Unit 10 TANGENT TO A CIRCLE

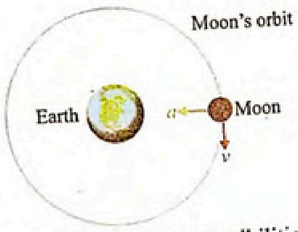
In this unit the students will be able to

To prove the following theorems along with corollaries and apply them to solve appropriate problems.

- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside it, are equal in length.
- If two circles touch externally or internally, the distance between their centres is respectively equal to the sum or difference of their radii.

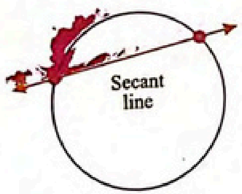
Why it's important

The moment of an object revolving around Earth in a circular orbit is in the direction of a line tangent to the orbit of the object. You will encounter such lines frequently.

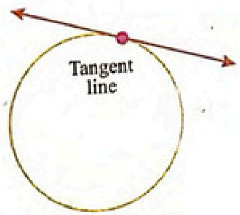


Secants and Tangents

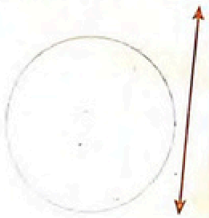
A line in the plane of a circle may or may not intersect the circle. There are three possibilities.



2 points of intersection



1 point of intersection

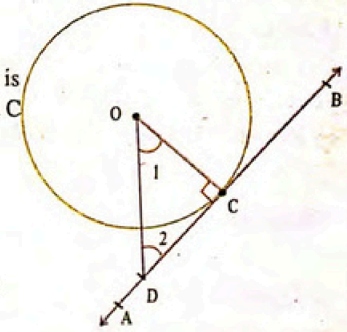


0 points of intersection

A **secant** to a circle is the line that intersects the circle at two distinct points. If a straight line and a circle have only one point of contact then that line is called a **tangent** and the point of intersection is known as point of tangency/contact.

Theorem 10.1

If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.



Given

A circle having centre O with a line $\overleftrightarrow{AB}$ which is perpendicular to the radial segment $\overline{OC}$ at its outer end C

i.e. $\overleftrightarrow{AB} \perp \overline{OC}$.

To prove $\overleftrightarrow{AB}$ is a tangent to the circle.

Construction

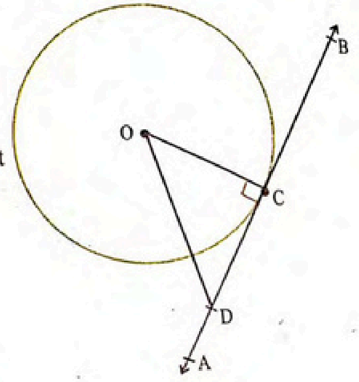
Take any point D on $\overleftrightarrow{AB}$ other than C. Join O to D.

Proof

Statements	Reasons
$\because \overline{OC} \perp \overleftrightarrow{AB}$	Given
So the ΔOCD is a right triangle	$\because m\angle OCD = 90°$
$\therefore \angle 1$ and $\angle 2$ are acute	i.e. $\angle 1 + \angle 2 = 90°$
Thus $m\angle OCD > m\angle 2$	$\because \angle OCD$ is a right angle and $\angle 2$ is an acute angle.
$\therefore m\overline{OD} > m\overline{OC}$ (i)	In a triangle, greater angle has greater side opposite to it.
But $\overline{OC}$ is the radial segment with C as its outer end.	Given
$\therefore$ D lies in the exterior of the circle.	
Hence $\overleftrightarrow{AB}$ meets the circle at one and only one point which is C.	Definition of tangent to a circle.
$\therefore \overleftrightarrow{AB}$ is tangent to the circle.	

Theorem 10.2

The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.



Given

A circle with centre O. $\overleftrightarrow{AB}$ is tangent to the circle at point C. $\overline{OC}$ is the radial segment which is obtained by joining O with the point of contact C of the tangent $\overleftrightarrow{AB}$.

