05

Algebraic Fractions

After studying this unit students will be able to:

- Describe rational expression.
- Factorize and simplify rational expression.
- Demonstrate manipulation of algebraic fractions.
- Performs operations on rational expressions.
- Solve rational equations.
- Apply the concept of rational equations to real world problems.

Use of letters to represent an unknown quantity was introduced by "Rene Descartes", a French Mathematician, in 1637. Today '*x*' is used by most nations as the standard letter for a single unknown. Infect x–rays were so named because the scientists who discovered them did not know what they were and thus labeled them the 'unknown rays' or x–rays. Algebra is basically an extension of Arithmetic.

X–ray

A factory worker wants to work overtime to fulfill his basic necessities of life. The factory owner has decided to give him Rs. 60 per hour for working extra. But sometimes he does not have enough time to work overtime daily. So he cannot find his accurate monthly income for working extra. If we represent overtime of the worker in hours for a month by an unknown quantity say 't', then we can say that his salary for working overtime will be '60t'.

Algebraic Expressions

A statement in which variables or constants or both are connected by arithmetic operations (i.e. $+, -, \times, \div$) is called an algebraic expression.
For example,

$$\frac{-5x^2 + 4}{4},\ 3(a+b)-4,\ 0,\ -5$$

$$z - \sqrt{2}\, t,\ \frac{1}{x},\ \frac{x}{y+1}\ \ \sqrt{b^2-4ac} \text{ etc.}$$

History:

Muhammad Bin Musa Al–Khawarizimi was the first Muslim mathematician who introduced Algebra and wrote a book entitled Hisab–Al–Jabr Wal Muqabala in 820 A.D. He was known as 'Father of Algebra'.

Key Fact:

An algebraic expression may contain:
- Numbers
- Signs of operations $(+, -, \times, \div)$
- Variables $(a, b, c, \dots, x, y, z)$
- Grouping symbols $\overline{\ \ }$, ( ), { }, [ ]

Kinds of Algebraic Expressions

Algebraic expressions are of three kinds.
- Polynomial Expressions
- Rational Expressions
- Irrational Expressions

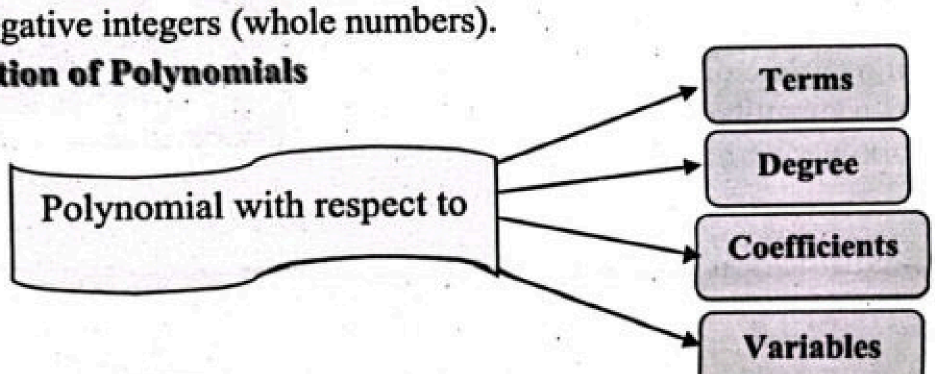
(a) Polynomial Expressions (Polynomials)

Polynomials are algebraic expressions consisting of one or more terms in which exponents of the variables involved are non-negative (either zero or positive) integers.
For example:

$$0,\ -2,\ \frac{3}{4}x + \frac{3}{4}y^2z,\ -\sqrt{\frac{3}{9}}\,y^3,\ \sqrt{2}x^4 - \pi x^2 - \sqrt{10} \text{ etc.}$$

The expressions x^{-3}, $y^2+\frac{1}{y^2}$, $\sqrt[3]{y^4}$, $2y^{\frac{1}{2}}$ are not polynomials because their exponents are not non negative integers (whole numbers).

Classification of Polynomials



(a) Types of Polynomials w.r.t. Terms

- A polynomial having one term is called a monomial. e.g., 3, x, 0 etc.
- A polynomial having two terms is called a binomial.

  e.g. $a+b$, $2x+y$, $7ab-\frac{1}{3}$, $x^{\frac{1}{3}}+2\frac{1}{4}$ etc.
- A polynomial having three terms is called a trinomial.

  e.g. $x^2+2xy+y^2$, $\frac{1}{8}x^3+\frac{3}{4}x^2-9$ etc.

All other polynomials having more than three non- zero terms are called polynomials of four terms, five terms, six terms and so on.

