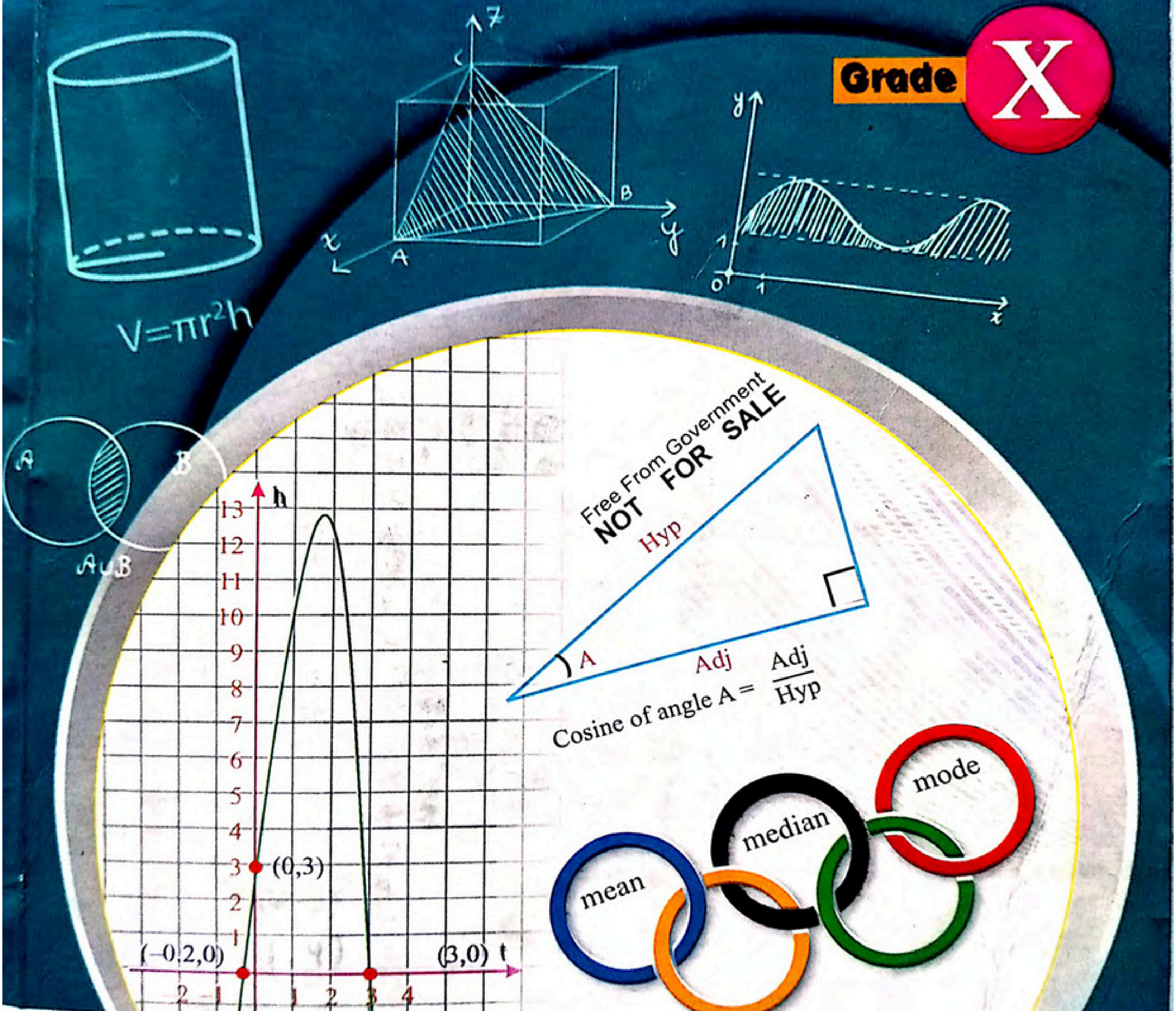
A TEXTBOOK OF

Mathematics

Grade X



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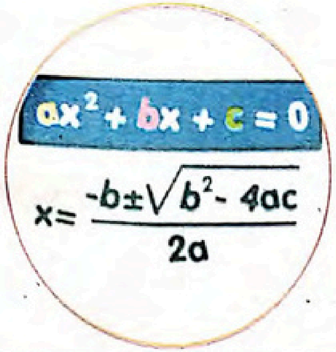
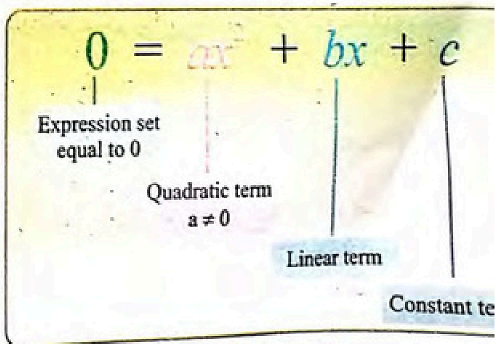
Unit 1

QUADRATIC EQUATIONS

In this unit the students will be able to

- Define a quadratic equation.
- Solve a quadratic equation in one variable by
  - Factorization
  - Completing square method
- Use method of completing square to derive the quadratic formula.
- Use the quadratic formula to solve quadratic equations.
- Solve equations, reducible to quadratic form, of the type: $ax^4+bx^2+c=0$.
- Solve the equations of the type: $a\,p(x)+\dfrac{b}{p(x)}=c$.
- Solve reciprocal equations of the type $a\left(x^2+\dfrac{1}{x^2}\right)+b\left(x+\dfrac{1}{x}\right)+c=0$.
- Solve exponential equations in which the variables occur in exponents.
- Solve equations of the type $(x+a)(x+b)(x+c)(x+d)=k$, where $a+b=c+d$.
- Solve equations of the type:
  - $\sqrt{ax+b}=cx+d$.
  - $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$.
  - $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$.





