Unit 2

THEORY OF QUADRATIC EQUATIONS

In this unit the students will be able to

- Define discriminant $(b^2 - 4ac)$ of the quadratic expression $ax^2 + bx + c$.
- Find discriminant of a given quadratic equation.
- Discuss the nature of roots of a quadratic equation through discriminant.
- Determine the nature of roots of a given quadratic equation and verify the result by solving the equation.
- Determine the value of an unknown involved in a given quadratic equation when the nature of its roots is given.
- Find cube roots of unity.
- Recognize complex cube roots of unity as ω and ω^2.
- Prove the properties of cube roots of unity.
- Use properties of cube roots of unity to solve appropriate problems.
- Find the relation between the roots and the coefficients of a quadratic equation.
- Find the sum and product of roots of a given quadratic equation without solving it.
- Find the value(s) of unknown(s) involved in a given quadratic equation when
  - ○ sum of roots is equal to a multiple of the product of roots,
  - ○ sum of the squares of roots is equal to a given number,
  - ○ roots differ by a given number,
  - ○ roots satisfy a given relation (e.g. the relation $2\alpha + 5\beta = 7$ where α and β are the roots of given equation),
  - ○ both sum and product of roots are equal to a given number.
- Define symmetric functions of roots of a quadratic equation.
- Evaluate a symmetric function of the roots of a quadratic equation in terms of its coefficients.
- Establish the formula, $x^2 -$ (Sum of roots)$x +$ (Product of roots) $= 0$, to find a quadratic equation from the given roots.
- Form the quadratic equation whose roots, for example, are of the type:
  - ○ $2\alpha + 1, 2\beta + 1,$
  - ○ $\alpha^2, \beta^2,$
  - ○ $\frac{1}{\alpha}, \frac{1}{\beta},$
  - ○ $\frac{\alpha}{\beta}, \frac{\beta}{\alpha},$

  - ○ $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta},$

    where α, β are the roots of a given quadratic equation..
- Describe the method of synthetic division.
- Use synthetic division to
  - ○ find quotient and remainder when a given polynomial is divided by a linear polynomial,
  - ○ find the value(s) of unknown(s) if the zeros of a polynomial are given,
  - ○ find the value(s) of unknown(s) if the factors of a polynomial are given,
  - ○ solve a cubic equation if one root of the equation is given,
  - ○ solve a biquadratic (quartic) equation if two of the real roots of the equation are given.
- Solve a system of two equations in two variables when
  - ○ one equation is linear and the other is quadratic,
  - ○ both the equations are quadratic.
- Solve the real life problems leading to quadratic equations.

Why it's important

The discriminant tells you the number and types of answers (roots) you will get. The discriminant can be +, –, or 0 which actually tells you a lot! Since the discriminant is under a radical, think about what it means if you have a positive or negative number or 0 under the radical.

2.1 The discriminant of a quadratic equation

In the quadratic formula, the expression $b^2 - 4ac$, is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

The value of the discriminant is used to determine the number of solutions of a quadratic equation and number of x intercepts of the graph of the related function. An x intercept of the graph is the x – coordinate of a point where the graph crosses the x – axis.

Cases	Case (i)	Case (ii)	Case (iii)
Discriminant	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Nature of roots	The roots are unequal and real. Roots are rational if $b^2 - 4ac$ is perfect square otherwise they are irrational.	The root are equal.	The roots are unequal and imaginary.

Example 1

Find discriminant of the quadratic equation $x^2+9x+2=0$.

Solution

Comparing the coefficient $x^2+9x+2=0$ with $ax^2+bx+c=0$, we have

$a=1, b=9, c=2$

$\therefore$ Discriminant $= b^2-4ac=(9)^2-4(1)(2)=8-18=73$

2.1.1 Nature of roots of a quadratic equation through discriminant

Example 2

Examine the nature of the roots of the following quadratic equations.

(i) $x^2-8x+16=0$ (ii) $x^2+9x+2=0$

(iii) $6x^2-x-15=0$ (iv) $4x^2+x+1=0$

Solution

(i) Comparing $x^2-8x+16=0$ with $ax^2+bx+c=0$, we have

$a=1, b=-8, c=16.$

Discriminant $= b^2-4ac=(-8)^2-4(1)(16)=64-64=0$

Since discriminant = 0, therefore, the roots of the given equation are real (rational) and equal.

(ii) In x^2+9x+2, we have,

Here $a=1, b=9, c=2$

Discriminant $= b^2-4ac=(9)^2-4(1)(2)=81-8=73$

Since discriminant > 0, but not a perfect square, therefore, the roots are real, unequal and irrational.

(iii) Here $a=6, b=-1, c=-15$

Discriminant $= b^2-4ac=(-1)^2-4(6)(-15)=1+360=361=(19)^2$

As the discriminant is a perfect square, therefore, the roots are real, unequal and rational.

(iv) Here $a=4, b=1, c=1$

$$\text{Discriminant} = b^2-4ac=(1)^2-4(4)(1)$$
$$=1-16$$
$$=-15$$

As the discriminant is negative, therefore, the roots are imaginary and unequal.

2.1.2 Determining and verifying nature of roots.

Example 3 Determine the nature of roots of the following equations and verify the results by solving them by factorization.

(i) $x^2-6x+9=0$ (ii) $x^2+5x+6=0$

Solution

(i) $x^2-6x+9=0$

Here $a=1, b=-6, c=9$. The discriminant is given by

$$b^2-4ac=(-6)^2-4(1)(9)=36-36=0$$

Since the discriminant is zero, the roots are real and equal.

$$x^2-6x+9=0$$
$$(x-3)(x-3)=0$$
$$x=3, 3$$

Which are real and equal. Hence the result is verified.

(ii) $x^2+5x+6=0$

Here $a=1, b=5, c=6$. The discriminant is given by

$$b^2-4ac=(5)^2-4(1)(6)=25-24=1=(1)^2$$

Since the discriminant is a perfect square, the roots are unequal and rational.

$$x^2+5x+6=0$$
$$(x+2)(x+3)=0$$
$$x=-2 \text{ and } x=-3$$

Which are real, unequal and rational. Hence the result is verified.

Example 4 Without solving, determine the nature of the roots of the quadratic equation.

