Unit

12

ANGLE IN A SEGMENT OF A CIRCLE

In this unit the students will be able to

To prove the following theorems along with corollaries and apply them to solve appropriate problems.

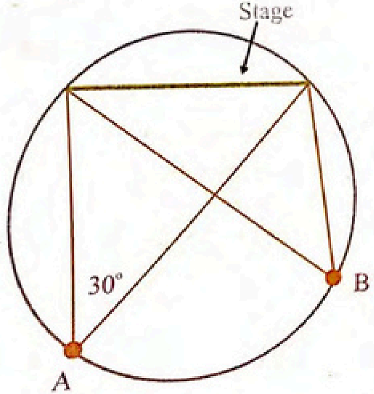
- The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segment of a circle are equal.
- The angle in a semi-circle is a right angle,
- The angle in a segment greater than a semi-circle is less than a right angle,
- The angle in a segment less than a semi-circle is greater than a right angle.
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Why it's important

A person's effective field of vision is about 30°. In the diagram of the amphitheater, a person sitting at point A can see the entire stage. What is the measure of $\angle B$? Can the person sitting at point B view the entire stage?

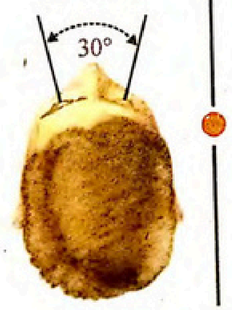
Angles A and B intercept the same arc. By Theorem 2 of this unit, the angles must have the same measure, so $m\angle A = m\angle B = 30°$.

The person sitting at point B can view the entire stage.



APPLICATION

OPTICS



Theorem 12.1

The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.

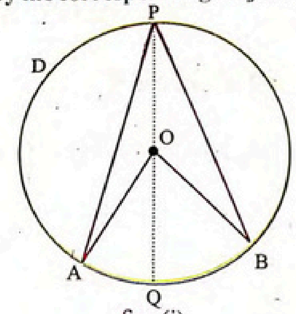


fig: (i)

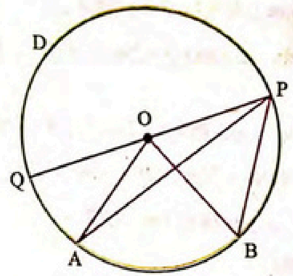


fig: (ii)

Given

A circle with centre O, $\widehat{AB}$ is a minor arc whose central angle is $\angle AOB$.

P is any point on the major arc $\widehat{ADB}$. $\angle APB$ is the angle subtended by the arc at P.

To prove

$m\angle AOB = 2m\angle APB$

Case 1: When P is an arbitrary point of the major arc (General Case) figure (i).

Case 2: When PQ is an extremity of the semi-circle containing the minor arc $\widehat{AB}$ figure (ii).

Construction

Draw $\overrightarrow{PO}$ to meet the circumference at Q (for both cases).

Proof

Statements	
Case (i)	
In the ΔOAP	
$m\overline{OA} = m\overline{OP}$	Radii of the same circle.
$\therefore$ ΔOAP is an isosceles triangle	Definition of an isosceles triangle.
and $m\angle OAP = m\angle OPA$	$\because$ If two sides of a triangle are equal, the angles which are opposite to them are also equal.

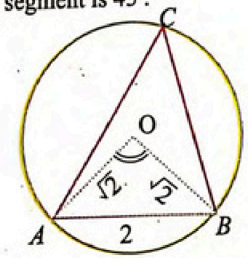
Also $m\angle AOQ = m\angle OAP + m\angle OPA$	$\because$ The measure of exterior angle of a triangle is equal to the sum of the measures of opposite interior angles. $\because$ $m\angle OAP = m\angle OPA$ (Proved above)
$\therefore \quad m\angle AOQ = 2m\angle OPA \quad (I)$ Similarly $m\angle BOQ = 2m\angle OPB \quad (II)$	Adding I and II.
$\therefore$ $m\angle AOQ + m\angle BOQ = 2m\angle OPA + 2m\angle OPB$ or $m\angle AOB = 2[m\angle OPA + m\angle OPB]$ or $m\angle AOB = 2m\angle APB$	Postulate of addition of angles.
Case (ii) All the statements and reasons are same up to obtaining the equations I and II. Subtracting I from II we get $m\angle BOQ - m\angle AOQ = 2m\angle OPB - 2m\angle OPA$ $m\angle BOQ - m\angle AOQ = 2[m\angle OPB - m\angle OPA]$ $\therefore \quad m\angle AOB = 2m\angle APB$	Postulate of subtraction of angles.

Corollary

The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.