Key Fact:

When a polynomial is written in descending order, then the first coefficient is called the leading coefficient. e.g., in $p(x)=3x^4-2x^2+1$, the leading coefficient is 3. Here $p(x)$ means that the variable in the polynomial is 'x'.

(b) Types of Polynomials w.r.t. Degree

- **Zero polynomial or no degree polynomial:**
  '0' is called a polynomial of no degree. Also, $0x^3+0x$ is a no degree polynomial, because coefficients are always zero in zero polynomial.
- **Constant polynomial:**
  A polynomial having degree zero is called a constant polynomial. e.g. $2, -5, \frac{1}{2}, \sqrt{5}$ are all constant polynomials.
- **Linear polynomial:**
  A polynomial having degree one is called a linear polynomial. e.g., x, $2x-y$, $-7xy^0$ etc.
- **Quadratic polynomial:** A polynomial having degree two is called a quadratic polynomial. e.g., $2x^2+7$, $ax+2xy+3$, $-\frac{3}{4}xyz^0$ etc.
- **Cubic polynomial:**

  A polynomial having degree three is called a cubic polynomial. e.g. $9x^3-7x+5$, $9xzy$, $3x^2y-\frac{3}{4}z$ etc.

All other polynomials have no specific name w.r.t. degree but simply, we call them polynomials of degree four, degree five and so on.

(c) Types of Polynomials w.r.t. Coefficients

Polynomials can also be classified according to the coefficients of the variables involved in each term of them. e.g., $-3x^2+4x-1$ is a polynomial with integers co-efficients because $-3, 4, -1 \in Z$.

Similarly, $\frac{1}{2}y^3 - \pi x^2 y + \frac{22}{7}$ is a polynomial with real numbers because $\frac{1}{2}, -\pi, \frac{22}{7} \in R$.

Just like this we can also check the coefficients in any polynomial and classify it accordingly.

(d) Types of Polynomials w.r.t. Variables

Classification of polynomials is also possible with respect to the variables involved in them. e.g., $x^2y - 7$ is a polynomial in two variables x, y. We can express it as $P(x, y) = x^2y - 7$.

Similarly, $6x^2yz + 12xy^6z^6 + 9z^4$ is a polynomial in three variables x, y, z. We may write it as $Q(x, y, z) = 6x^2yz + 12xy^6z^6 + 9z^4$.

Recall

The highest exponent of the variable involved in a polynomial is called its degree. If more than one variable are being multiplied in terms of a polynomial. Then the degree of that polynomial is the sum of exponent of all the variables present in a term.

(b) Rational Expression

An algebraic expression of the form $\frac{P(x)}{Q(x)}$,where $P(x)$and $Q(x)$ are polynomials and $Q(x) \neq 0$

(i.e. it is not a zero polynomial) is called a rational expression.

For example: $1\frac{1}{2}$, $\frac{3}{4x^2}$, $\frac{2x-1}{x^2+3}$, $\frac{2x+4}{x^2+5x+6}$, $\frac{x^2+x}{x^3+x}$, 5 etc.

Memory Plus:

Rational expression are the expression in which exponent of variable are integers.

Reduction / Simplification of Rational Expressions

In algebraic rational expressions, we can multiply or divide the numerator and denominator by a same non–zero expression.

e.g. $$\frac{x^2+x}{x^3+x} = \frac{(x^2+x)\div x}{(x^3+x)\div x} = \frac{x+1}{x^2+1}$$

Hence, $\frac{x+1}{x^2+1}$ is the lowest or reduced form of $\frac{x^2+x}{x^3+x}$.

Reduction of a rational expression is done on the same rules as in reduction of a rational number. To reduce a rational expression to its lowest form, we have to

- Factorize the numerator and denominator completely.
- Divide the numerator and denominator by their common factors.
- Rewrite the remaining terms as a fraction, computing all products. This will be the reduced form.

Example: Reduce each rational expression to its lowest form.

(a) $\dfrac{x^2y^3}{x^5y^3}$ (b) $\dfrac{b-a}{a^3-b^3}$

Solution: (i) By using laws of exponents, we have

$$\frac{x^2y^3}{x^5y^3}=\frac{y^{3-3}}{x^{5-2}}$$

$$=\frac{y^o}{x^3}=\frac{1}{x^3}$$

(ii) Since, $(a^3-b^3)=(a-b)(a^2+ab+b^2)$

So, $$\frac{b-a}{a^3-b^3}=\frac{(b-a)}{(a-b)(a^2+ab+b^2)}$$

$$=\frac{-\cancel{(a-b)}^{1}}{\cancel{(a-b)}_{1}(a^2+ab+b^2)}$$

$$=\frac{-1}{a^2+ab+b^2}$$

Check Point:

What will be the degree of polynomial $\dfrac{x^3y^2-\sqrt{3}y^2z^3x}{xy^2}$ after simplification?

