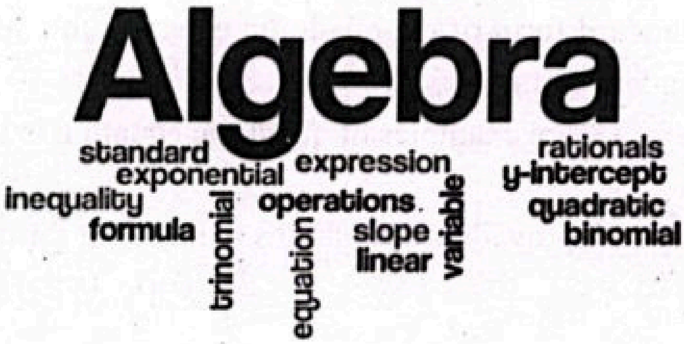
02

Quadratic Equations

After studying this unit students will be able to:

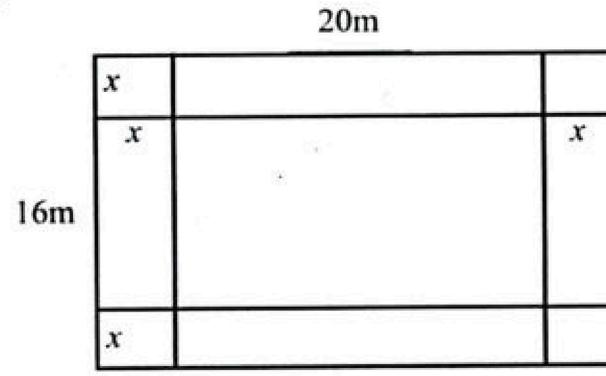
- Define quadratic equations.
- Solve quadratic equations by graphical method, factorization and completing squares.
- Derive quadratic formula and use it to solve quadratic equations.
- Solve different types of equations which are reducible to quadratic form.
- Solve simultaneous equations. When one is homogenous and one is linear.
- Know nature of roots of a quadratic equation.
- Find relation between roots and coefficients of a quadratic equation.
- Define and evaluate symmetric functions of roots of a quadratic equation.
- Form a quadratic equation when its roots are given.



Algebra is your trusted tool that helps you carry out of various activities of daily importance. There is hardly any line of work that does not employ the concept of algebra. We will look at the variables and equation, you surely will not be wondered about why you need to leave there in a nutshell. Algebra and equations prepare you for hardly all aspects of life and stays with you right from your infanthood to your adulthood.

Quadratic Equations

Saleem wants to build a swimming pool surrounded by a sidewalk of uniform width. He wants the dimensions of the pool and sidewalk to be 16 meters by 20 meters. The pool has an area of 192 square meters. How wide should the sidewalk be?



Let x meters be the width of the sidewalk.
The length of a pool is $20 - 2x$ meters.
The width of the pool is $16 - 2x$ meters.
The area of the pool can be expressed as the product of the length and width.
Area = Length × Width

$$= (20 - 2x)(16 - 2x)$$

$$= 4x^2 - 72x + 320$$

The area of the pool can be expressed as $4x^2 - 72x + 320$ square meters. Since the area of the pool is 192 square meters, replacing area with 192 results in the equation $192 = 4x^2 - 72x + 320$. Saleem would probably never use a quadratic equation like $192 = 4x^2 - 72x + 320$ to express area of a pool. However, in this unit we will study many other formulas in science and business that involve this type of equations.

Quadratic Equation

It is an equation that can be written in the form of $ax^2 + bx + c = 0$, where $a \neq 0$. We say that equations like this have a degree of 2. Since the greatest exponent of the variable is 2. We observe that this quadratic equation only has one variable and all the exponents are positive.

Standard Quadratic Equation

$ax^2 + bx + c = 0$ is a standard form of the quadratic equation in one variable x. The equation $x^2 + 6x + 5 = 0$ is in standard form.

$(x+1)^2 = 4$ and $x^2 + 3x = 18$ are examples of quadratic equation in one variable x but they are not in standard form.

Example: Write the following quadratic equations in standard form.

(i). $ax^2 + bx = c$ (ii). $(3x - 1)^2 = 4$ (iii). $(x + 1)^2 - (2x - 1)^2 = 2$

Solution:

i. $ax^2 + bx = c$

$ax^2 + bx - c = 0$

ii. $(3x - 1)^2 = 4$

$9x^2 - 6x + 1 = 4$

$9x^2 - 6x + 1 - 4 = 0$

$9x^2 - 6x - 3 = 0$

$3(3x^2 - 2x - 1) = 0$, as $3 \neq 0$

$3x^2 - 2x - 1 = 0$

iii. $(x + 1)^2 - (2x - 1)^2 = 2$

$(x^2 + 2x + 1) - (4x^2 - 4x + 1) = 2$

$x^2 + 2x + 1 - 4x^2 + 4x - 1 - 2 = 0$

$-3x^2 + 6x - 2 = 0$

$-1(3x^2 - 6x + 2) = 0$

$3x^2 - 6x + 2 = 0$ as $-1 \neq 0$

Solution of Quadratic Equations

The values of variables for which an equation becomes a true sentence are called solutions or roots of the equation. i.e. if $P(a) = c$, then $x = a$ is root of the equation $P(x) = c$. The set of roots of an equation is called the solution set. A quadratic equation in one variable has two roots. These roots of the quadratic equation can be found in many ways. In this section we will study four methods.

i. Graphing method
ii. Factorization Method
iii. Completing Square Method
iv. Using Quadratic Formula

i. Graphical Method

One way to determine the roots of a quadratic equation is to graph the related quadratic function. Write the equation in general form like:

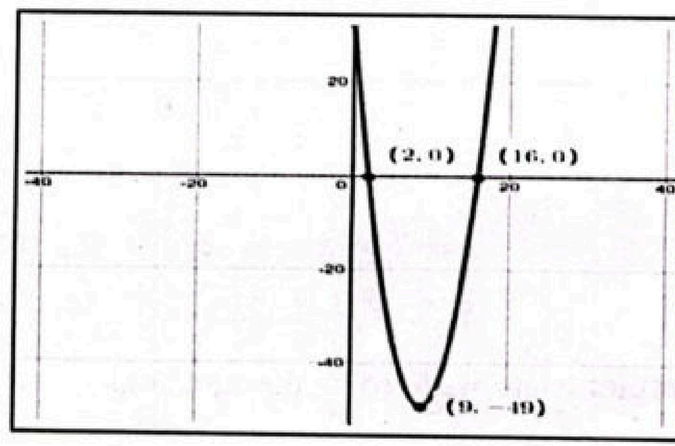
$192 = 4x^2 - 72x + 320$

$0 = 4x^2 - 72x + 128$ subtract 192 from each side.

$0 = x^2 - 18x + 32$ dividing each side by 4.

The related function for this equation is

$f(x) = y = x^2 - 18x + 32.$

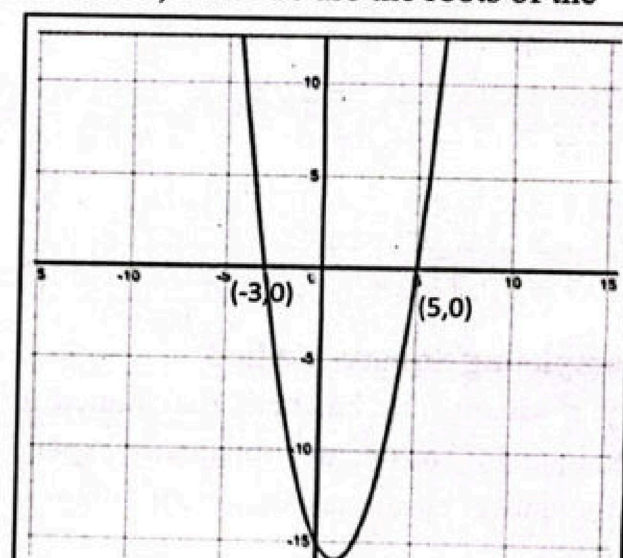


x	2	4	16	20
y	32	–24	0	72

Notice that the function crosses the x-axis twice and the function has a degree of 2. These intercepts points are called the zero of a function. The x values, 2 and 16 are the roots of the equation $x^2 - 18x + 32 = 0$

Example: Graph the related quadratic function $y = x^2 - 2x - 15$ by finding and graphing the ordered pairs that satisfy the function.

