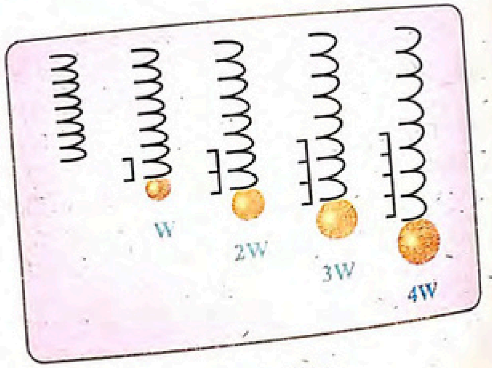
Unit 3

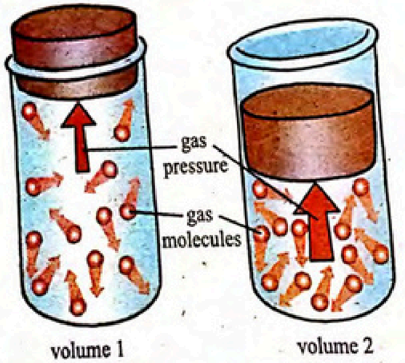
VARIATIONS

In this unit the students will be able to

- Define ratio, proportions and variations (direct and inverse).
- Find 3^{rd}, 4^{th} mean and continued proportion.
- Apply theorems of invertendo, alternendo, componendo, dividendo and componendo and dividendo to find proportions.
- Define joint variation.
- Solve problems related to joint variation.
- Use K-method to prove conditional equalities involving proportions.
- Solve real life problems based on variations.



This is Direct Variation

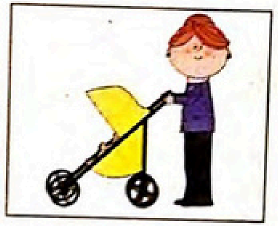


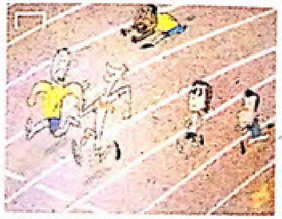
This is Inverse Variation

Why it's important

Direct variation is the relationship between two quantities, whereby if one quantity increases or decreases the other also increases or decreases. If one quantity increases, the other decreases or if one quantity decreases the other increases, it is called inverse variation. We often make use of one type of variation or another in our daily life. Thus knowing of variation is necessary for us. For example,

- If a person pushes a car, he has to apply more force.
- If a mother pushes the cart of her child, she has to apply less force.





- In a marathon race an athlete who has more speed than other participants will reach the finishing line in less time.
- On motorway from Peshawar to charsadda, from the same two cars if one car has to reach in less time, then its speed must be increased.

3.1 Ratio, Proportions and Variations

3.1.1 Ratio

A ratio is used to compare two or more quantities of the same kind which are measured in same unit. The ratio of two quantities a, b of the same unit can be shown as:

Example 1 Write the following ratio in simplified form:

(i) $3 : 12$ (ii) $6a : 18b$

Solution (i) $3 : 12 = 1 : 4$ (ii) $6a : 18b = a : 3b$

Example 2 Divide Rs. 5070 among three persons in the ratio $2 : 5 : 6$.

Solution Given ratio $= 2:5:6$

Sum of the ratios $= 13$

Share of 1^{st} person $= 5070 \times \frac{2}{13} =$ Rs. 780

Share of 2^{nd} person $= 5070 \times \frac{5}{13} =$ Rs. 1950

Share of 3^{rd} person $= 5070 \times \frac{6}{13} =$ Rs. 2340

Tidbit

A ratio is said to be in its simplest form a : b when a and b are integers with no common factors(other than 1)

3.1.2 Proportions

A proportion is an equation that states that two ratios are equivalent.
If a, b, c, d are four quantities then the general form of a proportion is given as

$\frac{a}{b} = \frac{c}{d}$ where b not equal to zero, d not equal to zero.

The above equations can also be written as:

$$a : b :: c : d$$
$$a : b = c : d$$

Note

Proportion is a comparison of the quantity of a part to the quantity of a whole.

The proportion $\frac{3}{4} = \frac{6}{8}$ can be written as $3:4 = 6:8$. In this form 4 and 6 are called means of the proportion and 3 and 8 are called the extremes of the proportion.

