RATIO AND PROPORTION

Unit 24

• Weightage = 5%

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with theircorollaries and apply them to solve allied problems.
  - A line parallel to one side of a triangle, intersectingthe other two sides, divides them proportionally.
  - If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.
  - The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the sides containing the angle.
  - If two triangles are similar, the measures of their corresponding sides are proportional.

24.1 Ratio and proportion

In this chapter, we will study the theorems related to the ratio and proportion of sides of a triangle along with the similarity of triangles. So, first of all we have to recall the concepts of ratio, proportion and similarity.

Ratio:

Ratio is the comparison of two quantities of same kind with same units. The ratio of a and b is written as $a:b$ or $\frac{a}{b}$ where a is called antecedent and b is called consequent.

For example the ratio of 25 litres and 5 litres is $25:5$ or 5:1.

Proportion:

Equality of two ratios is called proportion.

If two ratios $a:b$ and $c:d$ are equal then we write it as $a:b=c:d$ or $a:b::c:d$ and call it as proportion.

In the proportion $a:b=c:d$, a and d are called extremes, whereas b and c are called means.

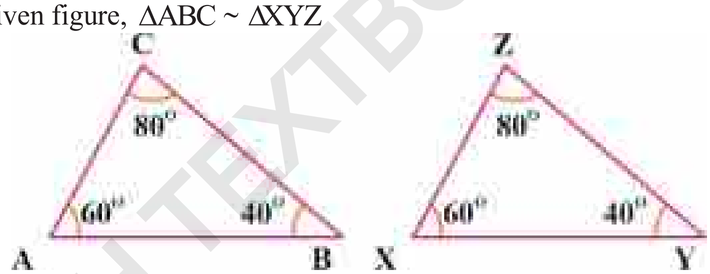
In a proportion, the product of means is always equal to the product of extremes.

Similar Triangles:

Two triangles ABC and PQR are called similar triangles if their corresponding angles are congruent.

Symbolically, we write it as $\Delta ABC \sim \Delta PQR$

In the given figure, $\Delta ABC \sim \Delta XYZ$



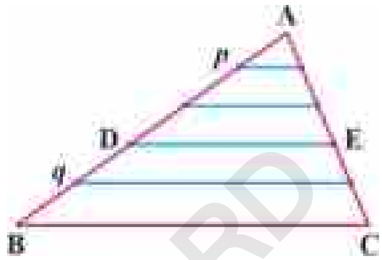
For similar triangles, following theorem is important

"If two triangles are similar then their corresponding sides are proportional". We will prove this theorem in last.

Theorem 24.1

A line parallel to one side of a triangle and intersecting the other two sides, divides them proportionally.

Given: In $\Delta ABC, \overline{DE} \parallel \overline{BC}$ and $\overline{DE}$ intersects the other sides $\overline{AB}$ and $\overline{AC}$ at points D and E respectively.



To prove: $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$

Construction:

Select such a unit of length so that $m\overline{AD} = p$ units, and $m\overline{BD} = q$ units where p and q are natural numbers.

Divide $\overline{AD}$ into p congruent segments and $\overline{BD}$ into q congruent segments.

So, $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{p}{q}$ .From the points of division, draw lines parallel to $\overline{BC}$.

Proof:

Statements	Reasons
$\overline{AD}$ is divided into p congruent segments by the parallel lines.	Construction
$\overline{AE}$ is also divided into p congruent segments by the same parallel lines.	Parallel lines make equal numbers of congruent intercepts on each transversal.
Similarly, $\overline{EC}$ is also divided into q congruent segments.	$\because$ $\overline{BD}$ is divided into q congruent segments by the parallel lines.
Now, $\dfrac{m\overline{AE}}{m\overline{EC}} = \dfrac{p}{q}$	$\because$ $\overline{AE}$ and $\overline{EC}$ are divided into p and q congruent segments respectively (Proved above)
But $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{p}{q}$	By construction
So $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$	By transitive property of equality

Q.E.D

Corollary 1:

From the figure of above theorem, it can be proved that $\dfrac{m\overline{AB}}{m\overline{DB}} = \dfrac{m\overline{AC}}{m\overline{EC}}$.

Corollary 2:

From the figure of the above theorem, it can be proved that $\frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$.