Solution:



x	–3	–1	1	5
y	0	–12	–16	0

The graphs tells that the solutions are

–3 and 5, where the curve intersects the x-axis.

ii. Factoring Method

Another way of solving a quadratic equation is by factoring. The factoring method depends on the zero-product property.

Zero Product Property:

For any real number a and b, if ab = 0, then either a = 0 or b = 0.

i. To solve the quadratic equation by factorization first write it in standard form, then factorize the polynomial on the left-hand side of the equation.

ii. Use zero product property.

iii. Now solve linear equations and write solution set.

Fr example, to solve $x^2 = 4$ by factorization we do the following steps:

$$x^2 = 4$$
$$x^2 - 4 = 0 \quad \text{(standard form)}$$
$$(x)^2 - (2)^2 = 0$$
$$(x-2)(x+2) = 0 \quad \text{(factorization)}$$
$$x-2 = 0 \text{ or } x+2 = 0 \quad \text{(linear equations)}$$
$$x = 2 \text{ or } x = -2 \quad \text{(roots)}$$
$$S.S. = \{-2, 2\} \quad \text{(solutionset)}$$

Example: Factorize to solve the equations.

i. $x^2 - 18x + 32 = 0$

ii. $x^2 - 2x - 15 = 0$.

Solution:

i.
$$x^2 - 18x + 32 = 0$$
$$x^2 - 16x - 2x + 32 = 0$$
$$x(x-16) - 2(x-16) = 0$$
$$(x-2)(x-16) = 0$$
$$x - 2 = 0 \text{ or } x - 16 = 0$$
$$x = 2 \text{ or } x = 16$$
$$S.S = \{2, 16\}$$

ii.
$$x^2 - 2x - 15 = 0$$
$$x^2 - 5x + 3x - 15 = 0$$
$$x(x-5) + 3(x-5) = 0$$
$$(x-5)(x+3) = 0$$
$$x - 5 = 0 \text{ or } x + 3 = 0$$
$$x = 5 \text{ or } x = -3$$
$$S.S = \{-3, 5\}$$

iii. Completing Square Method

In a city, a square field has been maintained for playing. Each side of the field is $110m$ long. The management has decided to equally expand two of the sides of the field for parking spaces so that the entire region has an area of 14000 m^2. What is the width of the parking strip being added?

Let x represents the width of each parking strip. After expanding each side of the field would be $110+x$ meters long.

The equation expresses the area of the new field, which is still square.

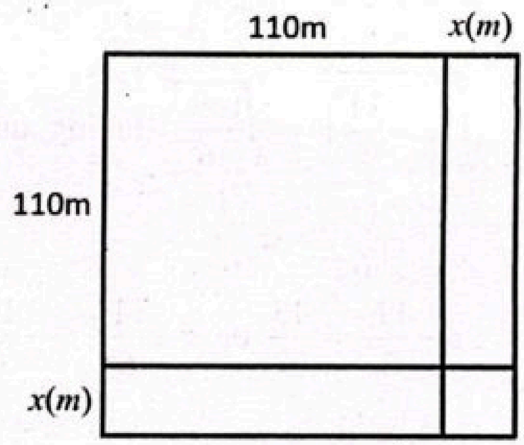
$$(x+110)^2 = 14000$$

$$\sqrt{(x+110)^2} = \sqrt{14000}$$

$$x+110 = \pm\sqrt{14000}$$

$$x = -110 \pm \sqrt{14000}$$

$$x = 8.32m$$



Where the negative value is neglected.

Quadratic equations can be solved in the same way as long as one side of the equation contains a perfect square. When the equation does not contain a perfect square, we can use process called completing the square to create a perfect square.

For an expression of the form $x^2 + bx$, we can add a constant c in the expression so that the expression $x^2 + bx + c$ is a perfect square trinomial. This process is called completing square.

Procedure

Step I: Write the equation in standard form.

Step II: Shift constant term to the right hand side of the equation.

Step III: To make the left hand side a complete square, add a suitable term to both sides of the equation. If the coefficient of x^2 is 1, then it is easy to complete the square. Just add the square of half the coefficient of x and get a complete square.

Step IV: Take the square root on both sides and form two linear equations.

Step V: Solve the linear equations and find roots.

Example:

Solve $3x^2 - 11x - 4 = 0$ by completing square.

Solution:

$$3x^2 - 11x - 4 = 0$$

$$3x^2 - 11x = 4 \quad \text{Shift 4 right side}$$

$$x^2 - \frac{11}{3}x = \frac{4}{3} \quad \text{dividing each side by 3.}$$

Check Point:

Describe the error.

$$x^2 - 14x = 11$$

$$x^2 - 14x + 49 = 11$$

$$(x-7)^2 = 11$$

$$x - 7 = \pm\sqrt{11}$$

$$x = 7 \pm \sqrt{11}$$

$$x^2-\frac{11}{3}x+\frac{121}{36}=\frac{4}{3}+\frac{121}{36} \text{ adding } \frac{121}{36} \text{ on both sides.}$$

$$\left(x-\frac{11}{6}\right)^2=\frac{169}{36} \text{ factor}$$

$$\sqrt{\left(x-\frac{11}{6}\right)^2}=\sqrt{\frac{169}{36}} \text{ taking square root}$$

$$x-\frac{11}{6}=\pm\frac{13}{6}$$

$$x-\frac{11}{6}=\frac{13}{6} \text{ or } x-\frac{11}{6}=-\frac{13}{6}$$

$$x=4 \text{ or } x=-\frac{1}{3} \qquad \text{S. S.}=\{-\frac{1}{3}, 4\}$$

Example:

Solve $3x^2+6x-9=0$ by the completing square method.

Solution:

$3x^2+6x-9=0$

$3x^2+6x=9$

$x^2+2x=3$ (dividing by 3)

$x^2+2x+1=3+1$ (adding 1 to both sides)

$(x+1)^2=4$

$x+1=\pm\sqrt{4}$ (taking square root)

$x+1=\pm 2$

$x=\pm 2-1$

S. S. $=\{1,-3\}$

iv. Quadratic Formula

We have learnt several ways to solve quadratic equations. Each has its limitations. The question we think is that is there any formula that will work for any quadratic equation? The answer is 'yes' and the formula is called the quadratic formula. The formula is derived from solving the general form of a quadratic equation for x.

Key Fact:

The solution of a quadratic equation of the form $ax^2+bx+c=0$ with $a\neq 0$ is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Derivation of the Quadratic Formula

Standard form of quadratic equation is: $ax^2+bx+c=0$

$$ax^2+bx=-c$$

$$x^2+\frac{b}{a}x=-\frac{c}{a} \text{ (Divide by a both sides)}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a} \text{ (Add } \left(\frac{b}{2a}\right)^2 \text{ both sides)}$$

$$x^2+2x\frac{b}{2a}+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$ (Simplify the left side)

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$ (Taking square root both sides)

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This equation is known as quadratic formula.

Example: Solve $2x^2+7x=4$ using quadratic formula.

Solution: $2x^2+7x=4$

$2x^2+7x-4=0$ General form

$a=2, b=7, c=-4$

We know that: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-7\pm\sqrt{(7)^2-4(2)(-4)}}{2(2)}$$

$$x=\frac{-7\pm\sqrt{49+32}}{4}$$

$$x=\frac{-7\pm\sqrt{81}}{4}\Rightarrow\frac{-7\pm9}{4}$$

$$x=\frac{-7+9}{4},\ x=\frac{-7-9}{4}$$

$$x=\frac{2}{4},\ x=\frac{-16}{4}$$

$$x=\frac{1}{2},\ x=-4$$ S. S. $=\{-4, \frac{1}{2}\}$

Procedure

To solve a quadratic equation by using the quadratic formula, we proceed as follows:

Step I: Write the equation in standard form.

Step II: Compare it with $ax^2+bx+c=0$, to get the values of a, b and c.

Step III: Write the quadratic formula for x.

Step IV: Put values of a, b and c, then simplify.

Step V: Write solution set.