The cross product of a proportion are equal. i.e $4 \times 6 = 24$ and $3 \times 8 = 24$.

Example 3

$a^3 - b^3$, $a^2 - b^2$, $a^2 + ab + b^2$ and x are in a proportion. Find the value of x.

Solution

According to the question

$$\frac{(a^3 - b^3)}{(a^2 - b^2)} = \frac{(a^2 + ab + b^2)}{x}$$

$$\frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)} = \frac{(a^2 + ab + b^2)}{x}$$

or $$\frac{1}{a+b} = \frac{1}{x}$$

or $$x = a + b$$

3.1.3 Variations

In Mathematics, we usually deal with two types of quantities: Variable quantities (or variables) and Constant quantities (or constants). If the value of a quantity remains unchanged under different situations, it is called a constant. On the other hand, if the value of a quantity changes under different situations, it is called a variable.

For example: 4, 2.718, $\frac{22}{7}$ etc. are constants while speed of a train, demand of a commodity, population of a town etc. are variables.

The change of variable parameters is called as variation.

Kinds of Variation

There are two types of variation. Direct Variation and Inverse Variation.

a. Direct Variation

If we borrow books from the school library and are late in returning the books, we will be fined Rs. 15 per day for each overdue book. The table below shows the fine for an overdue book.

No. of days (x)	1	2	3	4	5
Fine (Rs. y)	15	30	45	60	75

From the table we notice that if the number of days (x) the book is overdue increases, the fine (Rs. y) will also increase proportionally, i.e. if x is doubled, y will also double; if x is halved, y will also halve. This is called direct proportion. We say that the fine (Rs. y) is directly proportional to the number of days (x) a book is overdue. We say that y varies directly as x, or y is directly proportional to x.

If y varies directly as x, this relation is written as

$$y \propto x \quad \text{or} \quad y = kx$$

It is clear from the above statement that $\frac{y}{x} = k$

Where k is a constant of a direct relation, and is called constant of variation.

Example 4

Given that y varies directly with x and $y = 27$ when $x = 3$. Find

(i) An equation connecting x and y. (ii) The value of y when $x = 11$.

Solution

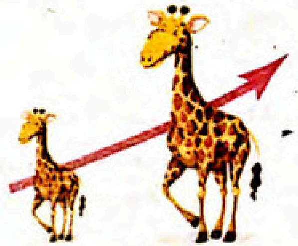
Since y is directly proportional to x, then

$$y \propto x \Rightarrow y = kx$$

(i) Putting the given values of $x = 3$ and $y = 27$

$$\frac{27}{3} = k \Rightarrow k = 9$$

So the equation connecting x and y is $y = 9x$



(ii) Putting $x = 11$ in above equation.

$$y = 9(11) = 99$$

Which is the required value.

Example 5

If $y \propto x$, then complete the following table.

x	4	5	8		
y	6			18	22.5

Solution

$y \propto x$

or $y = kx$ (i)

when $x = 4$, $y = 6$

Putting the values in (i) $6 = 4k$ or $k = \frac{3}{2}$

Equation (i) becomes $y = \frac{3x}{2}$ (ii)

Putting $x = 5$ in equation (ii)

$$y = \frac{3}{2} \times 5 = \frac{15}{2} = 7.5$$

Which is the corresponding value of y

Putting $x = 8$ in equation (ii)

$$y = \frac{3}{2} \times 8 = 12$$

Which is the corresponding value of y.

Now putting $y = 18$ in equation (ii)

$$18 = \frac{3x}{2} \quad \text{or} \quad x = 12$$

Which is the corresponding value of x.

Putting $y = 22.5$ in equation (ii)

$$22.5 = \frac{3x}{2} \quad \text{or} \quad x = 15$$

Which is the corresponding value of x.

So complete table is given as under.

x	4	5	8	12	15
y	6	7.5	12	18	22.5

b. Inverse Variation

The following table shows the time taken by a car to travel a distance of 120 km at different speeds.