(c) Irrational Expression

An algebraic expression which cannot be expressed in the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials but Q is not a zero polynomial, is called an irrational expression.

For example: $\dfrac{x}{\sqrt{y}}$, $x^{\frac{3}{2}}y-7$, $\sqrt{x}-\dfrac{1}{\sqrt[3]{y}}$, $\sqrt[5]{x^2+y^2}$, $24x^{\frac{3}{2}}y^{-2}+\dfrac{9}{y^2}-7$ etc.

Evaluation of Algebraic Expressions

The process of replacing the variables by numbers in an algebraic expression is called 'substitution'. The process of getting a numerical answer by substituting numbers for each variable is called 'evaluation'.

Example: Find the value of $\dfrac{3}{4}x^2-\dfrac{2}{3}x+1$ at $x=\dfrac{-1}{2}$.

Solution: $\dfrac{3}{4}x^2-\dfrac{2}{3}x+1$

By substituting the value of x, we have

$$=\frac{3}{4}\left(\frac{-1}{2}\right)^2-\frac{2}{3}\left(\frac{-1}{2}\right)+1$$

$$=\frac{3}{4}\left(\frac{1}{4}\right)-\frac{\cancel{2}^{1}}{3}\left(\frac{-1}{\cancel{2}}\right)+1$$

Memory Plus:

Irrational expressions are those expression in which exponent of variable are non-integers.

$$= \frac{3}{16} + \frac{1}{3} + 1$$

$$= \frac{9+16+48}{48} = \frac{73}{48} = 1\frac{25}{48}$$

Example: If p = –5 and q = 2 , find the value of $p^2q + pq^2 + 2p \cdot pq$

Solution: $p^2q + pq^2 + 2p \cdot pq$

$= p^2q + pq^2 + 2p^2q$ ← simplify the given expression

$= 3p^2q + pq^2$ ← combine like terms to reduce number of terms

$= 3(-5)^2(2) + (-5)(2)^2$ ← substitute the given values

$= 3(25)(2) + (-5)(4)$ ← simplify exponents

$= 150 - 20 = 130$

Example: Find the value of the expression

$3.1x^4 - 0.2x^3y^2 + 0.3x^2y^2 + 1.1xy^3 - y^4$ at $x = 1$, $y = -1$.

Solution: $3.1x^4 - 0.2x^3y^2 + 0.3x^2y^2 + 1.1xy^3 - y^4$

$= 3.1(1)^4 - 0.2(1)^3(-1)^2 + 0.3(1)^2(-1)^2 + 1.1(1)(-1)^3 - (-1)^4$

$= 3.1 - 0.2(1)(+1) + 0.3(1)(+1) + 1.1(1)(-1) - (+1)$

$= 3.1 - 0.2 + 0.3 - 1.1 - 1$

$= 3.4 - 2.3 = 1.1$

Exercise 5.1

1. Reduced the following expressions to their lowest form.

(i) $\frac{15ax^3y^2}{25a^2xy^6}$ (ii) $\frac{38k^2p^3m^4}{57k^3pm^2}$ (iii) $\frac{mn^4pq}{m^2n^3p^4}$

(iv) $\frac{3abc}{15a^2b^2c}$ (v) $\frac{46l^3m^4n^5}{69l^2m^3n^4}$ (vi) $\frac{x-3}{3-x}$

(vii) $\frac{x^2-81}{x+9}$ (viii) $\frac{(r+3)(r+4)}{r^2-16}$

2. Evaluate the following expressions for the given value of each variable.

(i) $3(r^2 - s^2)$, if r = 2 , s = –1 (ii) $\frac{1}{2}mv^2$ at $m = 18.75$ and $v = 5.6$

(iii) $\sqrt{2gs}$ when g = 32.2 and s = 144.9

(iv) $3x - y + \frac{1}{z}$ if $x = \frac{-1}{2}$, $y = 3$, $z = \frac{-1}{3}$

(v) $0.1d^2 + 0.01d + 1$ if $d = -0.2$ (vi) $\frac{4}{7}b^3 - 3\frac{1}{2}b^2 + b - 3$ if $b = \frac{1}{2}$

3. If nth triangular number is represented by $T(n) = \frac{n(n+1)}{2}$, then find 100th triangular number.

4. If $P(x) = x^2 + 2x - 15$, $D(x) = x - 3$ and $Q(x) = x + 5$, show that $\frac{P(2)}{Q(2)} = D(2)$.

5. If $g(x) = \frac{1}{2x^3} + \frac{x}{2} + 2$, find $g\left(-\frac{1}{3}\right)$.

6. The volume of a basketball (sphere) is approximately 38808 cm^3. The radius 'r' of the ball is given by

$r = \sqrt[3]{\frac{3v}{4\pi}}$, where v is its volume.

