ve integer and $x > 0$, we take different integral values of n. The graphs for $n = -2, -1, 0, 1, 2$ are shown in the given graph.



> **Enlighten Yourself!**
> The graph of $y = x^n$(where n is +ve or −ve integer and $x > 0$):
> - Is a horizontal line at n = 0.
> - Bisects first quadrant at n = 1.
> - Will remain in first quadrant.
> - Passes through (1, 1).

Solution of a Linear and a Quadratic Equation Graphically

As we know that the graph of a linear equation is a straight line and of a quadratic equation is a curve. The point of intersection of these two graphs is a point where both the graphs intersect each other.

Example:

Solve $f(x) = 2x + 1$ and $g(x) = x^2 - x + 3$ graphically.

Solution:

Table of values for $f(x) = 2x + 1$

x	-2	-1	0	1	3
$f(x)$	-3	-1	1	3	7

For the graph of $g(x) = x^2 - x + 3$, we compare it with $y = ax^2 + bx + c$, we have: $a = 1, b = -1$ and $c = 3$.

Here, $a = 1 > 0$, so the curve will open upward.

Also $h = \frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2} = 0.5$

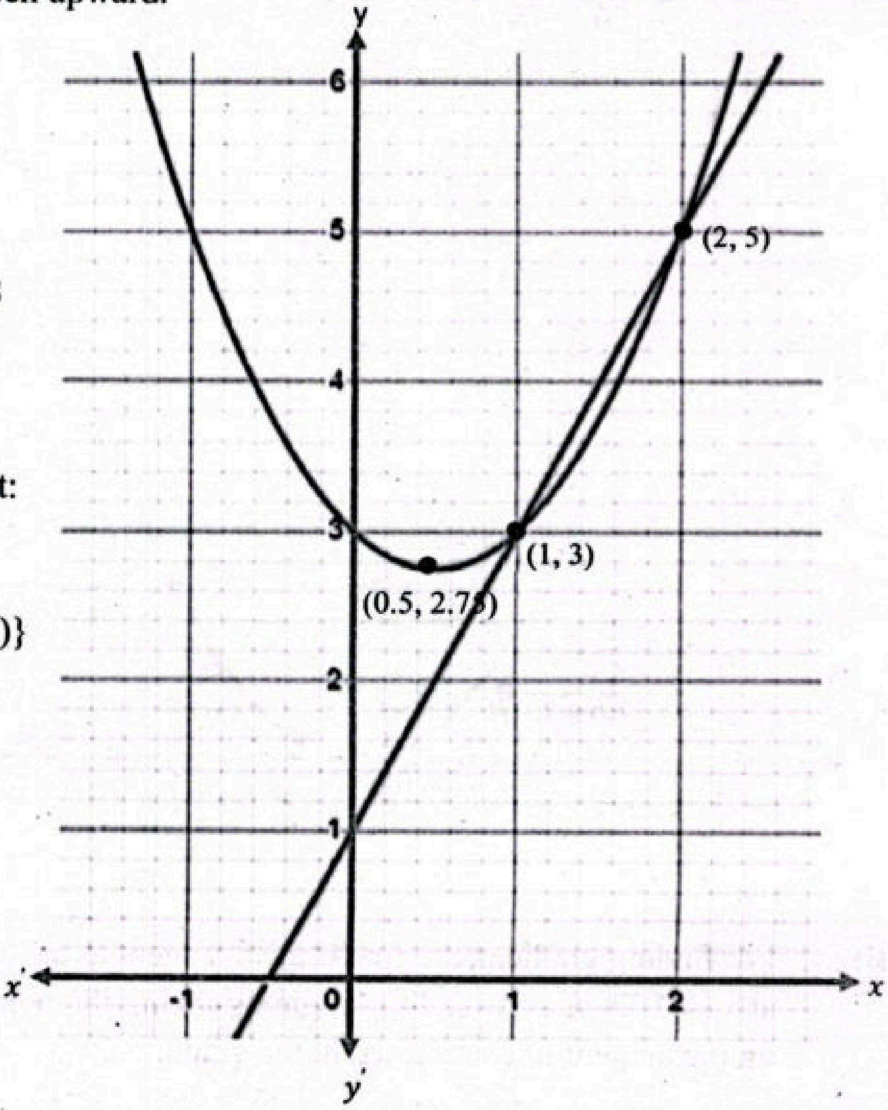
and $k = c - \frac{b^2}{4a} = 3 - \frac{(-1)^2}{4(1)} = 2.75$

So, the vertex of parabola is:

$(h, k) = (0.5, 2.75)$.

