Unit 9

CHORDS OF A CIRCLE

In this unit the students will be able to

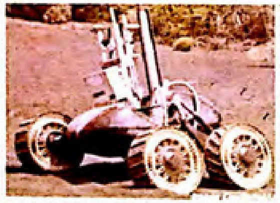
To prove the following theorems along with corollaries and apply them to solve appropriate problems.

- One and only one circle can pass through three non- collinear points.
- A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
- Perpendicular from the centre of a circle on a chord bisects it.
- If two chords of a circle are congruent then they will be equidistant from the centre.
- Two chords of a circle which are equidistant from the centre are congruent.

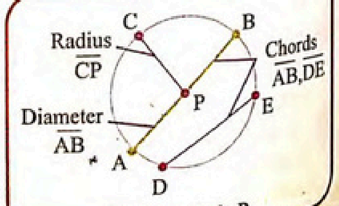
Why it's important

One of the most useful applications of the circles is the wheel. From ancient times until today, the wheel has made an enormous contribution to progress in travel, transportation, industry, and other elements of civilization.

A **circle** is the set of all points in a plane that are equidistant from a given point in the plane known as the center of the circle. A **radius** (plural, radii) is a line segment from the centre of the circle to a point on the circle.



A **chord** is a line segment whose endpoints lie on the circumference of a circle. It divides a circle into two segments. The larger part is called major segment and the smaller part, the minor segment. If the chord happens to be a diameter, each segment is a semicircle. A **diameter** is a chord that contains the centre of a circle. A circle can be named by using the symbol ⊙ and the center of the circle. The circle in the illustration is ⊙ P, or circle P.



⊙ P, or circle P

Theorem 9.1

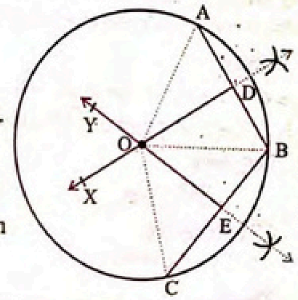
One and only one circle can pass through three non-collinear points.

Given

A, B and C are three non-collinear points.

To prove

One and only one circle can pass through the points A, B and C.



Construction

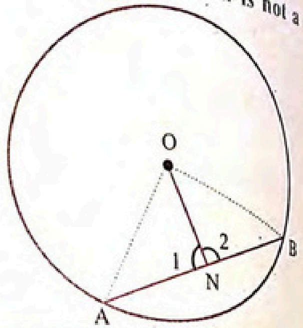
Join B to A and C. Draw perpendicular bisectors $\overleftrightarrow{XD}$ and $\overleftrightarrow{YE}$ of $\overline{AB}$ and $\overline{BC}$ respectively, which intersect at O. Join O to A, B and C.

Proof

Statements	Reasons
In $\Delta OAD \longleftrightarrow \Delta OBD$	
$\angle ODA \cong \angle ODB$	Both are right angles.
$\overline{AD} \cong \overline{BD}$	D is the mid-point of $\overline{AB}$.
$\overline{OD} \cong \overline{OD}$	Common.
$\therefore \Delta OAD \cong \Delta OBD$	S.A.S postulate.
$\therefore \overline{OA} \cong \overline{OB} \longrightarrow (i)$	Corresponding sides of congruent triangles.
Again in $\Delta OBE \longleftrightarrow \Delta OCE$	
$\angle OEB \cong \angle OEC$	Both are right angles.
$\overline{BE} \cong \overline{EC}$	E is the midpoint of $\overline{BC}$
$\overline{OE} \cong \overline{OE}$	Common.
$\therefore \Delta OBE \cong \Delta OCE$	S.A.S postulate.
$\therefore \overline{OB} \cong \overline{OC} \longrightarrow (ii)$	Corresponding sides of congruent triangles.
From (i) and (ii), $\overline{OA} \cong \overline{OB} \cong \overline{OC}$ $\Rightarrow m\overline{OA} = m\overline{OB} = m\overline{OC}$ It means that the point O is equidistant from the three points A, B, C. Therefore a circle with centre O and radius $m\overline{OA}$ or $m\overline{OB}$ or $m\overline{OC}$ will pass through the points A, B and C. Since O is the only point which is equidistant from the points A,B and C. Therefore one and only one circle can pass through these three non-collinear points.	Transitive property of congruence.

Theorem 9.2

A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.



Given

A circle with centre at O, $\overline{AB}$ is a chord of the circle.
N is the mid point of $\overline{AB}$ which is joined to O.

