Summary

- An angle is a union of two rays which have a common point (vertex). One of the ray is called 'initial side' and other ray is called 'terminal side'.
- Sexagesimal system (degrees, minutes, seconds) is the system of measurement of an angle in which one complete rotation is divided into 360 parts called degrees, written as $360°$. One degree is divided into 60 parts called minutes, written as $60'$ and one minute is again divided into 60 parts called seconds, written as $60''$.
- In Circular system (radians) unit of measure of angle is radian. One radian is an angle subtended at the centre of a circle an arc whose length is equal to radius of the circle.
- Coterminal angles are angles having the same initial and terminal sides and differ by a multiple of 2 radians or $360°$. They are also called general angles.
- Angles are in standard position if the vertex of an angle lies at the origin, and initial side lies on positive x-axis.
- Quadrants are obtained when XY-plane is divided into four equal parts.
- Quadrant angles are $0°, 90°, 180°, 270, 360°$.
- Relationship between radian and degree measure

  $1° = \frac{\pi}{180}$ radian $= 0.0175$ radian and 1 radian $= \left(\frac{180}{\pi}\right)^{\circ} = 57.295$ degrees.

  Relation between central angle and arc length of a circle: $l = r\theta$

  Area of a circular sector, $A = \frac{1}{2}r^2\theta$

  two or more than two angles with the same initial and terminal sides are called coterminal angles.
- There are six trigonometric ratios: $\sin\theta, \cos\theta, \tan\theta, \sec\theta, \cot\theta, \operatorname{cosec}\theta$.
- Trigonometric identities are
  - (a) $\cos^2\theta + \sin^2\theta = 1$
  - (b) $1 + \tan^2\theta = \sec^2\theta$
  - (c) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
- Angle of elevation is the angle between the horizontal line and observer's line of sight when looking at the top of a wall.
- Angle of depression is the angle between the horizontal line and observer's line of sight when looking at the bottom of a wall.

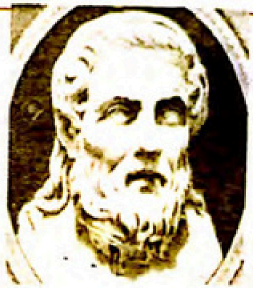
Unit 8 PROJECTION OF A SIDE OF A TRIANGLE

In this unit the students will be able to

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).

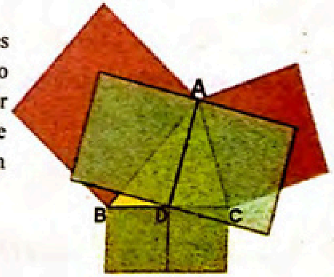
Why it's important



In geometry, **Apollonius's theorem** is a theorem relating the length of a median of a triangle to the lengths of its side. It states that "the sum of the squares of any two sides of any triangle equals twice the square on half the third side, together with twice the square on the median bisecting the third side".

Specifically, in any triangle ABC, if AD is a median, then

$$|AB|^2 + |AC|^2 = 2(|AD|^2 + |BD|^2)$$



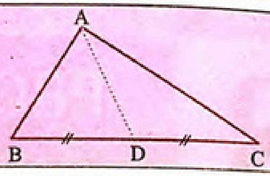
It is a special case of Stewart's theorem. For an isosceles triangle with $|AB| = |AC|$, the median AD is perpendicular to BC and the theorem reduces to the Pythagorean Theorem for triangle ADB (or triangle ADC). From the fact that the diagonals of a parallelogram bisect each other, the theorem is equivalent to the parallelogram law.

The theorem is named after Apollonius of Perga.

Math formula (Apollonius theorem)

If AD is the median, then:

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$



Theorem 8.1

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

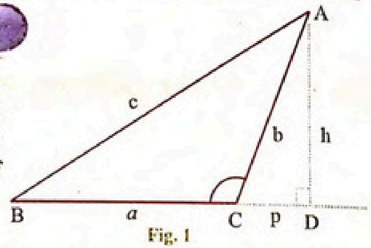


Fig. 1

Given

In ΔABC, $\angle C$ is the obtuse angle. Let $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$. $\overline{AD}$ is the perpendicular from A to $\overline{BC}$ (produced) so that $\overline{CD}$ is the projection of $\overline{AC}$ on $\overrightarrow{BC}$. Let $m\overline{CD} = p$ and $m\overline{AD} = h$.