Table of values for $g(x) = x^2 - x + 3$

x	-1	0	0.5	1	2	3
$g(x)$	5	3	2.75	3	7	9



Both the graphs intersect each other at: $(1, 3)$ and $(2, 5)$.

Hence, solution set = $\{(1, 3) \text{ and } (2, 5)\}$

Determining the Gradient of Curves through Tangents

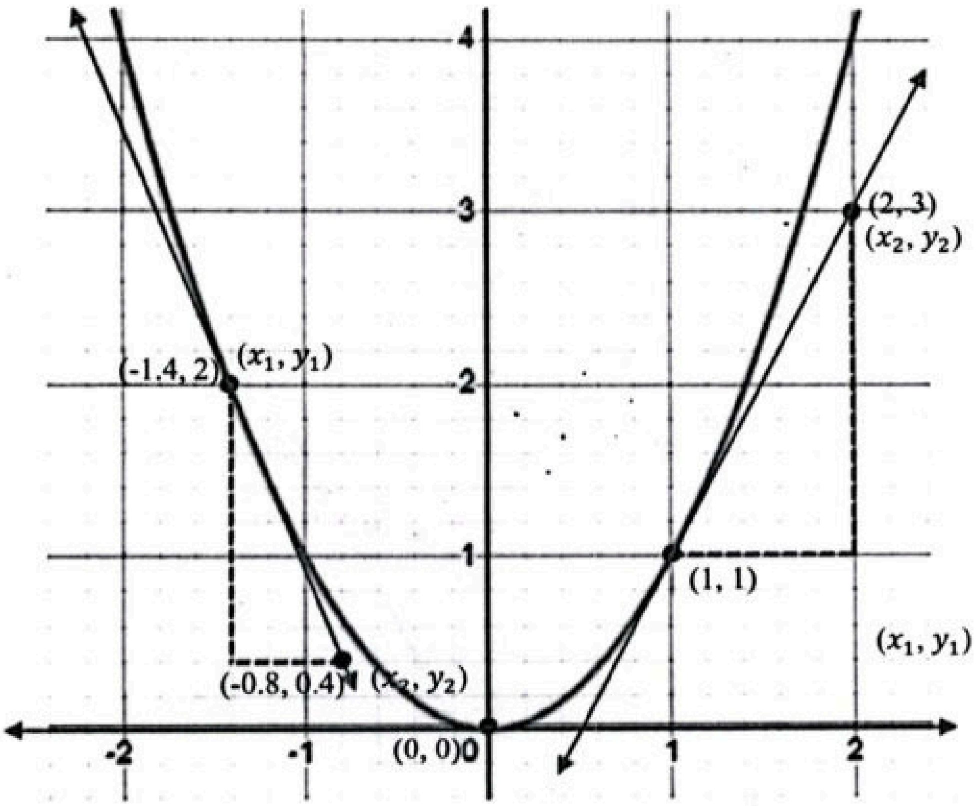
The gradient of a curve at a given point is the slope of the tangent line drawn to the curve at that point. It is obtained by drawing a tangent to the curve at that point. The tangent to a curve is a line which touches the curve onle at one point externally. The slope of tangent to the curve is its gradient. The slope of the tangent $= \frac{\text{change in } y}{\text{change in } x}$ $(\frac{rise}{run})$.

So, the gradient of the curve $= \frac{\text{change in } y}{\text{change in } x} = \frac{y_2-y_1}{x_2-x_1}$

Caution!
Make sure that the angle of tangent line with the curve below and above the given point must be same. The gradient of the curve can also be found by taking any two points on the tangent line.