Speed x km/h	10	20	30	40	60	120
Time taken (y hours)	12	6	4	3	2	1

You can notice that as the speed of car increases, the time taken decreases proportionally i.e, if x is doubled y will be halved. If x is tripled then y will be reduced to $\frac{1}{2}$ of its original value. Similarly as the speed of car decreases the time taken increases proportionally. This relationship is known as inverse proportion. We say that the speed of car, x km/h varies inversely to the time y hours taken.

If y varies inversely to x

i.e $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ or $xy = k$

Where k is a constant and k is not equal to zero.

Example 6 If x varies inversely to y and $x = 3$, when $y = 12$ Find the value of y when $x = 6$.

Solution

$y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ (i)

Putting $x = 3$, $y = 12$ in equation (i)

$$12 = \frac{k}{3}$$

or $36 = k$

Putting the value of k in equation (i)

$$y = \frac{36}{x} \quad \text{(ii)}$$

Putting $x = 6$ in equation (ii)

$y = \frac{36}{6} = 6$ which is the required corresponding value of y.

> **Tidbit**
>
> **Inverse variation**
>
> when one quantity increases / decreases while another quantity decreases / increases.
>
> $xy = k$ or $y = \frac{k}{x}$

Example 7 Given that Pressure 'p' on the quantity of gas in a container varies inversely to its volume 'V', i.e. $P \propto \frac{1}{V}$. When pressure on gas in 10 N/m², its volume is 25m³. Find the pressure when the volume is 20m³.

Solution $P \propto \frac{1}{V}$ or $P = \frac{k}{V}$ (i)

Putting the values P = 10 N/m² and V = 25m³ in equation (i)

$$10 = \frac{k}{25} \quad \text{or} \quad k = 250$$

For equation (i)

$$P = \frac{250}{V} \qquad \text{(ii)}$$

Putting $V = 20\text{m}^3$ in equation (ii)

$$P = \frac{250}{20} = 12.5\,N/m^2$$

> **Tidbit**
> For direct proportion, $y_2/y_1 = x_2/x_1$, but for inverse proportion, we have $y_2/y_1 = x_1/x_2$ or $x_1y_1 = x_2y_2$. Note the order of x_1 and x_2.

Exercise 3.1

1. Which is the greater ratio, 5 : 7 or 151 : 208 ?
2. Gold and silver are mixed in the ratio 7:4. If 36 grams of silver is used. How much gold is used.
3. Divide the annual profit of Rs. 40,000 of a factory among 3 partners in the ratio of 5 : 8 : 12.
4. If 11 : x–1 = 22 : 27, find the value of x.
5. There is a direct variation between x^2 and y. When $x = 7$, $y = 49$ find:
   (i). y when $x = 9$ (ii). x when $y = 100$
6. There is an inverse variation between x and y, and when $x = 4$, $y = 6$, find:
   (i). y when $x = 12$ (ii). x when $y = 24$.
7. $r \propto \frac{1}{p^3}$ and $p = 9$ when $r = 2$. Find:
   (i). r when p = 3. (ii). p when $r = \frac{1}{4}$
8. If $y \propto x$, then complete the following table.

x	4	6		15
y	2		3.5	

3.2 Third, Fourth Mean and Continued Proportion

Three quantities are said to be in continued proportion if the ratio of the first term and secor term are equal to the ratio of the second term and third term. If a, b and c are in continued proportion then b is called the mean proportional (or geometr mean) of a and c. If $a : b :: b : c$ then $ac = b^2$

c is called the third proportional.

For example, the numbers 4, 6 and 9 are in continued proportion because

$$4 : 6 :: 6 : 9$$
$$4 \times 9 = 6^2$$
$$36 = 36$$

The numbers 2, 4 and 6 are not in continued proportion because

$$2 \times 6 \neq 4^2$$
$$12 \neq 16.$$

Example 8 Find the mean proportional of 5 and 15.

Solution Let x is the mean proportional

$$\frac{15}{x} = \frac{x}{5}$$

or $x^2 = 75$

or $x = \pm\sqrt{75}$ taking square roots on both sides

or $x = 5\sqrt{3}$

Example 9 Find the third proportion of a^2b^2 and abc.

Solution Let x be the third proportion.

Therefore, $\frac{a^2b^2}{abc} = \frac{abc}{x}$ or $x = c^2$.