To prove

$\overline{OC} \perp \overleftrightarrow{AB}$

Construction

Take any point D on $\overleftrightarrow{AB}$ except C. Join O and D.

Proof

Statements	Reasons
$\because \overleftrightarrow{AB}$ is tangent to the circle at the point C.	Given
So C is the only point common to the circle and the line $\overleftrightarrow{AB}$.	Definition of tangent to a circle.
D is a point on $\overleftrightarrow{AB}$ other than C	Construction
$\therefore$ D is an exterior point of the circle.	Except C every point of $\overleftrightarrow{AB}$ is outside the circle.
It means, $m\overline{OD} > m\overline{OC}$	
Hence $m\overline{OC}$ is the shortest distance between the point O and the line $\overleftrightarrow{AB}$.	
$\therefore \quad \overrightarrow{OC} \perp \overleftrightarrow{AB}$	By definition of shortest distance.

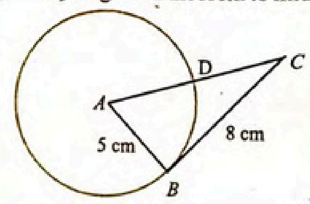
Corollaries

1. At any point on the circumference of a circle one and only one tangent to the circle can be drawn.
2. Perpendicular to a tangent at the point of contact passes through the centre of the circle.

Example 1 $\overline{CB}$ is tangent in the figure to the circle. Find the length of segment CD.

Solution Since CB is tangent to the circle, then $\overline{AB} \perp \overline{CB}$.

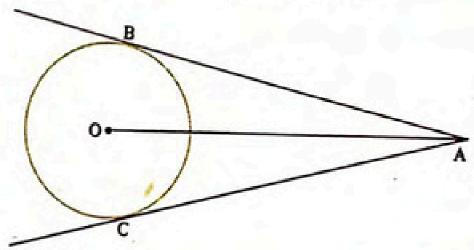
This means that ΔABC is a right triangle and we can apply the Pythagorean theorem to find the length of line segment AC.



$$AC^2 = AB^2 + BC^2$$
$$AC^2 = 25 + 64 = 89$$
$$AC = \sqrt{89} \approx 9.43 \text{ cm.}$$
$$CD = AC - AD = 9.43 \text{ cm} - 5 \text{ cm} = 4.43 \text{ cm.}$$

Theorem 10.3

The two tangents drawn to a circle from a point outside it, are equal in length.



Given

A circle with centre O. A is any point outside the circle. $\overline{AB}$ and $\overline{AC}$ are drawn two tangents from point A.

To prove $m\overline{AB} = m\overline{AC}$

Construction

Join O to A, B and C.

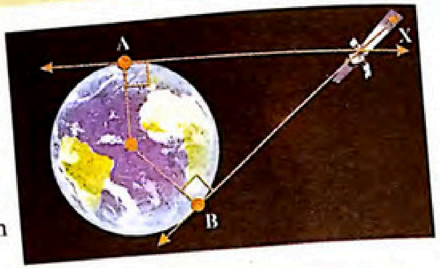
Proof

Statements	Reasons
In $\Delta AOB \leftrightarrow \Delta AOC$	
$\overline{AO} \cong \overline{AO}$	Common
$\overline{OB} \cong \overline{OC}$	Radial segments.
$\angle ABO \cong \angle ACO = 90^\circ$	Right angles because tangent to a circle is perpendicular to the radial segment at the point of contact.
$\therefore \quad \Delta AOB \cong \Delta AOC$	H.S $\cong$ H.S
$\therefore \quad \overline{AB} \cong \overline{AC}$	Corresponding sides of two congruent triangles.
Or $\quad m\overline{AB} = m\overline{AC}$	

Corollary

The two tangents drawn to a circle from an external point subtend equal angles at the centre.

Example 2
Global positioning satellites are used in navigation. If the range of the satellite. AX is 16,000 miles. What is the range of BX?