Example:
Find the gradient to the curve $y = x^2$ drawn below at:
(i) $x = 1$ **(ii)** 0 **(iii)** -1.4.

Solution: The graph of the given curve is drawn here.



(i) For finding gradient, we first draw a tangent to the curve with equal space on either side at $x = 1$.Then draw right triangle by taking the point at which $x = 1$ and any other point on the tangent line as shown in the graph.

The gradient of the tangent $= \frac{y_2-y_1}{x_2-x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{3-1}{2-1} = 2$

Hence, the gradient of the curve at $x = 1$ is 2.

(ii) The tangent at $x = 0$ is a horizontal line, so its slope is zero. Hence, the gradient of the curve at $x = 0$ is 0.

(iii) Similarly by drawing a tangent to the curve at $x = -1.4$, we have a right triangle as shown in the graph. By taking the point $(-1.4, 2)$ and any other point $(-0.8, 0.4)$ on the tangent line, we have,

The gradient of the tangent $= \frac{y_2-y_1}{x_2-x_1}$

$$= \frac{0.4-2}{-0.8+1.4} = \frac{-1.6}{0.6} \approx -2.7$$

Key Fact:

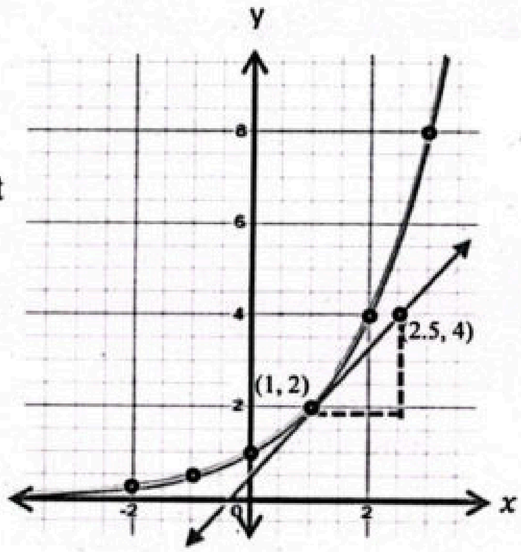
The slope of the vertical line is undefined.

Yo Be Noted!

- If the tangent line comes down from left to right, the slope of tangent line will be negative.
- If the tangent line goes up from left to right, the slope of tangent line will be positive.

Example:

Estimate gradient to the given curve at (1, 2).



Solution: We first draw a tangent to the curve with equal space on either side at (1, 2).Then draw right triangle at the point at (1, 2).and any other point on the tangent line as shown in the graph.

The gradient of the tangent $= \frac{y_2-y_1}{x_2-x_1}$

$$= \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{4-2}{2.5-1} = 1.33$$

Hence, the gradient of the curve at (1, 2) is estimated as 1.33.

Real Life Problems Related to Functions and Graphs

Graphs and functions are essential tools in various aspects of daily life. They have wide applications often unconsciously in solving many real life problems including distance and time, supply and demand, temperature and time, population growth, health and fitness, weather forecasting, financial planning, time management, medical research, environmental monitoring, fitness tracking etc. By applying graphs and functions to daily life problems, we can develop a more analytical approach to decision making and problem solving.

Here are some problems involving functions and their graphs.