Exercise 2.1

1. Write the following quadratic equation in standard form:
   i. $(x+2)(x-3)=5$ ii. $(x-5)^2-(2x+4)^2=7$ iii. $x=x(x-1)$
2. Solve the following equations by factoring method.
   i. $(x-1)(x-4)=0$ ii. $x^2-2x+1=0$

iii. $x^2-7x-8=0$

iv. $x^2-4x+4=(2x-7)^2$

v. $\left(2x+\frac{7}{4}\right)^2=\frac{48x^2+529}{16}$

3. Solve the following equations by completing the square method:

i. $x^2+4x-32=0$ ii. $x^2+8x=0$ iii. $x^2+6x-9=0$

iv. $3x^2+12x+8=0$ v. $x^2+x+1=0$ vi. $4x^2-8x-5=0$

4. Solve the following equations by quadratic formula:

i. $x^2-9=0$

ii. $2x^2+5x+1=0$

iii. $x^2-23x-24=0$

iv. $(x+1)^2=(2x-1)^2$

v. $\frac{x+1}{2}-\frac{x(x+2)}{3}=0$

vi. $(x-2)(x-6)=(2x+1)(x+1)$

5. Solve $x^2+6x=-9$ by graphing and factoring method.

6. Graph the function $y=x^2+2x+4$ and verifying solution by completing square method.

7. Explain each term:

i. Solution

ii. root

iii. Zero of a function

iv. x-intercept

8. Can a quadratic equation have more than two solutions? Why or why not?

9. Plans for rectangular ice-skating rink that is 30m by 60m have to be revised to double the area of the playground by adding strips of the same width to a side and an end of the area to form a rectangle. Find the width of the strips. What are the dimensions of the playground now?

10. If a car has an initial speed of 20m/s and a constant acceleration of 2m/s^2, determine the amount of time it takes to travel 145m. (Use: $S=v_it+\frac{1}{2}at^2$)

Equation Reducible to Quadratic Form

Some equations are not quadratic, but to solve them we can reduce them to quadratic form.

The example below illustrates solving an equation that is not a quadratic equation itself, but contains a factor that is a quadratic. The method we use to solve the quadratic part of that equation.

Example: A cuboid box has a base that is x cm long. The width of the base is 2 cm less than its length. The height is 3 cm greater than the length of the base. The measure of the volume is 6 times the measure of the length of the base. Find the dimensions of the box.

Solution:

The volume of the rectangular box is the product of the length, width and height of box.

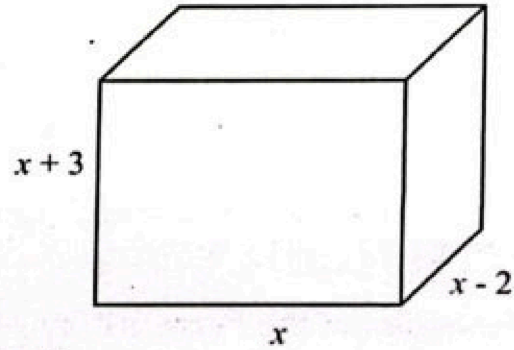
Here, Length = x, Width = $x-2$, Height = $x+3$

$\therefore$ The volume is:

V = Length × Width × Height = $l\ w\ h$

V = 6 × the measure of the length of the base ..... (i)

As, $V = x(x-2)(x+3)$ .... (ii)



$\therefore$ $6x = x(x-2)(x+3)$ (from (i) and (ii))

$6x = x^3 + x^2 - 6x \Rightarrow x^3 + x^2 - 12x = 0$

$x(x^2 + x - 12) = 0 \Rightarrow x(x-3)(x+4) = 0$

We use the zero-product property

$x = 0,\ x+4 = 0,\ x-3 = 0$

$x = 0,\ x = -4,\ x = 3$

Since we are dealing with the dimensions of a box, we ignore 0 and -4, the solution is:

length $x = 3$

width $= x - 2 = 3 - 2 = 1$

height $= x + 3 = 3 + 3 = 6$

Hence, the dimensions of the box are $3\text{cm} \times 1\text{cm} \times 6\text{cm}$.

We have many types of equations that are not quadratic, but we can write in a form that resembles a quadratic equation. For example, the equation $9x^4 - 8x^2 - 1 = 0$ can be written as: $9(x^2)^2 - 8x^2 - 1 = 0$. Equations that can be written in this way are said to be equations reducible to quadratic form.

Some of equations that can be reduced to quadratic equations, are given below.

Equations of the Form $ax^4 + bx^2 + c = 0$

It is an equation of degree 4. It can be reduced to quadratic form by using the substitution $x^2 = y$. In this way the above equation becomes $ay^2 + by + c = 0$. Solve it by any of the three methods discussed earlier. Replace y with its substitute x^2. Find values of x by taking the square root and write the solution set.

Example: Solve $9x^4 - 8x^2 - 1 = 0$.

Solution:

$9x^4 - 8x^2 - 1 = 0$

Put $x^2 = y$ which gives $x^4 = y^2$

$\therefore 9y^2 - 8y - 1 = 0$

$9y^2 - 9y + y - 1 = 0$

$9y(y-1) + 1(y-1) = 0$

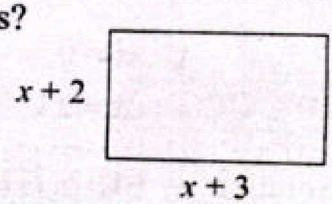
$(y-1)(9y+1) = 0$

$y - 1 = 0$ or $9y + 1 = 0$

Back substitution gives:

Challenge

The rectangle has an area of 42 square centimeters. What is the value of length of sides?



$x^2 - 1 = 0$ or $9x^2 + 1 = 0$

$x = \pm\sqrt{1}$ or $x = \pm\sqrt{\dfrac{-1}{9}}$

$x = \pm 1$ or $x = \pm\dfrac{i}{3}$

Equations of the Form $a\mathrm{P}(x) + \frac{b}{\mathrm{P}(x)} = c$

To solve this type of equation, use a substitution $\mathrm{P}(x) = y$. Multiply the equation by y and form a quadratic equation involving the variable y. Solve the new equation for y and replace y by $\mathrm{P}(x)$. In this way, you will get two simple equations involving variable x. Solve these equations and write the solution set.

Example: Solve: $3x + \dfrac{4}{x} = 7$

Solution: $3x + \dfrac{4}{x} = 7$

$3x^2 + 4 = 7x$

$3x^2 - 7x + 4 = 0$

$3x^2 - 3x - 4x + 4 = 0$

$3x(x-1) - 4(x-1) = 0 \Rightarrow x - 1 = 0$ or $3x - 4 = 0$

$x = 1$ or $x = \frac{4}{3}$

$\therefore S.S. = \left\{1, \frac{4}{3}\right\}$

Challenge

(i) Solve for real roots, $y^3 - 27 = 0$

(ii) Solve:

$$x^2 + 3x + 7 = \frac{6}{x^2 + 3x + 2}$$

Example: Solve $2(x^2+1) - \dfrac{2}{x^2+1} = 3$

Solution: $2(x^2+1) - \dfrac{2}{x^2+1} = 3$

Put $x^2 + 1 = y$

So, $2y - \frac{2}{y} = 3 \Rightarrow 2y^2 - 2 = 3y$

$2y^2 - 3y - 2 = 0 \Rightarrow 2y^2 - 4y + y - 2 = 0$

$2y(y-2) + 1(y-2) = 0 \Rightarrow (y-2)(2y+1) = 0$

$y - 2 = 0$ or $2y + 1 = 0 \Rightarrow y = 2$ or $y = \frac{-1}{2}$

Back substitution: When $y = 2$

$x^2 + 1 - 2 = 0$

$x^2 - 1 = 0$

$x = \pm 1$

When $y = \frac{-1}{2}$

$x^2 + 1 = -\frac{1}{2}$

$2x^2 = -3 \Rightarrow x^2 = \frac{-3}{2}$

$x = \pm\sqrt{\frac{3}{2}}\,i$

Solution $= \pm 1, \pm\sqrt{\frac{3}{2}}\,i$

Reciprocal Equations

If an equation is not affected by the replacement of a variable by its reciprocal, then it is called a reciprocal equation. The reciprocal of a root of such an equation is also its root.

Consider the equation $ax^4+bx^3+cx^2+bx+a=0.$ $\rightarrow$ (i)

$$a\left(\frac{1}{x}\right)^4+b\left(\frac{1}{x}\right)^3+c\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)+a=0 \qquad (x \text{ is replaced by } \tfrac{1}{x})$$

$$\frac{a}{x^4}+\frac{b}{x^3}+\frac{c}{x^2}+\frac{b}{x}+a=0$$

$$a+bx+cx^2+bx^3+ax^4=0 \quad (\text{equation is multiplied by } x^4)$$

$$ax^4+bx^3+cx^2+bx+a=0. \qquad \rightarrow \text{(ii)}$$

Equations (i) and (ii) are same.