Example 10 Find fourth proportional of $a^3 - b^3$, $a + b$ and $a^2 + ab + b^2$

Solution Let x be the fourth proportional,

then $(a^3 - b^3) : (a + b) :: (a^2 + ab + b^2) : x$

i.e., $x(a^3 - b^3) = (a + b)(a^2 + ab + b^2)$

$$x = \frac{(a+b)(a^2+ab+b^2)}{a^3 - b^3} = \frac{(a+b)(a^2+ab+b^2)}{(a-b)(a^2+ab+b^2)}$$

$$x = \frac{a+b}{a-b}$$

3.3 Theorems on Proportion

1 Alternendo Property

If $a:b = c:d$ then $a:c = b:d$

that is, if the second and third term interchange their places, then also the four terms are in proportion.

Example If $3:5 = 6:10$ then $3:6 = 1:2 = 5:10$

2 Invertendo Property

If $a:b :: c:d$ then $b:a :: d:c$.

that is, if two ratios are equal, then their inverse ratios are also equal.

Example $6:10 = 9:15$

therefore, $10:6 = 5:3 = 15:9$

3 Componendo Property

If $a:b = c:d$ then $(a+b):b :: (c+d):d$

Example $4:5 = 8:10$

therefore, $(4+5):5 = 9:5 = 18:10$

$= (8+10):10$

4 Dividendo Property

If $a:b :: c:d$ then $(a-b):b :: (c-d):d$.

Example $5:4 = 10:8$

$(5-4):4 = 1:4 = (10-8):8$

5 Componendo-Dividendo Property

If $a:b :: c:d$ then $(a+b):(a-b) :: (c+d):(c-d)$.

Example $7:3 = 14:6$

$7+3):(7-3) = 10:4 = 5:2$

Again, $(14+6):(14-6) = 20:8 = 5:2$

Therefore, $(7+3):(7-3) = (14+6):(14-6)$

Example 11 If $\frac{a}{b} = \frac{c}{d}$ then prove that $2a+3b : b = 2c+3d : d$

Solution

$$\frac{a}{b} = \frac{c}{d} \qquad \text{multiplying both sides by } \frac{2}{3}$$

$$\frac{2a}{3b} = \frac{2c}{3d}$$

$$\frac{2a+3b}{3b} = \frac{2c+3d}{3d} \qquad \text{using componendo property}$$

$$\frac{2a+3b}{b} = \frac{2c+3d}{d}$$

$$2a + 3b : b = 2c + 3d : d$$

Example 12 Prove that if $\frac{(3a-4b)}{(3a+4b)} = \frac{(3c-4d)}{(3c+4d)}$.

Then $\frac{a}{b} = \frac{c}{d}$

Solution $\frac{3a-4b}{3a+4b} = \frac{3c-4d}{3c+4d}$

Then $$\frac{(3a-4b)+(3a+4b)}{(3a-4b)-(3a+4b)} = \frac{(3c-4d)+(3c+4d)}{(3c-4d)-(3c+4d)}$$

(Componendo – Dividendo)

or $$\frac{6a}{-8b} = \frac{6c}{-8d}$$

or $$\frac{a}{b} = \frac{c}{d}$$

Example 13 If $\frac{(x+3)^2+(x-4)^2}{(x+3)^2-(x-4)^2} = \frac{13}{12}$ then find the value of x.

Solution $$\frac{(x+3)^2+(x-4)^2}{(x+3)^2-(x-4)^2} = \frac{13}{12}$$

By componendo – dividend property

$$\frac{(x+3)^2+(x-4)^2+(x+3)^2-(x-4)^2}{(x+3)^2+(x-4)^2-\left[(x+3)^2-(x-4)^2\right]} = \frac{13+12}{13-12}$$

or $$\frac{\cancel{2}(x+3)^2}{\cancel{2}(x-4)^2} = \frac{25}{1}$$

Taking square root of both sides $\frac{x+3}{x-4} = \pm 5$

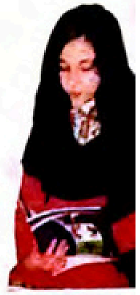
When $\frac{x+3}{x-4} = 5$ or, $x+3 = 5x-20$

or $x = \frac{23}{4}$

When $\frac{x+3}{x-4} = -5$ or, $x+3 = -5x+20$

or $x = \frac{17}{6}$

Solution set $= \left\{\frac{23}{4}, \frac{17}{6}\right\}$.