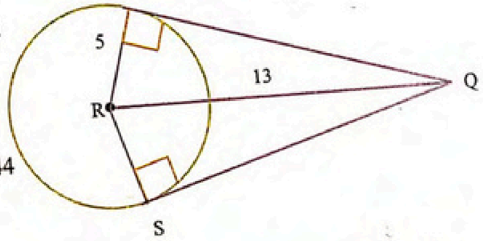
Solution
$\overline{AX}$ and $\overline{BX}$ and are tangents to a circle from the same external point, so they are equal.

$$AX = BX = 16,000 \text{ miles}$$

Example 3 QT and QS are tangent segments to circle R. $TR = 5$. And $RQ = 13$. Find QT and QS.

Solution
Since QRT is a right triangle.

$$\therefore (QR)^2 = (QT)^2 + (TR)^2$$
$$(13)^2 = (QT)^2 + 5^2$$
$$169 = (QT)^2 + 25$$
$$(QT)^2 = 169 - 25 = 144$$
$$QT = 12$$
$$QT = QS = 12$$

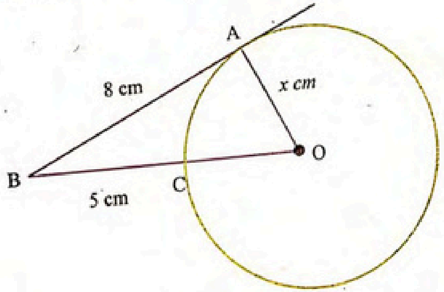


Example 4 In the figure $\overline{AB}$ is a tangent to the circle, with centre O. Given that AB = 8cm, BC = 5cm and OA = x cm, find the value of x, and ∠AOB.

Solution

$$\angle AOB = 90^\circ$$
$$OB = (x+5)\text{cm}$$

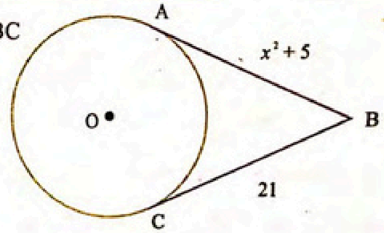
Now
$$(x+5)^2 = x^2 + 8^2$$
$$x^2 + 10x + 25 = x^2 + 64$$
$$10x = 64 - 25$$
$$10x = 39$$
$$x = 3.9$$



In ΔAOB, $\tan \angle AOB = \dfrac{8}{3.9}$

Therefore $\angle AOB = 64.01^\circ$

Example 5 In the adjacent figure, AB and BC are two tangents. Evaluate x.

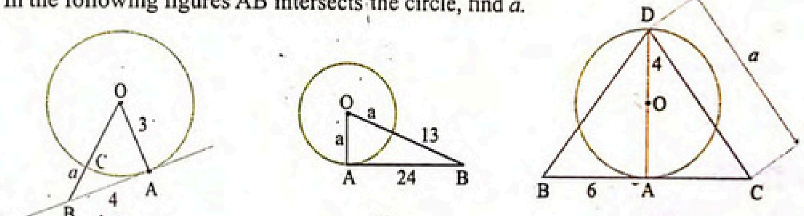


Solution We know that tangent segments from an external point are equal.

So, AB = BC
$$x^2 + 5 = 21$$
$$x^2 = 21 - 5$$
$$x^2 = 16$$
$$x = 4$$

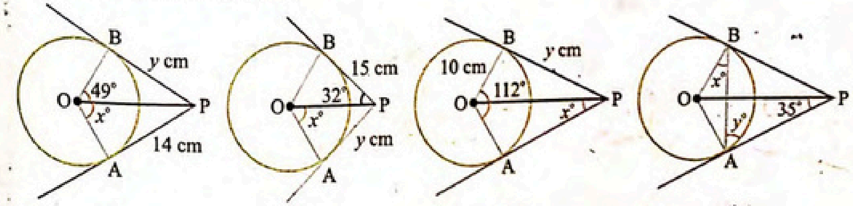
Exercise 10.1

1. Find the length of a tangent drawn to a circle of radius 6 cm from a point at a distance of 10 cm from the centre of the circle.
2. Find the radius of a circle if the length of the tangent drawn is 8 cm and distance from the centre of the circle to the point outside the circle is 9 cm.
3. A chord $\overline{AC}$ of a circle is produced to P. From P a tangent $\overline{PB}$ to the circle is drawn. Prove that $m\angle PBC = m\angle BAP$.
4. In the following figures $\overline{AB}$ intersects the circle, find a.