To solve the reciprocal equation (i), arrange it in the form

$$a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$$

Put $y=x+\frac{1}{x}$ and find the value of $x^2+\frac{1}{x^2}=y^2-2$.

In this way equation (i) is reduced to quadratic form. Solve this equation for y and then replace y with its substitute $x+\frac{1}{x}$. Finally find values of x and write solution set.

Example: Solve the equation:

$$6x^4-35x^3+62x^2-35x+6=0.$$

Solution:

$$6x^4-35x^3+62x^2-35x+6=0.$$

$$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0 \quad (\text{dividing by } x^2)$$

$$6(x^2+\frac{1}{x^2})-35(x+\frac{1}{x})+62=0 \quad \rightarrow \text{(i)}$$

Let $x+\frac{1}{x}=y$

$$\left(x+\frac{1}{x}\right)=y^2$$

$$x^2+\frac{1}{x^2}+2=y^2$$

$$x^2+\frac{1}{x^2}=y^2-2$$

Put in equation (i)

$$6\left(y^2-2\right)-35y+62=0$$

$$6y^2-12-35y+62=0$$

Critical Thinking

Example: Solve $y-7\sqrt{y}-8=0,\ y\in R$

Solution: $y-7\sqrt{y}-8=0$

Let $\sqrt{y}=z$

Then $y=z^2$

Equation will become:

$$z^2-7z-8=0$$

$$z=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(-8)}}{2(1)}$$

$$z=\frac{+7\pm\sqrt{49+32}}{2}=\frac{+7\pm\sqrt{81}}{2}$$

$$z=\frac{+7\pm9}{2}$$

$$z=\frac{+7+9}{2}, z=\frac{+7-9}{2}$$

$$z=8, z=-1$$

Back substitution gives:

$\sqrt{y}=8, \sqrt{y}=-1$

$y=64$, (No real roots for $\sqrt{y}=-1$)

So, the solution is $=64$

$6y^2-35y+50=0$

$6y^2-15y-20y+50=0$

$3y(2y-5)-10(2y-5)=0$

$(2y-5)(3y-10)=0 \quad \Rightarrow \quad 2y-5=0$ or $3y-10=0$

$2\left(x+\frac{1}{x}\right)-5=0$ or $3\left(x+\frac{1}{x}\right)-10=0$ (Substituting the value of y)

$2x+\frac{2}{x}-5=0$ or $3x+\frac{3}{x}-10=0$

$2x^2+2-5x=0$ or $3x^2+3-10x=0$

$2x^2-5x+2=0$ or $3x^2-10x+3=0$

$2x^2-4x-x+2=0$ or $3x^2-9x-x+3=0$

$2x(x-2)-1(x-2)=0$ or $3x(x-3)-1(x-3)=0$

$(x-2)(2x-1)=0$ or $(x-3)(3x-1)=0$

$x-2=0$ or $2x-1=0$ or $x-3=0$ or $3x-1=0$

$x=2$ or $x=\frac{1}{2}$ or $x=3$ or $x=\frac{1}{3} \quad \Rightarrow \quad S.S.=\{2, \frac{1}{2}, 3, \frac{1}{3}\}$

Exponential Equations

An equation involving expressions of the form k^x is called an exponential equation where k is a constant. In exponential equations the exponents are variables. We will discuss only those exponential equations which can be reduced to quadratic form.

Exponential equations reducible to quadratic form are of the type:

$ak^{2x}+bk^x+c=0, a\neq 0.$

We use the substitution $k^x=y$ to reduce the above equation to quadratic form. Then we find y and replace it with k^x. In this way, roots of the exponential equations are found.

Example: Solve the following exponential equations

i. $2^{2x}-2\times 2^x+1=0$ **ii.** $9^x-3^{x+1}-4=0$

Solution:

i. $2^{2x}-2\times 2^x+1=0 \rightarrow$ (i)

Put $2^x=y$, so $2^{2x}=y^2$

$\therefore$ Equation (i) becomes:

$y^2-2y+1=0$

$(y-1)^2=0$

$y-1=0$

$y=1$

Replace y by its substitute 2^x

ii. $9^x-3^{x+1}-4=0 \rightarrow$ (i)

$3^{2x}-3^x\times 3-4=0$

Put $3^x=y$, so $3^{2x}=y^2$

$\therefore$ Equation (i) becomes:

$y^2-3y-4=0$

$y^2-4y+y-4=0$

$y(y-4)+1(y-4)=0$

$(y-4)(y+1)=0$

$y=4$ or $y=-1$

$3^x=4$ or $3^x=-1$

$2^x = 1 = 2^0$
$x = 0$
$S.S = \{0\}$

Converting $3^x = 4$ into logarithmic form, we get
$x = \log_3 4$
Solution of $3^x = -1$ is not possible.
$\therefore$ S.S. = $\{\log_3 4\}$

Equations of the Form $(x+a)(x+b)(x+c)(x+d) = k;\ a+b = c+d$

In this type, we multiply the pairs of factors after rearranging them in such a way that the variable terms of both products are the same or one of them is multiple of the other. Choose a substitute for the common terms and form a quadratic equation. Solve the new equation and then replace the substitute. Find the values of original variable involved in the given equation and write the solution set.

Example:

Solve the following equations by using suitable substitution.

$$(x+1)(x+2)(x+3)(x+4) = 24$$

Solution:

$$(x+1)(x+2)(x+3)(x+4) = 24$$
$$(x+1)(x+4)\ (x+2)(x+3) = 24$$
$$\left(x^2+5x+4\right)\left(x^2+5x+6\right) = 24$$

Put $x^2+5x = y$

$$(y+4)(y+6) = 24$$
$$y^2+10y+24 = 24$$
$$y^2+10y = 0$$
$$y(y+10) = 0$$
$$y = 0 \text{ or } y+10 = 0$$

For $y = 0$

$$x^2+5x = 0$$
$$x(x+5) = 0$$
$$x = 0 \text{ or } x+5 = 0$$
$$x = 0 \text{ or } x = -5$$

For $y+10 = 0$

$$x^2+5x+10 = 0$$

Here, $a = 1, b = 5$ and $c = 10$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$x = \frac{-5 \pm \sqrt{5^2-4(1)(10)}}{2(1)}$$
$$x = \frac{-5 \pm \sqrt{25-40}}{2}$$
$$x = \frac{-5 \pm \sqrt{-15}}{2}$$
$$x = \frac{-5 \pm \sqrt{15}\,i}{2}$$

$$S.S. = \left\{0, -5, \frac{-5 \pm \sqrt{15}\,i}{2}\right\}$$

Exercise 2.2

1. Reduce the following equations to quadratic form using suitable substitution.

i. $ax^4+bx^2+c=0$

ii. $9x^6-3x^3+7=0$

iii. $3x+\dfrac{4}{6x-2}=-1$

iv. $(x+1)^2+\dfrac{3}{(x+1)^2}=4$

v. $3\left(x^2+\dfrac{1}{x^2}\right)+8\left(x+\dfrac{1}{x}\right)+11=0$

vi. $3x^4+7x^3+5x^2-7x+3=0$

vii. $ak^{2x}+bk^{x}+c=0$

viii. $8\times 4^x-7\times 2^x-1=0$

ix. $(x-1)(x-2)(x+3)(x-6)=6$

x. $(2x-1)(2x-7)(x-3)(x-1)=8$

2. Solve the following equations by reducing them to quadratic form.

i. $x^4-20x^2+64=0$

ii. $x^4+16x^2-225=0$

iii. $x^{\frac{2}{5}}+5x^{\frac{1}{5}}+6=0$

iv. $3x^2+\dfrac{4}{x^2}=7$

v. $5(x+1)+\dfrac{3}{x+1}=8$

vi. $5x^2+\dfrac{36}{5x^2+4}=16$

vii. $2x^4-x^3-6x^2-x+2=0$

viii. $12x^4+11x^3-146x^2+11x+12=0$

ix. $4\left(x^2+\dfrac{1}{x^2}\right)-\left(x-\dfrac{1}{x}\right)-11=0$

x. $2^{2x}-34\times 2^x+64=0$

xi. $3^{2x}-12\times 3^x+27=0$

xii. $5^{2x}-150\times 5^x+3125=0$

xiii. $(x+2)(x-3)(x+10)(x+5)=-396$

xiv. $x(x-1)(x+2)(x+3)=40$

xv. $(x-2)(x-6)(x+4)(x+8)+256=0$

3. Solve the following equations by factoring without using substitution.

i. $2x^4-3x^2+1=0$

ii. $8x^6-7x^3-1=0$ (Find only real roots)

iii. $x^2+\dfrac{1}{x^2}=2$

iv. $4\times 2^{2x}-4\times 2^x+1=0$

4. Write two examples of equations that are not quadratic, but can be written in quadratic form.
5. How would you solve the equation $(y-4)^2-7(y-4)=-6$? Write explanation and then solve the equation.
6. Solve $y^3=125$.