Exercise 3.2

1. Which of the following quantities are in continued proportion?

   (i). 4, 12, 36 (ii). 3, 12, 39 (iii). 72, 24, 8

2. Find the mean proportional of 12, 3.
3. If 5 : 15 : x are in continued proportion, find the value of x.
4. If 3x − 1, 4, 35 are continued proportion, find the value of x.
5. Find the mean proportional of $a^2 - b^2$ and $\frac{a+b}{a-b}$
6. If $\frac{a}{b}=\frac{c}{d}$ then prove that $\frac{ac+ad}{ac-bd}=\frac{a^2+b^2}{a^2-b^2}$
7. Solve the following equations.

   (i). $\frac{\sqrt{3x+2}+\sqrt{x}}{\sqrt{3x+2}-\sqrt{x}}=\frac{4}{1}$ (ii). $\frac{(x-1)^2+(x+2)^2}{(x-1)^2-(x+2)^2}=-\frac{17}{8}$

   (iii). $\frac{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}=\frac{1}{3}$

Math Fun

6 x 7 = 42

66 x 67 = 4422

666 x 667 = 444222

6666 x 6667 = 44442222

66666 x 66667 = 4444422222

666666 x 666667 = 444444222222

6666666 x 6666667 = 44444442222222

66666666 x 66666667 = 4444444422222222

666666666 x 666666667 = 444444444222222222

3.4 Joint Variation

Joint variation is the same as direct variation with two or more quantities, i.e. Joint variation is a variation where a quantity varies directly as the product of two or more other quantities. If x is jointly proportional to y and z, we can write $xkyz$=for some constant k. We can also write this relationship as

$$\frac{x}{yz}=k.$$

For example,

Area of a triangle = $\frac{1}{2}bh$.

Here the constant k is $\frac{1}{2}$.

Area of a triangle varies jointly with base 'b' and height 'h'.

Area of a rectangle = L × W.

Here the constant k is 1.

Area of a rectangle varies jointly with length 'L' and width 'W'.

Newton's law of motion; Force = mass × acceleration.

The force exerted on an object varies jointly as the mass of the object and the acceleration produced.

Example 14 If y varies jointly as x and z, and y = 12 when x = 9 and z = 3, find z when y = 6 and x = 15.

Solution Write the equation. $y = kxz$ (i)

Substituting y = 12 , x = 9 , z = 3 in (i).

$$12 = k(9)(3)$$

$$12 = 27k \Rightarrow \frac{4}{9}=k$$

So (i) becomes,

$$y=\frac{4}{9}xz \quad \text{(ii)}$$

Substituting $y = 6$, $x = 15$ in (ii).

$$6=\frac{4}{9}(15)(z)$$

$$6=\frac{60}{9}z \Rightarrow 54 = 60z \Rightarrow z=\frac{9}{10}$$

Try This

If z varies jointly as x and y and z = 24, when x = 2 and y = 4, find z when x = 2 and y = 5.

Types of Variation

y varies directly as x.
As x increases, y also increases.
As x decreases, y also decreases.
Equation: $y = kx$

y varies inversely as x.
As x increases, y also decreases.
As x decreases, y also increases.
Equation: $xy = k$ or $y = \frac{k}{x}$

y varies directly as x.
As x increases, y also increases.
As x decreases, y also decreases.
Equation: $y = kx$

y varies inversely as x.
As x increases, y also decreases.
As x decreases, y also increases.
Equation: $xy = k$ or $y = \frac{k}{x}$

Exercise 3.3

1. If y varies jointly as x and z, and y = 33 when x = 9 and z = 12, find y when x = 16 and z = 22.
2. If f varies jointly as g and the cube of h, and f = 200 when g = 5 and h = 4, find f when g = 3 and h = 6.
3. Suppose a is jointly proportional to b and c. If a = 4 when b = 8 and c = 9, then what is a when b = 2 and c = 18?
4. If p varies jointly as q and r squared, and p = 225 when q = 4 and r = 3, find p when q = 6 and r = 8.
5. If a varies jointly as b cubed and c, and a = 36 when b = 4 and c = 6, find a when b = 2 and c = 14.
6. If z varies jointly as x and y and z = 12, when x = 2 and y = 4, find the constant of variation.
7. If y varies jointly as x^2 and z and y = 6 when $x = 4$, z = 9. Write y as a function of x and z and determine the value of y, when $x = -8$ and z = 12.
8. If p varies jointly as q and r^2 and inversely as s and t^2, $p = 40$, when $q = 8, r = 5, s = 3, t = 2$. Find p in terms of q, r, s and t. Also find the value of p when $q = -2, r = 4, s = 3$ and $t = -1$.