To prove

$$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CA})^2 + 2(m\overline{BC})\cdot(m\overline{CD})$$

or $c^2 = a^2 + b^2 + 2ap$

Proof

Statements	Reasons
In a right angled triangle ΔADB	$\overline{AD} \perp \overrightarrow{BC}$
$\therefore$ $(m\overline{AB})^2 = (m\overline{BD})^2 + (m\overline{AD})^2$	Pythagoras theorem
or $(m\overline{AB})^2 = (m\overline{BC} + m\overline{CD})^2 + (m\overline{AD})^2$	$m\overline{BD} = m\overline{BC} + m\overline{CD}$
or $(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CD})^2$ $+2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{AD})^2 \longrightarrow$(i)	$(x+y)^2 = x^2 + y^2 + 2xy$
Again in the right angled triangle ΔADC $(m\overline{AC})^2 = (m\overline{CD})^2 + (m\overline{AD})^2$	Pythagoras theorem.
or $(m\overline{AC})^2 - (m\overline{CD})^2 = (m\overline{AD})^2 \longrightarrow$(ii)	Equal can be subtracted from the equal without changing the value.
so that (i) becomes $(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{CD})^2$	Putting the value of $(m\overline{AD})^2$ from (ii) in (i).

$+2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{AC})^2 - (m\overline{CD})^2$	Commutative property of addition of real numbers.
$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2$ $+2(m\overline{BC})\cdot(m\overline{CD}) + (m\overline{CD})^2 - (m\overline{CD})^2$	The difference in two equal numbers is zero.
$\therefore (m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 + 2(m\overline{BC})(m\overline{CD})$	
$c^2 = a^2 + b^2 + 2ap$	

Corollary – I:

In a ΔABC with obtuse angle at C. If $\overline{BD}$ is perpendicular on $\overline{AC}$ produced and $m\overline{BC} = m\overline{AC}$ then prove that

$(\overline{AB})^2 = 2(\overline{AC})(\overline{AD})$

or

$c^2 = 2(b)(b + p)$

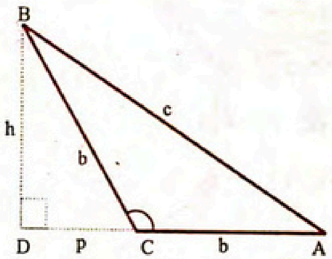


Fig. 2

Given

A ΔABC having on obtuse angle $m\angle ACB$, $m\overline{BC} = m\overline{AC}$ and $\overline{BD}$ is perpendicular on $\overleftrightarrow{AC}$ produced.