Determine the radius of that ball. (take $\pi = \frac{22}{7}$)

Basic Operations on Algebraic Fractions

Operations with rational numbers and rational algebraic expressions are very similar. A rational number can be expressed as the quotient of two integers. A rational algebraic expression can be expressed as the quotient of two polynomials. In either case, the denominator can never be 0.

Rational Numbers: $\frac{3}{11}, \frac{27}{100}, \frac{-7}{25}, \frac{61}{-89}$

Rational Algebraic Expressions: $\frac{5}{x}, \frac{y}{10}, \frac{2x}{x-5}, \frac{y^2-36}{y+6}, \frac{x^2-6x+5}{\frac{5x}{x+1}}$

If P and Q are any two algebraic expressions then $\frac{P}{Q}$ is called an algebraic fraction, where $Q \neq 0$ e.g., $\frac{a-b}{a^2+2ab+b^2}$

To write a fraction in simplest form, we divide both the numerator and denominator by their greatest common factor (GCD).

Addition of Algebraic Expressions

Algebraic expressions can be added as explained in the following examples.

Example: Add the following expressions.

(i) $\frac{x^2-1}{2}, \frac{4(2x^2-x+1)}{3}$ **(ii)** $4x+7, \frac{(x+3)(x-1)}{x-2}$

Solution:

(i)
$$\frac{x^2-1}{2}+\frac{4(2x^2-x+1)}{3}$$
$$=\frac{3(x^2-1)+8(2x^2-x+1)}{6}$$
$$=\frac{3x^2-3+16x^2-8x+8}{6}$$
$$=\frac{19x^2-8x+5}{6}$$

Key Fact:
For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$:
$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$, $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$

(ii)
$$4x+7+\frac{(x+3)(x-1)}{x-2}$$
$$=\frac{(4x+7)(x-2)+(x+3)(x-1)}{x-2}$$
$$=\frac{4x^2+7x-8x-14+x^2+3x-x-3}{x-2}$$
$$=\frac{5x^2+x-17}{x-2}$$

Subtraction of Algebraic Expressions

In subtraction of algebraic expressions, we have to change the signs of the expression which is being subtracted and all other steps are the same as in addition of algebraic expressions.

Example: Subtract $\frac{3x^2-7x+4}{x^2-4}$ from $\frac{3x-1}{x+2}$.

Solution: $\frac{3x-1}{x+2}-\frac{3x^2-7x+4}{x^2-4}$

$$=\frac{3x-1}{x+2}-\frac{3x^2-7x+4}{(x+2)(x-2)} \longleftarrow$$ **By using the formula $a^2-b^2=(a+b)(a-b)$**

$$=\frac{(3x-1)(x-2)-(3x^2-7x+4)}{(x+2)(x-2)}$$
$$=\frac{3x^2-7x+2-3x^2+7x-4}{(x+2)(x-2)}=\frac{-2}{(x+2)(x-2)}=\frac{-2}{x^2-4}$$

Multiplication of Algebraic Expressions

Multiplication of algebraic expression can be explained with the help of following examples.

Example: Find the product of $\frac{x^2-9}{x^2+2x}$ and $\frac{x}{x+3}$.

Solution: $\frac{x^2-9}{x^2+2x} \times \frac{x}{x+3}$

$$=\frac{(x+3)(x-3)}{x(x+2)} \times \frac{x}{x+3} = \frac{x-3}{x+2}$$

Example: Multiply x^2-x-20 by $\frac{3}{x^3+4x^2}$.

Solution: $(x^2-x-20) \times \frac{3}{x^3+4x^2}$

$$=(x^2-5x+4x-20) \times \frac{3}{x^2(x+4)}$$

$$=(x-5)(x+4) \times \frac{3}{x^2(x+4)} = \frac{3x-15}{x^2}$$

Key Fact:

For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$:

$\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd}$, $\quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b}.\frac{d}{c} = \frac{ad}{bc}$

Division of Algebraic Expressions

The process of dividing any two algebraic expressions is explained with the help of following examples.

Example: Divide $\frac{2x^2-11x-21}{x^2-49}$ by $\frac{2x+3}{x+7}$.

Solution: $\frac{2x^2-11x-21}{x^2-49} \div \frac{2x+3}{x+7}$

$$=\frac{2x^2-14x-3x-21}{(x)^2-(7)^2} \times \frac{x+7}{2x+3}$$

$$=\frac{2x(x-7)+3(x-7)}{(x+7)(x-7)} \times \frac{x+7}{2x+3}$$

$$=\frac{(x-7)(2x+3)}{(x+7)(x-7)} \times \frac{x+7}{2x+3} = 1$$

Complex Algebraic Expressions

For simplifying the complex algebraic expression involving addition, subtraction, multiplication and division, we apply the BODMAS rule similar to arithmetic expression.