To prove

$\overline{ON} \perp \overline{AB}$

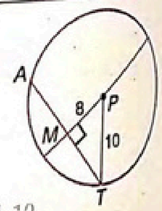
Construction

Join O to A and B.

Proof

Statements	Reasons
In $\Delta OAN \longleftrightarrow \Delta OBN$	
$\overline{OA} \cong \overline{OB}$	Radii of a given circle
$\overline{AN} \cong \overline{BN}$	Given
$\overline{ON} \cong \overline{ON}$	Common
$\therefore \Delta OAN \cong \Delta OBN$	$S.S.S \cong S.S.S$
$\therefore \angle 1 \cong \angle 2$	Corresponding angles of the congruent triangles
But $m\angle 1 + m\angle 2 = 180^\circ \longrightarrow (i)$	Supplementary angles postulate
$\therefore m\angle 1 = m\angle 2 = 90^\circ$	
Hence $\overline{ON} \perp \overline{AB}$	$\because m\angle 1 = m\angle 2$

Example 1 In $\odot P$, if $\overline{PM} \perp \overline{AT}$, $PT = 10$, and $PM = 8$, find MT.

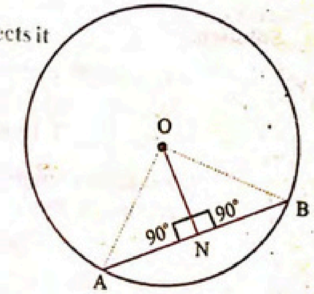


Solution

$\angle PMT$ is a right angle.	*Def. of perpendicular*
ΔPMT is a right triangle.	*Def. of right triangle*
$(MT)^2 + (PM)^2 = (PT)^2$	*Pythagorean Theorem*
$(MT)^2 + 8^2 = 10^2$	*Replace PM with 8 and PT with 10.*
$(MT)^2 + 64 = 100$	*$8^2 = 64$; $10^2 = 100$*
$(MT)^2 + 64 - 64 = 100 - 64$	*Subtract 64 from each side.*
$(MT)^2 = 36$	*Simplify.*
$\sqrt{(MT)^2} = \sqrt{36}$	*Take the square root of each side.*
$MT = 6$	*Simplify.*

Theorem 9.3

Perpendicular from the centre of a circle on a chord bisects it



Given

A circle with centre at O, $\overline{AB}$ is a chord. $\overline{ON} \perp \overline{AB}$.

To prove

$\overline{AN} \cong \overline{BN}$

Construction

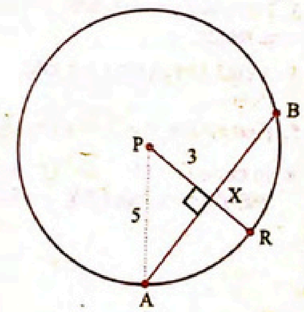
Join O to A and B.

Proof

Statements	Reasons
In $\Delta AON \longleftrightarrow \Delta BON$	
$\angle ONA \cong \angle ONB$	Right angles
$\overline{ON} \cong \overline{ON}$	common
$\overline{OA} \cong \overline{OB}$	Radii of the same circle
$\therefore \Delta AON \cong \Delta BON$	H.S $\cong$ H.S
Hence $\overline{AN} \cong \overline{BN}$	Corresponding sides of congruent triangles

Example 2

$\odot$P has a radius of 5 cm. and PX is 3 cm
$\overline{PR}$ is perpendicular to $\overline{AB}$ at point X. Find AB.



Solution

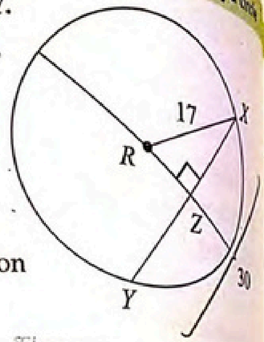
By the Pythagorean theorem:

$$(AX)^2 + 3^2 = 5^2$$
$$(AX)^2 = 5^2 - 3^2$$
$$(AX)^2 = 16$$
$$AX = 4$$

By the radius and chord theorem, $\overline{PR}$ bisects $\overline{AB}$, so BX = AX = 4.

Therefore, AB = AX + BX = 4 + 4 = 8 cm.

Example 3 In $\odot R$, $XY = 30$, $RX = 17$, and $\overline{RZ} \perp \overline{XY}$. Find the distance from R to $\overline{XY}$.