To prove

$(m\overline{AB})^2 = 2(m\overline{AC})(m\overline{AD})$

or

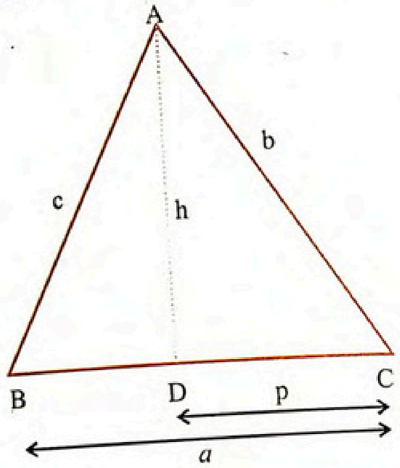
$c^2 = 2b(b + p)$

Proof

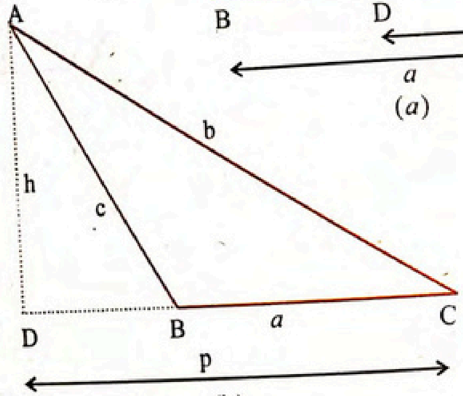
Statement	Reasons
$(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2 + 2(m\overline{AC})(m\overline{CD})$	By Theorem 8.1
$= (m\overline{AC})^2 + (m\overline{AC})^2 + 2(m\overline{AC})(m\overline{CD})$	
$= 2(m\overline{AC})^2 + 2(m\overline{AC})(\overline{CD})$	
$= 2(m\overline{AC})(m\overline{AC} + m\overline{CD})$	
$= 2(m\overline{AC})(m\overline{AD})$	Point C is inbetween $\overline{AD}$
$(m\overline{AB})^2 = 2(m\overline{AC})(m\overline{AD})$ or $c^2 = 2b(b + p)$	

Theorem 8.2

In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished twice the rectangle contained by one of those sides, and the projection on it of the other.



(a)



(b)

Given

The given triangle is either acute as in figure (a) or obtuse as in figure (b). In the ΔABC, $m\angle C$ is acute. $\overline{AD}$ is the perpendicular from A to $\overline{CB}$ (produced if necessary as in figure b) so that $\overline{CD}$ is the projection of $\overline{CA}$ on $\overline{CB}$. Let us denote $m\overline{BC}=a$, $m\overline{AC}=b$, $m\overline{AB}=c$, $m\overline{CD}=p$, the projection and $m\overline{AD}=h$.

To prove

$$(m\overline{AB})^2=(m\overline{BC})^2+(m\overline{AC})^2-2(m\overline{BC})\cdot(m\overline{CD})$$

or $c^2=a^2+b^2-2ap$

Proof

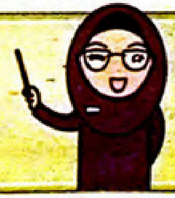
Statements	Reasons
ΔADB is a right-triangle	$\because \overline{AD}\perp\overline{BC}$ (produced where necessary)
$\therefore (m\overline{AB})^2=(m\overline{BD})^2+(m\overline{AD})^2$	Pythagoras theorem
Also $m\overline{BD}+m\overline{CD}=m\overline{BC}$ ………	Segment addition postulate (fig. a)
$\therefore (m\overline{AB})^2=(m\overline{BC}-m\overline{CD})^2$ $+(m\overline{AD})^2 \longrightarrow$ (I)	$\because m\overline{BD}=m\overline{BC}-m\overline{CD}$
Again $m\overline{BD}+m\overline{BC}=m\overline{CD}$	Segment addition postulate (fig. b)
$\therefore (m\overline{AB})^2=(m\overline{CD}-m\overline{BC})^2$ $+(m\overline{AD})^2 \longrightarrow$ (II)	Since $m\overline{BD}=m\overline{CD}-m\overline{BC}$
But the R.H.S of (i) and (ii) are the same so we can select any one of the above equations. Thus from (i), we have	$\because (m\overline{BC}-m\overline{CD})^2=(m\overline{CD}-m\overline{BC})^2$
$(m\overline{AB})^2=(m\overline{BC})^2+(m\overline{CD})^2$ $-2(m\overline{BC})\cdot(m\overline{CD})+(m\overline{AD})^2$	
or $(m\overline{AB})^2=(m\overline{BC})^2+(m\overline{CD})^2+(m\overline{AD})^2$ $-2(m\overline{BC})\cdot(m\overline{CD}) \longrightarrow$ (III)	Commutative property of addition of real numbers.
But $(m\overline{CD})^2+(m\overline{AD})^2=(m\overline{AC})^2 \longrightarrow$ (IV)	By Pythagoras theorem.
$\therefore (m\overline{AB})^2=(m\overline{BC})^2+(m\overline{AC})^2$ $-2(m\overline{BC})\cdot(m\overline{CD})$	Using IV in III.
or $c^2=a^2+b^2-2ap$	

Did You Know?