$3x^2-4x+6=0.$

Solution We evaluate b^2-4ac using $a=3$, $b=-4$, and $c=6$:

$$b^2-4ac=(-4)^2-4(3)(6)=16-72= 56$$

The discriminant is negative, so the equation has two complex roots.

Example 5 Without solving, determine the nature of the roots of the equation.

$2x^2-7x=-1$

Solution We rewrite the equation in the standard form

$2x^2-7x+1=0$

and then substitute $a=2$, $b=-7$, and $c=1$ in the discriminant. Thus,

$$b^2-4ac=(-7)^2-4(2)(1)=49-8=41$$

The discriminant is positive and is not a perfect square; thus, the roots are real, unequal, and irrational.

2.1.3 Determining and verifying the value of an unknown nature of roots.

Example 5 Determine the set of values of k for which the given quadratic equations have real roots.

(i) $kx^2+4x+1=0$ (ii) $2x^2+kx+3=0$

Solution

(i) Comparing $kx^2+4x+1=0$ with the quadratic equation

$ax^2+bx+c=0$, we have

$a=k,\ b=4,\ c=1$

Discriminant $= b^2-4ac$

$= 4^2-4k$

Since roots are real, so $b^2-4ac\geq 0$

$\Rightarrow$ $4^2-4k\geq 0$

$\Rightarrow$ $16-4k\geq 0$

$\Rightarrow$ $16\geq 4k$

$\Rightarrow$ $4\geq k$

or $k\leq 4$

(ii) Comparing $2x^2+kx+3=0$ with the quadratic equation

$ax^2+bx+c=0$, we have

$a=2,\ b=k,\ c=3$

Since roots are real, so $b^2-4ac\geq 0$

$\Rightarrow$ $k^2-4(2)(3)\geq 0$

Discriminant $=b^2-4ac=k^2-4(2)(3)\geq 0$

$\Rightarrow$ $k^2-24\geq 0$

or $k^2\geq 24$

$\sqrt{k^2}\geq\sqrt{24}$

$|k|\geq 2\sqrt{6}$

$\pm k\geq 2\sqrt{6}$

$k\geq 2\sqrt{6},\ -k\geq 2\sqrt{6}$

$k\geq 2\sqrt{6}$ *or* $k\leq -2\sqrt{6}$

Exercise 2.1

1. Find the discriminant of the following quadratic equations:

(i). $x^2-4x+13=0$ (ii). $4x^2-5x+1=0$ (iii). $x^2+x+1=0$

2. Examine the nature of the roots of the following equations:

(i). $3x^2-5x+1=0$ (ii). $6x^2+x-2=0$ (iii). $3x^2+2x+1=0$

3. For what value of k the roots of the following equations are equal.

(i). $x^2+kx+9=0$ (ii). $12x^2+kx+3=0$ (iii). $x^2-5x+k=0$

4. Determine whether the following quadratic equations have real roots and if so, find the roots.

(i). $x^2+5x+5=0$ (ii). $4x^2+12x+9=0$ (iii). $6x^2+x-2=0$

5. Determine the nature of roots of the following quadratic equations and verify the results by solving them.

(i). $3x^2-10x+3=0$ (ii). $x^2-6x+4=0$ (iii). $x^2-3=0$

6. For what value of k the roots of the following equations are:

(a) real (b) imaginary

(i). $2x^2+3x+k=0$ (ii). $kx^2+2x+1=0$ (iii). $x^2+5x+k=0$

Math Fun

$1\times 8+1=9$

$12\times 8+2=98$

$123\times 8+3=987$

$1234\times 8+4=9876$

$12345\times 8+5=98765$

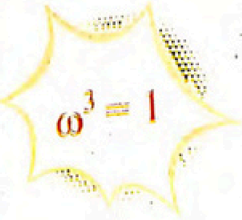
$123456\times 8+6=987654$

$1234567\times 8+7=9876543$

$12345678\times 8+8=98765432$

$123456789\times 8+9=987654321$

2.2 Cube roots of unity and their properties



2.2.1 Cube root of unity

Let x be a cube root of unity,

then $x = \sqrt[3]{1} = (1)^{\frac{1}{3}}$

$\Rightarrow \quad x^3 = 1$

$\Rightarrow \quad x^3 - 1 = 0$

$\Rightarrow \quad (x-1)(x^2+x+1) = 0$

$\Rightarrow \quad x - 1 = 0 \quad or \quad x^2 + x + 1 = 0$

gives $x = 1$ or $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

where $i = \sqrt{-1} \Rightarrow i^2 = -1$

Thus the cube roots of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$.

Here 1 is the real root and $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$ are complex roots.

Let one of the complex roots be denoted by the Greek letter ω (read as omega).

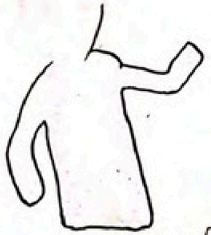
Suppose $\omega = \dfrac{-1+i\sqrt{3}}{2}$

Then $\omega^2 = \left(\dfrac{-1+i\sqrt{3}}{2}\right)^2$

$$= \frac{(-1)^2 + (i\sqrt{3})^2 + 2(-1)(i\sqrt{3})}{4}$$

$$= \frac{1-3-2i\sqrt{3}}{4}$$

$$= \frac{-2-2i\sqrt{3}}{4} = \frac{-1-i\sqrt{3}}{2}$$



Similarly, if $\omega = \dfrac{-1-i\sqrt{3}}{2}$, then $\omega^2 = \dfrac{-1+i\sqrt{3}}{2}$

2.2.2 Properties of the cube roots of unity

1. The sum of the cube roots of unity is zero i.e. $1 + \omega + \omega^2 = 0$.

If $\omega = \dfrac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$

Then we have $1 + \omega + \omega^2 = 1 + \dfrac{-1+i\sqrt{3}}{2} + \dfrac{-1-i\sqrt{3}}{2}$

$$= \frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}$$

$$= \frac{0}{2}$$

$$= 0$$

Thus, the sum of the cube roots of unity is zero.

2. **The product of the cube roots of unity is 1, i.e. $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$**

If $\omega = \dfrac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$,

then $1 \cdot \omega \cdot \omega^2 = 1 \cdot \left(\dfrac{-1+i\sqrt{3}}{2}\right)\left(\dfrac{-1-i\sqrt{3}}{2}\right)$

$$= \frac{(-1)^2 - (i\sqrt{3})^2}{4} = \frac{1-3i^2}{4}$$

$$= \frac{1-(-3)}{4} \qquad (\because i^2 = -1)$$

$$= \frac{1+3}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

Thus, the product of the cube roots of unity is 1.