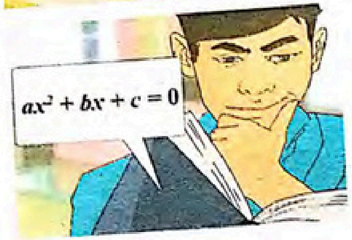
Why it's important

Balls, Arrows, Missiles and Stones

When you throw a ball (or shoot an arrow, fire a missile or throw a stone) it goes up into the air, slowing as it travels, then comes down again faster and faster ...

... and a Quadratic Equation tells you its position at all times!

Quadratic Equations are useful in many other areas:
For a parabolic mirror, a reflecting telescope or a satellite dish, the shape is defined by a quadratic equation.



Quadratic equations are also needed when studying lenses and curved mirrors.
Many questions involving time, distance and speed need quadratic equations.

1.1 Quadratic equation

A quadratic equation in one variable is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c, are real numbers and $a \neq 0$.

$$5x^2 - 3x + 3 = 0$$

this makes it Quadratic

Tidbit
The name Quadratic comes from "quad" meaning square because the highest power of the variable is 2.

Examples of quadratic equations include
$x^2 + x + 1 = 0$, $2x^2 - 3 = 0$, $x^2 + 2x = 0$, and $x^2 = 3x + 1$.

1.2 Solution of quadratic equations

All those values of the variable for which the given equation is true are called solution or roots of the equation, and the set of all solutions is called solution set.

For example, the quadratic equation $x^2 - 9 = 0$ is true only for $x = 3$ and $x = -3$, hence $x = 3$ and $x = -3$ are the solutions or roots of the quadratic equation $x^2 - 9 = 0$ and $\{3, -3\}$ is the solution set.

We may solve quadratic equations by the following methods.

(a) by factorization
(b) by completing the square
(c) by quadratic formula

(a) Solution of a quadratic equation by factorization to its

A quadratic equation can easily be solved by splitting it in factors. The factorization method is illustrated in the following examples:

Example 1 Solving a quadratic equation with factorization.

Solve each quadratic equation by factorization. Check your results

(i) $2x^2 + 2x - 11 = 1$ (ii) $12t^2 = t + 1$

Solution

(i) Start by writing the equation in the form $ax^2 + bx + c = 0$.

	$2x^2 + 2x - 11 = 1$		given equation
	$2x^2 + 2x - 12 = 0$		subtract 1 from each side
	$x^2 + x - 6 = 0$		divide each side by 2 (optional step)
	$(x + 3)(x - 2) = 0$		factor
$x + 3 = 0$	or	$x - 2 = 0$	zero-product property
$x = -3$	or	$x = 2$	solve

These solution can be checked by substituting them in the given equation.

$2(-3)^2 + 2(-3) - 11 = 1$ $2(2)^2 + 2(2) - 11 = 1$
$1 = 1$ (true) $1 = 1$ (true)

(ii) Write the equation in the form $at^2 + bt + c = 0$.

	$12t^2 = t + 1$		given equation
	$12t^2 - t - 1 = 0$		subtract t and 1
	$(3t - 1)(4t + 1) = 0$		factorize
$3t - 1 = 0$	or	$4t + 1 = 0$	zero-product property
$t = \frac{1}{3}$	or	$t = \frac{-1}{4}$	solve

To check these solutions, substitute them into the given equation.

$$12\left(\frac{1}{3}\right)^2 = \frac{1}{3} + 1 \qquad 12\left(-\frac{1}{4}\right)^2 = -\frac{1}{4} + 1$$

$$\frac{4}{3} = \frac{4}{3} \text{ (true)} \qquad \frac{3}{4} = \frac{3}{4} \text{ (true)}$$

Try This
Solve the following equations
(i). $a^2 + 7a - 8 = 0$
(ii). $6p^2 + 7p - 20 = 0$

A ball is thrown straight up, from 3 m above the ground with a velocity of 14 m/s. When does it hit the ground?

Solution

We can work out its height by adding up these three things:

The height starts at 3 m: 3

It travels upwards at 14 meters per second (14 m/s): 14t

Gravity pulls it down, changing its position by *about* 5 m per second squared: ...$-5t^2$

Add them up and the height h at any time t is:

Note t is time in seconds

$$h = 3 + 14t - 5t^2$$

And the ball will hit the ground when the height is zero:

$$3 + 14t - 5t^2 = 0$$

Or $\quad 5t^2 - 14t - 3 = 0.$

Which is a quadratic equation.

Let us solve it ...

Replace the middle term with -15 and 1

i.e $\quad 5t^2 - 15t + t - 3 = 0$

$$5t(t-3) + 1(t-3) = 0$$

$$(5t+1)(t-3) = 0$$

And the two solutions are:

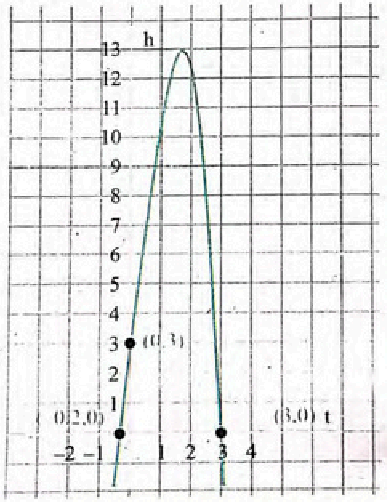
$$5t + 1 = 0 \text{ or } t - 3 = 0$$

$$t = -0.2 \text{ or } t = 3$$

The "$t = -0.2$" is a negative time which is impossible in our case.