Some odd multiplications

$37037037037\times 9=333333333333$

$13717421\times 9=123456789$



Exercise 8.1

1. In ΔABC, $m\overline{AB}=6cm$, $m\overline{BC}=10cm$, $m\angle B=120°$. The projection of $\overline{BC}$ on $\overline{AB}$ is 5 cm. Find $m\overline{AC}$.
2. In ΔABC, $m\overline{AB}=3cm$, $m\overline{BC}=5cm$, $m\overline{AC}=7cm$. Find the projection of $\overline{BC}$ on $\overline{AB}$.
3. In ΔABC, $a=7$, $b=11$, $c=8$. Calculate the projection of $\overline{AC}$ on $\overline{AB}$.
4. In a parallelogram ABCD, m $\overline{AB}=4$cm, m $\overline{AC}=7$cm, m $\overline{AD}=5$cm. Find which of the angles of the parallelogram are obtuse.

Theorem 8.3

(Apollonius Theorem)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' theorem)

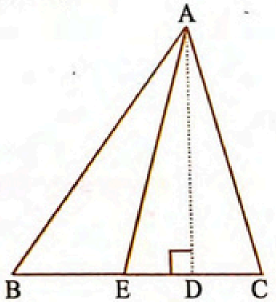


Fig. (a)

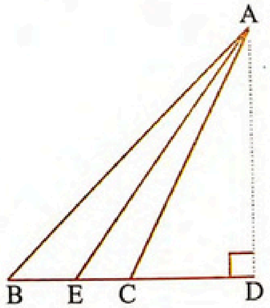


Fig. (b)

Given

In the ΔABC, $\overline{AE}$ is the median drawn from the vertex A to $\overline{BC}$.

To prove

$$(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{EB})^2+2(m\overline{EA})^2$$

Construction

From A, draw

$\overline{AD}\perp\overline{BC}$ (produced if necessary)

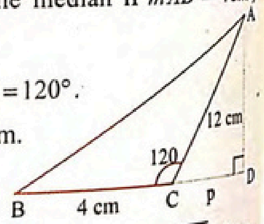
Proof

Statements	Reasons
Since ΔABD and ΔADC are right triangles,	Construction
$\therefore (m\overline{AB})^2=(m\overline{AD})^2+(m\overline{BD})^2 \longrightarrow (1)$	Pythagoras theorem
$(m\overline{AC})^2=(m\overline{AD})^2+(m\overline{DC})^2 \longrightarrow (2)$	Pythagoras theorem
Therefore, $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$ $+(m\overline{BD})^2+(m\overline{DC})^2$	Adding (1) and (2)
But $m\overline{BD}=m\overline{BE}+m\overline{ED}$	Segments addition postulate
So that the last result becomes $m(\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$ $+(m\overline{BE}+m\overline{ED})^2+(m\overline{DC})^2 \longrightarrow (3)$	
Since $m\overline{DC}=m\overline{EC}-m\overline{ED}$	From construction in figure (a)
and $m\overline{DC}=m\overline{ED}-m\overline{EC}$	From construction in figure (b)
$\therefore (m\overline{DC})^2=(m\overline{EC}-m\overline{ED})^2$ $=(m\overline{ED}-m\overline{EC})^2 \longrightarrow (4)$	From algebra $(a-b)^2=(b-a)^2$
so any one of these values can be put in place of $(m\overline{DC})^2$ and thus (3) becomes	
$(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$ $+(m\overline{BE}+m\overline{ED})^2+(m\overline{EC}-m\overline{ED})^2$	Using (4) in (3)
or $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$ $+(m\overline{BE})^2+(m\overline{ED})^2+2(m\overline{BE})\cdot(m\overline{ED})$ $+(m\overline{EC})^2+(m\overline{ED})^2-2(m\overline{EC})\cdot(m\overline{ED})$	$\because$ from algebra $(a+b)^2=a^2+b^2+2ab$ and $(a-b)^2=a^2+b^2-2ab$
or $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$ $+(m\overline{ED})^2+(m\overline{ED})^2+(m\overline{BE})^2+(m\overline{EC})^2$	Using commutative property of addition for real numbers.