3. **Each complex cube root of unity is reciprocal of the other** i.e. $w = \dfrac{1}{w^2}$ and $w^2 = \dfrac{1}{w}$

By property 2, we have $w^3 = 1$.

$\Rightarrow \quad w \cdot w^2 = 1$

$\Rightarrow \quad w = \dfrac{1}{w^2}$

$\Rightarrow \quad w^2 = \dfrac{1}{w}$ Thus, $w = \dfrac{1}{w^2}$ and $w^2 = \dfrac{1}{w}$.

2.2.3 Using properties of cube roots of unity to solve problems

Example 7 Show that $x^3+y^3=(x+y)(x+wy)(x+w^2y)$

Solution R.H.S $=(x+y)(x+wy)(x+w^2y)$

$$=(x+y)\left[(x+wy)(x+w^2y)\right]$$
$$=(x+y)\left[x^2+(w+w^2)xy+w^3y^2\right]$$
$$=(x+y)(x^2-xy+y^2) \qquad (\because 1+w+w^2=0 \text{ and } w^3=1)$$
$$=x^3+y^3$$
$$=L.H.S$$

Example 8 Evaluate $w^{15},\ w^{24},\ w^{90},\ w^{101},\ w^{-2},\ w^{-13}$

Solution

$$w^{15}=(w^3)^5=(1)^5=1$$
$$w^{24}=(w^3)^8=(1)^8=1$$
$$w^{90}=(w^3)^{30}=(1)^{30}=1$$
$$w^{101}=w^{99}\cdot w^2=(w^3)^{33}.w^2=(1)^{33}\cdot w^2=1\cdot w^2=w^2$$
$$w^{-2}=\frac{1}{w^2}=w \qquad (by\ property\ 3)$$
$$w^{-13}=\frac{1}{w^{13}}=\frac{1}{w^{12}\cdot w}=\frac{1}{(w^3)^4\cdot w}=\frac{1}{(1)^4\cdot w}=\frac{1}{1\cdot w}=\frac{1}{w}=w^2$$

Example 9 Show that $\left(-1+i\sqrt{3}\right)^3+\left(-1-i\sqrt{3}\right)^3=16$

Solution Since $w=\dfrac{-1+i\sqrt{3}}{2}$ and $w^2=\dfrac{-1-i\sqrt{3}}{2}$

Then $2w=-1+i\sqrt{3}$ and $2w^2=-1-i\sqrt{3}$

$$L.H.S=\left(-1+i\sqrt{3}\right)^3+\left(-1-i\sqrt{3}\right)^3$$
$$=(2w)^3+(2w^2)^3$$
$$=8w^3+8w^6$$
$$=8w^3+8(w^3)^2$$
$$=8(1)+8(1)^2$$
$$=8+8$$
$$=16$$
$$=R.H.S$$

Exercise 2.2

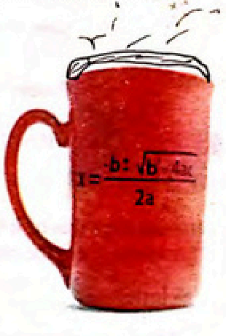
1. Find the cube roots of the following numbers.
   (i). -1 (ii). 8 (iii). -27
2. Evaluate:
   (i). $w^{12}+w^{58}+w^{95}$ (ii). $(1+w-w^2)^7$ (iii). $(1+3w-w^2)(1+w-2w^2)$
3. Prove that:
   (i). $(1+2w)(1+2w^2)(1-w-w^2)=6$ (ii). $\left(-1+i\sqrt{3}\right)^4\left(-1-i\sqrt{3}\right)^5=512w^2$
4. Show that:
   (i). $x^3-y^3=(x-y)(x-wy)(x-w^2y)$ (ii). $(1+w)(1+w^2)(1+w^4)(1+w^8)=1$

2.3 Roots and coefficients of a quadratic equation

2.3.1 Relation between the roots and the co-efficient of a quadratic equation

We express the sum and the product of the roots of the quadratic equation in terms of its co-efficient. Let α,β be the roots of the quadratic equation $ax^2+bx+c=0;\ a\neq 0$ where

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$



Then sum of the roots

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-2b}{2a}=\frac{-b}{a}$$

and product of the roots

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\cdot\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$
$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{c}{a}$$

Thus sum of the roots $=\alpha+\beta=\dfrac{-b}{a}=-\dfrac{co-efficient\ of\ x}{co-efficient\ of\ x^2}$

Product of the roots $=\alpha\beta=\dfrac{c}{a}=\dfrac{constant\ term}{co-efficient\ of\ x^2}$

2.3.2 The sum and product of roots of a given quadratic equation without solving it

Example 10

Without solving, find the sum and product of the roots of the equation.

(i) $2x^2-3x-4=0$ (ii) $3x^2+6x-2=0$

Solution

(i) In the equation $2x^2-3x-4=0$ $a=2$, $b=-3$ and $c=-4$

$$\text{Sum of the roots} = \frac{-b}{a} = -\frac{(-3)}{2} = \frac{3}{2}$$

$$\text{Product of the roots} = \frac{c}{a} = \frac{-4}{2} = -2$$

(ii) $3x^2+6x-2=0$

Here $a=3, b=6, c=-2$

$$\text{Sum of the roots} = \frac{-b}{a} = \frac{-6}{3} = -2$$

$$\text{Product of the roots} = \frac{c}{a} = \frac{-2}{3} = -\frac{2}{3}$$

2.3.3 The values of unknown(s) involved in a given quadratic equation

Example 11

Find the value of k so that the sum of the roots of the equation $2x^2+kx+6=0$ is equal to three times the product of its roots.

Solution

The given equation is

$$2x^2+kx+6=0$$

Here $a=2, b=k, c=6$

$$\therefore \text{ sum of the roots} = \frac{-b}{a} = -\frac{k}{2}$$

$$\text{product of the roots} = \frac{c}{a} = \frac{6}{2} = 3$$

$\because$ sum of the roots = three times the product of roots

$$\Rightarrow \frac{-k}{2} = 3(3) = 9$$

$$\Rightarrow k=-18$$

Example 12

Find the value of a if the sum of the square of the roots of $x^2-3ax+a^2=0$ is 7.