Solution

The measure of the distance from R to $\overline{XY}$ is RZ. Since $\overline{RZ} \perp \overline{XY}$, $\overline{RZ}$ bisects $\overline{XY}$, by Theorem 9.3. Thus, $XZ = \frac{1}{2}(30)$ or 15.

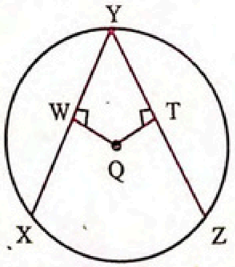
For right triangle RZX, the following equation can be written.

$(RZ)^2 + (XZ)^2 = (RX)^2$	*Pythagorean Theorem*
$(RZ)^2 + 15^2 = 17^2$	*Replace XZ with 15 and RX with 17.*
$(RZ)^2 + 225 = 289$	*$15^2 = 225$; $17^2 = 289$*
$(RZ)^2 + 225 - 225 = 289 - 225$	*Subtract 225 from each side.*
$(RZ)^2 = 64$	*Simplify.*
$\sqrt{(RZ)^2} = \sqrt{64}$	*Take the square root of each side.*
$RZ = 8$	*Simplify.*

The distance from R to $\overline{XY}$, or $\overline{RZ}$, is 8 units.

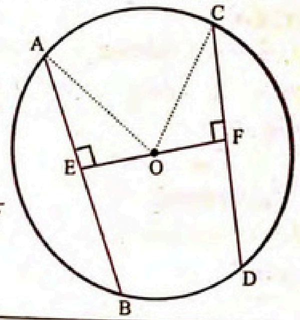
Exercise 9.1

1. If the radius of a circle is 30 cm, find the length of a chord which is 10 cm from the centre.
2. If a chord of a circle is 48 cm long and its distance from the centre is 18 cm. Find the diameter of the circle.
3. The diameter of a circle is 5 units long. How far from the centre is a chord which is 4 units in length.
4. A chord of a circle is 8 cm in length is drawn 5 cm from the centre. Find the length of the radius.
5. Find the length of a chord that is at a distance of 5cm from the centre of a circle with radius 8cm.
6. In circle Q, $\overline{QW} \perp \overline{XY}$, $\overline{QT} \perp \overline{YZ}$, $QW = 5$, $QT = 5$ and $YT = 12$. Find XY.



Theorem 9.4

If two chords of a circle are congruent then they will be equidistant from the centre.



Given

A circle with centre O, $\overline{AB}$ and $\overline{CD}$ are two congruent chords of the circle.

To prove

$\overline{AB}$ and $\overline{CD}$ are equidistant from the centre O.

Construction

Join O to A and C. Also draw perpendiculars $\overline{OE}$ and $\overline{OF}$ on the given chords $\overline{AB}$ and $\overline{CD}$ respectively.

Proof

Statements	Reasons
Since $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$	Construction
$\therefore \overline{AE} \cong \overline{EB}$ and $\overline{CF} \cong \overline{DF}$	By the use of Theorem 9.3.
Or $m\overline{AE} = m\overline{EB}$ and $m\overline{CF} = m\overline{DF}$	
But $m\overline{AB} = m\overline{CD}$	Given
or $m\overline{AE} + m\overline{EB} = m\overline{CF} + m\overline{DF}$	Segment addition postulate
$m\overline{AE} + m\overline{AE} = m\overline{CF} + m\overline{CF}$	$\because m\overline{EB} = m\overline{AE}$ and $m\overline{DF} = m\overline{CF}$
$2m\overline{AE} = 2m\overline{CF}$	Adding equal quantities
$m\overline{AE} = m\overline{CF}$	Dividing both sides by 2.
or $\overline{AE} \cong \overline{CF} \longrightarrow (i)$	
Now in $\Delta AOE \longleftrightarrow \Delta COF$	
$\overline{OA} \cong \overline{OC}$	Radii of the same circle
$\overline{AE} \cong \overline{CF} \longrightarrow (ii)$	From (i) proved above
$\angle AEO \cong \angle CFO$	Right angles
$\therefore \Delta AOE \cong \Delta COF$	H.S $\cong$ H.S
$\therefore \overline{OE} \cong \overline{OF}$ or $m\overline{OE} = m\overline{OF}$	Corresponding sides of the congruent triangles.
$\therefore \overline{AB}$ and $\overline{CD}$ are equidistant from the centre of the circle.	

Corollary

In congruent circles, congruent chords are equidistant from the centres.

Theorem 9.5

chords of a circle which are equidistant from the centre are congruent.