Activity

Use online calculator.
Give examples of joint variation / direct and inverse variation from daily life. Write down these examples on flip chart / chart paper and present your work in class in groups.

3.5 K – Method

Let: a : b : c : d be a proportion then $\frac{a}{b} = \frac{c}{d} = k(say)$

Thus a = bk, c = dk

These equation are used to evaluate certain expressions more easily. This method is called K-method.

K – Method is explained with the help of following examples.

Example 15

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ when b, d, f are non -zero numbers, then prove that each of the ratios is equal to the following ratios.

$$\frac{\ell a + mc + ne}{\ell b + md + nf}$$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$a = bk, \; c = dk, e = fk$

$$\frac{\ell a + m + cne}{\ell b + md + nf} = \frac{\ell bk + mdk + nfk}{\ell b + md + nf} = \frac{k(\ell b + md + nf)}{(\ell b + md + nf)} = k$$

Example 16

Prove that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f}$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$a = bk, c = dk, \; e = fk$

or $a + c + e = bk + dk + fk$

or $a + c + e = k(b + d + f)$

$$\frac{a+b+c}{b+d+f} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$$

Did You Know?

Beautiful numbers relationship

$1^3 + 5^3 + 3^3 = 153$

$1 + 125 + 27 = 153$

Example 17 Prove that $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$

Solution $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$

Let $a=bk,\ c=dk,\ e=fk$

or $a^2=b^2k^2,\ c^2=d^2k^2,\ e^2=f^2k^2$

$$a^2+c^2+e^2=b^2k^2+d^2k^2+f^2k^2=k^2(b^2+d^2+f^2)$$

$$\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=k^2$$

$$\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$$

Example 18 If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$ where a, b, c and x, y, z are non-zero numbers then prove that.

Solution

$$\frac{x^3}{a^3}=\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}$$

$$a=xk,\ b=yk,\ c=zk$$

or $\frac{x}{a}=\frac{1}{k},\ \frac{y}{b}=\frac{1}{k},\ \frac{z}{c}=\frac{1}{k}$ (i)

or $\frac{x^3}{a^3}=\frac{1}{k^3},\ \frac{y^3}{b^3}=\frac{1}{k^3},\ \frac{z^3}{c^3}=\frac{1}{k^3}$ (ii)

For equations in (ii) sum of L.H.S = Sum of R.H.S i.e

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{1}{k^3}+\frac{1}{k^3}+\frac{1}{k^3}=\frac{3}{k^3} \quad \text{(iii)}$$

For equations given in (i), Product of L.H.S = Product of R.H.S.

$$\frac{x}{a}\cdot\frac{y}{b}\cdot\frac{z}{c}=\frac{1}{k}\cdot\frac{1}{k}\cdot\frac{1}{k}=\frac{1}{k^3}$$

or $\frac{3xyz}{abc}=\frac{3}{k^3}$ (iv)

From (iii) and (iv)

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}$$

Exercise 3.4

1. If $a:b=c:d$ then prove that
   (i) $\frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}$. (ii) $pa+qb:ma-nb=pc+qd:mc-nd$.
2. Prove that $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\frac{pa^2+qc^2+e^2}{pb^2+qd^2+f^2}}$
3. If $\frac{x-y}{z}=\frac{y-z}{x}=\frac{z-x}{y}$ then prove that: $x=y=z$ where x, y, z are non-zero numbers and $x+y+z\neq 0$.
4. If $\frac{2y+2z-x}{a}=\frac{2z+2x-y}{b}=\frac{2x+2y-z}{c}$, then prove that
   $$\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}$$
5. Prove that each of the fraction in:
   $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$ is equal to $\frac{x+y+z}{a+b+c}$
6. If $\frac{bz+cy}{b-c}=\frac{cx+az}{c-a}=\frac{ay+bx}{a-b}$ then $(a+b+c)(x+y+z)=az+by+cz$.
7. If $\frac{x}{(b+c-a)}=\frac{y}{(c+a-b)}=\frac{z}{(a+b-c)}$, then $(b-c)x+(c-a)y+(a-b)z=0$.
8. If $2x+3y:3y+4z:4z+5x=4a-5b:3b-a:2b-3a$, then $7x+6y+8z=0$.