Nature of Roots of a Quadratic Equation

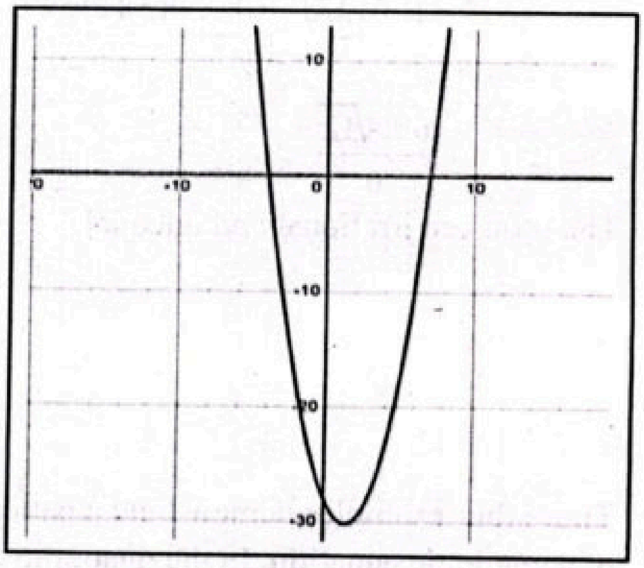
We have already derived the quadratic formula by solving the general form of quadratic equation for x. We know that the solutions of a quadratic equation of the form $ax^2+bx+c=0$ with $a\neq 0$, are given by this formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To define discriminant, first we solve the following examples by using the quadratic formula:

Example A: Solve. $x^2-3x-28=0$

Solution: We have $a=1$, $b=-3$ and $c=-28$



$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-28)}}{2(1)}$$

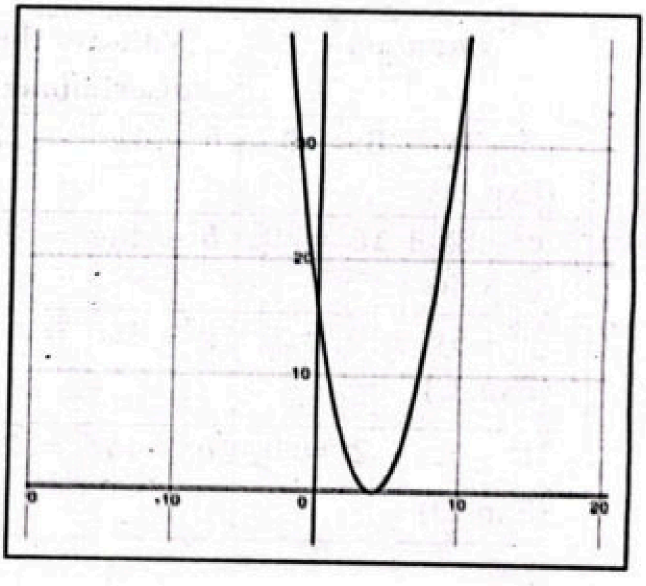
$$=\frac{3\pm\sqrt{121}}{2}$$

$$x=\frac{3\pm 11}{2}=7,-4$$

The roots of equation are rational and unequal.

Example B: Solve $x^2-8x+16=0$

Solution: We have $a=1$, $b=-8$ and $c=16$



$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(16)}}{2(1)}$$

$$=\frac{8\pm 0}{2}$$

$$x=\frac{8+0}{2},\ x=\frac{8-0}{2}$$

$x=4, x=4$

The roots of equation are rational and equal.

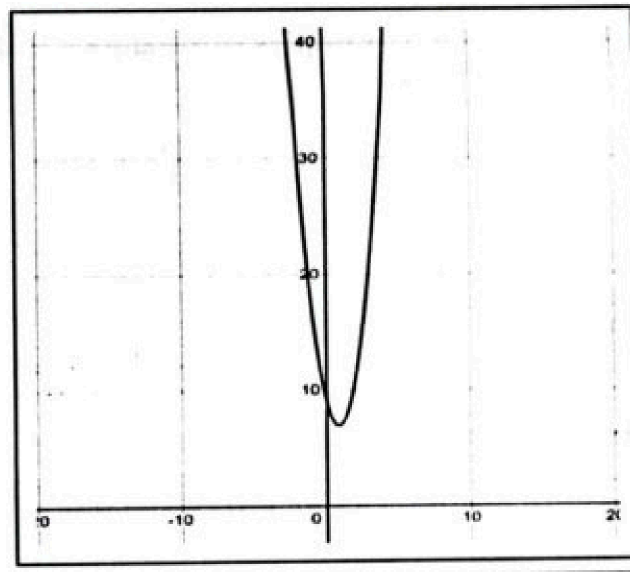
Example C: Solve $3y^2 - 5y + 9 = 0$

Solution: We have $a = 3$, b = –5 and c = 9

$$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(9)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{-83}}{6} \qquad = \frac{5 \pm \sqrt{83}\,i}{6}$$

Since radical contains a negative value, therefore, the roots of equation are imaginary.



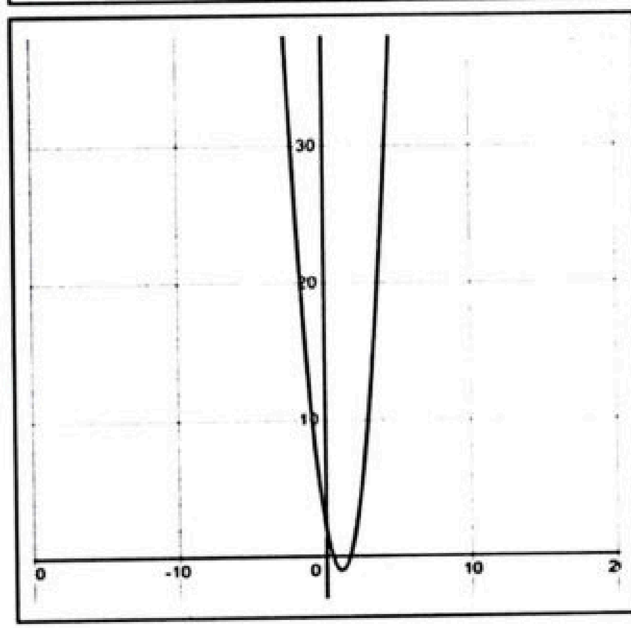
Example D: Solve $3t^2 - 6t + 2 = 0$

Solution: We have $a = 3$, b = – 6 and c = 2

$$t = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{6 \pm \sqrt{12}}{6}$$

The roots are irrational and unequal.



These four examples demonstrate a pattern that is useful in determining the nature of the roots of a quadratic equation. In the quadratic formula, the expression under the sign '$b^2 - 4ac$', is called the discriminant. The discriminant tells the nature of roots of a quadratic equation.

Equation	Value of the discriminant	Roots	Nature of Roots
$x^2 - 3x - 28 = 0$ (Exp. A)	$b^2 - 4ac = 121$	$7, -4$	Roots are rational and unequal.
$x^2 - 8x + 16 = 0$ (Exp. B)	$b^2 - 4ac = 0$	4, 4	Roots are rational and equal.
$3y^2 - 5y + 9 = 0$ (Exp. C)	$b^2 - 4ac = -83$	$\frac{5 \pm \sqrt{83}\,i}{6}$	Roots are imaginary/complex.
$3t^2 - 6t + 2 = 0$ (Exp. D)	$b^2 - 4ac = 24$	$\frac{3 \pm \sqrt{3}}{3}$	Roots are irrational and unequal.

The above chart shows that if the value of the discriminant is a perfect square or 0, the roots are real and rational. Other positive discriminant will yield irrational roots. A negative discriminant means roots will be imaginary/complex.