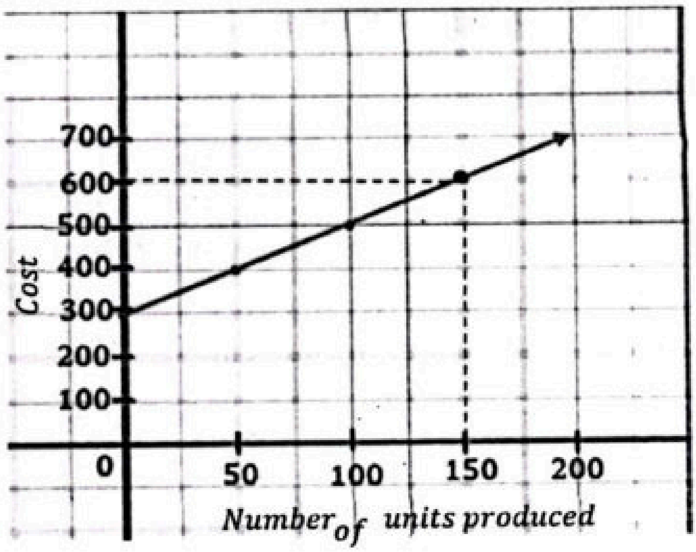
Example: A company's cost function is $C(x) = 2x + 300$, where x is the number of units produced and cost is given in dollars. Graph the cost function and find the cost of producing 150 units.

Solution:

Here, $C(x) = 2x+300$

Table of values

x	0	50	100	150
$C(x)$	300	400	500	600



From the graph, cost of producing 150 units is 600 dollars.

Example: A researcher studies the spread of a disease and finds that the number of infected people grows according to the function $f(t) = 10(1.2)^t$, where t is time in days. Determine the number of people to become infected in next 6 days.

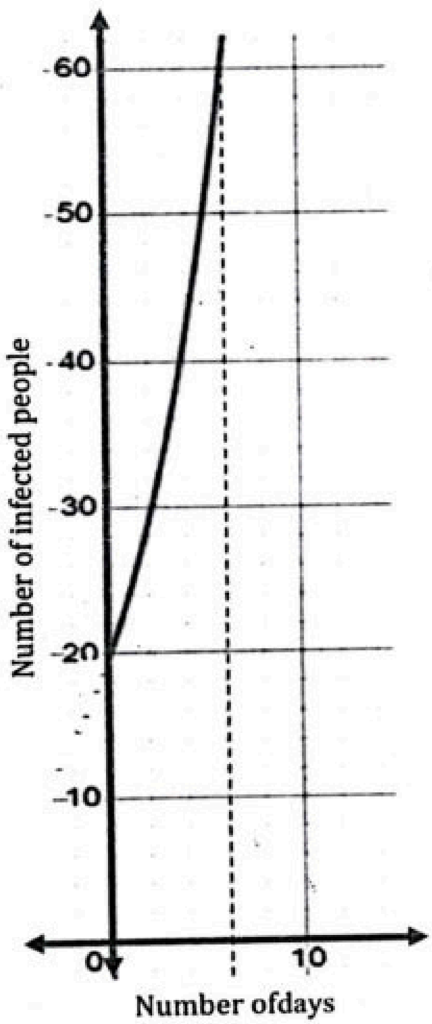
Solution:

Here, $f(t) = 20(1.2)^t$

Table of values

t	0	2	4	6
$f(t)$	20	29	41	60

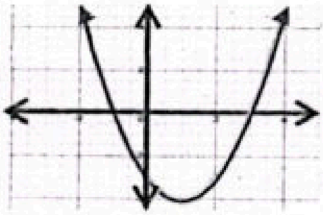
From the graph, number of people to be infected in next 6 days = 60



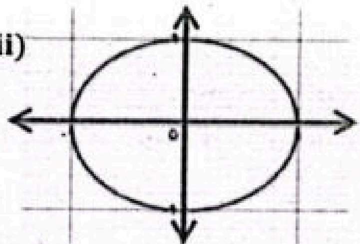
Exercise 6.3

1. Identify the graphs of linear and non-linear functions from the following? Also indicate the graphs which are graphs of functions by applying vertical line test.