Corollary 3:

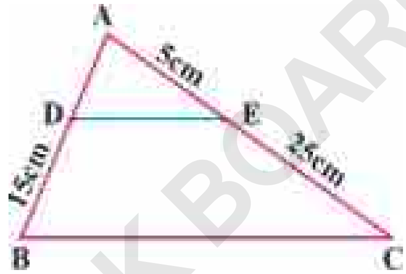
If a line is parallel to the base of an isosceles triangle and intersects other two congruent sides then the corresponding intercepts on the sides will be congruent.

Example:

In the adjacent figure, $\overline{DE} \parallel \overline{BC}$ in ΔABC.

Find $m\overline{AD}$ if $m\overline{AE} = 5cm$,

$m\overline{BD} = 15cm$ and $m\overline{CE} = 25cm$.



Solution:

$\because \quad \overline{DE} \parallel \overline{BC}$

$\therefore \quad \frac{m\overline{AD}}{m\overline{BD}} = \frac{m\overline{AE}}{m\overline{EC}}$

i.e. $\quad \frac{m\overline{AD}}{15} = \frac{5}{25}$

$\Rightarrow \quad m\overline{AD} = \frac{15 \times 5}{25}$

$\Rightarrow \quad m\overline{AD} = 3cm$

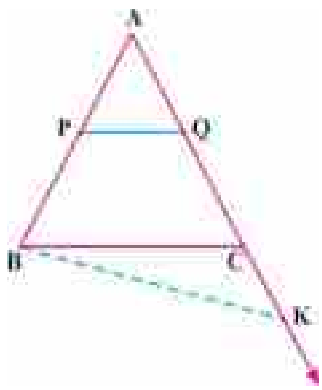
Theorem 24.2

If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.

Given:

In ΔABC, $\overline{PQ}$ cuts $\overline{AB}$ and $\overline{AC}$ at points P and Q respectively.
such that

$$\frac{m\overline{AP}}{m\overline{PB}} = \frac{m\overline{AQ}}{m\overline{QC}}$$



To prove:

$\overline{PQ} \parallel \overline{BC}$

Construction:

If $\overline{PQ}$ is not parallel to $\overline{BC}$, draw $\overline{BK}$ meeting $\overrightarrow{AC}$ at point K other than C such that $\overline{PQ} \parallel \overline{BK}$

Proof:

Statements	Reasons
In ΔABK	
$\overline{PQ} \parallel \overline{BK}$	Construction
$\dfrac{m\overline{AP}}{m\overline{PB}} = \dfrac{m\overline{AQ}}{m\overline{QK}}$	Line parallel to one side of triangle divides other sides proportionally
But	
$\dfrac{m\overline{AP}}{m\overline{PB}} = \dfrac{m\overline{AQ}}{m\overline{QC}}$	Given
So	
$\dfrac{m\overline{AQ}}{m\overline{QK}} = \dfrac{m\overline{AQ}}{m\overline{QC}}$	Transitive property of equality
$\Rightarrow$ $m\overline{QK} = m\overline{QC}$	If antecedents are equal then consequents are also equal in equal ratios.
i.e. $\overline{QK} \cong \overline{QC}$	By definition of congruent segments.
This is possible only when K coincides with C.	Q is common point in both
Hence	
$\overline{PQ} \parallel \overline{BC}$	Our assumption is wrong

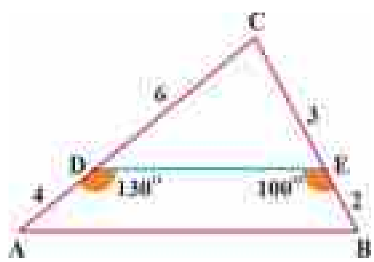
Q.E.D

Corollary:

The line segment joining the mid points of two sides of a triangle is parallel to the third side.

Example:

In adjacent figure, $\overline{DE}$ cuts two sides $\overline{AC}$ and $\overline{BC}$ of ΔABC at points D and E respectively. Find the base angles A and B if $m\angle ADE = 130^\circ$ and $m\angle BED = 100^\circ$ where lengths of sides are given as shown in the figure.