Example 1

The radius of a circle is $\sqrt{2}$ cm. A chord 2cm in length divides the circle into two segments. Prove that the angle of larger segment is 45°.



Solution

Given

a circle with centre O and radius $m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm,
length of chord $\overline{AB} = 2$ cm divides the circle into two segments with ACB as
larger one.

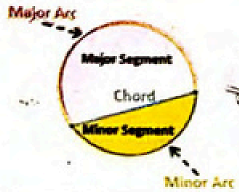
To prove

$m\angle ACB = 45°$

Construction

Join O with A and B.

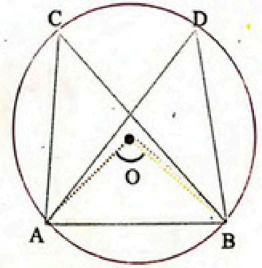
Proof

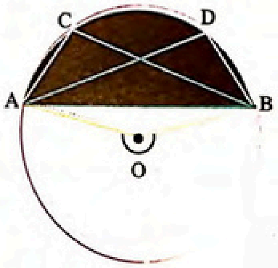


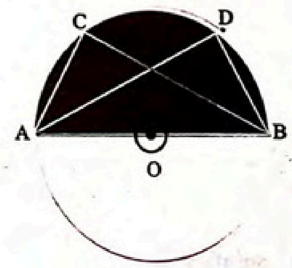
Statements	Reasons
In ΔOAB	
$(OA)^2 + (OB)^2 = (\sqrt{2})^2 + (\sqrt{2})^2$ $= 2 + 2$	$m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm
$= 4 = (2)^2 = (AB)^2$	$\because$ $m\,\overline{AB} = 2$ cm
$\therefore$ ΔOAB is right angled triangle.	
With $m\angle AOB = 90°$	Which being a central angle standing on an arc AB.
$m\angle ACB = \frac{1}{2} m\angle AOB$	By Theorem 12.1
$= \frac{1}{2}(90°) = 45°$	Circum-angle is half of the central angle.

Theorem 12.2

Any two angles in the same segment of a circle are equal.







Given

A circle with centre O. $\angle ACB$ and $\angle ADB$ are any two angles in the same segment ACDB (Shaded) of the circle.

To prove

$$m\angle ACB = m\angle ADB$$

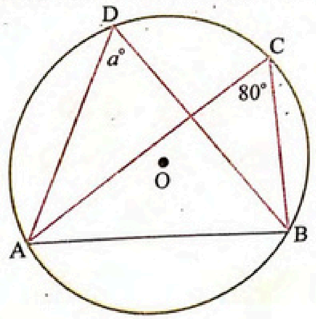
Construction

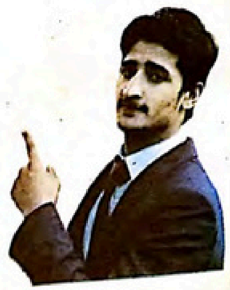
Join O to A and B respectively.

Proof

Statements	Reasons
In all the three figures, we have $m\angle AOB = 2m\angle ACB$ (I)	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
Similarly $m\angle AOB = 2m\angle ADB$ (II)	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$\therefore 2m\angle ACB = 2m\angle ADB$	The RHS of (I) and (II) are equal to the same quantity.
$\therefore m\angle ACB = m\angle ADB$	Halves of equal quantities arc again equal.

Example 2 Find the value of a in the following circle centred at O.

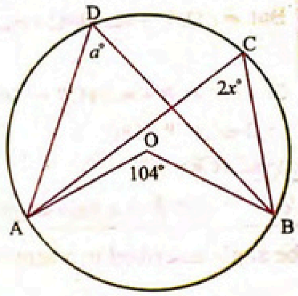




Solution

a = 80 (Angles in the same segment).

Example 3 Find the value of each of the variables in the following circle centred at O.



Solution

$2a = 104$ (Angle at centre theorem)

$\frac{2a}{2} = \frac{104}{2}$

$a = 52$

Also, $2x = a$ (Angles in the same segment)

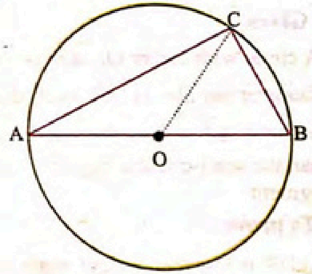
$\therefore 2x = 52$

$\frac{2x}{2} = \frac{52}{2}$

$x = 26$

Theorem 12.3(a)

The angle in a semi-circle is a right angle.



Given

A circle with centre O, $\overline{AB}$ is a diameter of the circle and $\angle ACB$ is any angle in the semi-circle.