Example: Find the value of discriminant and describe the nature of roots.

a. $2x^2+x-3=0$ b. $x^2+8=0$

Solution:

a.	b.
$2x^2+x-3=0$ $a=2, b=1, c=-3$ $\text{Disc}=b^2-4ac$ $=(1)^2-4(2)(-3)=25$ The value of discriminant is positive and a perfect square. So, the given equation has two real roots and they are rational and unequal.	$x^2+8=0$ $a=1, b=0, c=8$ $\text{Disc}=b^2-4ac$ $=(0)^2-4(1)(8)=-32$ The value of discriminant is negative. So, the given equation has two imaginary roots.

Check Point:

Verify the nature of roots of $9x^2-12x+4=0$, by solving it.

Example: Find the value of m, when $x^2-3x+m=0$ has equal roots.

Solution: $x^2-3x+m=0$

Here $a=1, b=3, c=m$

It is given that equation has equal roots (same root) so the discriminant is zero.

i.e. $\text{Disc}=b^2-4ac=0$

$$(3)^2-4(1)(m)=0$$

$$9-4m=0 \quad \text{or} \quad 4m=9 \quad \text{or} \quad m=\frac{9}{4}$$

Exercise 2.3

1. Find the discriminant of the following quadratic equations.
   (i) $x^2+6x-27=0$ (ii) $x^2-x-12=0$
   (iii) $8x^2+2x+1=0$ (iv) $12x^2-11x-15=0$
2. Discuss the nature of roots of the following quadratic equations.
   (i) $x^2-2x-15=0$ (ii) $x^2+3x-4=0$ (iii) $12x^2+x-20=0$
   (iv) $x^2+2x+8=0$ (v) $x^2+3x-9=0$
3. For what value of k, $9x^2-kx+16=0$ is a perfect square?
4. If roots of $x^2+kx+9=0$ are equal, find k?
5. Show that the roots of $2x^2+(mx-1)^2=3$, are equal if $3m^2+4=0$.

6. Find the value of "m" when roots of the following quadratic equations are equal.
   (i) $x^2 - 6x + m = 0$ (ii) $m^2x^2 + (2m+1)x + 1 = 0$
   (iii) $(m+3)x^2 + (m+1)x + m + 1 = 0$
7. Show that the roots of the equation $(a^2 + b^2)x^2 + 2(ac + bd)x + c^2 + d^2 = 0$ are imaginary. Moreover, it shows repeated roots if $ad = bc$.
8. Show that the roots of the equation $(ax + c)^2 = 4bx$ will be equal, if $b = ac$.
9. Show that the roots of the following equations are real.
   (i) $mx^2 - 2mx + m - 1 = 0$ (ii) $bx^2 + ax + a - b = 0$
10. Show that the roots of the following equation are real.
    $(a+b)x^2 - ax - b = 0$

Roots and Coefficients of the Quadratic Equation

Let S_1, S_2 be the roots of $ax^2 + bx + c = 0$, then using the quadratic formula:

$$S_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad S_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\text{Sum of roots} = S_1 + S_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$$\text{and product of roots} = S_1S_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - (\sqrt{b^2-4ac})^2}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore$ Sum of the roots $S_1 + S_2 = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coeficient of } x^2}$

and product of roots $S_1S_2 = \frac{c}{a} = \frac{\text{contant term}}{\text{coefficient of } x^2}$

Example: Write down the sum and product of the roots of quadratic equation:

(i) $x^2 - x + 2 = 0$ (ii) $x^2 - 7x + 5 = 0$
(iii) $3x^2 - 11x - 4 = 0$ (iv) $6s^2 + 2s + 3 = 0$

Solution:

(i) $x^2 - x + 2 = 0$ Here $a = 1$, $b = -1$, $c = 2$ Sum of the roots $= S_1 + S_2$ $= \frac{-b}{a} = \frac{-(-1)}{1} = 1$ Product of roots $= S_1 S_2$ $= \frac{c}{a} = \frac{2}{1} = 2$	(ii) $3x^2 - 11x - 4 = 0$ $a = 3, b = -11, c = -4$ Sum of roots $= S_1 + S_2$ $= \frac{-b}{a} = \frac{-(-11)}{3} = \frac{11}{3}$ Product of roots $= S_1 S_2$ $= \frac{c}{a} = \frac{-4}{3}$
(iii) $x^2 - 7x + 5 = 0$ $a = 1, b = -7, c = 5$ $S_1 + S_2 = \frac{-b}{a} = \frac{-(-7)}{1} = 7$ $S_1 S_2 = \frac{c}{a} = \frac{5}{1} = 5$	(iv) $6s^2 + 2s + 3 = 0$ $a = 6, b = 2, c = 3$ $S_1 + S_2 = \frac{-b}{a} = \frac{-(2)}{6} = \frac{-1}{3}$ $S_1 S_2 = \frac{c}{a} = \frac{3}{6} = \frac{1}{2}$

Example: If S_1, S_2 are the roots of $x^2 - 8x + 20 = 0$. Find the values of

(i) $S_1^2 + S_2^2$ (ii) $(S_1 - S_2)^2$ (iii) $\frac{1}{S_1+1} + \frac{1}{S_2+1}$

Solution: Here $a = 1, b = -8, c = 20$

Since S_1, S_2 are the roots of $x^2 - 8x + 20 = 0$,

$$\therefore \quad S_1 + S_2 = \frac{-b}{a} = \frac{-(-8)}{1} = 8 \quad \text{and} \quad S_1 S_2 = \frac{c}{a} = \frac{20}{1} = 20$$

(i) Now
$$\begin{aligned} S_1^2 + S_2^2 &= S_1^2 + S_2^2 + 2S_1S_2 - 2S_1S_2 \\ &= (S_1 + S_2)^2 - 2S_1S_2 \\ &= (8)^2 - 2(20) \\ &= 64 - 40 = 24 \end{aligned}$$

(ii)
$$\begin{aligned} (S_1 - S_2)^2 &= S_1^2 + S_2^2 - 2S_1S_2 \\ &= S_1^2 + S_2^2 + 2S_1S_2 - 4S_1S_2 \\ &= (S_1 + S_2)^2 - 4S_1S_2 \\ &= (8)^2 - 4(20) \\ &= 64 - 80 = -16 \end{aligned}$$

(iii)
$$\begin{aligned} \frac{1}{S_1+1} + \frac{1}{S_2+1} &= \frac{S_2+1+S_1+1}{(S_2+1)(S_1+1)} \\ &= \frac{S_1+S_2+2}{S_1S_2+(S_1+S_2)+1} \\ &= \frac{8+2}{20+8+1} = \frac{10}{29} \end{aligned}$$

Note:
Instead of S_1, S_2, we can use α and β.

Formation of an Equation whose Roots are Given

We have solved several quadratic equations and found the roots. Here, we may know the roots of the quadratic equation but we do not know the equation itself. Consider the roots of a quadratic

quadratic equation are 4 and $-\frac{1}{3}$.

let $x=4$ and $x=-\frac{1}{3}$ are roots and $x-4=0$ and $x+\frac{1}{3}=0$ are factors of required equation.

$(x-4)(3x+1)=0$ or $3x^2+x-12x-4=0$

$3x^2-11x-4=0$

Which is the required equation.

If S_1, S_2 are the roots of required quadratic equation, then

Let $x=S_1$ and $x=S_2$

$\Rightarrow x-S_1=0, x-S_2=0$

$(x-S_1)(x-S_2)=0$

$x^2-(S_1+S_2)x+S_1S_2=0,$

$S_1+S_2=$ Sum of roots $=S$ and $S_1S_2=$ Product of roots $=P$

Then, $x^2-Sx+P=0$.........(i)

Above formula (i) is used to form a quadratic equation when its roots are given.

Let us solve some examples to understand the concept.

Example: Form a quadratic equation whose roots are $\frac{1}{4}$ and -1.

Solution: The roots are $\frac{1}{4}$ and -1.

Let $x=\frac{1}{4}$ and $x=-1$

$x-\frac{1}{4}=0$ and $x+1=0$

$4x-1=0$, $x+1=0$

Factors of quadratic equation are $(4x-1)$ and $(x+1)$, we have

$(4x-1)(x+1)=0$

$4x^2+4x-x-1=0$

$4x^2+3x-1=0$, required quadratic equation.