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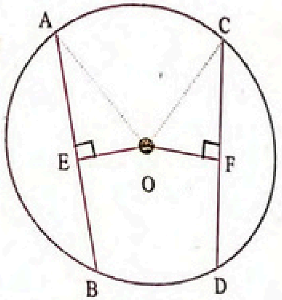
rcle with centre at O, $\overline{AB}$ and $\overline{CD}$ are two chords of the e. $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$ and $\overline{OE} \cong \overline{OF}$.

prove

$\overline{AB} \cong \overline{CD}$

nstruction

O with A and C.



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Statements	Reasons
$\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$	Given
$\therefore m\overline{AE} = m\overline{EB}$ and $m\overline{CF} = m\overline{FD} \longrightarrow (i)$	Theorem 9.3
Now in $\Delta AEO \longleftrightarrow \Delta CFO$	
$\angle AEO \cong \angle CFO$	Both are right angles
$\overline{OA} \cong \overline{OC}$	Radii of the same circle
$\overline{OE} \cong \overline{OF}$	Given
$\therefore$ $\Delta AEO \cong \Delta CFO$	H.S ≅ H.S
$\therefore$ $\overline{AE} \cong \overline{CF}$	Corresponding sides of congruent triangles
or $m\overline{AE} = m\overline{CF}$	
$\therefore$ $2m\overline{AE} = 2m\overline{CF} \longrightarrow (ii)$	Double of equal lengths
Also $m\overline{AB} = m\overline{AE} + m\overline{EB}$	Segment addition postulate
$m\overline{AB} = m\overline{AE} + m\overline{AE} = 2m\overline{AE} \longrightarrow (iii)$	$\because m\overline{EB} = m\overline{AE}$
Similarly $m\overline{CD} = 2m\overline{CF} \longrightarrow (iv)$	From (ii)
Since $2m\overline{AE} = 2m\overline{CF}$	
Therefore $m\overline{AB} = m\overline{CD}$	From (iii) and (iv)
or $\overline{AB} \cong \overline{CD}$	

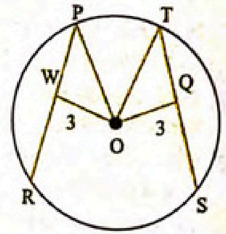
Hence the two chords are congruent.

ollary

ongruent circles, chords equidistant from the centre are also congruent.

Example 4 Circle O has a radius of 5 centimeters. Chords ST and RP are each 3 centimeters from 0. Compare the lengths of $\overline{ST}$ and $\overline{RP}$.

Solution



Draw $\overline{OT}$ and $\overline{OP}$. $\qquad$ $\overline{OT}$ and $\overline{OP}$ are radii of circle O.

OT = OP = 5 $\qquad$ Radii of the same circle have the same measure.

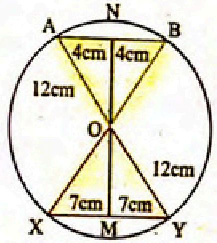
$$(OW)^2 + (WP)^2 = (OP)^2 \qquad (OQ)^2 + (QT)^2 = (OT)^2$$
$$3^2 + (WP)^2 = 5^2 \qquad 3^2 + (OT)^2 = 5^2$$
$$(WP)^2 = 25 - 9 \qquad (OT)^2 = 25 - 9$$
$$(WP)^2 = 16 \qquad (OT)^2 = 16$$
$$WP = 4 \qquad QT = 4$$

PR = 2(WP) = B ← Theorem 9.3→ ST = 2(QT) = 8

Chord ST and RP are congruent.

Example 5 The length of two parallel chords of a circle of radius 12cm are 14cm and 8cm respectively. Calculate the distance between the chords.

Solution

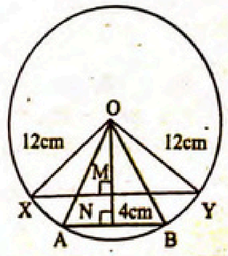


In ΔAON, $(ON)^2 = 12^2 - 4^2$
$= 144 - 16$
$ON = \sqrt{128}$
$ON = 11.31$cm

In ΔYOM, $(OM)^2 = 12^2 - 7^2$
$= 144 - 49$
$OM = \sqrt{95}$
$OM = 9.747$

In first case NM = ON + OM
= 11.31 +9.747
= 21.1 cm

In second case, NM = ON − OM
= 11.31− 9.747
= 1.56cm



Therefore the distance between the chords can either be about 21.1cm or 1.56 cm.