Solution:

$\because$ $\angle ADE$ and $\angle CDE$ are supplementary

$\therefore$ $m\angle CDE = 50^\circ \left(\quad m\angle ADE = 130^\circ \right)$

Similarly

$m\angle DEC = 80^\circ$

$\because$ $\dfrac{4}{6} = \dfrac{2}{3}$ i.e. $\overline{DE}$ cuts two sides $\overline{AC}$ and $\overline{BC}$ proportionally

$\therefore$ $\overline{DE} \parallel \overline{AB}$

So, $m\angle A = m\angle CDE$, because corresponding angles of parallel lines are equal

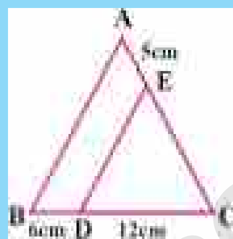
i.e. $m\angle A = 50^\circ$

Similarly

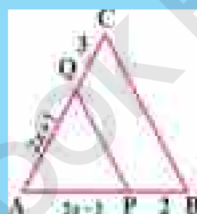
$m\angle B = m\angle DEC = 80^\circ$

EXERCISE 24.1

1. In ΔABC, $\overline{DE} \parallel \overline{AB}$. Find $m\overline{CE}$ if $m\overline{BD} = 6cm$, $m\overline{DC} = 12cm$ and $m\overline{AE} = 5cm$.



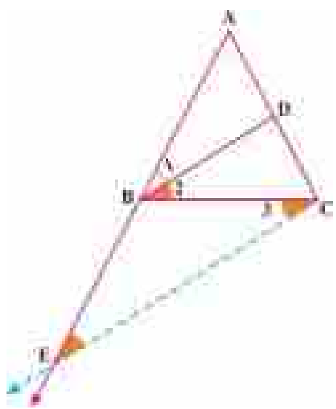
2. In ΔABC, $\overline{PQ} \parallel \overline{BC}$. Find x if $m\overline{AP} = 5x - 3$, $m\overline{PB} = 2$, $m\overline{AQ} = 2x + 1$ and $m\overline{QC} = 3$



3. Prove that the line drawn parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.
4. Prove that the line segment drawn through the mid point of a side of a triangle and parallel to another side bisects the third side.
5. Prove that the line which divides the non-parallel sides of a trapezium proportionally is parallel to the third side.

Theorem 24.3

The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the sides containing the angle.



Given:

$\overline{BD}$ is the bisector of $\angle ABC$ of ΔABC.

i.e $\angle 1 \cong \angle 2$

To prove:

$$\frac{m\overline{AD}}{m\overline{DC}} = \frac{m\overline{AB}}{m\overline{BC}}$$

Construction:

Draw $\overrightarrow{CE}$ parallel to $\overline{BD}$ meeting $\overrightarrow{AB}$ at point E

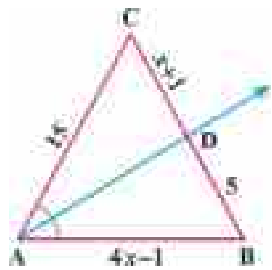
Proof:

Statements	Reasons
$\because$ $\overline{EC} \parallel \overline{BD}$	Construction
$\therefore$ $m\angle E = m\angle 1 \ldots$(i)	Corresponding angles of parallel lines.
Also	
$m\angle 2 = m\angle 3$	Alternate angles of parallel lines
But $m\angle 1 = m\angle 2$	Given
So $m\angle 1 = m\angle 3$	Transitive property
$m\angle E = m\angle 3$	Using eq (i)
In ΔBCE,	Sides opposite to congruent angles of a triangle are congruent
$\overline{BC} \cong \overline{BE}$	
In ΔACE,	
$\because$ $\overline{BD} \parallel \overline{EC}$	Construction
$\therefore$ $\dfrac{m\overline{AD}}{m\overline{DC}} = \dfrac{m\overline{AB}}{m\overline{BE}}$	Line parallel to one side of triangle divides other sides proportionally
Or	
$\dfrac{m\overline{AD}}{m\overline{DC}} = \dfrac{m\overline{AB}}{m\overline{BC}}$	$\overline{BC} \cong \overline{BE}$ (Proved above)

Q.E.D

Corollary:

If the internal bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles.