Solution Let α, β be the roots of the equation $x^2-3ax+a^2=0$

then $\alpha+\beta = -\frac{(-3a)}{1} = 3a$ and $\alpha\beta = \frac{a^2}{1} = a^2$.

Given $\alpha^2+\beta^2=7$

$\Rightarrow (\alpha+\beta)^2-2\alpha\beta=7$

$\Rightarrow (3a)^2-2(a^2)=7$

$\Rightarrow 9a^2-2a^2=7$

$\Rightarrow 7a^2=7$

$\Rightarrow a^2=1$

$\Rightarrow a=\pm 1$

Example 13 Find the value of k if the roots of $x^2-7x+k=0$ differ by unity.

Solution Let α, $\alpha+1$ be the roots of $x^2-7x+k=0$

Then $\alpha+(\alpha+1) = \frac{-(-7)}{1}$

or $2\alpha+1=7$

or $\alpha=3$

and $\alpha(\alpha+1) = \frac{k}{1} = k$

$\Rightarrow 3(1+3)=k$

$\Rightarrow k=12$

Example 14 If α, β are the roots of $9x^2-27x+k=0$, find the value of k such that $2\alpha+5\beta=7$.

Solution If α, β are the roots of $9x^2-27x+k=0$, then

$\alpha+\beta = \frac{27}{9} = 3$ (i)

and $\alpha\beta = \frac{k}{9}$ (ii)

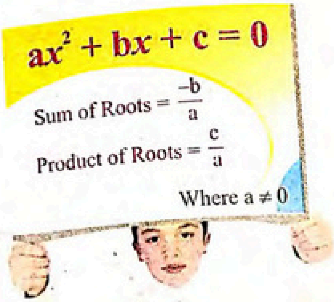
also $2\alpha+5\beta=7$ (iii)

Solving (i) and (iii), we get $\alpha=\frac{8}{3}$, $\beta=\frac{1}{3}$

Putting these values in (ii) we obtain $\left(\frac{8}{3}\right)\left(\frac{1}{3}\right)=\frac{k}{9}$

$\Rightarrow k=8$

Example 15 Find the value of m and n if both sum and product of roots of the quadratic equation $mx^2-5x+n=0$ are equal to 10.



Solution The given equation is $mx^2-5x+n=0$

Here sum of the roots $=\frac{5}{m}$

and product of the roots $=\frac{n}{m}$

According to given condition

$$\frac{5}{m}=10$$

$$\Rightarrow \quad m=\frac{1}{2}$$

$$\text{and} \quad \frac{n}{m}=10$$

$$\Rightarrow \quad n=10\,m$$

$$=10\left(\frac{1}{2}\right) \qquad \left(\because m=\frac{1}{2}\right)$$

$$n=5$$

$$\therefore \quad m=\frac{1}{2} \text{ and } n=5.$$

Exercise 2.3

1. Without solving the equation, find the sum and products of the roots of the following quadratic equations.
   (i). $4x^2-4x-3=0$ (ii). $2x^2+5x+6=0$ (iii). $3x^2+2x-5=0$
2. Find the value of k if sum of the roots of $2x^2+kx+6=0$ is equal to the product of its roots.
3. Find the value of k if the sum of the square of the roots of $x^2-5kx+6k^2=0$ is equal to 13.
4. Find the value of k if the roots of $x^2-5x+k=0$ differ by unity.
5. Find the value of k if the roots of $x^2-9x+k+2=0$ differ by three.
6. If α, β are the roots of $x^2-5x+k=0$, find k such that $3\alpha+2\beta=12$.
7. Find the value of m and n if both sum and product of roots of the equation $mx^2-3x-n=0$ are equal to $\frac{3}{5}$.

2.4 Symmetric functions of roots of a quadratic equation

Let α, β be the roots of a quadratic equation, then the expressions of the form $\alpha+\beta$, $\alpha\beta$, $\alpha^2+\beta^2$ are called the functions of the roots of the quadratic equation. By symmetric function of the roots of an equation, we mean that the function remains invariant (unchanged) in values when the roots are interchanged. For example, the functions $\alpha+\beta$, $\alpha^2+\beta^2$, $\alpha^3+\beta^3$ are symmetric function of α and β.

2.4.1 Symmetric function of the roots of a quadratic equation in terms of its coefficients

Example 16 If α, β are the roots of $ax^2+bx+c=0$, then find the values of the symmetric functions of the roots of a given quadratic equation in terms of its coefficients.

(i) $\alpha+\beta$ (ii) $\alpha\beta$ (iii) $\alpha^2+\beta^2$ (iv) $\alpha^3+\beta^3$

(v) $\frac{1}{\alpha}+\frac{1}{\beta}$ (vi) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solution

(i) As α, β are the roots of the equation $ax^2+bx+c=0$,

$$\therefore \text{ sum of the roots } =\alpha+\beta=-\frac{b}{a}$$

(ii) Product of the roots $=\alpha\beta=\frac{c}{a}$

(iii) Since $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)$$

$$=\frac{b^2}{a^2}-\frac{2c}{a}$$

$$=\frac{b^2-2ac}{a^2}$$

(iv) Since $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$=\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)$$

$$=-\frac{b^3}{a^3}+3\frac{bc}{a^2}$$

$$=\frac{3bc}{a^2}-\frac{b^3}{a^3}=\frac{3abc-b^3}{a^3}$$

Did You Know?

Beautiful Number Relationships
$81=(8+1)^2=9^2$
$4913=(4+9+1+3)^3=17^3$

(v) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}$

$$=\frac{\alpha+\beta}{\alpha\beta}$$

$$=\frac{-\frac{b}{a}}{\frac{c}{a}} \qquad \left(\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}\right)$$

$$=-\frac{b}{c}$$

(vi) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2} \qquad =\frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)^2}$$

$$=\frac{\frac{b^2-2ac}{a^2}}{\frac{c^2}{a^2}} \qquad =\frac{b^2-2ac}{c^2}$$

2.5 Formation of a quadratic equation whose roots are given

Let α, β be the roots of the quadratic equation $ax^2+bx+c=0$,

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Now $ax^2+bx+c=0;\quad a\neq 0$

$$\Rightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$\Rightarrow \quad x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0$$

$$\Rightarrow \quad x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\Rightarrow \quad x^2-Sx+P=0$$

Where $S=\alpha+\beta=$ sum of the roots

and $P=\alpha\beta=$ product of the roots

Thus, the formula for forming a quadratic equation whose roots are given is

x^2- (sum of the roots) $x+$ product of the roots $=0$

Example 17 Form a quadratic equation whose roots are $1+\sqrt{5}$ and $1-\sqrt{5}$.