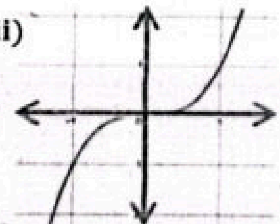
(i)



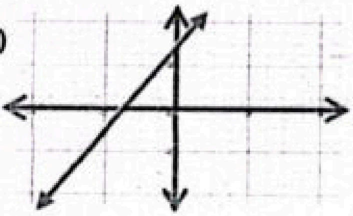
(ii)



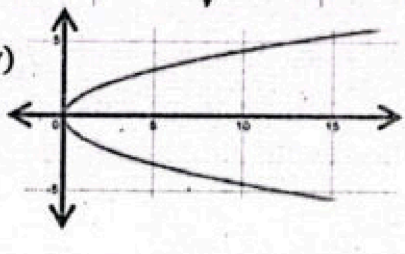
(iii)



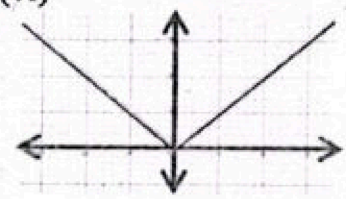
(iv)



(v)



(vi)



2. Sketch the graph of the following functions and identify the graphs which are linear.
   (i) $y = 3x + 2$ (ii) $y = x$ (iii) $y = x^2$
3. Sketch the graph of the following functions.
   (i) $y = 3^x$ (ii) $y = |x|$ (iii) $y = x^3$
   (iv) $y = \sqrt{x+2}$ (v) $y = \frac{1}{x-1}$
4. Check whether the curve (parabola) drawn for the following functions will open upwards or downwards. Also find x-intercept, y-intercept and sketch the graph of the given functions.
   (i) $f(x) = (x-1)^2$ (ii) $f(x) = -x^2 + 2$
   (iii) $f(x) = 3 - (x+2)^2$ (iv) $f(x) = -x^2 + 2x + 3$
5. Find the values of h and k if $y = hx^2 + 2x + k$ cuts x-axis and y-axis at $(-3, 0)$ and $(0, 2)$ respectively.
6. Solve the following equations graphically.
   (i) $f(x) = x + 2$, $g(x) = x^2 - 3x + 2$ (ii) $f(x) = 2x + 5$, $g(x) = 2x^2 + 1$
7. If demand function of a product is $D(x) = 100 - 5x$ and supply function is $S(x) = x - 200$. Draw graphs of both functions on same graph by using same scale and find the value of x for which the supply will become equal to demand?
8. The growth of the population of a town is given by the function, $P(t) = 50{,}000(1.05)^t$, where t is the time in years. Draw the graph of this function and find the population growth over 5years.
9. Draw gradient of the curve $y = \frac{1}{2}x^2$ by drawing tangent line at $x = 2$.
10. A logistics company charges a base fee of 50 dollars and additional charges of 0.5 dollars per mile for transporting goods. If a shipment travels 250 miles, what is the total cost?

I have Learnt

- Cartesian product of any two non-empty sets A and B is a set containing all ordered pairs (x, y) such that $x \epsilon A$ and $y \epsilon B$.
- A function f from set A to set B is a relation, rule or mapping which maps each element of set A to a unique element of set B.
  - A function f : A→B will be into if range(f) ⊂ B.
  - A function f : A→B will be onto if range(f) = B.
  - A function f : A→B will be one-one if unique element of A have unique image in B.
  - A function f : A→B will be injective if it is both into and one-one.
  - A function f : A→B will be bijective if it is both onto and one-one.
- Every bijective function has one-one correspondence between its elements.
- The process of finding the value of dependent variable by substituting any specific value of the independent variable is called evaluation of a function.
- For any two functions f and g:
  $(f+g)(x) = f(x) + g(x)$, $(f-g)(x) = f(x) - g(x)$
  $(f \times g)(x) = f(x) \times g(x)$, $(f \div g)(x) = f(x) \div g(x)$
- The composition of any two functions, $f(x)$ and $g(x)$ is denoted by $(fog)(x) = f(g(x))$.
- If $y = f(x)\ \forall\ x \in X$, then $x = f^{-1}(y)\ \forall\ y \in Y$.
- The domain of $f(x)$ = range of $f^{-1}(x)$ and range of $f(x)$ = domain of $f^{-1}(x)$.
- A polynomial function of no degree is called a zero polynomial function.
- A polynomial function of degree zero is called a constant function.
- A polynomial function of degree one is called a linear function.
- A polynomial function of degree two is called a quadratic function.
- The graph of the curve, $y = ax^2 + bx + c$ is a parabola with vertex at (h , k), where $h = \frac{-b}{2a}$ and $k = c - \frac{b^2}{4a}$. It will open upward when $a > 0$ and open downward when $a < 0$.
- A polynomial function of degree three is called a cubic function.
- The function of the form, $f(x) = \sqrt{x}$, where $x \geq 0$ is called a square root function.
- The function of the form, $f(x) = |x|$ is called the absolute valued function.
- The function defined by $f(x) = \frac{1}{x}$, where $x \neq 0$ is called a reciprocal function.
- The exponential function is of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
- The point of intersection of two graphs is called their solution point.