Example: Simplify.

$$\frac{3x+2}{x+2}+\frac{x-2}{2x+10}\div\frac{2x^2-4}{x^2+2x-15}$$

Solution: $\frac{3x+2}{x+2}+\frac{x-2}{2x+10}\times\frac{x^2+2x-15}{2x^2-8}$

$$=\frac{3x+2}{x+2}+\frac{x-2}{2(x+5)}\times\frac{x^2+5x-3x-15}{2(x^2-4)}$$

$$=\frac{3x+2}{x+2}+\frac{x-2}{2(x+5)}\times\frac{x(x+5)-3(x+5)}{2(x+2)(x-2)}$$

$$=\frac{3x+2}{x+2}+\frac{x-2}{2(x+5)}\times\frac{(x+5)(x-3)}{2(x+2)(x-2)}$$

$$=\frac{3x+2}{x+2}+\frac{x-3}{4(x+2)}$$

$$=\frac{4(3x+2)+(x-3)}{4(x+2)}=\frac{12x+8+x-3}{4(x+2)}=\frac{13x+5}{4x+8}$$

Example: Simplify the following.

$$\left[\frac{a}{(a+b)^2-2ab}\div\frac{(a+b)^3-3ab(a+b)}{a^4-b^4}\right]\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

Solution:

$$\left[\frac{a}{(a+b)^2-2ab}\div\frac{(a+b)^3-3ab(a+b)}{a^4-b^4}\right]\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{(a+b)^2-2ab}\times\frac{a^4-b^4}{(a+b)^3-3ab(a+b)}\right]\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{a^2+b^2+2ab-2ab}\times\frac{(a^2-b^2)(a^2+b^2)}{a^3+b^3+3ab(a+b)-3ab(a+b)}\right]\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{a^2+b^2}\times\frac{(a^2-b^2)(a^2+b^2)}{a^3+b^3}\right]\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{a^2+b^2}\times\frac{(a+b)(a-b)(a^2+b^2)}{(a+b)(a^2-ab+b^2)}\right]\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a(a-b)}{a^2-ab+b^2}\div\frac{(a+b)^2-4ab}{(a+b)^2-3ab}\right]$$

$$=\left[\frac{a(a-b)}{a^2-ab+b^2}\times\frac{(a+b)^2-3ab}{(a+b)^2-4ab}\right]$$

$$=\left[\frac{a(a-b)}{a^2-ab+b^2}\times\frac{a^2+2ab+b^2-3ab}{a^2+2ab+b^2-4ab}\right]$$

$$=\left[\frac{a(a-b)}{a^2-ab+b^2}\times\frac{a^2+b^2-ab}{a^2+b^2-2ab}\right]$$

$$=\left[\frac{a(a-b)}{a^2-2ab+b^2}\right]$$

$$=\left[\frac{a(a-b)}{(a-b)^2}\right]$$

$$=\frac{a}{a-b}$$

Another method for simplifying a complex rational expression involves first adding or subtracting, as necessary, to get one rational expression in the numerator and one rational expression in the denominator. The problem is thereby simplified to one involving the division of two expressions.

Example: Simplify. $\dfrac{\frac{3}{x}-\frac{2}{x^2}}{\frac{3}{x-2}+\frac{1}{x^2}}$

Solution: $\dfrac{\frac{3}{x}-\frac{2}{x^2}}{\frac{3}{x-2}+\frac{1}{x^2}}=\dfrac{\frac{3x-2}{x^2}}{\frac{3x^2+x-2}{x^2(x-2)}}$, taking LCM

$=\dfrac{3x-2}{x^2}\times\dfrac{x^2(x-2)}{3x^2+x-2}$ to divide, multiply, by the reciprocal of the divisor.

$=\dfrac{3x-2}{x^2}\times\dfrac{x^2(x-2)}{3x^2+3x-2x-2}=\dfrac{(3x-2)}{x^2}\times\dfrac{x^2(x-2)}{(3x-2)(x+1)}$ factoring

$=\dfrac{x-2}{x+1}$ removing a common factor from numerator and denominator.

Exercise 5.2

1. Add the following expressions.

(i) $\frac{x}{2}, \frac{x}{5}$ (ii) $\frac{x-2}{2}, \frac{x+10}{9}$ (iii) $\frac{4+x}{4}, \frac{x-1}{7}, \frac{5x}{2}$

(iv) $\frac{3x}{x+5}, \frac{10}{5x+25}$ (v) $\frac{24x}{6x-18}, \frac{3(1+x)}{x-3}$

2. Subtract.

(i) $\frac{23-x}{5}$ from 7 (ii) $\frac{6(x-8)}{7}$ from $\frac{5(x-7)}{3}$

(iii) $2x^2-2x+1$ from $\frac{x+1}{x}$.