Solution Roots of the required equation are $1+\sqrt{5}$ and $1-\sqrt{5}$

Sum of the roots $=1+\sqrt{5}+1-\sqrt{5}=2$

Product of the roots $=(1+\sqrt{5})(1-\sqrt{5})=1^2-(\sqrt{5})^2=1-5=-4$

$\therefore$ the required quadratic equation is

x^2- (sum of the roots) $x+$ product of the roots $=0$

$$\Rightarrow \quad x^2-2x+(-4)=0$$

$$\Rightarrow \quad x^2-2x-4=0$$

Forming an equation with the given roots is the reverse process of solving an equation.

Example 18 Form the quadratic equation whose roots are:

(i) $2a+1, 2b+1$ (ii) a^2, b^2 (iii) $\frac{1}{a},\frac{1}{b}$ (iv) $\frac{2}{3},\frac{3}{2}$

Solution (i) The roots of the required equation are $2a+1, 2b+1$

$\therefore$ sum of the roots $=2a+1+2b+1$

$$=2a+2b+2$$

and product of the roots $=(2a+1)(2b+1)$

$$=4ab+2a+2b+1$$

The required equation is given by

x^2- (sum of the roots) $x+$ product of the roots $=0$

$$x^2-(2a+2b+2)x+(4ab+2a+2b+1)=0$$

Which is the required quadratic equation.

(ii) The roots of the required quadratic equation are a^2, b^2 therefore,

$$x=a^2 \quad \text{or} \quad x=b^2$$

$$x-a^2=0 \quad \text{or} \quad x-b^2=0$$

$$(x-a^2)(x-b^2)=0$$

$$x(x-b^2)-a^2(x-b^2)=0$$

$$x^2-xb^2-a^2x+a^2b^2=0$$

$$x^2-(a^2-b^2)x+a^2b^2=0$$

Which is the required quadratic equation.

(iii) The roots of the required quadratic equation are $\frac{1}{a}, \frac{1}{b}$

Since $x = \frac{1}{a}$ or $x = \frac{1}{b}$

$ax = 1$ or $bx = 1$

$ax - 1 = 0$ or $bx - 1 = 0$

$(ax - 1)(bx - 1) = 0$

$ax(bx - 1) - 1(bx - 1) = 0$

$abx^2 - ax - bx + 1 = 0$

$abx^2 - (a - b)x + 1 = 0$

Which is the required quadratic equation.

(iv) The roots of the required equation are $\frac{2}{5}, \frac{5}{2}$

Therefore $x = \frac{2}{5}$ or $x = \frac{5}{2}$

$5x = 2$ or $2x = 5$

$5x - 2 = 0$ or $2x - 5 = 0$

$(5x - 2)(2x - 5) = 0$

$5x(2x - 5) - 2(2x - 5) = 0$

$10x^2 - 25x - 4x + 10 = 0$

$10x^2 - 29x + 10 = 0$

Which is the required quadratic equation.

$$9 - 24 = 25 - 40$$

$$9 - 24 + 16 = 25 - 40 + 16$$

$$(3 - 4)^2 = (5 - 4)^2$$

$$3 - 4 = 5 - 4$$

$$-1 = 1$$

Exercise 2.4

1. If α, β are the roots of $ax^2 + bx + c = 0$, find the values of

   (i). $\alpha^3\beta + \beta^3\alpha$ (ii). $(\alpha - \beta)^2$

2. Find the quadratic equation whose roots are

   (i). $1, \frac{1}{2}$ (ii). $-3, 4$ (iii). $3 + \sqrt{2},\ 3 - \sqrt{2}$ (iv). $a, -2a$

3. Form a quadratic equation whose roots are square of the roots of the equation $ax^2 + bx + c = 0;\ a \neq 0$.

4. If α, β are the roots of $2x^2 + 3x + 1 = 0$, then find the values of

   (i). $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii). $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (iii). $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

5. If α, β are the roots of $3x^2 - 2x + 5 = 0$, find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.

6. If α, β are the roots of $x^2 - 4x + 2 = 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}$.

2.6 Synthetic Division

We are familiar with the process of long division for dividing one polynomial by another. We noticed that if the degrees of the polynomials in the numerator and denominator differ considerably, then the process of long division indeed becomes very long. However if the polynomial in the denominator is of the form $x - a$, then there is a shortcut method called synthetic division.

Steps of Synthetic Division:

Step 1 If the divisor is $x - r$, write r in the box. Arrange the coefficient of the dividend by descending powers of x, supplying a zero coefficient for every missing power.

Step 2. Copy the leading coefficient in the third row.

Step 3. Multiply the latest entry in the third row by the number in the box and write the result in the second row under the next coefficient. Add the number in that column.

Step 4. Repeat step 3 until there is an entry in the third row for each entry in the first row. the last numbers are the coefficient of the quotient in descending order.

WARNING

(a) Synthetic division can be used only when the divisor is a linear factor. Don't forget to write a zero for the coefficient of each missing term.

(b) When dividing by $x - r$. place r in the box. For example, when the divisor is $x + 3$, place -3 in the box, since $x + 3 - x - (-3)$. Similarly, when the divisor is $x - 3$, place $+3$ in the box, since $x - 3 = x - (+3)$.

(iii) The roots of the required quadratic equation are $\frac{1}{a}, \frac{1}{b}$

Since $x = \frac{1}{a}$ or $x = \frac{1}{b}$

$ax = 1$ or $bx = 1$

$ax - 1 = 0$ or $bx - 1 = 0$

$(ax-1)(bx-1) = 0$

$ax(bx-1) - 1(bx-1) = 0$

$abx^2 - ax - bx + 1 = 0$

$abx^2 - (a-b)x + 1 = 0$

Which is the required quadratic equation.