MISCELLANEOUS EXERCISE-6

1. Encircle the correct option in each of the following.

   i. If A = {−2, 0, 2} and B = {0, 2} and f : A→B is defined as f = {(−2, 2), (0, 0), (2, 0)} what type of function is f ?

   (a) into (b) onto (c) injective (d) bijective

ii. If number of elements in set X is 3 and number of elements in set Y is 2, how many binary relations are possible from Y to X?

(a) 4 (b) 6 (c) 2^6 (d) 2^9

iii. What is the domain of the relation g = {(1, 0), (2, 2), (3, 4)}?

(a) {0, 1, 2, 3} (b) {0, 2, 4} (c) {1, 2, 3} (d) {0, 1, 2, 3, 4}

iv. What is the x-intercept of every point on y-axis?

(a) 0 (b) 1 (c) −1 (d) undefined

v. At what point will the graph of $y = 2x^2 - 1$ cuts y-axis?

(a) $(\pm\frac{1}{\sqrt{2}}, 0)$ (b) −1 (c) (0, −1) (d) (−1, 0)

vi. If $y = 2x - 1$, what is $f^{-1}(x)$?

(a) $\frac{1+y}{2}$ (b) $2y-1$ (c) $\frac{1+x}{2}$ (d) $y+1$

vii. If $f(x) = \frac{1}{2}x$, what is $f^2(x)$?

(a) $\frac{1}{4x}$ (b) $\frac{1}{4}x$ (c) $2x$ (d) $\frac{1}{4}x^2$

viii. If $y = \frac{x}{x-2}$, for what value of x will the function become undefined?

(a) 0 (b) −2 (c) 2 (d) ±2

ix. Which of the following is an exponential function?

(a) $\left(\frac{1}{3}\right)^x$ (b) e^x (c) 2^x (d) All of these

x. If $f(x) = \frac{2}{3}x^2 - 5$, what is the value of f(−3)?

(a) 0 (b) 1 (c) −3 (d) −1

2. Find inverse of $f(x) = \frac{3x}{2x-1}$. At what value of the variable, $f^{-1}(x)$ will be undefined? Also find $f^{-1}(-1)$.
3. Sketch the graph of the quadratic function $y = -x^2 - 4x$. Will it open upward or downward?
4. Find the equation in the form of $y = ax^2 + bx + c$ which cuts x-axis at (−1, 0) and (1, 0) and y-axis at (0, 10).
5. For the function f defined by $f(x) = x^2 - x - 6$, find the value of x if f(x) = f(3).
6. Find the inverse function to convert Celsius to Fahrenheit, if $C = \frac{5}{9}(F-32)$ and use it to convert $25^\circ C$ to Celsius graphically.
7. Find the point of intersection of the graphs of $f(x) = x - 2$ and $g(x) = x^2 - 4x + 2$.
8. When medication is taken by a patient, it is slowly used by the body. After n hours, the amount of medicine remaining in the body is given by $f(n) = 120\times(0.9)^n$mg. Draw the graph of f(n) against n and use the graph to find when there is 20mg of the medicine left in the body.
9. A sound wave's amplitude (A) is given by the function, $A(t) = |2\cos(t)|$, where t is time in seconds. If t = 5 seconds, what is the amplitude?