3. Divide the first expression by the second.

(i) x^4-10x^2+9, x^2-2x-3 (ii) $x^3-3x^2y+3xy^2-y^3, x-y$

(iii) $\frac{4x^2-16}{5x}, \frac{2x+4}{15}$ (iv) $\frac{x^2+5x}{x-3}, \frac{x^2-25}{x-3}$

4. Simplify the following.

(i) $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}$ (ii) $\frac{1}{2}\left(4-\frac{x}{3}\right)-\frac{5}{6}+\frac{1}{3}\left(11-\frac{x}{2}\right)$

(iii) $\frac{2}{x+1}+\frac{x}{x-1}-\frac{x+2}{x-1}$ (iv) $\frac{x^2-25}{5}-\frac{x}{4}\div\frac{3x}{20}$

(v) $\frac{45a^2b^3c^4}{27x^4y^3z}\times\frac{243xy^2z^3}{180a^2bc^3}$ (vi) $\frac{m^2}{8n}\times\frac{36p^3q^2}{81mn}\div\frac{15mpx^5}{270n^2x^3y}$

(vii) $3x\div\frac{3x^2-27}{x+3}+\frac{1}{x-3}$ (viii) $\frac{5x+5}{3(2x-1)}+\frac{6-2x}{2(1-2x)}$

(ix) $\frac{2a}{2a-3}-\frac{5}{6a+9}-\frac{4(3a+2)}{3(4a^2-9)}$ (x) $\frac{5}{5+x-18x^2}-\frac{2}{2+5x+2x^2}$

(xi) $\frac{m+3}{24m}-\frac{m+1}{24m}+\frac{3m-1}{6m^2+18m}\div\frac{12m-4}{m+3}$

(xii) $\frac{1-p^2}{1+q}\times\frac{1-q^2}{p+p^2}\times\left(1+\frac{p}{1-p}\right)$ (xiii) $\left(1-\frac{x}{1+\frac{x}{1-x}}\right)\div(1+x^3)$

Solving Rational Equations

An equation that contains one or more rational expressions is called a rational equation. Here are some examples:

$$\frac{2}{5}-\frac{x}{4}=\frac{1}{x}, \qquad \frac{x-4}{x+1}-\frac{6}{x-2}=\frac{7}{x+1}, \qquad x^2+\frac{5}{x}=9$$

Equations of this type occur frequently in applications. It is easiest to solve a rational equation if the fractions are eliminated. This can be done by multiplying each side of the equation by the least common denominator (LCD). Remember that when you multiply each side by the LCD, each term on each side must be multiplied by the LCD.

Example: Solve $\frac{3}{x-7}+1=\frac{8}{x^2-9x+14}$. Verify the solution.

Solution: $$\frac{3}{x-7}+1=\frac{8}{x^2-9x+14}$$

$$(x-2)(x-7).\frac{3}{x-7}+(x-2)(x-7).1=(x-2)(x-7).\frac{8}{(x-2)(x-7)} \quad \text{Multiply by LCD } (x-2)(x-7).$$

$$3(x-2)+(x^2-9x+14) = 8 \quad \text{Simplify.}$$

$$x^2-6x+8 = 8 \quad \text{Combine the like terms.}$$

$$x^2-6x = 0 \quad \text{Simplify.}$$

$$x(x-6)=0$$

$x=0$, or $x=6$. The solutions are 0 and 6.

Check: if $x=0$, $\frac{3}{0-7}+1=\frac{8}{0^2-9.0+14}$, we have: $\frac{4}{7}=\frac{4}{7}$, True.

if $x=6$, $\frac{3}{6-7}+1=\frac{8}{6^2-9.6+14}$, we have: $-2=-2$, True.

S . S = $\{0 , 6\}$

Note: Remember that a rational expression is undefined when the value for variables results in a denominator of zero. When solving rational equations, we check that for solution denominator should be non-zero. If denominator is zero in any value of x, these values be excluded from list of solutions.

Example: Solve $\frac{x}{x-2}+\frac{1}{5}=\frac{2}{x-2}$.