(iv) The roots of the required equation are $\frac{2}{5}, \frac{5}{2}$

Therefore $x = \frac{2}{5}$ or $x = \frac{5}{2}$

$5x = 2$ or $2x = 5$

$5x - 2 = 0$ or $2x - 5 = 0$

$(5x-2)(2x-5) = 0$

$5x(2x-5) - 2(2x-5) = 0$

$10x^2 - 25x - 4x + 10 = 0$

$10x^2 - 29x + 10 = 0$

Which is the required quadratic equation.

Math Fun

$$9 - 24 = 25 - 40$$

$$9 - 24 + 16 = 25 - 40 + 16$$

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$$3 - 4 = 5 - 4$$

$$-1 = 1$$

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   (i). $\alpha^3\beta + \beta^3\alpha$ (ii). $(\alpha-\beta)^2$
2. Find the quadratic equation whose roots are
   (i). $1, \frac{1}{2}$ (ii). $-3, 4$ (iii). $3+\sqrt{2},\ 3-\sqrt{2}$ (iv). $a, -2\dot{a}$
3. Form a quadratic equation whose roots are square of the roots of the equation $ax^2 + bx + c = 0;\ a \neq 0$.
4. If α, β are the roots of $2x^2 + 3x + 1 = 0$, then find the values of
   (i). $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii). $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (iii). $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$
5. If α, β are the roots of $3x^2 - 2x + 5 = 0$, find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.
6. If α, β are the roots of $x^2 - 4x + 2 = 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}$.

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Example 19 Use synthetic division to find the quotient $Q(x)$ and the remainder R when the polynomial $3x^3-2x^2-150$ is divided by $x-4$.

Solution Let $P(x)=3x^3-2x^2-150$

$=3x^3-2x^2+0x-150$

and $x-a=x-4 \Rightarrow a=4$

Then by synthetic, division we have

$$\begin{array}{r|rrrr} 4 & 3 & -2 & 0 & -150 \\ & & 12 & 40 & 160 \\ \hline & 3 & 10 & 40 & 10 \end{array}$$

Coefficients of quotient

↑ Remainder

Therefore, $Q(x)=3x^2+10x+40$ and $R=10$

Example 20 Use synthetic division to find the value of k if 2 is a zero of the polynomial $2x^4+x^3+kx^2-8$.

Solution Let $P(x)=2x^4+x^3+kx^2-8$

$=2x^4+x^3+kx^2+0x-8$

Since 2 is a zero of $P(x) \Rightarrow P(2)=0$

$\therefore$ $x-2$ is a factor of the polynomial $P(x)$

Now use synthetic division to divide $P(x)$ by $x-2$.

$$\begin{array}{r|rrrrr} 2 & 2 & 1 & k & 0 & -8 \\ & & 4 & 10 & 2k+20 & 4k+40 \\ \hline & 2 & 5 & k+10 & 2k+20 & 4k+32 \end{array}$$

↑ Remainder

Since the remainder is $4k+32$.

$\Rightarrow$ $P(2)=4k+32$ by Remainder theorem

$\Rightarrow$ $4k+32=0$

$\Rightarrow$ $4k=-32=0$

$\Rightarrow$ $k=-\frac{32}{4}$

$\Rightarrow$ $k=-8$

Example 21 Use synthetic division to find the values of m and n if $x-1$ and $x+2$ are the factors of the polynomial $x^3-mx^2+nx+12$.

Solution Here $x-a=x-1 \Rightarrow a=1$

and $x-a=x+2 \Rightarrow a=-2$

Let $P(x)=x^3-mx^2+nx+12$

Now use synthetic division to divide $P(x)$ by $x-1$ and $Q(x)$ by $x+2$.

$$\begin{array}{r|rrr|r} 1 & 1 & -m & n & 12 \\ & & 1 & 1-m & 1-m+n \\ \hline -2 & 1 & 1-m & 1-m+n & 13-m+n \\ & & -2 & 2+2m & \\ \hline & 1 & -1-m & 3+m+n & \end{array}$$

↑ Remainder

↑ Remainder

Since $x-1$ and $x+2$ are the factors of $P(x)$, we have

$13-m+n=0=$ (i)

$3+m+n=0=$ (ii)

Adding (i) and (ii), we get

$16+2n=0$

$\Rightarrow$ $2n=-16$

$\Rightarrow$ $n=-8$

Putting n = –8 in equation (i), we get

$13-m-8=0$

$\Rightarrow$ $-m+5=0$

$\Rightarrow$ $-m=-5$

$\Rightarrow$ $m=5$

Thus $m=5$ and $n=-8$.

Note

Synthetic division is a method of performing Euclidean division of polynomials with less writing and fewer calculations.

The advantages of synthetic division are that it allows one to calculate without writing variables, it uses few calculations, and it takes significantly less space on paper than long division. Also, the subtractions in long division are converted to additions by switching the signs at the very beginning, preventing sign errors.

Example 22 If -1 and 2 are roots of the quartic equation $x^4 - 5x^2 + 4 = 0$, use synthetic division to find the other roots.

Solution Let $P(x) = x^4 - 5x^2 + 4$

$$= x^4 + 0x^3 - 5x^2 + 0x + 4$$

Since -1 and 2 are two roots of the equation $P(x) = 0$, therefore $x+1$ and $x-2$ are factors of $P(x)$. To find the quotient we use synthetic division.

$$\begin{array}{r|rrrrr}
-1 & 1 & 0 & -5 & 0 & 4 \\
 & & -1 & 1 & 4 & -4 \\
\hline
2 & 1 & -1 & -4 & 4 & \multicolumn{1}{|r}{0} \\
 & & 2 & 2 & -4 & \\
\hline
 & 1 & 1 & -2 & \multicolumn{1}{|r}{0} &
\end{array}$$

Other factor is $x^2 + x - 2 = Q(x)$

Other roots will be the roots of the equation $x^2 + x - 2 = 0$

$\Rightarrow \quad (x+2)(x-1) = 0$

$\Rightarrow \quad x = -2 \ or \ x = 1$

Thus, the other roots of $P(x) = 0$ are -2, 1.