Challenge

9. If $\frac{(a-b)}{(d-e)}=\frac{(b-c)}{(e-f)}$, then each of them is equal to $\frac{b(f-d)+(cd-af)}{e(f-d)}$.

Did You Know?

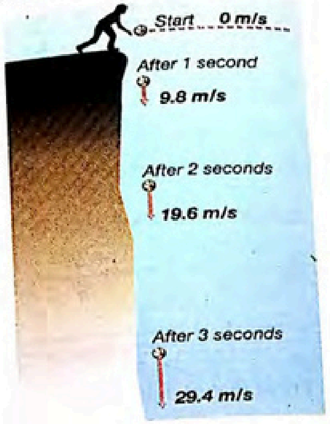
Do you notice anything interesting in the following multiplication?

$138\times 42=5796$

Answer: All digits are used

3.6 Real life problems based on variations

Example 19 A stone is dropped from the top of a hill. The distance it falls is proportional to the square of the time of fall.
The stone falls 19.6 m after 2 seconds, how far does it fall after 3 seconds?



Solution We can use: $d = kt^2$

Where: d is the distance fallen and t is the time of fall

When $d = 19.6$ then $t = 2$

$$19.6 = k \times 2^2$$

$$19.6 = 4k \quad \Rightarrow k = 4.9$$

So now we know:

$$d = 4.9t^2$$

And when $t = 3$:

$$d = 4.9 \times 3^2 = 44.1$$

So it has fallen 44.1 m after 3 seconds.

Example 20 Height of an image y on a screen varies directly as distance x of the projector from the screen. Height of the image is 20cm when distance of the projector from the screen is 100cm. At what distance should the projector be kept from the screen so that the height of an image on the screen be 15cm.

Solution $y \propto x$ or $\dfrac{y}{x} = k$ (i)

when $y = 20\text{cm}$, $x = 100$ cm

Putting these values in (i)

$$\frac{20}{100} = k \text{ or } \frac{1}{5} = k$$

$$\frac{y}{x} = \frac{1}{5}$$

Putting $y = 15$ cm and $k = 1$ in equation (i)

$$\frac{15}{x} = \frac{1}{5}$$

or $$x = 75\text{cm}$$

Hence the distance of projector from the screen = 75cm.

Example 21 The ratio of the mass of sand to cement in a particular type of concrete is 4.8 : 2. If 6 kg of sand are used, how much cement is needed?

Solution Let the amount required of cement be x kg. Then the ratio is,

sand : cement
4.8 : 2
6 : x

This is direct proportion.

$$\therefore \quad \frac{4.8}{6} = \frac{2}{x}$$

Cross-multiply: $4.8 \times x = 2 \times 6$, or more simply, $4.8x = 12$.

Solve this equation for x: $x = \dfrac{12}{4.8}$, which gives $x = 2.5$.

So 2.5 kg of cement are needed.

Example 22 4 people can paint a fence in 3 hours.

(i) How long will it take 6 people to paint it?

(ii) How many people are needed to complete the job in half an hour? (Assume everyone works at the same rate)

Solution It is an Inverse Proportion:

As the number of people goes up, the painting time goes down.
As the number of people goes down, the painting time goes up.

(i) We can use: $t = \dfrac{k}{n}$ Where: t = number of hours
k = constant of proportionality
n = number of people

"4 people can paint a fence in 3 hours" means that $t = 3$ when $n = 4$.

Therefore, $3 = \dfrac{k}{4} \quad \Rightarrow k = 12$

So $t = \dfrac{12}{n}$ And when $n = 6$, $t = \dfrac{12}{6} = 2$ hours

So 6 people will take 2 hours to paint the fence.