The "$t = 3$" is the answer we want:

The ball hits the ground after 3 seconds!



Note

(0,3) says when $t = 0$ (at the start) the ball is at 3 m.

(3,0) says that at 3 seconds the ball is at ground level.

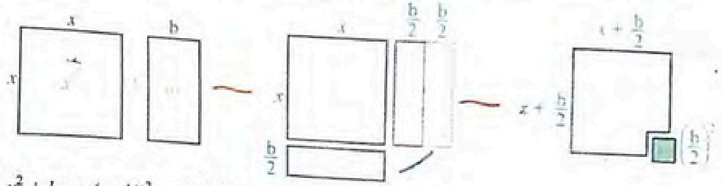
Study Tip

The examples in the text are carefully chosen to prepare you for success with the exercise sets. Study the step-by-step solutions of the examples, noting the substitutions and explanations. The time you spend studying the examples will save your valuable time when you do your homework.

(b) Solution of a quadratic equation by completing square

Some quadratic equations are not easily factorized and hence it is not easy to find their solution by factorization.

For example, the quadratic equation $x^2 + bx = -c$ cannot be easily factorized. Such quadratic equation can be solved by completing the square method



$x^2 + bx = (x + ½)^2 - (½ b)^2$

$$\left(x + \frac{1}{2}b\right)^2 = \left(\frac{1}{2}b\right)^2 - c.$$

$$x + \frac{1}{2}b = \pm\sqrt{\left(\frac{1}{2}b\right)^2 - c}$$

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{2}b\right)^2 - c}$$

Try This

What must be added to obtain a perfect square? (i). $x^2 + 5x$ (ii). $q^2 - 4q$

Steps involved in completing the square

(i). Write the quadratic equation in its standard form.

(ii). Divide both sides of the equation by the co-efficient of x^2 if it is other than 1.

(iii). Shift the constant term to the right-hand side of the equation.

(iv). Square half the co-efficient of x and add the square to both sides.

(v). Write the left-hand side of the equation as a perfect square and simplify the right-hand side.

(vi). Take square root of both sides of the equation and solve the resulting equation to find the solutions of the equation.

Example 3 Completing the square.

Solve each equation

$x^2 - 8x + 9 = 0$

Solution

Start by writing the equation in the form $x^2 + kx = d$.

$x^2 - 8x + 9 = 0$	given equation
$x^2 - 8x = -9$	subtract 9 from each side
$x^2 - 8x + 16 = -9 + 16$	add $\left(\frac{x}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = 16$
$(x-4)^2 = 7$	the perfect square
$x - 4 = \pm\sqrt{7}$	square root property
$x = 4 \pm \sqrt{7}$	add 4 to each side

Avoid Errors

When completing the square to solve an equation make sure that

- the coefficient of x^2 is 1
- you add the term $(b/2)^2$ to both sides of the equation.

1.3 Quadratic formula

The general form of quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and "a" is not equal to zero. By using completing the square method we can derive the quadratic formula for the solution of all quadratic equations.

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$ shift the constant term to the right of the equation

$x^2 + \frac{b}{a}x = -\frac{c}{a}$ divide all terms of both sides by a.

Add $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ to both sides of the equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Which is the required quadratic formula.

Example 4 Using the quadratic formula.

Solve the equation $3x^2 - 6x + 2 = 0$

Solution

Let a = 3, b = −6 and c = 2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \quad \text{substitute for a, b and c}$$

$$x = \frac{6 \pm \sqrt{12}}{6} \quad \text{simplify}$$

$$x = 1 \pm \frac{1}{\sqrt{3}}$$

Activity

Use an online calculator To solve the following quadratic equations step by step

$2x^2 - 9x + 4 = 0$

Try This

Solve

(i). $4x^2 + 3x - 2 = 0$

(ii). $9x^2 - 42x + 49 = 0$

(iii). $5x^2 - 10x + 13 = 0$

Example 5

A company is making frames as part of a new product they are launching. The frame will be cut out of a piece of steel. To keep the weight down, the final area should be 28 cm². The inside of the frame has to be 11 cm by 6 cm. What should the width x of the metal be?

Solution Area of steel before cutting:

Area = $(11 + 2x) \times (6 + 2x)$ cm² $= 4x^2 + 34x + 66$cm²

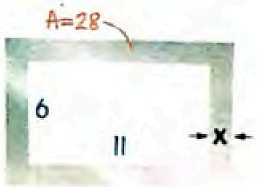
Area of steel after cutting out the 11 × 6 middle:

Area = $4x^2 + 34x + 66 - 66 = 4x^2 + 34x$

Since the area equals 28 cm².

$4x^2 + 34x = 28.$

Or $2x^2 + 17x - 14 = 0$. Here a = 2, b = 17 and c = −14



$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$

$$x = \frac{-(17) \pm \sqrt{(17)^2 - 4(2)(-14)}}{2(2)} \quad \text{substitute the values}$$

$x = \frac{-b \pm \sqrt{b^2 - 4...}}{2a}$

$$x = \frac{-17 \pm \sqrt{401}}{4} \quad \text{simplify}$$

x is about −9.3 or 0.8

The negative value of x make no sense, so the answer is:

x = 0.8 cm (approx.)