Exercise 9.2

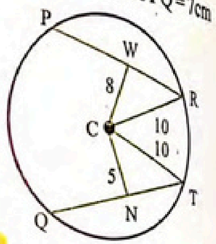
In a circle of radius 5cm, there are two parallel chords of length 8cm and 6cm respectively. Calculate the distance between the chords.

Two Parallel chords $\overline{PQ}$ and $\overline{MN}$ are 3cm apart on the same side of a circle where PQ = 7cm and MN = 14 cm. Calculate the radius of the circle.

Circle C has a radius of 10. Chord QT is 5 units from C and chord PR is 8 units from C.

a. Compare the lengths of $\overline{PR}$ and $\overline{QT}$.

b. Compare the distances of $\overline{PR}$ and $\overline{QT}$ from C.



Review Exercise 9

At the end of each question, four circles are given. Fill in the correct circle only.

(i). In a circle, two chords are equally distant from the centre of the circle. The chords are

○ congruent ○ not congruent ○ parallel ○ non parallel

(ii). $\overline{AB}$ and $\overline{CD}$ are two chords of the same circle ⊙ and $AB < CD$. Then

○ AB is closer to O ○ $\overline{AB}$ must be parallel to $\overline{CD}$
○ CD is closer to O ○ Can't decide

(iii). A chord is 5 cm from the centre of a circle of radius 13cm. The length of the chord is

○ 6 centimeters ○ 12 centimeters
○ 24 centimeters ○ 30 centimeters

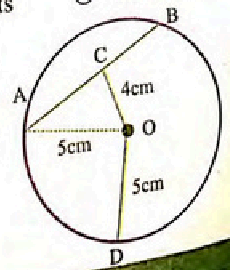
(iv). A chord 40 units long is contained in a circle of radius 25. The distance of the chord from the centre of the circle is

○ 15 units ○ 31.2 units ○ 47.1 units ○ 50.1 units

(v). A chord $8\sqrt{3}$ units long is 4 units from the centre of a circle. The length of the radius of the circle is

○ 14.4 units ○ 8 units ○ $8\sqrt{2}$ units ○ $2\sqrt{8}$ units

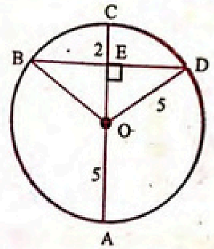
(vi). In the adjacent circular figure with centre O ad radius 5cm, the length of the chord intercepted at 4cm away from the centre of this circle is:



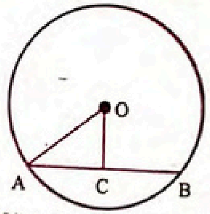
○ 4 cm ○ 6 cm
○ 7 cm ○ 9 cm

(vii). In the diagram, circle O has a radius of 5, and CE = 2. Diameter $\overline{AC}$ is perpendicular to chord $\overline{BD}$ at E. What is the length of $\overline{BD}$?

○ 12 ○ 8
○ 10 ○ 4



2. In circle O, $\overline{AB}$ is a chord, $\overline{OA}$ is a radius and $\overline{AB} \perp \overline{OC}$. (Leave radicals in your answers in simplest form).



(i). If AO= 13m and OC = 5m, find AC and AB.

(ii). If AB is 16 cm long and is 6 cm from O. Find the radius and diameter of the circle.

(iii). If the diameter of circle O is 34 cm, how far from the centre is a chord 30 cm long?

(iv). The radius of circle O is 25 units long. How long is $\overline{AB}$ if it is 7 units from point O?

(v). Chord AB is 10 m long and is 5 m from O. Find OA.

3. The perpendicular bisector of a chord $\overline{XY}$ cuts $\overline{XY}$ at N and the circle at P. If XY = 16cm and NP = 2cm, calculate the radius of the circle.

Summary

- A **circle** is the set of all points in a plane that are equidistant from a given point in the plane known as the center of the circle.
- A **radius** (plural, radii) is a line segment from the centre of the circle to a point on the circle.
- A **chord** is a line segment whose endpoints lie on the circumference of a circle. It divides a circle into two segments. The larger part is called major segment and the smaller part, the minor segment. If the chord happens to be a diameter, each segment is a semicircle.
- A **diameter** is a chord that contains the centre of a circle.
- One and only one circle can pass through three non-collinear points.
- A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
- Perpendicular from the centre of a circle on a chord bisects it.
- If two chords of a circle are congruent then they will be equidistant from the centre.
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