Example: In adjacent figure, $\overrightarrow{AD}$ is the bisector of $\angle A$ of ΔABC. Find x if $m\overline{AC} = 15$ cm, $m\overline{AB} = 4x - 1$ cm, $m\overline{CD} = x + 1$ cm and $m\overline{BD} = 5$cm. Also specify the type of triangle where $x \in N$.

Solution:

$\because$ $\overrightarrow{AD}$ is the bisector of $\angle A$

$\therefore$ $\dfrac{m\overline{CD}}{m\overline{DB}} = \dfrac{m\overline{AC}}{m\overline{AB}}$

i.e. $\dfrac{x+1}{5} = \dfrac{15}{4x-1}$

$\Rightarrow$ $4x^2 - x + 4x - 1 = 75$

$\Rightarrow$ $4x^2 + 3x - 76 = 0$

$\Rightarrow$ $4x^2 + 19x - 16x - 76 = 0$

$\Rightarrow$ $x(4x+19) - 4(4x+19) = 0$

$\Rightarrow$ $(x-4)(4x+19) = 0$

$\Rightarrow$ $x - 4 = 0$ or $4x + 19 = 0$

$\Rightarrow \quad x = 4$

$\Rightarrow \quad x = \frac{-19}{4}$

$\because \quad \frac{-19}{4} \notin N$

$\therefore$ we neglect $\frac{-19}{4}$

Hence $x = 4$

Now $m\overline{AB} = 4x - 1$

$= 4 \times 4 - 1$

$= 15$

$m\overline{AB} = m\overline{AC} = 15$ cm

and $m\overline{BC} = x + 1 + 5 = 10$ cm

$\therefore$ ΔABC is an isosceles triangle.

Theorem 24.4:

If two triangles are similar, the measures of their corresponding sides are proportional.

Given:

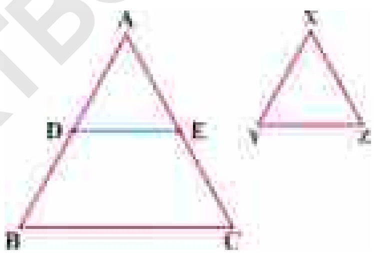
ΔABC and ΔXYZ are similar triangles.

i.e. In $\Delta ABC \leftrightarrow \Delta XYZ$

$\angle A \cong \angle X$

$\angle B \cong \angle Y$

and $\angle C \cong \angle Z$



To prove:

$$\frac{m\overline{AB}}{m\overline{XY}} = \frac{m\overline{BC}}{m\overline{YZ}} = \frac{m\overline{AC}}{m\overline{XZ}}$$

Construction:

From $\overline{AB}$, cut off $\overline{AD} \cong \overline{XY}$ and from $\overline{AC}$, cut off $\overline{AE} \cong \overline{XZ}$. Draw $\overline{DE}$.

Proof:

Statements	Reasons
In $\Delta ADE \leftrightarrow \Delta XYZ$	
$\overline{AD} \cong \overline{XY}$	Construction
$\angle A \cong \angle X$	Given
$\overline{AE} \cong \overline{XZ}$	Construction
$\Delta ADE \cong \Delta XYZ$	By S.A.S postulate
$\angle ADE \cong \angle Y$	Corresponding angles of congruent triangles
But $\angle B \cong \angle Y$	Given
So $\angle ADE \cong \angle B$	Transitive property
Hence	

$\overline{DE} \parallel \overline{BC}$	Corresponding angles $\angle ADE$ and $\angle B$ are congruent
$\dfrac{m\overline{AB}}{m\overline{AD}} = \dfrac{m\overline{AC}}{m\overline{AE}}$	By Theorem 1 (Corollary)
Or $\dfrac{m\overline{AB}}{m\overline{XY}} = \dfrac{m\overline{AC}}{m\overline{XZ}}$ ...(i)	$m\overline{AD} = m\overline{XY}$ and $m\overline{AE} = m\overline{XZ}$
Similarly $\dfrac{m\overline{AB}}{m\overline{XY}} = \dfrac{m\overline{BC}}{m\overline{YZ}}$ ...(ii)	By the above process
So $\dfrac{m\overline{AB}}{m\overline{XY}} = \dfrac{m\overline{BC}}{m\overline{YZ}} = \dfrac{m\overline{AC}}{m\overline{XZ}}$	From (i) and (ii)

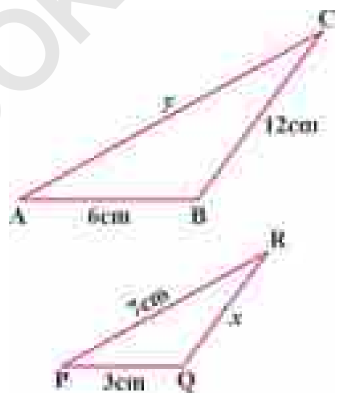
Q.E.D

Corollary:

In a correspondence of two triangles, if two angles of a triangle are congruent to the corresponding two angles of other triangle then their corresponding sides are proportional.