Exercise 1.1

1. Solve each of the following equations by factorization.
   - (i). $x^2+5x+4=0$ (ii). $(x-3)^2=4$
   - (iii). $x^2+3x-10=0$ (iv). $6x^2-13x+5=0$
   - (v). $3(x^2-1)=4(x+1)$ (vi). $x(3x-5)=(x-6)(x-7)$
2. Solve each of the following equations by completing the square.
   - (i). $x^2+6x-40=0$ (ii). $x^2-10x+11=0$
   - (iii). $4x^2+12x=0$ (iv). $5x^2-10x-840=0$
   - (v). $9x^2-6x+\frac{5}{9}=0$ (vi). $(x-1)(x+3)=5(x+2)-3$
3. Solve each of the following equations by quadratic formula.
   - (i). $x^2-8x+15=0$ (ii). $x^2-2x-4=0$
   - (iii). $4x^2+3x=0$ (iv). $3x(x-2)+1=0$
   - (v). $6x^2-17x+12=0$ (vi). $\frac{x^2}{3}-\frac{x}{12}=\frac{1}{24}$
4. Find all solutions to the following equations
   - (i). $t^2-8t+7=0$ (ii). $72+6x=x^2$
   - (iii). $r^2+4r+1=0$ (iv). $x(x+10)=10(-10-x)$
5. The equation $(y+13)(y+a)$ has no linear term. Find value of a.
6. The equation $ax^2+5x=3$ has $x=1$ as a solution. What is the other solution?
7. What is the positive difference of the roots of $x^2-7x-9=0$?

Activity

Verify the answers of Exercise 1.1 by using an online calculator.

Activity

Solve the following quadratic equation $x^2-8x+15=0$, by

(i) Factorization (ii) Completing square

Verify the answer by using an online calculator.

1.4 Solution of equations reducible to quadratic form

The following types of equations can be reduced to the quadratic form and can be solved by any of the method learnt above.

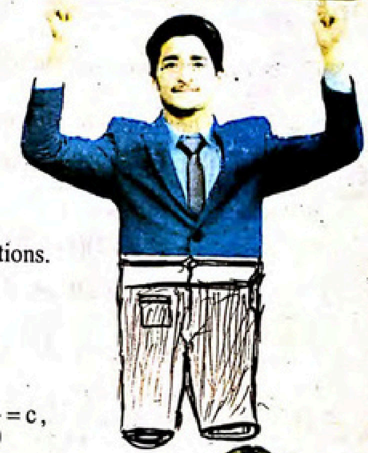
Type Equation of the form $ax^4+bx^2+c=0$.

Example 6 Solve $12x^4-11x^2+2=0$

Solution By making substitution $y=x^2$ the equation becomes $12y^2-11y+2=0$, which is a quadratic equation in terms of y and can be solved by factorization,

$$(3y-2)(4y-1)=0$$
$$3y-2=0 \text{ or } 4y-1=0$$
$$\Rightarrow \quad y=\frac{2}{3} \text{ or } y=\frac{1}{4}$$

Polynomial of degree four is called biquadratic.



To find x, use the fact that $y=x^2$, therefore,

$$x^2=\frac{2}{3} \quad or \quad x^2=\frac{1}{4}$$
$$\Rightarrow \quad x=\pm\sqrt{\frac{2}{3}} \quad or \quad x=\pm\sqrt{\frac{1}{4}}$$
$$\Rightarrow \quad x=\pm\sqrt{\frac{2}{3}} \quad or \quad x=\pm\frac{1}{2}$$

The given equation $12x^4-11x^2+2=0$ has four solutions.

Thus the solution set is $\left\{\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{2}\right\}$.

Type Equation of the form $a\,p(x)+\frac{b}{p(x)}=c$,

Example 7 Solve $2x+\frac{4}{x}=9$

Solution

$$2x+\frac{4}{x}=9$$

Multiplying both sides by x, we get

$$x\left(2x+\frac{4}{x}\right)=9x$$
$$\Rightarrow \quad 2x^2+4=9x$$

$$\Rightarrow \quad 2x^2-9x+4=0$$

which is a quadratic equation in x and can be solved by factorization

$$(2x-1)(x-4)=0$$

$$\Rightarrow \quad x=\frac{1}{2} \text{ or } x=4$$

Hence the solutions are $\frac{1}{2}, 4$ and the solution set is $\left\{\frac{1}{2}, 4\right\}$.

Example 8 Solve $\frac{x-1}{x+3}+\frac{x+3}{x-1}=\frac{13}{6}$

WARNING

When you perform a substitution of variable, you must remember to go back and to express the answers in terms of the original variable

Solution

$$\frac{x-1}{x+3}+\frac{x+3}{x-1}=\frac{13}{6}$$

Let $\quad y=\frac{x-1}{x+3}$. Then $\frac{1}{y}=\frac{x+3}{x-1}$

Therefore, the given equation reduces to $y+\frac{1}{y}=\frac{13}{6}$.

Multiplying both sides by $6y$, we get

$6y^2-13y+6=0$, which is a quadratic equation in terms of x and can be solved by factoring as follows:

$$(2y-3)(3y-2)=0$$

$$\Rightarrow \quad 2y-3=0 \text{ or } 3y-2=0$$

$$\Rightarrow \quad y=\frac{3}{2} \quad \text{or} \quad y=\frac{2}{3}$$

If $\quad y=\frac{3}{2}$, then $\frac{x-1}{x+3}=\frac{3}{2}$

$$\Rightarrow \quad 2x-2=3x+9$$

$$\Rightarrow \quad x=-11$$

If $\quad x=\frac{2}{3}$, then $\frac{x-1}{x+3}=\frac{2}{3}$

$$\Rightarrow \quad 3x-3=2x+6$$

$$\Rightarrow \quad x=9$$

Thus the solution set is $\{-11,9\}$.

Type III Reciprocal equation of the form: $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$

Example 9 Solve the following equations.