Solution: $$\frac{x}{x-2}+\frac{1}{5}=\frac{2}{x-2}.$$

$$5(x-2).\frac{x}{x-2}+5(x-2).\frac{1}{5}=5(x-2).\frac{2}{x-2} \quad \text{Multiply by LCD } 5(x-2).$$

$$5x+x-2=10 \quad \text{Simplify.}$$

$$6x-2=10 \quad \text{Combine the like terms.}$$

$$x=2$$

The solution appears to be 2, but the expression $\frac{x}{x-2}+\frac{1}{5}=\frac{2}{x-2}$ are undefined when $x=2$.

We see that 2 is extraneous solution.

There is no solution.

Example: If a certain number is added 5 times the reciprocal of 2 more than that number, the result is 4.

Solution: Suppose a number is x. According to condition, added to 5 times the reciprocal of 2 more than that number, the result is 4:

$$x+5(\frac{1}{x+2})=4 \quad \text{As per statement, written in mathematical form}$$

$$(x+2).x+(x+2).5(\frac{1}{x+2})=(x+2).4 \quad \text{Multiply by LCD on both sides.}$$

$$x^2+2x+5=4x+8 \quad \text{Simplify}$$

$$x^2-2x-3=0 \quad \text{Collecting all like terms on one side}$$

$$(x-3)(x+1)=0 \quad \text{Factoring}$$

$$x=3 \text{ or } x=-1, \text{ the solution can be checked.}$$

The solution can be verified:

$$x+5(\frac{1}{x+2})=4 \quad \text{(As per statement, written in mathematical form)}$$

$$(x+2).x+(x+2).5(\frac{1}{x+2})=(x+2).4 \quad \text{(Multiply by LCD on both sides.)}$$

$$x^2+2x+5=4x+8 \quad \text{(Simplify)}$$

$$x^2-2x-3=0 \quad \text{(Collecting all like terms on one side)}$$

$$(x-3)(x+1)=0 \quad \text{(Factoring)}$$

$$x=3 \text{ or } x=-1, \text{ the solution can be checked.}$$

Result: The required number is -1 or 3.

Example: A car travels 300 km in the same time that a freight train travels 200 km. The speed of the car is 20 km/h more than the speed of the train. Find the speed of the car and the speed of the train.

Solution: We know that distances both vehicles traveled. We also observe that both vehicles traveled the same amount of time and that the speed of the car is 20 km/h faster than that of the train.

Suppose the distance is S and speed is V, we know that S=VT, where T is time. If V be the speed of the train, then car moves 20 km/h faster than the speed of the train. So, the speed of the car is V+20. We can equate the formula for the two vehicles in terms of T. We have:

$$T = \frac{S}{V}$$

car's time = train's time

$$\frac{S}{V} = \frac{300}{V+20} = \frac{200}{V}$$

$$\frac{300}{V+20} = \frac{200}{V} \quad \text{(Solve for V)}$$

$$(V + 20)\, V.\frac{300}{V+20} = V(V + 20).\frac{200}{V} \quad \text{(Multiply each side by LCD)}$$

$$300V = 200(V + 20) \quad \text{(Simplify)}$$

$$100V = 4000$$

$$V = 40$$

The speed of the train is 40 km/h and the speed of the car is 60 km/h.

Exercise 5.3

Solve and check your solutions (1-7).

1. $\dfrac{6x}{x-11}+1=\dfrac{3}{x-11}$
2. $\dfrac{2y}{y+3}=\dfrac{-4}{y-7}$
3. $\dfrac{x+7}{x+4}-1=\dfrac{x+10}{2x+8}$
4. $\dfrac{3y}{y+1}=\dfrac{12}{y^2-1}+\dfrac{y+4}{y+1}$
5. $x+\dfrac{5}{x}=-6$
6. $\dfrac{y+2}{y^2+6y-7}=\dfrac{8}{y^2+3y-4}$
7. $\dfrac{5}{y+1}+\dfrac{3y+5}{y^2+4y+3}=\dfrac{2}{y+3}$
8. Kaleem can mow a lawn in 4 hours. Moiz can mow the same lawn in 5 hours. How long would it take both of them, working together, to mow the lawn.
9. You have an 8-pint mixture of paint that is made up of equal amounts of yellow paint and blue paint. To create a certain shade of green, you need a paint mixture that is 80 % yellow. How many pints of yellow paint do you need to add to the mixture?
10. Waqar takes 9 hours longer to build a wall than it takes Wasi. If they work together, they can build the wall in 20 hours. How long would it take each, working alone, to build the wall?

I have Learnt

- Describing rational expression.
- Factorizing and simplifying rational expression.
- Demonstrating manipulation of algebraic fractions.
- Performing operations on rational expressions.
- Solving rational equations.
- Applying the concept of rational equations to real world problems.