Example 1:

In the given figure, ΔABC and ΔPQR are similar. Find the values of x and y if $m\overline{AB} = 6\text{cm}$, $m\overline{BC} = 12\text{cm}$, $m\,\overline{PQ} = 3\text{cm}$, and $m\overline{PR} = 7\text{cm}$



Solution:

ΔABC and ΔPQR are similar

$\therefore$ Their corresponding sides are equal

i.e. $\dfrac{m\overline{AB}}{m\overline{PQ}} = \dfrac{m\overline{BC}}{m\overline{QR}} = \dfrac{m\overline{AC}}{m\overline{PR}}$

$\Rightarrow \dfrac{6}{3} = \dfrac{12}{x} = \dfrac{y}{7}$ (Using the given measures)

$\Rightarrow \dfrac{6}{3} = \dfrac{12}{x}$ and $\dfrac{6}{3} = \dfrac{y}{7}$

or $2 = \dfrac{12}{x}$ or $2 = \dfrac{y}{7}$

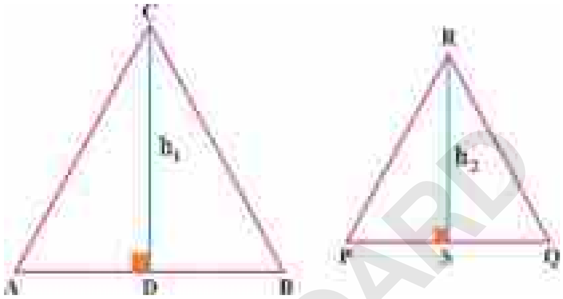
or $2x = 12$ or $14 = y$

$\Rightarrow x = 6$ cm or $y = 14$ cm

So the values of x and y are 6cm and 14cm respectively.

Example 2

In the given figure, ΔABC and ΔPQR are similar and $\frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y}$ where h_1 and h_2 are the altitudes of the given triangles.



Prove that:

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{x^2}{y^2}$$

Proof:

we have $\frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y}$

$\because$ ΔABC and ΔPQR are similar

$\therefore$ $\frac{m\overline{AC}}{m\overline{PR}} = \frac{m\overline{BC}}{m\overline{QR}} = \frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y}$ ... (i)

In $\Delta ADC \leftrightarrow \Delta PSR$

$m\angle A = m\angle P$ (Given)

$\because$ and $m\angle D = m\angle S = 90^\circ$

$\therefore$ $\Delta ADC \sim \Delta PSR$

Hence $\frac{h_1}{h_2} = \frac{m\overline{AC}}{m\overline{PR}} = \frac{x}{y}$ (using eq: i)

Now

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{\frac{1}{2}\left(m\overline{AB}\right)h_1}{\frac{1}{2}\left(m\overline{PQ}\right)h_2}$$

$$= \frac{m\overline{AB}}{m\overline{PQ}} \times \frac{h_1}{h_2}$$

$$= \frac{x}{y} \times \frac{x}{y} \qquad \left(\frac{m\overline{AB}}{m\overline{PQ}} = \frac{x}{y} \text{ and } \frac{h_1}{h_2} = \frac{x}{y}\right)$$

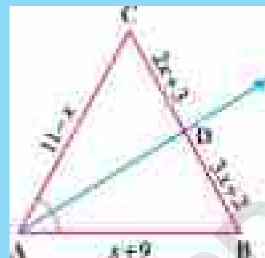
$$= \frac{x^2}{y^2}$$

Hence proved.