(i) $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$ **(ii)** $8\left(x^2+\frac{1}{x^2}\right)-42\left(x-\frac{1}{x}\right)+29=0$

Solution

(i) The given equation is $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$

Let $x+\frac{1}{x}=y$, square both sides $x^2+\frac{1}{x^2}+2=y^2$ or $x^2+\frac{1}{x^2}=y^2-2$

Therefore, the given equation reduces as follows:

$$2(y^2-2)-9y+14=0$$

$$\Rightarrow \quad 2y^2-4-9y+14=0$$

$$\text{or} \quad 2y^2-9y+10=0$$

The equation can be factorized

$$\therefore \quad (2y-5)(y-2)=0$$

$$\therefore \quad y=\frac{5}{2} \quad \text{or} \quad y=2$$

Now $y=\frac{5}{2}$

$$\Rightarrow \quad x+\frac{1}{x}=\frac{5}{2}$$

$$\Rightarrow \quad 2x^2+2=5x$$

$$\text{or} \quad 2x^2-4x-x+2=0$$

$$\Rightarrow \quad 2x^2-4x-x+2=0$$

$$\Rightarrow \quad 2x(x-2)-1(x-2)=0$$

$$\Rightarrow \quad (x-2)(2x-1)=0$$

and $\quad y=2$

$$\Rightarrow \quad x+\frac{1}{x}=2$$

$$\Rightarrow \quad x^2+1=2x$$

$$\Rightarrow \quad x^2-2x+1=0$$

$$\Rightarrow \quad (x-1)^2=0$$

$$\therefore \quad x=1,1$$

Given $\quad x=2$ or $x=\frac{1}{2}$

Hence the solutions are $2, \frac{1}{2}, 1$ and 1. The solution set is $\left\{2, \frac{1}{2}, 1\right\}$.

(ii) The given equation is $8\left(x^2+\frac{1}{x^2}\right)-42\left(x-\frac{1}{x}\right)+29=0$

Let $x-\frac{1}{x}=y$. Then $\left(x-\frac{1}{x}\right)^2=y^2 \Rightarrow x^2+\frac{1}{x^2}-2=y^2$

$$\Rightarrow x^2+\frac{1}{x^2}=y^2+2.$$

Therefore, the given equation reduces to

$$8(y^2+2)-42y+29=0$$
$$8y^2+16-42y+29=0$$
$$\Rightarrow \quad 8y^2-42y+45=0$$
$$\text{or} \quad 8y^2-42y+30y+45=0$$
$$\Rightarrow \quad 4y(2y-3)-15(2y-3)=0$$
$$\Rightarrow \quad (2y-3)(4y-15)=0$$
$$\Rightarrow \quad 2y-3=0 \text{ or } 4y-15=0$$

give $y=\frac{3}{2}$ *or* $y=\frac{15}{4}$

Now $y=\frac{3}{2}$

$$\Rightarrow \quad x-\frac{1}{x}=\frac{3}{2}$$
$$\therefore \quad \frac{x^2-1}{x}=\frac{3}{2}$$
$$\text{or} \quad 2(x^2-1)=3x$$
$$\Rightarrow \quad 2x^2-3x-2=0$$
$$(x-2)(2x+1)=0$$
$$(x-2)=0 \text{ or } (2x+1)=0$$
$$x=2 \quad \text{or} \quad x=-\frac{1}{2}$$

and $y=\frac{15}{4}$

$$\Rightarrow \quad x-\frac{1}{x}=\frac{15}{4}$$
$$\therefore \quad \frac{x^2-1}{x}=\frac{15}{4}$$
$$\text{or} \quad 4(x^2-1)=15x$$
$$\Rightarrow \quad 4x^2-15x-4=0$$
$$(x-4)(4x+1)=0$$
$$(x-4)=0 \quad (4x+1)=0$$
$$x=4 \quad \text{or} \quad x=-\frac{1}{4}$$

Hence the required solution set is 2, $\left\{2, \frac{-1}{2}, 4, \frac{-1}{4}\right\}$.

Type IV

Equations that involve terms of the form a^x where $a > 0$, $a \neq 1$ are called **exponent equations**. These equations can be reduced to quadratic equations by making the substituti $y=a^x$, which changes the equations into quadratic equations in term of y.

Example 10 Solve $4.2^{2x}-10.2^x+4=0$

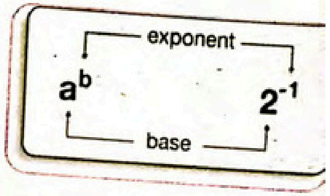
Solution

$$4.2^{2x}-10.2^x+4=0$$

We may write the given equation as

$$4.(2^x)^2-10.2^x+4=0$$

Let $2^x=y$. The above equation reduces to



$$4y^2-10y+4=0$$
$$\Rightarrow \quad 2y^2-5y+2=0 \qquad \textit{(Taking 2 common)}$$
$$\Rightarrow \quad (2y-1)(y-2)=0 \qquad \textit{(Factorizing)}$$
$$\Rightarrow \quad 2y-1=0 \quad or \quad y-2=0$$
$$gives \quad y=\frac{1}{2} \quad or \quad y=2$$
$$\Rightarrow \quad 2^x=\frac{1}{2} \quad or \quad 2^x=2$$
$$\Rightarrow \quad 2^x=2^{-1} \quad or \quad 2^x=2^1$$
$$\Rightarrow \quad x=-1 \quad or \quad x=1$$

Thus the solution set is $\{-1,1\}$.