Example: Form a quadratic equation with roots:

(i) 5 and 6 (ii) $-\frac{5}{4}$ and $\frac{16}{5}$ (iii) $3+2i$ and $3-2i$

Solution:

(i) Since 5 and 6 are roots then (iii) $3+2i$ and $3-2i$

S = sum of roots = 5 + 6 = 11

P = product of the roots = 5(6) = 30

The required equation is

$x^2 - Sx + P = 0$..........(i)

Substituting these values of S and P in (*i*), we have

$$x^2 - 11x + 30 = 0$$

(ii) $-\frac{5}{4}$ and $\frac{16}{5}$

$$\text{Sum} = S = -\frac{5}{4} + \frac{16}{5} = \frac{13}{3}$$

$$\text{Product} = P = -\frac{5}{4} \times \frac{16}{5} = -4$$

The equation is: $x^2 - \frac{13}{3}x + 4 = 0$

or $3x^2 - 13x + 12 = 0$

$S = \text{Sum} = 3 + 2i + 3 - 2i = 6$

$P = \text{Product} = (3 + 2i)(3 - 2i)$

$= 9 - 4i^2 = 13$

The equation is:

$$x^2 - 6x + 13 = 0$$

Example: If S_1, S_2 are the roots of $ax^2 + bx + c = 0$. Form the equation whose roots are twice to the roots of the given equation.

Solution: If S_1, S_2 are the roots of $ax^2 + bx + c = 0$ then

$$S_1 + S_2 = \frac{-b}{a}, \quad S_1S_2 = \frac{c}{a}$$

The new roots are $2S_1$ and $2S_2$

Sum of new roots = $S = 2S_1 + 2S_2$

$$= 2(S_1 + S_2) = 2\left(\frac{-b}{a}\right) = \frac{-2b}{a}$$

Product of new roots = $P = (2S_1)(2S_2) = 4S_1S_2$

$$= 4\left(\frac{c}{a}\right) = \frac{4c}{a}$$

Now substituting the values of S and P in $x^2 - Sx + P = 0$, we get

$$x^2 - \left(\frac{-2b}{a}\right)x + \frac{4c}{a} = 0$$

or $ax^2 + 2bx + 4c = 0$

Exercise 2.4

1. Find the sum and product of the roots of the quadratic equation.

   (i) $x^2-5x+2=0$ (ii) $-4x^2-6x-2=0$ (iii) $5x^2-2x+2=0$

   (iv) $-4x^2-8x-9=0$ (v) $16y^2-17y-12=0$ (vi) $0.3x^2-7.7x+1.8=0$

2. Form a quadratic equation with roots:

   (i) $1, -\frac{8}{3}$ (ii) $\sqrt{3}, 2\sqrt{3}$

   (iii) $2+\sqrt{3}, 2-\sqrt{3}$ (iv) $5i, -5i$

   (v) $7+2i, 7-2i$

3. If S_1, S_2 are the roots of $3x^2-2x+4=0$, find the value of:

   (i) $\frac{1}{S_1^2}+\frac{1}{S_2^2}$ (ii) $S_1^2+S_2^2$ (iii) $2S_1+2S_2+4$ (iv) $\frac{1}{S_1}+\frac{1}{S_2}$

   (v) $\frac{S_1}{S_2}+\frac{S_2}{S_1}$ (vi) $S_1S_2^2+S_1^2S_2$ (vii) $S_1^3S_2+S_1S_2^3$ (viii) $(S_1-3)(S_2-3)$

4. if S_1, S_2 are the roots of $7x^2+10x+7=0$, form the equations whose roots are:

   (i) S_1^2, S_2^2 (ii) $\frac{1}{S_1}, \frac{1}{S_2}$ (iii) $S_1^3S_2, S_1S_2^3$ (iv) $S_1-\frac{1}{S_1}, S_2-\frac{1}{S_2}$

   (v) $2S_1+1, 2S_2+1$ (vi) $\frac{S_1}{S_2}, \frac{S_2}{S_1}$ (vii) $S_1+S_2, \frac{1}{S_1}+\frac{1}{S_2}$ (viii) S_1^2+1, S_2^2+1

   (ix) $S_1^2+S_2, S_1+S_2^2$

5. If S_1, S_2 are the roots of $x^2+6x+3=0$, form the equation whose roots are

   $(S_1+S_2)^2, (S_1-S_2)^2$

6. If S_1, S_2 are the roots of $2x^2+6x-3=0$, form the equation whose roots are

   $S_1-\frac{3}{S_2^2}, S_2-\frac{3}{S_1^2}$

7. Find k if S_1 and S_1-5 are the roots of $x^2-3kx+5=0$.

8. Find k such that 3 is a root of $x^2+kx-21=0$.

Simultaneous Equations

A system of equations having a common solution is called a system of simultaneous equations. The set of all the ordered pairs (x, y) which satisfies the system of equations is called the solution of the system. System of two equation involving two variables.

System of two Equations Involving two Variables

Case I: When one Equation is Linear and One Quadratic

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following example.

Example: Solve the system of equations $x+y=7$ and $x^2-xy+y^2=13$

Solution: Given Equations are

$$x+y=7 \text{......}(i) \qquad x^2-xy+y^2=13 \text{.............}(ii)$$

From equation (i)

$$x=7-y$$

Substituting in equation (ii)

$$(7-y)^2-(7-y)y+y^2=13$$
$$49-14y+y^2-7y+y^2+y^2=13$$
$$3y^2-21y+36=0$$
$$y^2-7y+12=0$$
$$y^2-4y-3y+12=0$$
$$y(y-4)-3(y-4)=0$$
$$(y-4)(y-3)=0$$
$$y-4=0 \text{ or } y-3=0$$
$$y=4,\ y=3$$

Putting $y=3$, in (i), we get $x=7-3=4$

Putting $y=4$, in (i), we get $x=7-4=3$

Hence solution set is $\{(4, 3), (3, 4)\}$

Case II: When both of the Equations are Quadratic

The equations in this case are classified as:

(i) Both the equations contain only x^2 and y^2 terms.

(ii) One of the equations, is homogeneous in x and y.

(iii) Both the equations are non-homogeneous.

The methods of solving these types of equations are explained through the following examples.

Example: Solve the system of equations $x^2+y^2=25$ and $2x^2+3y^2=66$

Solution: Given equations are

$$x^2+y^2=25 \quad (i) \qquad 2x^2+3y^2=66 \quad (ii)$$

Multiplying equation (i) by 2 and subtracting from equation (ii).

$$\begin{array}{r} 2x^2+3y^2=66 \\ \pm 2x^2 \pm 2y^2=50 \\ \hline y^2=16 \end{array}$$

Putting $y^2=16$ in equation (i)

$$x^2+16=25$$
$$x^2=9$$

Now from $x^2=9$, we have $x=\pm 3$

and from $y^2=16$, we have $y=\pm 4$

Hence, solution set is $\{(\pm 3, \pm 4)\}$

Example:

Solve the system of equations $x^2+y^2=20$ and $6x^2+xy-y^2=0$

Solution:

Given equations are

$$x^2+y^2=20 \qquad \text{(i)}$$
$$6x^2+xy-y^2=0 \qquad \text{(ii)}$$

From equation (ii)

$$-y^2+xy+6x^2=0$$
$$y^2-xy-6x^2=0$$
$$y^2-3xy+2xy-6x^2=0$$
$$y(y-3x)+2x(y-3x)=0$$
$$(y-3x)(y+2x)=0$$
$$y-3x=0 \quad \text{or} \quad y+2x=0$$
$$y=3x\ldots\ldots\text{(iii)} \quad y=-2x\ldots\ldots\text{(iv)}$$

Putting $y=3x$ in (i)

$$x^2+(3x)^2=20$$
$$x^2+9x^2=20$$
$$10x^2=20$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

When $x=\sqrt{2}$

From (iii) $y=3\sqrt{2}$

When $x=-\sqrt{2}$,

From (iii) $y=-3\sqrt{2}$

Putting $y=-2x$ in (i)

$$x^2+(-2x)^2=20$$
$$x^2+4x^2=20$$
$$5x^2=20$$
$$x^2=4$$
$$x=\pm 2$$

When $x=2$

From (iv) $y=-4$

When $x=-2$

From (iv) $y=4$

Hence, solution set is $\{(\sqrt{2},\ 3\sqrt{2}), (-\sqrt{2},\ -3\sqrt{2}), (2,\ -4), (-2,\ 4)\}$

Exercise 2.5

Solve the following simultaneous equations:

1. $2x+y=1; \quad x^2+y^2=10$
2. $3x-2y=1; \quad x^2+xy-y^2=1$
3. $3x+y+3=0; \quad (x+1)^2-4(x+1)-6=y$
4. $(x+3)^2-(y-2)^2=10; \quad x+y=4$
5. $2x^2-8xy+6y^2=0; \quad x^2+y^2=45$
6. $6x^2-5xy-y^2=0; \quad y^2+4xy=30$
7. $4x^2+4y^2=65; \quad 6x-2y=5$
8. $x^2+y=0; \quad x+y=-2$
9. $x^2+3y^2=14; \quad 3x^2+y^2=6$
10. $2x^2-5y^2=8; \quad x^2+2y^2=13$

Test Taking Tip:
Most standardized tests have a time limit, so you must budget your time carefully. Some questions will be much easier than others. If you cannot answer a question within a few minutes go on to the next one. If there is still time left when you get to the end of the test, go back to the ones that you skipped.