MISCELLANEOUS EXERCISE-5

1. **Tick the correct option.**

i. An expression which is the ratio of two polynomials but the polynomial in denominator is non–zero is called

(a) polynomial (b) rational Expression
(c) compound Expression (d) irrational Expression

ii. The degree of $x^2y^3 - \frac{xy^2z^3}{y} - \sqrt{25}z^5$ is

(a) 5 (b) 6 (c) 7 (d) none

iii. Constant polynomial is also called

(a) linear polynomial (b) no degree polynomial
(c) expression (d) zero degree polynomial

iv. Ali is 2 years younger than his sister Ayesha. If Ayesha's present age is x years, then the age of Ali after 5 years will be

(a) $(x+7)$ years (b) $(x-2)$ years
(c) $(x+3)$ years (d) $(x-7)$ years

v. The value of $2\left\{x^3-\left(x^2-\overline{3-2x^2}\right)\right\}$ at $x=2$ is

(a) 2 (b) 14 (c) –2 (d) 6

vi. Reduced form of the expression $\frac{x^2y^3-y^2x^3+x^2y^2z}{x-y-z}$ is

(a) x^2y^2 (b) not possible (c) $\frac{x^2y^2(x-y-z)}{y-x+z}$ (d) $-x^2y^2$

vii. If $y = 2 - \frac{1}{y}$, then the value of $y^2 + \frac{1}{y^2}$ is

(a) 4 (b) zero (c) not possible (d) 2

viii. Simplified from of $\frac{(a+b)^2 - (a-b)^2}{8ab}$ is

(a) $\frac{2(a^2 + b^2)}{8ab}$ (b) 2 (c) $\frac{a^2 + b^2}{+ab}$ (d) $\frac{1}{2}$

ix. Difference of the sum of a and b from the product of a and b is

(a) $ab - a - b$ (b) $a + b - ab$ (c) $2ab - b$ (d) none

x. $\frac{x^3y^3 + y^3z^3 + z^3x^3}{x^3y^3z^3} =$

(a) $x^3 + y^3 + z^3$ (b) $x^6 + y^6 + z^6$

(c) $\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3}$ (d) $\frac{1}{x^6} + \frac{1}{y^6} + \frac{1}{z^6}$

xi. Leading coefficient in $\frac{x^2}{2} - \frac{1}{8} - \frac{x^4}{4} + \frac{x^3}{7}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{7}$ (d) $-\frac{1}{4}$

xii. Coefficients in the polynomial $\sqrt{16}x^2y - \frac{1}{2}y^3 + \frac{22}{7}z$ are the elements of the set of:

(a) Integers (b) Irrational numbers
(c) Odd numbers (d) Rational numbers

xiii. The degree of the quotient in $(x-y)^3 \div (x-y)^2$ will be

(a) 3 (b) 2 (c) 1 (d) no

2. Find the value of the circumference of a circle whose radius is $12.5cm$. (Take $\pi = \frac{22}{7}$)

3. The surface S of a sphere of radius 'r' is given by the formula $S = 4 \times \frac{22}{7} r^2$, find

(i) The surface of a sphere whose radius is 1.4 inches;

(ii) The radius of a sphere whose surface is $38\frac{1}{2}$ square feet.

4. In a right-angled triangle, if a and b denote the lengths of the sides containing the right angle and c denotes the length of the hypotenuse, it is known that $c^2 = a^2 + b^2$. By substitution find which of the following sets of numbers represent the sides of right-angle triangle.

(i) 7 , 24 , 25 (ii) 1.6 , 6.3 , 6.5

(iii) 12 , 35 , 36 (iv) 3 , 4 , 5

5. If $a = 3$, $b = 4$ and $c = 1$, find the value of : $\sqrt{2ab + 4ac} + \sqrt{9b} + \dfrac{2abc}{3}$

6. Subtract the sum of:

$2x^3 - 3x + 4$ and $-3x^2 + 2x - 7$ from $4x^3 - 3x^2 + x - 6 - \{2x^3 - (x - 6)\}$

7. Divide the product of $x - 2$, $x + 3$ and $2x - 7$ by the sum of $3\,(x^2 - 2x - 2)$ and $5x - x^2 - 15$.

8. Simplify the following.

(i) $\dfrac{x}{x+2} - \dfrac{5x+3}{x-2} + \dfrac{1}{2}$ (ii) $\dfrac{x}{x^2 - y^2} \times \dfrac{x^2 + 2xy + y^2}{x+y} \div \dfrac{3x}{x-y}$

9. Solve.

(i) $\dfrac{12}{x^2 - 16} - \dfrac{24}{x-4} = 3$ (ii) $\dfrac{y}{2y-6} - \dfrac{3}{x^2 - 6x + 9} = \dfrac{y-2}{3y-9}$

10. When checking a possible solution of a rational equation, is it necessary to check that the solution does not make any denominator equal 0? Why or why not?