Example 11 Solve the equation $2^{2+x}+2^{2-x}=10$

One-to-One Property of Exponential Functions

If $b^n=b^m$
then $n=m$

Solution

$$2^{2+x}+2^{2-x}=10$$
$$\Rightarrow \quad 2^2\cdot 2^x+2^2\cdot 2^{-x}-10=0$$

Let $2^x=y$. Then $2^{-x}=\frac{1}{y}$

The above equation reduces to

$$4y+\frac{4}{y}-10=0$$
$$\Rightarrow \quad 4y^2-10y+4=0 \qquad \text{taking 2 common}$$
$$\Rightarrow \quad 2y^2-5y+2=0$$

$\Rightarrow \quad (2y-1)(y-2)=0$ factorizing

$\Rightarrow \quad 2y-1=0 \quad or \quad y-2=0$

$\Rightarrow \quad y=\frac{1}{2} \quad or \quad y=2$

$\Rightarrow \quad 2^x=\frac{1}{2} \quad or \quad 2^x=2$

$\Rightarrow \quad 2^x=2^{-1} \quad or \quad 2^x=2^1$

$\Rightarrow \quad x=-1 \quad or \quad x=1$

Thus the solution set is $\{-1,1\}$.

Type V Equations of the form $(x+a)(x+b)(x+c)(x+d)=k$ where $a+b=c+d$

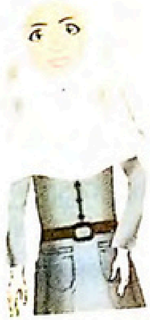
Example 12 Solve $(x+1)(x+3)(x-2)(x-4)=24$

Solution As $1+(-2)=3+(-4)$

So re-arranging the factors on the left side of the given equation, we have

$$[(x+1)(x-2)][(x+3)(x-4)]=24$$

$$\Rightarrow \quad (x^2-x-2)(x^2-x-12)-24=0$$



Let $x^2-x=y$. The above equation becomes

$$(y-2)(y-12)-24=0$$

$$\Rightarrow \quad y^2-14y+24-24=0$$

$$\Rightarrow \quad y^2-14y=0$$

$$\Rightarrow \quad y(y-14)=0$$

$$\Rightarrow \quad y=0 \quad or \quad y=14$$

If $y=0$, then $x^2-x=0$

$\Rightarrow \quad x(x-1)=0$

gives $x=0$ or $x=1$. If $y=14$, then $x^2-x=14$

$$\Rightarrow x^2-x-14=0$$

$$\Rightarrow x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-14)}}{2(1)}=\frac{1\pm\sqrt{1+56}}{2}=\frac{1\pm\sqrt{57}}{2}=\frac{1+\sqrt{57}}{2},\frac{1-\sqrt{57}}{2}$$

Thus the solution set is $\left\{0, 1, \frac{1+\sqrt{57}}{2}, \frac{1-\sqrt{57}}{2}\right\}$.

Exercise 1.2

1. Solve the following equations.

(i). $x^4-5x^2+4=0$

(ii). $x^4-7x^2+12=0$

(iii). $6x^4-13x^2+5=0$

(iv). $x+2-\frac{1}{x+2}=\frac{3}{2}$

(v). $x-\frac{4}{x}=2$

(vi). $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{5}{6}$

(vii). $3\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+26=0$

(viii). $\left(x+\frac{1}{x}\right)^2-10\left(x+\frac{1}{x}\right)+16=0$

(ix). $\left(x^2+\frac{1}{x^2}\right)-\left(x-\frac{1}{x}\right)-4=0$

(x). $3^{2x}-10.3^x+9=0$

(xi). $3.3^{2x+1}-10.3^x+1=0$

(xii). $5^{x+1}+5^{2-x}=126$

(xiii). $(x-3)(x+9)(x+5)(x-7)=385$

(xiv). $(x+1)(x+2)(x+3)(x+4)+1=0$

(xv). $(x+1)(x+3)(x+5)(x+7)+16=0$

2. Solve the equation $x^4-x^3-2x^2+2x+1=0$

Math Fun

Take any four digit number, follow these steps, and you'll end up with 6174.

1. Choose a four digit number (the only condition is that it has at least two different digits).
2. Arrange the digits of the four digit number in descending then ascending order.
3. Subtract the smaller number from the bigger one.
4. Repeat.

Eventually you'll end up at 6174, which is known as Kaprekar's constant. If you then repeat the process you'll just keep getting 6174 over and over again.

1.5 Radical equations

An equation in which the variable appears in one or more radicands is called a radical equation.

For example,

$\sqrt{x+2}=3$, $\sqrt{2x+3}=2x+5$, $\sqrt{x+5}=\sqrt{2x-1}$, $3\sqrt{x^2+x+1}=2$ are radical equations.

To solve radical equations, we transform the given equation into an equation that contains no radicals by squaring it.
A solution of the transformed equation that does not satisfy the original radical equation is called an extraneous solution.

Note
Check all roots (solutions) of the transformed equation in the original equation to exclude extraneous roots.

Type I Equation of the form $\sqrt{ax+b}=cx+d$

Example 13 Solve $\sqrt{27-3x}=x-3$

Solution

$\sqrt{27-3x}=x-3$

Squaring both sides, we get $\left(\sqrt{27-3x}\right)^2=(x-3)^2$

Tidbit
We can get rid of a square root by squaring.

$$27-3x=x^2-6x+9$$
$$\Rightarrow \quad 0=x^2-6x+9-27+3x$$
$$\text{or} \quad x^2-3x-18=0$$
$$(x-6)(x+3)=0 \qquad \text{factorizing}$$
$$\Rightarrow \quad x-6=0 \quad or \quad x+3=0$$
$$\text{gives} \quad x=6 \quad \text{or} \quad x=-3$$

Now it is necessary to check the solutions in the original equation,

$$\sqrt{27-3x}=x-3$$

If $x=6$, then $\sqrt{27-3x}=x-3$

$$\sqrt{27-3(6)}=6-3$$
$$\sqrt{27-18}=3$$
$$\sqrt{9}=3$$
$$3=3 \quad \text{(true)}$$

If $x=-3$, then $\sqrt{27-3x}=x-3$

$$\sqrt{27-3(-3)}=-3-3$$
$$\sqrt{27+9}=-6$$
$$\sqrt{36}=-6$$
$$6=-6 \quad \text{(false)}$$

On checking, we find that $x=-3$ is an extraneous root.
Thus the solution set is $\{6\}$.