(i) (ii) (iii)

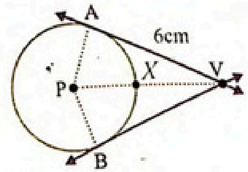
5. If the radius of a circle is 8 cm. Tangents drawn from an external point P make an angle of 60°. Find the distance between the centre of the circle and the point P.
6. Given that $\overline{PA}$ and $\overline{PB}$ are tangents to the circle, with centre O, find the values of x and y in each of the following figure.



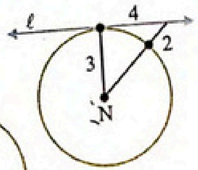
(i) (iii) (iv)

. In the figure below, $\overline{VA}$ and $\overline{VB}$ are tangent to P, the radius of P is 3cm, and VA = 6 cm. Find the following.

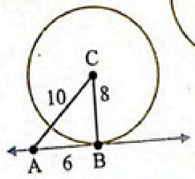
(i). VB (ii). AP (iii). PV (iv). XV



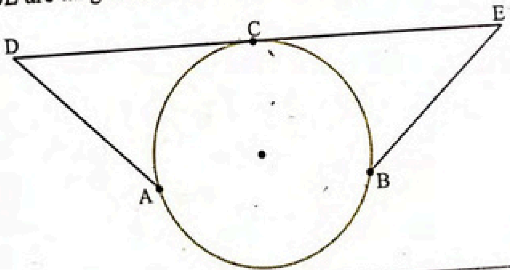
8. In circle N. verify that line ℓ is a tangent.



9. Verify that $\overline{AB}$ is tangent to C at B.



10. If AD, DE and BE are tangents to a circle as shown prove that $AD + BE = DE$.



Note

Two circles that intersect in one point are called tangent circles.

Two circles that have a common centre are called concentric circles.

Theorem 10.4(a)

If two circles touch externally, the distance between their centres is equal to the sum of their radii.

Given

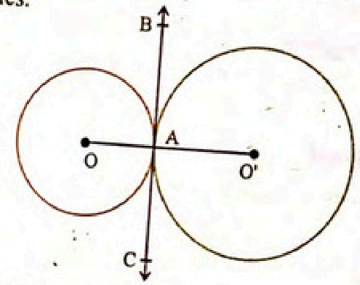
Two circles with centres O and O' which are touching each other externally at the point A, $\overline{OA}$ and $\overline{O'A}$ are the radial segments of the circles.

To prove

$$m\overline{OO'} = m\overline{OA} + m\overline{AO'}$$

Construction

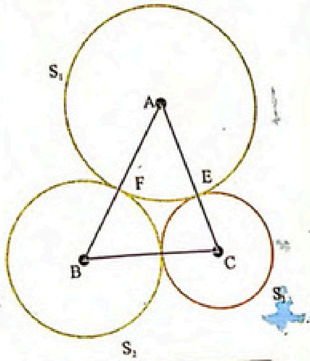
Draw a common tangent $\overleftrightarrow{BC}$ at the point A which is the common point of contact of the given two circles.



Proof

Statements	Reasons
$m\angle BAO = 90°$ (i)	$\because$ A tangent to a circle is perpendicular to the radial segment at the point of contact.
Similarly $m\angle BAO' = 90°$ (ii)	$\because$ A tangent to a circle is perpendicular to the radial segment at the point of contact.
$\because m\angle BAO + m\angle BAO' = 90° + 90° = 180°$	Adding (i) and (ii).
$m\angle BAO$ and $m\angle BAO'$ are supplementary angles with the common vertex A.	$\because$ Their sum is 180°.
$\therefore \overrightarrow{AO}$ and $\overrightarrow{AO'}$ are opposite rays.	Postulate of supplementary angles.
This makes O, A and O', the three different collinear points.	Consequence of being $\overrightarrow{AO}$ and $\overrightarrow{AO'}$, the opposite rays
$\therefore m\overline{OO'} = m\overline{OA} + m\overline{AO'}$	segments addition postulate.