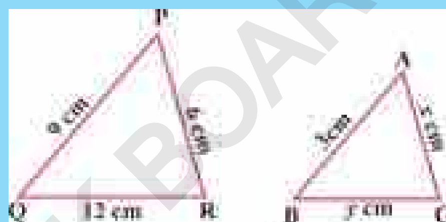
From above example, we conclude that

The ratio of areas of two similar triangles is equal to the square of the ratio of any two corresponding sides.

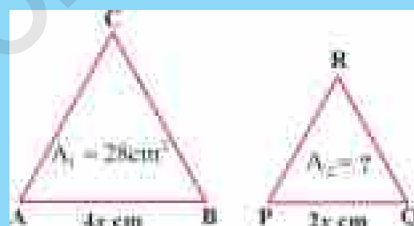
EXERCISE 24.2



1. In the adjacent figure, $\overrightarrow{AD}$ is the bisector of $\angle A$ of ΔABC. Find the value of x if $m\overline{AB} = x + 9, m\overline{AC} = 11 - x, m\overline{CD} = 2x + 3$ and $m\overline{BD} = 3x + 2$. Also specify the type of triangle.



2. In the adjacent figure, ΔPQR and ΔABC are similar. Find the values of x and y if lengths of sides are indicated in the figure.



3. Let A_1 and A_2 be the areas of two similar triangles ΔABC and ΔPQR respectively as shown in the figure. Find A_2 if $A_1 = 28\text{cm}^2, m\overline{AB} = 4x\,\text{cm}$ and $m\overline{PQ} = 2x\,\text{cm}$

4. Ratio of corresponding sides of two similar triangles is $2 : x - 5$ and the ratio of their areas is $1 : 9$. Find the value of x.
5. Prove that two right triangles have their sides proportional if an acute angle of the one is congruent to an acute angle of the other.
6. In a right triangle the perpendicular drawn from the right angle to the hypotenuse divides the triangle into two triangles. Prove that each of these triangles is similar to the original one.

REVIEW EXERCISE 24

1. Tick the correct option.

i. In a proportion, the product of means is equal to __________ of extremes.
   (a) sum (b) difference (c) quotient (d) product
ii. __________ triangles are always similar.
   (a) right (b) scalene (c) acute angled (d) equilateral
iii. The symbol of similarity of triangles is __________.
   (a) $=$ (b) $\cong$ (c) $\sim$ (d) $\leftrightarrow$

iv. If a line parallel to base of a triangle and divides one side in 2:3 then it will divide other side in __________.

(a) 3:2 (b) 2:3 (c) 2:6 (d) 5:3

v. If a line intersects two sides of a triangle in same ratio then it is ________ to other side.

(a) parallel (b) non-parallel (c) coincident (d) all of these

vi. In ΔABC, the bisector of $\angle A$ divides $\overline{BC}$ in ratio __________ if $m\overline{AB} = 6\text{cm}$ and $m\overline{AC} = 8\text{cm}$.

(a) 5:8 (b) 3:4 (c) 1:1 (d) 5:7

vii. The bisector of an angle of equilateral triangle divides the opposite side in

(a) 2:3 (b) 3:2 (c) 1:1 (d) 5:2

viii. The corresponding sides of two similar triangles are__________.

(a) equal (b) un- equal (c) proportional (d) None of these

ix. If the ratio of two corresponding sides of similar triangle is 5:7 then ratio of their areas is equal to __________.

(a) 5:7 (b) 7:5 (c) 25:7 (d) 25:49

x. If the ratio of areas of two similar triangles is 36:121 then the ratio of its corresponding sides will be __________.

(a) 6:10 (b) 6:11 (c) 11:6 (d) 10:6

xi. If the ratio of corresponding sides of similar triangles is 2:x and that of areas is 4:9 then $x =$ _______.

(a) 3 (b) -3 (c) both a and b (d) none of these

xii. Two equilateral triangles are also __________.

(a) congruent (b) similar (c) proportional (d) equivalent

SUMMARY

- Ratio is the comparison of two similar quantities.
- In ratio $a:b$, a is called antecedent and b is called consequent.
- Proportion is the equality of two ratios.
- In proportion, the product of means is equal to the product of extremes.
- Two triangles are similar if they are equiangular.
- If two triangles are similar then their corresponding sides are proportional.
- A line parallel to one side of a triangle and intersecting the other two sides, divides them proportionally.
- If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.
- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the sides containing the angle.