Type II Equation of the form $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Example 14 Solve $\sqrt{x+2}+\sqrt{x+7}=\sqrt{x+23}$

Solution Solve

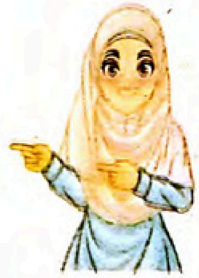
$$\sqrt{x+2}+\sqrt{x+7}=\sqrt{x+23}$$

Squaring both sides, we get

$$\left(\sqrt{x+2}+\sqrt{x+7}\right)^2=\left(\sqrt{x+23}\right)^2$$
$$x+2+x+7+2\sqrt{x+2}\cdot\sqrt{x+7}=x+23$$
$$\Rightarrow \quad 2x+9+2\sqrt{x+2}\cdot\sqrt{x+7}=x+23$$
$$\Rightarrow \quad 2\sqrt{(x+2)(x+7)}=14-x$$

Squaring both sides again, we get

$$\Rightarrow \quad 4(x+2)(x+7)=(14-x)^2$$
$$\Rightarrow \quad 4(x^2+9x+14)=196-28x+x^2$$
$$\Rightarrow \quad 4x^2+36x+56=196-28x+x^2$$
$$\Rightarrow \quad 4x^2-x^2+36x+28x+56-196=0$$
$$\Rightarrow \quad 3x^2+64x-140=0$$
$$\Rightarrow \quad 3x^2-6x+70x-140=0$$
$$\Rightarrow \quad 3x(x-2)+70(x-2)=0$$
$$\Rightarrow \quad (x-2)(3x+70)=0$$
$$\Rightarrow \quad x-2=0 \quad \text{or} \quad 3x+70=0$$
$$x=2 \quad \text{or} \quad x=-\frac{70}{3}$$



On checking, we find that $-\frac{70}{3}$ is an extraneous root.
Thus the solution set is $\{2\}$.

WARNING
You can perform addition only with identical radical forms. Adding unlike radicals is one of the most common mistakes made by students in algebra! You can easily verify that

$$\sqrt{9}+\sqrt{16}=3+4=7$$
$$\sqrt{9+16}=\sqrt{25}=5$$
$$7=\sqrt{9}+\sqrt{16}\neq\sqrt{9+16}=5$$

Type III Equation of the form $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 13

Solve the equation $\sqrt{x^2+3x+5}+\sqrt{x^2+3x+1}=2$

Solution

$$\sqrt{x^2+3x+5}+\sqrt{x^2+3x+1}=2$$

The given equation can be written as

$$\sqrt{x^2+3x+5}=2-\sqrt{x^2+3x+1}$$

Squaring both sides, we get

$$x^2+3x+5=4+x^2+3x+1-2(2)\sqrt{x^2+3x+1}$$

$$\Rightarrow\quad x^2+3x+5=x^2+3x+5-4\sqrt{x^2+3x+1}$$

$$\Rightarrow\quad 0=-4\sqrt{x^2+3x+1}$$

$$\Rightarrow\quad \sqrt{x^2+3x+1}=0$$

squaring both sides

$$\Rightarrow\quad x^2+3x+1=0$$

$$\Rightarrow\quad x=\frac{-3\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$$

$$=\frac{-3\pm\sqrt{9-4}}{2}$$

$$=\frac{-3\pm\sqrt{5}}{2}$$

Thus the solution set is $\left\{\frac{-3\pm\sqrt{5}}{2}\right\}$.

so easy

Did You Know?

Beautiful Number Relationships.

$135=1^1+3^2+5^3$

$175=1^1+7^2+5^3$

$518=5^1+1^2+8^3$

$598=5^1+9^2+8^3$

WOW!

Exercise 1.3

1. Solve the following equations.

(i). $\sqrt{5x+21}=x+3$ (ii). $\sqrt{2x-1}=x-2$

(iii). $\sqrt{4x+5}=2x-5$ (iv). $\sqrt{29-4x}=2x+3$

(v). $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$ (vi). $\sqrt{x}+\sqrt{3x+1}=\sqrt{5x+1}$

(vii). $\sqrt{6x+40}-\sqrt{x+21}=\sqrt{x+5}$ (viii). $\sqrt{2x-3}+\sqrt{2x+4}=\sqrt{6x+13}$

(ix). $\sqrt{x^2+2x+4}+\sqrt{x^2+2x+9}=5$ (x). $\sqrt{2x^2+3x+5}+\sqrt{2x^2+3x+1}=2$

2. Find $2x+5$ if x satisfies $\sqrt{40-9x}-2\sqrt{7-x}=\sqrt{-x}$

Review Exercise 1

1. At the end of each question, four circles are given. Fill in the correct circle only.

(i). If $(x+1)(x-5)=0$, then the solutions are

○ $x=1,-5$ ○ $x=1,5$ ○ $x=-1,-5$ ○ $x=-1,5$

(ii). If $x^2-x-1=0$, then $x=$

○ $\frac{-1\pm\sqrt{5}}{2}$ ○ $-1\pm\frac{\sqrt{5}}{2}$ ○ $\frac{1\pm\sqrt{5}}{2}$ ○ $1\pm\frac{\sqrt{5}}{2}$