Example 6 Three circles touch in pairs externally. Prove that the perimeter of a triangle formed by joining centres is equal to the sum of their diameters.



Given

Three circles have centres A, B and C their radii are r_1, r_2, and r_3 respectively. They touch in pairs externally at D, E and F. So that ΔABC is formed by joining the centres of these circles.

To prove

Perimeter of ΔABC = Sum of the diameters of these circles

Proof

Statements	Reasons
Three circles with centres A, B and C touch in pairs externally at the points, D, E and F.	Given
$\therefore \quad m\overline{AB} = m\overline{AF} + m\overline{FB}$ (i)	
$m\overline{BC} = m\overline{BD} + m\overline{DC}$ (ii)	
and $m\overline{CA} = m\overline{CE} + m\overline{EA}$ (iii)	
$m\overline{AB} + m\overline{BC} + m\overline{CA} = m\overline{AF} + m\overline{FB} + m\overline{BD} + m\overline{DC} + m\overline{CE} + m\overline{EA}$ $= (m\overline{AF} + m\overline{EA}) + (m\overline{FB} + m\overline{BD}) + (m\overline{CD} + m\overline{CE})$	Adding (i), (ii) and (iii).
Perimeter of $\Delta ABC = 2r_1 + 2r_2 + 2r_3$ $= d_1 + d_2 + d_3$ = Sum of diameters of the circles.	$d_1 = 2r_1$, $d_2 = 2r_2$ and $d_3 = 2r_3$ are diameters of the circles.

Math Fun

Using only addition, how do you add eight 8's and get the number 1000?

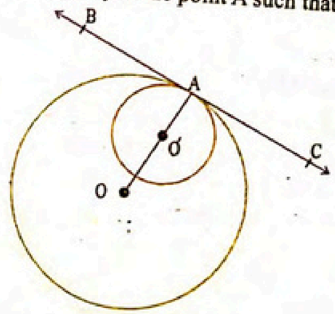
Answer: 888+88+8+8+8 = 1000.

Theorem 10.4(b)

If two circles touch internally, the distance between their centres is the difference of their radii.

Given

Two circles with centre O and O' touch each other internally at the point A such that $\overline{OA}$ and $\overline{O'A}$ are radial segments of these circles.



To prove

$$m\overline{OO'} = m\overline{OA} - m\overline{O'A}$$

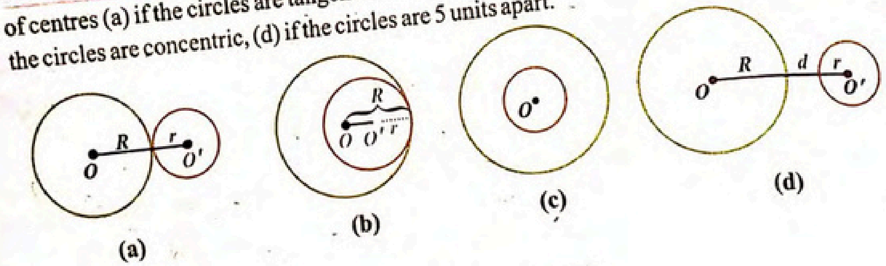
Construction

Draw a common tangent $\overleftrightarrow{BC}$ at the point A which is the common point of contact of the two circles.

Proof

Statements	Reasons
Since the circles touch each other at A therefore both $\overline{OA}$ and $\overline{O'A}$ are perpendicular to $\overline{BC}$.	$\because$ A tangent to a circle is perpendicular to the radius drawn through the point of contact.
$\therefore \quad m\angle BAO = m\angle BAO' = 90°$	
i.e. $\overline{OA} \perp \overleftrightarrow{BC}$ at A.	
and $\overline{O'A} \perp \overleftrightarrow{BC}$ at A.	
$\therefore$ O, O' and A lie on the same straight line.	At a point on a line a unique perpendicular can be drawn on it.
$\therefore \quad m\overline{OA} = m\overline{OO'} + m\overline{O'A}$	Segment addition postulate.
or $m\overline{OO'} = m\overline{OA} - m\overline{O'A}$	From a law of equation.