Real Problems Involving Quadratic Equations

There are many problems which lead to quadratic equations. To form an equation, we consider symbols for unknown quantities in the problems.
In order to solve problems, we must have

(i) Suppose the unknown quantities to be x or y.
(ii) Translate the problem into symbols and form the equation satisfying the conditions.

The method of solving the problem will be illustrated through the following examples.

Example: The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and breadth of the room.

Solution: Let the breadth of room $= x$ metres
and the length of room $= x + 3$ metres
Area of the room $= x(x + 3)$ square metres
By the conditions of the equation

$$x(x+3)=180 \quad \text{or} \quad x^2+3x-180=0$$
$$(x+15)(x-12)=0 \quad \text{or} \quad x=-15, x=12$$

As breadth cannot be negative, therefore, we take $x = 12$.
Length $= x+3=12+3=15$
$\therefore$ Breadth of the room $=12m$ and length of room $=15m$

Example:

The sum of the Cartesian coordinates of a point is 6 and the sum of their squares is 20. Find the coordinate of the point.

Solution:

Let (x, y) be the coordinates of required point, then by the given condition we get

$$x+y=6 \qquad (i)$$

$$x^2+y^2=20 \qquad (ii)$$

From(i) $x+y=6 \quad \Rightarrow y=6-x$ putting in (ii)

$$x^2+y^2=20 \quad \Rightarrow x^2+(6-x)^2=20$$

$$x^2+36-12x+x^2=20$$

$$x^2-6x+8=0$$

$$x^2-4x-2x+8=0$$

$$x(x-4)-2(x-4)=0$$

$$(x-4)(x-2)=0$$

$$x-4=0, \quad x-2=0$$

$$x=4, \quad x=2$$

Problem Solving Plan:	
• Explore the problem.	• Plan the solution.
• Solve the problem.	• Examine the solution.

When $x=4,\ y=6-x=6-4=2$

When $x=2,\ y=6-x=6-2=4$

$\therefore$ The coordinates of the point are 4 and 2, the points may be $(4, 2)$ and $(2, 4)$.

Exercise 2.6

1. Product of two consecutive positive even numbers is 120. Find the numbers.
2. The difference of a positive number and its square is 380. Find the number.
3. The difference of cubes of two consecutive positive numbers is 91. Find them.
4. The sum of the Cartesian coordinates of a point is 9 and the sum of their squares is 45. Find the coordinates of the point.
5. The sum of two numbers is 11 and the product is 30. Find the numbers.
6. The sum of the squares of two consecutive positive odd integers is 34. Find the integers.
7. The sum of ages of a father and his son is 50 years. Ten years ago, the father was 9 times as old as his son. Find the present age of father and son.
8. A two-digit number is decreased by 45 when the digits are reversed. If the sum of the digits is 11, find the number.
9. Sum of two numbers is 20. Find the numbers if the sum of first number and square of other is 40.
10. The reciprocal of the sum of reciprocals of two numbers is $\frac{12}{5}$. Find the numbers if their sum is 10.

11. A group of 1025 students form two square patterns during morning assembly. One square pattern contains 5 more students than the other. Find the number of students in each pattern.
12. Sum of squares of two positive consecutive numbers is 145. The difference of their squares is 17. Find the numbers.
13. A ball is thrown upwards its height $h(t)$ (in meters) after t seconds is modeled by $d(t) = -5t^2 + 20t + 2$. Find the time when the ball hit the ground.
14. The stopping distance $d(x)$ (in meters) of a car traveling at x km/h is modeled by $d(x) = 0.05x^2 + 0.4x$. If the stopping distance is 30 meters, find the speed of the car.
15. A valuable stamp 4cm wide and 5cm long. The stamp is to be mounted on a sheet of paper that is $5\frac{1}{2}$ times the area of the stamp. Determine the dimensions of the paper that will ensure a uniform border around the stamp.

I have Learnt

- $ax^2 + bx + c = 0, a \neq 0$ is standard form of quadratic equation in one variable x.
- There are three methods for solving a quadratic equation.
  i. Factorization Method ii. Completing Square Method iii. The Quadratic Formula
- Some equations are not quadratic but they can be reduced in quadratic form.
- In the quadratic formula, the expression under the radical sign '$b^2 - 4ac$', is called the discriminant. The discriminant tells the nature of roots of quadratic equation.
- Sum of the roots of a quadratic equation = $S = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coeficient of } x^2}$
- Product of roots of a quadratic equation = $P = \frac{c}{a} = \frac{\text{contant term}}{\text{coefficient of } x^2}$
- Formula used to form a quadratic equation when its roots are given is $x^2 - Sx + P = 0$
- A system of equations having a common solution is called a system of simultaneous equations.
- The set of all the ordered pairs (x, y) which satisfies the system of equations is called the solution of the system.

MISCELLANEOUS EXERCISE-2

1. Encircle the correct option in the following.
   i. Which of the following is a quadratic equation?
   (a) $ax + b = c$ (b) $ax^2 + bx + c$
   (c) $ax^2 + bx + c = 0, a \neq 0$ (d) $ax^2 + bx + c = 0, a = 0$

ii. How many roots of $(x-3)(x-2)=6$ exist?
(a) no (b) 0 (c) 1 (d) 2

iii. What should be added to x^2+x to make it a complete square?
(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 4

iv. Solution set of $x^2-4=0$ is:
(a) $\{0, 4\}$ (b) $\{2, -2\}$ (c) $\{4, -4\}$ (d) $\{\ \}$

v. Roots of the equation $(x-1)^2=9$ are:
(a) $-2, 4$ (b) $2, 4$ (c) $-4, 2$ (d) $-2, -4$

vi. Solution set of $2^{2x}-2^{x+1}+1=0$?
(a) $\{0\}$ (b) $\{1\}$ (c) $\{0, 1\}$ (d) $\{0, -1\}$

vii. Solution set of $x+\frac{1}{x}=2$ is:
(a) $\{0\}$ (b) $\{-1\}$ (c) $\{-1, 1\}$ (d) $\{1\}$

viii. Which of the following is a reciprocal equation?
(a) $x^2+2x+2=0$ (b) $x^4+x^3+x^2+x+1=0$
(c) $\sqrt{2x+3}=0$ (d) $x^4+2x^3+x^2+4x=0$

ix. 2 and -3 are roots of:
(a) $(x-2)(x-3)=0$ (b) $(x+2)(x+3)=0$
(c) $(x-2)(x+3)=0$ (d) $(x+2)(x-3)=0$

x. The discriminant of $ax^2+bx+c=0$ is:
(a) b^2+4ac (b) b^2-4ac
(c) $4ac-b^2$ (d) $-b^2-4ac$

xi. If S_1, S_2 are the roots of $ax^2+bx+c=0$, then sum of roots is:
(a) $\frac{c}{a}$ (b) $\frac{a}{c}$ (c) $\frac{-b}{a}$ (d) $\frac{a}{b}$

xii. Roots of the equations $x^2-5x+5=0$ are:
(a) Imaginary (b) rational (c) equal (d) irrational

xiii. Sum and product of roots of a quadratic equation are respectively 2 and 5. The equation is:
(a) $x^2-2x+5=0$ (b) $x^2+2x+5=0$
(c) $x^2-2x-5=0$ (d) $x^2+2x-5=0$

2. Find all roots of $8x^6-7x^3-1=0$.
3. For what values of m the roots of the equation $(m-1)x^2+2mx+m+3=0$ are equal? Also solve the equation.
4. If S_1, S_2 are the roots of $ax^2+bx+c=0$, find the value of $(S_1-3)(S_2-3)$.
5. If roots of $25x^2-5ax-b=0$ are equal then find the values of a and b if $a^2+b=6$.
6. A rectangular chocolate box has volume as x^3+2x^2-5x-6. Find the length and width of the box if its height is $x-2$.