To prove

$\angle ACB$ is a right angle i.e. $m\angle ACB = 90^\circ$.

Construction

Join O to C.

Proof

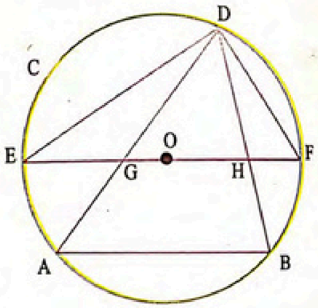
Statements	Reasons
In ΔOAC,	
$m\overline{OA} = m\overline{OC}$	Radii of the same circle
$\therefore \Delta OAC$ is an isosceles triangle	Definition of an isosceles triangle.
and $m\angle OAC = m\angle OCA$ (I)	If two sides of a triangle are equal, the angles which are opposite to them are also equal.
Similarly in the ΔOCB	
$m\overline{OB} = m\overline{OC}$	Radii of a circle.
$\therefore m\angle OBC = m\angle OCB$ (II)	
$\therefore m\angle OAC + m\angle OBC = m\angle OCA + m\angle OCB$	Adding (I) and (II).

$m\angle OAC + m\angle OBC = m\angle ACB$ (*III*)	$\because m\angle OCA + m\angle OCB = m\angle ACB$
But $m\angle OAC + m\angle OBC + m\angle ACB = 180^\circ$	$\because$ The sum of three angles of a triangle is equal to 180°.
	$\because m\angle OAC + m\angle OBC = m\angle ACB$
Or $m\angle ACB + m\angle ACB = 180^\circ$	Adding two equal numbers.
$\Rightarrow 2m\angle ACB = 180^\circ$	
$\therefore m\angle ACB = 90^\circ$	
or $\angle ACB$ is a right-angle.	

The angle inscribed in a semicircle is always a right angle (90°).

Theorem 12.3(b)

The angle in a segment greater than a semi-circle is less than a right angle.



Given

A circle with centre O, $\overline{AB}$ is a chord and $\overline{EF}$ is the diameter parallel to $\overline{AB}$ such that $\overline{AB}$ lies below the diameter $\overline{EF}$. Thus the segment AECDFB is greater than the semi-circular region. $\angle ADB$ is any angle in the segment.

To prove

$\angle ADB$ is less than a right angle i.e. $m\angle ADB < 90^\circ$.

Construction

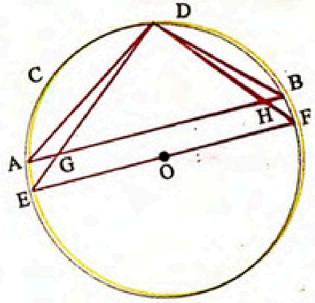
Join D with E and F such that $\angle EDF$ is in the semi-circle.

Proof

	Reasons
$m\angle EDF = 90^\circ$	The measure of an angle in a semi-circle is a right angle.
$\therefore m\angle EDG + m\angle GDH + m\angle HDF = 90^\circ$	$\because m\angle EDG + m\angle GDH + m\angle HDF = m\angle EDF$
$\therefore m\angle GDH = 90^\circ - [m\angle EDG + m\angle HDF]$	
$\Rightarrow m\angle GDH < 90^\circ$	
or $m\angle ADB < 90^\circ$	$\because \angle GDH$ and $\angle ADB$ are same.

Theorem 12.3(c)

The angle in a segment less than a semi-circle is greater than a right angle.



Given

A circle with centre O, $\overline{AB}$ is a chord and $\overline{EF}$ is the diameter parallel to $\overline{AB}$ such that $\overline{AB}$ lies above the diameter $\overline{EF}$. The segment ACDB is less than the semi-circular region. $\angle ADB$ is any angle in the segment.

To prove

$\angle ADB$ is greater than a right angle i.e. $m\angle ADB > 90^\circ$.

Construction

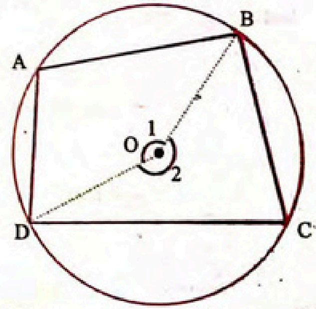
Join D to E and F. $\angle EDF$ is in the semi-circle.