Exercise 2.5

1. Use synthetic division to find the quotient $Q(x)$ and the remainder R when the first polynomial is divided by the second binomial in each case:
   (i). $3x^3 + 2x^2 - x - 1;\ x+3$ (ii). $2x^3 - 7x^2 + 12x - 27;\ \ x-3$
   (iii). $2x^4 - 3x^2 + 5x - 7;\ x+2$
2. Use synthetic division to find the value of k if -2 is a zero of the polynomial $x^3 + 4x^2 + kx + 8$.
3. Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of $x^3 + px^2 + qx + 6$.
4. If $x+1$ and $x-2$ are factors of the polynomial $x^3 + ax^2 + bx + 2$, then using synthetic division, find the values of a and b.
5. One root of the cubic equation $x^3 - 7x - 6 = 0$ is 3. Use synthetic division of find the to find the other roots.
6. If -1 and 2 are roots of the quartic equation $x^4 - 5x^3 + 3x^2 + 7x - 2 = 0$, use synthetic use synthetic division to find other roots.

2.7 Simultaneous Equations

More than one equation which are satisfied by the same values of the variables involved are called simultaneous equations.

Note

A system of Linear equations consists of two or more linear equations in the same variables. A solution of system of linear equations in two variables is an ordered pair that satisfies each equation in the system.

2.7.1 Solution of one linear equation and one quadratic equation

Example 23

Solve the system.

$$2x + y = 10$$
$$4x^2 + y^2 = 68$$

Solution

$$2x + y = 10 \qquad \text{(i)}$$
$$4x^2 + y^2 = 68 \qquad \text{(ii)}$$

From equation (i) we have

$$y = 10 - 2x \qquad \text{(iii)}$$

Substituting this value of y in equation (ii) we have

$$4x^2 + (10 - 2x)^2 = 68$$
$$4x^2 + 100 + 4x^2 - 40x = 68$$

or $\quad 8x^2 - 40x + 32 = 0$

$$x^2 - 5x + 4 = 0 \qquad \text{(Dividing by 8)}$$
$$(x-1)(x-4) = 0$$

$\Rightarrow \quad x - 1 = 0$ or $x - 4 = 0$

which gives $x = 1$, or $x = 4$.

Substituting these values in (iii) we have

For $x = 1$, $y = 8$ and for $x = 4$, $y = 2$.

$\therefore$ The solutions of the given system are (1,8) and (4,2).

or solution set $= \{(1, 8), (4, 2)\}$

Example **24** Solve the system

$$x-y=7,\ x^2+3xy+y^2=-1$$

Solution $x-y=7$ (i)

$x^2+3xy+y^2=-1$ (ii)

From equation (i) we have

$x=7+y$ (iii)

Substituting this value of x in equation (ii) we have

$$(7+y)^2+3(7+y)y+y^2=-1$$

$$49+14y+y^2+21y+3y^2+y^2+1=0$$

or $5y^2+35y+50=0$

$y^2+7y+10=0$ (Dividing by 5)

$(y+2)(y+5)=0$

$\Rightarrow$ $y+2=0$ or $y+5=0$

$\Rightarrow$ $y=-2$ or $y=-5$

Substituting these values in (iii) we have for $y=-2$, $x=5$ and for $y=-5$, $x=2$.

$\therefore$ The solutions of the given system are $(5,-2)$ and $(2,-5)$

or solution set $=\{(5,-2),(2,-5)\}$

WARNING

The expression for x or y obtained from an equation must not be substituted in the same equation. From the first equation of the system

$$x+2y=-1$$
$$3x^2+y=2$$

We obtain

$$x=-1-2y$$

Substituting (incorrectly) in the same equation would result in

$$(-1-2y)+2y=-1$$
$$-1=-1$$

The substitution $x=-1-2y$ must be made in the second equation.

2.7.2 Solution when both equations are quadratic

Example **25** Solve the system

$$x^2+y^2=4$$
$$2x^2-y^2=8$$

Solution $x^2+y^2=4$ (i)

$2x^2-y^2=8$ (ii)

Add (i) and (ii) to eliminate y^2.

$$\begin{array}{r} x^2+y^2=4 \\ 2x^2-y^2=8 \\ \hline 3x^2\quad =12 \end{array}$$

$$x^2=4$$
$$x=\pm 2$$

Substitute the values of x in (i) gives the corresponding values of y.

When $x=2$, we get $(2)^2+y^2=4$

$y^2=0$

$y=0$

When $x=-2$, we get $(-2)^2+y^2=4$

$4+y^2=4$

$y^2=0$

$y=0$

$\therefore$ the solutions of the given system are $(-2,0)$ and $(2,0)$ and the solution set is

$$\{(-2,0),(2,0)\}$$

Exercise 2.6

1. Solve the following system of equations.

(i). $2x-y=3$
$x^2+y^2=2$

(ii). $x+2y=0$
$x^2+4y^2=32$

(iii). $2x-y=-8$
$x^2+4x=y$

(iv). $2x+y=4$
$x^2-2x+y^2=3$

(v). $4x^2+5y^2=4$
$3x^2+y^2=3$

(vi). $5x^2=y^2+9$
$x^2=-y^2+45$

(vii). $4x^2+3y^2\boxed{-5=0}$
$2x^2+3y^2-4=0$

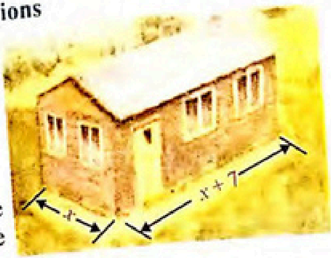
Challenge !

2. Solve the system of equations.

(i). $x+y=9$
$x^2+3xy+2y^2=0$

(ii). $y-x=4$
$2x^2+xy+y^2=8$

2.7.3 Real Life Application of Quadratic Equations



Example 26 Suppose a rectangular shed is being built that has an area of 120 square feet and is 7 feet longer than it is wide. Determine its dimensions.

Solution
Let x be the width of the shed. Then $x+7$ represents the length. Since area equals width times length, we solve the following equation.

$x(x+7)=120$ area is equal to 120 square foot
$x^2+7x=120$ distributive property
$x^2+7x-120=0$ subtract 120 from each side
$(x+15)(x-8)=0$ factorize
$x+15=0$ or $x-8=0$ zero-product property
$x=-15$ or $x=8$ solve

Since dimensions cannot be negative, the solution that has meaning is $x=8$. The length is 7 feet longer than the width, so the dimensions of the shed are 8 feet by 15 feet.