Example 7 Two circles have radii of 9 and 4, respectively. Find the length of their line of centres (a) if the circles are tangent externally, (b) if the circles are tangent internally, (c) if the circles are concentric, (d) if the circles are 5 units apart.



Solution

Let R = radius of larger circle, r = radius of smaller circle.

(a) Since $R = 9$ and $r = 4$, $OO' = R + r = 9 + 4 = 13$.

(b) Since $R = 9$ and $r = 4$, $OO' = R - r = 9 - 4 = 5$.

(c) Since the circles have the same centre, their line of centres has zero length.

(d) Since $R = 9$ and $r = 4$, and $d = 5$, $OO' = R + d + r = 9 + 5 + 4 = 18$.

Did You Know?

$$32 = 2^{(3+2)}$$

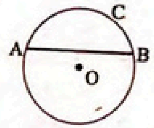
Exercise 10.2

1. If two circles with radii 8 cm and 3 cm respectively touch each other externally, then what is the distance between their centres?
2. The distance between the centres of two circles touching internally is 5 cm. The radius of the bigger circle is 17 cm. What is the radius of the smaller circle?
3. In a circle a 10 cm long chord is at a distance of 12 cm from the centre. What is the length of a chord at a distance of 5 cm from the centre?
4. A chord is 18 cm long. The radius of the circle is 15 cm. What is the distance of the midpoint of the chord from the centre of the circle?
5. What is the length of a chord at distance of 6cm from the centre of a circle of radius 10 cm?
6. The distance of a chord from the centre of a circle is 3 cm, and the length of the chord is 8 cm. What is the diameter?
7. The radius of a circle is 8 cm and one of the chords is of 12 cm. What is the distance of the chord from the centre?
8. The length of a chord of a circle of radius 7.5 cm is 9 cm. What is its distance from the centre of the circle?

Review Exercise 10

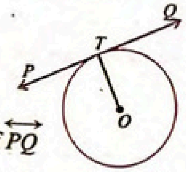
1. At the end of each question, four circles are given. Fill in the correct circle only.

   (i) In the figure, $\overset{\frown}{ACB}$ is called

   - ○ an arc
   - ○ a chord
   - ○ a secant
   - ○ a diameter

   (ii) In a circle with centre O, if $\overline{OT}$ is the radial segment and $\overleftrightarrow{PTQ}$ is the tangent line, then

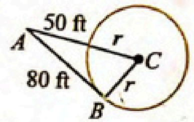
   

   - ○ $\overline{OT} \perp \overleftrightarrow{PQ}$
   - ○ $\overline{OT} \parallel \overleftrightarrow{PQ}$
   - ○ $\overline{OT} \not\perp \overleftrightarrow{PQ}$
   - ○ $\overline{OT}$ is right bisector of $\overleftrightarrow{PQ}$

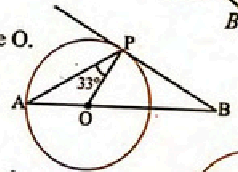
   (iii) Two tangents drawn to a circle from a point outside it are of.......... in length.

   - ○ half
   - ○ equal
   - ○ double
   - ○ triple

2. In the diagram, B is a point of tangency. Find the radius r of ⊙ C.

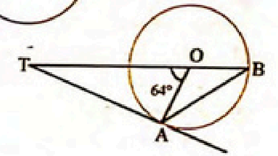


3. In the figure, $\overline{BP}$ is a tangent to the circle with centre O. Given that $\angle APO = 33°$, find $\angle PBA$.



4. In the figure, O is the centre of the circle passing through the points A and B. $\overline{TA}$ is a tangent to the circle at A and TOB is a straight line.

   Given that $\angle AOT = 64°$, Find (i) $\angle ATB$, (ii) $\angle TAB$



5. Given that $\overline{PA}$ and $\overline{PB}$ are tangents to each of the following circles with centre O, find the values of the unknowns.