Proof

Statements	Reasons
$m\angle EDF = 90^\circ$	The measure of an angle in a semi -circle is a right angle.
or $m\angle GDH = 90^\circ$	$\because \angle EDF$ and $\angle GDH$ are same.
$m\angle ADG + m\angle GDH + m\angle HDB =$ or $m\angle ADG + 90^\circ + m\angle HDB$	Adding equal quantities on both sides of an equation.
$\Rightarrow m\angle ADB = 90^\circ + m\angle ADG + m\angle HDB$	Postulate of addition of angles.
$\Rightarrow m\angle ADB > 90^\circ$	

Theorem 12.4

The opposite angles of any quadrilateral inscribed in a circle are supplementary.



Given

A circle with centre at O and ABCD is a quadrilateral inscribed in the given circle.

To prove

$m\angle BCD + m\angle BAD = 180^\circ$ and $m\angle ABC + m\angle ADC = 180^\circ$

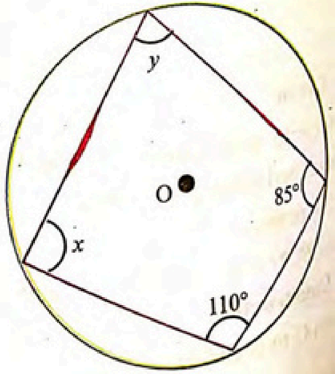
Construction

Join O to B and D.

Proof

Statements	Reasons
$m\angle 1 = 2m\angle BCD$	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$m\angle 2 = 2m\angle BAD$	The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.
$m\angle 1 + m\angle 2 = 2m\angle BCD + 2m\angle BAD$ (I)	Adding the above two results.
But $m\angle 1 + m\angle 2 = 360°$	$\because$ circumference of a circle subtends an angle equal t o four right angles at the centre.
$\therefore$ (I) becomes $2[m\angle BCD + 2m\angle BAD] = 360°$	
$\therefore\ m\angle BCD + m\angle BAD = \dfrac{360°}{2}$	Dividing both sides of an equation by 2.
or $m\angle BCD + m\angle BAD = 180°$ $\therefore \angle BCD$ and $\angle BAD$ are supplementary.	Definition of supplementary angles.
Similarly we can prove that $m\angle ABC$ and $m\angle ADC$ are supplementary.	

Example 4 O is the centre of the circle. Find the unknowns in the figure.

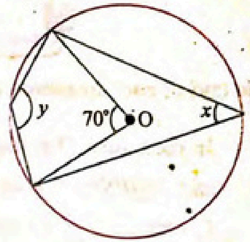


Solution Since

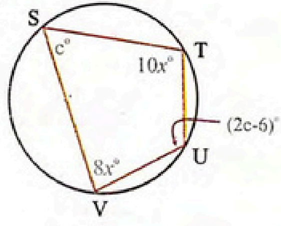
$$x + 85° = 180°$$
$$\therefore x = 180° - 85° = 95°$$
$$y + 110° = 180°$$
$$\therefore y = 180° - 110° = 70°$$

Exercise 12

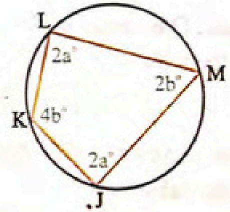
1. O is the centre of the circle.
   Find the unknowns in the figure.



2. A regular hexagon is inscribed in a circle. Each side of the hexagon is $5\sqrt{3}$ units fro centre of the circle. Find the radius of the circle.
3. Find the value of each variable.

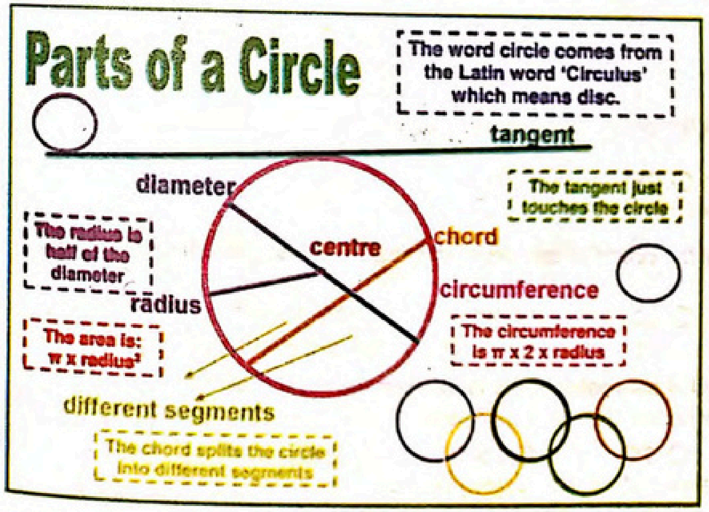


(a)



(b)

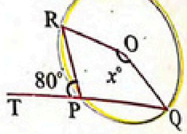
4. Show that a parallelogram inscribed in a circle will be a rectangle.