Example 27 A man purchased a number of shares of stock for an amount of Rs. 6000. If he had paid Rs. 20 less per share, the number of shares that could have been purchased for the same amount of money would have increased by 10. How many shares did he buy?

Solution Suppose the number of shares purchased $= x$
The amount paid per share $= y$ then $xy = \text{Rs. } 6000 \longrightarrow$ (i)
If the man had paid Rs. 20 less per share i.e. Rs. $(y-20)$, the number of shares would have been $x+10$
therefore $(x+10)(y-20) = Rs.\ 6000 \longrightarrow$ (ii)
From (i), we get $y=\dfrac{6000}{x}$. Substituting in (ii) we have

$$(x+10)\left(\frac{6000}{x}-20\right)=6000$$

$\Rightarrow\quad 6000x-20x^2+60000-200x=6000x$
$\Rightarrow\quad 20x^2+200x-60000=0$ (Dividing by 20)
$\Rightarrow\quad x^2+10x-3000=0$
$\Rightarrow\quad (x-50)(x+60)=0$
$\Rightarrow\quad x=50$ or $x=-60$

Since $x=-60$ is not admissible, so we neglect it. Thus the number of shares purchased is 50.

Exercise 2.7

1. Find two consecutive positive integers whose product is 72.
2. The sum of the squares of three consecutive integers is 50. Find the integers.
3. The length of a hall is 5 meters more than its width. If the area of the hall is 36 sq. m. Find the length and width of the hall.
4. The sum of two numbers is 11 and sum of their square is 65. Find the numbers.
5. The sum of the squares of two numbers is 100. One number is 2 more than the other. Find the numbers.
6. The area of a rectangular field is 252 square meters. The length of its side is 9 meter longer than its width. Find its sides.
7. One side of a rectangle is 3 centimeters less than twice the other. If the area of the rectangle is 54 square centimeters, then find the sides of the rectangle.
8. The length of one side of right triangle exceeds the length of the other by 3 centimeters. If the hypotenuse is 15 centimeters, then find the length of the sides of the triangle.
9. The sides of a right triangle in cm are $(x-1)$, x, $(x+1)$. Find the sides of the triangle.
10. A shepherd bought some goats for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 goats more for the same amount of money.
How many goats did he buy, when the rate in each case is uniform?

Review Exercise 2

1. At the end of each question, four circles are given. Fill in the correct circle only.
   - (i). If the sum of the roots of
   $(a+1)x^2+(2a+3)x+(3a+4)=0$
   ○ 0 ○ 1 ○ 2 ○ 3
   - (ii). The sum of the roots of a quadratic equation is 2 and the sum of the cubes of the roots in 98. The equations is
   ○ $x^2-2x-15=0$ ○ $x^2-2x+15=0$
   ○ $x^2-4x+15=0$ ○ none of these
   - (iii). If a, b, c are positive real number, then both the roots of the equation $ax^2+bx+c=0$, are always
   ○ real and positive ○ real and negative
   ○ rational and unequal ○ none of these

(iv). If a and b are the roots of $4x^2-3x+7=0$, then the value of $\frac{1}{a}+\frac{1}{b}$

○ $-\frac{3}{4}$ ○ $\frac{3}{7}$ ○ $-\frac{3}{7}$ ○ $\frac{4}{7}$

2. For what value of k the roots of the equation $3x^2-5x+k=0$ are equal.

3. Evaluate $\left(-1+\sqrt{-3}\right)^7+\left(-1-\sqrt{-3}\right)^7$.

4. Without solving the equation, find the sum and products of the roots of the following quadratic equations.

(i). $4x^2-1=0$ (ii). $3x^2+4x=0$

5. Find the value of k so that the sum of the roots of the equation $3x^2+(2k+1)x+k-5=0$ is equal to the product of roots.

6. Find the value of k if the roots of $x^2-3x+k+1=0$ differ by unity.

7. Find the quadratic equation whose roots are the multiplicative inverses of the roots of $12x^2-17x+6=0$.

8. If one of the roots of the quadratic equation $2x^2+kx+4=0$ is 2, find the other root. Also find the value of k.

9. One root of the cubic equation $x^3+6x^2+11x+6=0$ is -3. Use synthetic division to find the other roots.

10. Solve the following system of equations.

(i). $x+y=3; x^2-3xy+y^2=137$ (ii). $7x^2-4=5y^2, 3x^2+2=4y^2$

11. The area of a rectangle is 48 cm². If length and width are each increased by 4 cm. the area of the larger rectangle is 120 cm². Find the length and width of the original rectangle.

Activity

Find the error:

Abuzar is trying to solve the quadratic equation $x^2+x+11=0$ He finds two-real solutions to the quadratic equation. Habiba immediately points out that Abuzar's answers are wrong. Habiba explains that she doesn't know what the solutions are right away, but she knows they are not real numbers. Who is correct?

Aetivity

Using online calculator.

Divide x^5+x^2+5x+7 *by* $x+2$ and find Quotient and Remainder. Verify your answer by using long division method.

Summary

- The solution to an equation are called the **roots** of the equation.
- The quadratic formula is ⇒ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- The part of the quadratic formula underneath the square root sign is called the **discriminant.**
- Discriminant = b^2-4ac

Discriminant	$b^2-4ac>0$	$b^2-4ac=0$	$b^2-4ac<0$
Nature of roots	The roots are unequal and real. Roots are rational if b^2-4ac is perfect square otherwise they are irrational.	The roots are equal.	The roots are unequal and imaginary.

- A function of the roots of an equation, which remains unaltered when any two of the roots are interchanged is called **Symmetric function** of the roots.
- For finding the equation $ax^2+bx+c=0, (a <> 0)$ when roots are given x^2-(sum of the roots) x + product of the roots = 0 where

  Sum of roots $=-\frac{b}{a}$

  Product of roots $=\frac{c}{a}$
- **Synthetic division** is a method of performing Euclidean division of polynomials with less writing and fewer calculations.
- The advantages of synthetic division are that it allows one to calculate without writing variables, it uses few calculations, and it takes significantly less space on paper than long division. Also, the subtractions in long division are converted to additions by switching the signs at the very beginning, preventing sign errors.
- A system of Linear equations consists of two or more linear equations in the same variables.
- A solution of system of linear equations in two variables is an ordered pair that satisfies each equation in the system.