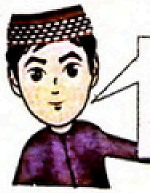
$+2(m\overline{BE})\cdot(m\overline{ED})-2(m\overline{EC})\cdot(m\overline{ED})$	
or $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AD})^2$	$\because m\overline{BE}=m\overline{EC}$ as $\overline{AE}$ is the median.
$+2(m\overline{ED})^2+(m\overline{BE})^2+(m\overline{BE})^2$	
$+2(m\overline{BE})\cdot(m\overline{ED})-2(m\overline{BE})\cdot(m\overline{ED})$	
or $(m\overline{AB})^2+(m\overline{AC})^2=2\left[(m\overline{AD})^2+(m\overline{ED})^2\right]$ $+2(m\overline{BE})^2 \longrightarrow (5)$	$\because x-x=0 \quad \forall x\in R$
But $(m\overline{AD})^2+(m\overline{ED})^2=(m\overline{AE})^2 \longrightarrow (6)$	ΔADE is a right-triangle. Using (6) in (5).
$\therefore (m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{AE})^2+2(m\overline{BE})^2$	$\because m\overline{EB}=m\overline{BE}$ and $m\overline{AE}=m\overline{EA}$
or $(m\overline{AB})^2+(m\overline{AC})^2=2(m\overline{EB})^2+2(m\overline{EA})^2$	

Exercise 8.2

1. Using Apollonius theorem find the length of the medians of a triangle having sides 10 cm, 12 cm and 16 cm respectively.
2. In ΔABC, D is the mid-point of $\overline{BC}$. Find the length of the median if $m\overline{AB}=4cm$, $m\overline{BC}=5cm$ and $m\overline{AC}=6cm$.
3. ΔABC is given with $m\overline{BC}=4cm$, $m\overline{AC}=12cm$ and $m\angle C=120°$. Find $m\overline{CD}$, $m\overline{AD}$, $m\overline{AB}$ and then verify Apollonius theorem.



4. ΔABC is a right triangle with $m\angle B=90°$ and $\overline{BD}\perp\overline{AC}$. If $m\overline{AB}=6cm$ and $m\overline{BC}=5cm$, find $m\overline{AD}$ and $m\overline{CD}$. Verify your answer with Pythagorean theorem.

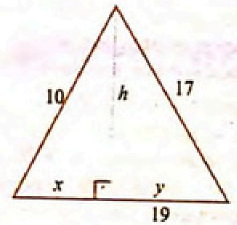


How to prove a, b, c form an acute, right or obtuse triangle?

$a^2+b^2<c^2$ obtuse triangle
$=c^2$ right triangle
$>c^2$ acute triangle
Where c is the largest side

Review Exercise 8

1. At the end of each question, four circles are given. Fill in the correct circle only.
   (i). Phythagoras was a /an __________ mathematician.
   ○ Indian ○ Greek ○ Pakistani ○ Chinese
   (ii). Apollonius was born in__________.
   ○ Perga ○ Istanbul ○ Islamabad ○ Kabul
   (iii). Apollonius is the name of a __________.
   ○ City ○ Town ○ Country ○ Mathematician
2. Find the projection of the side of measure 10 units upon the side of measure 17 units in a triangle whose sides respectively have measures 10, 17 and 21 units.
3. ΔABC is a right -triangle with $m\angle A=90°$. From the vertex A perpendicular $\overline{AD}$ is drawn on $\overline{BC}$. If $m\overline{AB}=5\ cm$, $m\overline{AC}=8\ cm$. Find $m\overline{BC}$, $m\overline{AD}$ and $m\overline{BD}$.
4. In the above question, $\overline{BE}$ is a median. Find $m\overline{BE}$ using
   a. Pythagoras theorem
   b. Apollonius theorem
5. Find h, x and y in the figure.



Summary

- A median of a triangle is a segment from one vertex of the triangle to the midpoint of the opposite side.
- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).