(iii). $\frac{-1\pm\sqrt{5}}{2}$ in simplified form is

○ $1\pm\sqrt{24}$ ○ $1\pm\sqrt{6}$ ○ $2\pm\sqrt{6}$ ○ cannot be simplified

(iv). To apply the quadratic formula to $2x^2-x=3$

○ a = 2, b = –1, c = 3 ○ a = 2, b = 1, c = 3

○ a = 2, b = –1, c = –3 ○ a = 2, b = –1, c = 0

(v). If $x^2-3x-4=0$, then the solutions are

○ $x=4,-1$ ○ $x=-4,1$ ○ $x=4,1$ ○ $x=-4,-1$

(vi). If $2x^2+4x-9=0$, the solutions are

○ $x=\frac{2\pm\sqrt{22}}{2}$ ○ $x=\frac{-2\pm\sqrt{22}}{2}$ ○ $x=2\pm\frac{\sqrt{22}}{2}$ ○ $x=-2\pm\frac{\sqrt{22}}{2}$

(vii). $x^2-\frac{1}{4}=0$, the solution are

○ $x=\pm\frac{1}{2}$ ○ $x=\pm\frac{1}{4}$ ○ $x=\pm\frac{1}{8}$ ○ $x=\pm\frac{1}{16}$

(viii). What are the solutions of the equation $x^2 + 7x - 18 = 0$?

○ 2 or −9 ○ −2 or 9 ○ −2 or −9 ○ 2 or 9

(ix). Which of the following values of x are roots of the equation $x^2 - 8x + 15 = 0$?

○ $x = 1$ or $x = -7$ ○ $x = 2$ or $x = 4$

○ $x = -2$ or $x = 4$ ○ $x = 3$ or $x = 5$

2. Solve $2w^4 - 5w^2 + 2 = 0$.
3. Find the constants a and b such that $x = -1$ and $x = 1$ are both solutions to the equation $ax^2 + bx + 2 = 0$.
4. Find all values of x such that $x^2 + 5x + 6$ and $x^2 + 19x + 34$ are equal.

Challenge !

5. Find the solutions to the equation: $49x^2 - 316x + 132 = 0$
If you can factorize this successfully, you have probably mastered the art of factorizing.

PROJECT

Find some internet sites which solve quadratic equations. Which site do you think is better and why?

Activity

Juwaria tried to solve the quadratic equation $x^2 + 5x - 2 = 0$ by completing the square, but she made a mistake. In which line of her working, shown below, did she make the mistake?

$x^2 + 5x - 2 = 0$

[Line 1] $\Rightarrow x^2 + 5x = 2$

[Line 2] $\Rightarrow x^2 + 5x + (5/2)^2 = 2 + (5/2)^2$

[Line 3] $\Rightarrow (x + 5/2)^2 = 2 + 10/4$

[Line 4] $\Rightarrow (x + 5/2)^2 = 18/4$

[Line 5] $\Rightarrow (x + 5/2) = \pm\sqrt{(18/4)} = \pm(\sqrt{18})/(\sqrt{4}) = \pm\sqrt{18}/2$

[Line 6] $\Rightarrow x = \pm\sqrt{18}/2 - 5/2$

[Line 7] $\Rightarrow x = -4.62$ or -0.38 to 2 decimal places.

Summary

The following table summarizes important topics related to quadratic equations.

Concept	Explanation	oles
Quadratic equation	$ax^2 + bx + c = 0$, where a, b, and c are constants with $a \neq 0$.	A quadratic equation can have zero, one, or two real solutions. $x^2 = -5$ No real solutions $(x-2)^2 = 0$ One real solution $x^2 - 4 = 0$ Two real solution
Factorizing	A symbolic technique for solving equations, based on the zero-product property: if $ab = 0$, then either $a = 0$ or $b = 0$.	$x^2 - 3x + 2 = 0$ $x^2 - 3x + 2 = 0$ $(x-1)(x-2) = 0$ $x - 1 = 0$ or $x - 2 = 0$ $x = 1$ or $x = 2$
Square root property	The solutions to $x^2 = k$ are $x = \pm\sqrt{k}$, where $k \geq 0$.	$x^2 = 9$ is equivalent to $x = \pm 3$. $x^2 = 11$ is equivalent to $x = \pm\sqrt{11}$.
Completing the square	To solve $x^2 + kx = d$ symbolically, add $\left(\frac{k}{2}\right)^2$ to each side to obtain to perfect square trinomial. Then apply the square root property.	$x^2 - 6x = 1$ $x^2 - 6x + 9 = 1 + 9$ $\left(\frac{-6}{2}\right)^2 = 9$ $x^2 - 6x + 9 = 1 + 9$ $(x-3)^2 = 10$ $x - 3 = \pm\sqrt{10}$ $x = 3 \pm \sqrt{10}$
Quadratic formula	The solutions to $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Always gives the exact solutions.	To solve $2x^2 - x - 4 = 0$, let $a = 2$, $b = -1$, and $c = -4$. $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-4)}}{2(2)}$ $= \frac{1 \pm \sqrt{33}}{4} \approx 1.69, -1.19$

The following table outlines important concepts in this section.

Solving radical equations	The solutions to $a = b$ are among the solutions to $a^n = b^n$ when n is a positive integer. Check your results.	Solve $\sqrt{2x+3} = x$ $2x + 3 = x^2$ Square each able. $x^2 - 2x - 3 = 0$ Rewrite equations $x = -1$ or $x = 3$ Factor and adobe. Checking reveals that 3 is the only solution.

To complete the square of a quadratic equation

Add and Subtract

$$\left(\frac{\text{Coefficient of "x"}}{2}\right)^2$$