Review Exercise 12

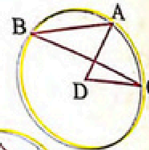
1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). In the figure, O is the centre of the circle, $\angle TPR = 80°$ and $\angle QPS = x°$. Find x.



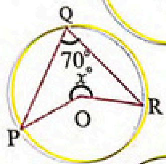
○ 80° ○ 160° ○ 100° ○ 120°

(ii). In the diagram, $\angle ADC$ is a central angle and $m\angle ADC = 60°$. What is $m\angle ABC$?



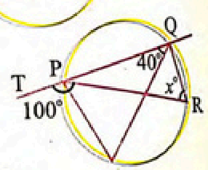
○ 15° ○ 60° ○ 120° ○ 30°

(iii). In the figure, O is centre of the circle and $\angle PQR = 70°$. Calculate the value of x.



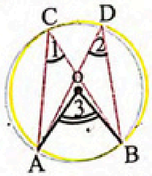
○ 140° ○ 220° ○ 290° ○ 110°

(iv). In the figure, $\angle SPT = 100°$, $\angle PQS = 40°$ and $\angle PRQ = x°$. Calculate the value of x.



○ 30° ○ 40° ○ 50° ○ 60°

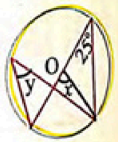
(v). In the adjacent figure if $m\angle 3 = 75°$, then find $m\angle 1$ and $m\angle 2$.



○ $37\frac{1°}{2}$, $37\frac{1°}{2}$ ○ $37\frac{1°}{2}$, 75°

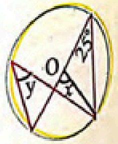
○ 75°, $37\frac{1°}{2}$ ○ 75°, 75°

(vi). Given that O is the centre of the circle. The angle marked x will be:



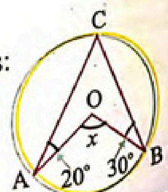
○ $12\frac{1°}{2}$ ○ 25° ○ 50° ○ 75°

(vii). Given that O is the centre of the circle, the angle marked y will be:



○ $12\frac{1°}{2}$ ○ 25° ○ 50° ○ 75°

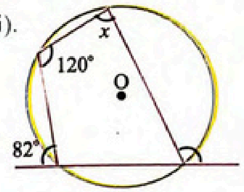
(viii). In the figure, O is the centre of the circle then the angle x is:



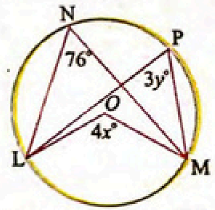
○ 50° ○ 75° ○ 100° ○ 125°

2. O is the centre of the circle. Find the unknowns in the figure.

(i).



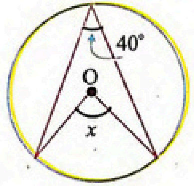
(ii).



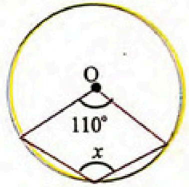
3. ABCD is a quadrilateral circumscribed about a circle. Show that $m\overline{AB} + m\overline{CD} = m\overline{BC} +$ ml

4. Find the value of x in each of the following figures where O is the centre of the circle.

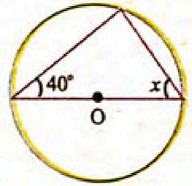
(i).



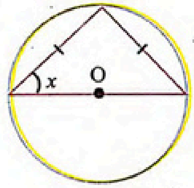
(ii).



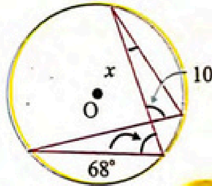
(iii).



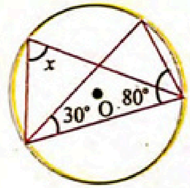
(iv).



(v).



(vi).



Summary

- The inscribed angle is an angle whose vertex is on a circle and whose sides cont chords of the circle.
- The arc that lies in the interior of an inscribed angle and has end points on the angl called the intercepted arc of the angle.
- The measure of an inscribed angle is one half the measure of its intercepted arc.
- If two inscribed angles of a circle intercept the same arc, then the angles are congruent
- A polygon is an inscribed polygon if all its vertices lie on a circle.
- The circle that contain the vertices is a circumscribed circle.
- A quadrilateral with its four vertices lying on the circumference of the circle is calle cyclic quadrilateral.
- A quadrilateral can be inscribed in a circle if and only if the opposite angles are supple
- In a circle:
  - the angle in a semicircle is right
  - the angle in a segment greater than a semicircle is acute
  - the angle in a segment less than a semicircle is obtuse.