(ii) $$\frac{1}{2} = \frac{12}{n} \quad \Rightarrow n = 24$$

So it needs 24 people to complete the job in half an hour.
(Assuming they don't all get in each other's way!)

Exercise 3.5

1. A hedge is made of wooden planks. The thickness (T) of the hedge varies directly as the number of planks(N). 4 planks make 12cm thick edge. Find
   (i) Thickness of the hedge when number of planks is 6.
   (ii) Number of planks when thickness of the hedge is 9cm.
2. In a fountain, the pressure 'P' of water at any internal point varies directly as depth 'd' from the surface. Pressure is 51 Newton/cm^2 when depth is 3cm. Find pressure when depth is 7cm.
3. Pressure P of gas in a container varies directly as temperature T. When pressure is 50 N/m^2, temperature is 75C°. Find pressure when temperature is 150°.
4. If 8 persons complete a work in 10 days then how many days would 10 persons take to complete the same work?
5. Volume of gas 'V' varies inversely as pressure 'P'. P = 300 N/m^2 when V = 4m^3. Find pressure when V = 3m^3
6. Attraction force 'F' between two magnets vary inversely as square of the distance 'd' between them. F is 18 Newton when d is 2cm. Find the distance when attraction force is 2 Newton.
7. The volume of a right circular cylinder varies jointly as the height and the square of the radius. The volume of a right circular cylinder, with radius 4 centimetres and height 7 centimetres, is 352 cm3. Find the volume of another cylinder with radius 8 centimetres and height 14 centimetres.

Review Exercise 3

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). Direct variation between a and b is expressed as.

○ $a=b$ ○ $a=\frac{1}{b}$ ○ $a\propto b$ ○ $a\propto\frac{1}{b}$

(ii). If $m\propto\frac{1}{n}$ then

○ m = kn ○ n = km ○ $\frac{m}{n}=k$ ○ mn = k

(iii). Identify the item that does not have the same ratio as the other three.

○ $\frac{30}{45}$ ○ 4 to 6 ○ 2:3 ○ 3 to 2

(iv). If $\frac{a}{b}=\frac{c}{d}$ then by alternendo property

○ $\frac{a-b}{b}=\frac{c-d}{d}$ ○ $\frac{a}{a+b}=\frac{c}{c+d}$

○ $\frac{a}{c}=\frac{b}{d}$ ○ $\frac{b}{a}=\frac{d}{c}$

(v). If 7 : 9 :: x : 27

○ $x=21$ ○ $x=3$ ○ $x=7$ ○ $x=81$

(vi). The third proportional of x and y is

○ xy ○ $\frac{x}{y}$ ○ $\frac{y^2}{x}$ ○ none of these

(vii). If $x\propto\frac{1}{y}$ and $y\propto\frac{1}{z}$ then

○ $y\propto\frac{1}{z}$ ○ $x\propto z$ ○ $xy\propto z$ ○ $xz\propto y$

(viii). If $2a+1:21::4:7$, then

○ $a=\frac{13}{2}$ ○ $a=\frac{11}{2}$ ○ $a=10$ ○ $a=\frac{9}{2}$

(ix). If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ then each fraction is equal to

○ $\frac{\ell a+mb+ne}{\ell d+me+nf}$ ○ $\frac{\ell a+mc+ne}{\ell b+md+nf}$

○ $\frac{\ell a+mc+ne}{md+nd+ef}$ ○ $\frac{\ell a+mb+nc}{\ell b+mc+nf}$

(x). Which of the follwing is a situation in which x varies directly as y?

○ $x=\frac{4}{y}$ ○ $xy=6$ ○ $x=xy$ ○ $x=\frac{7}{16}y$

2. Find the constant of variation, when $s\propto t^2$ and t = 10 when s = 5.
3. $y\propto\frac{1}{x^2}$. If $y=4$ when $x=3$, find the value of x when $y=9$.
4. Pressure of gas in a closed vessel varies directly to temperature. If pressure is 150 units then the temperature is 70 units. What will be the pressure if temperature rises to 140 units.
5. In an electric circuit, current varies inversely as the resistance. When current is 44 amp. The resistance is 30 ohm. How much current will flow if resistance becomes 22 ohm.