PARTIAL FRACTIONS

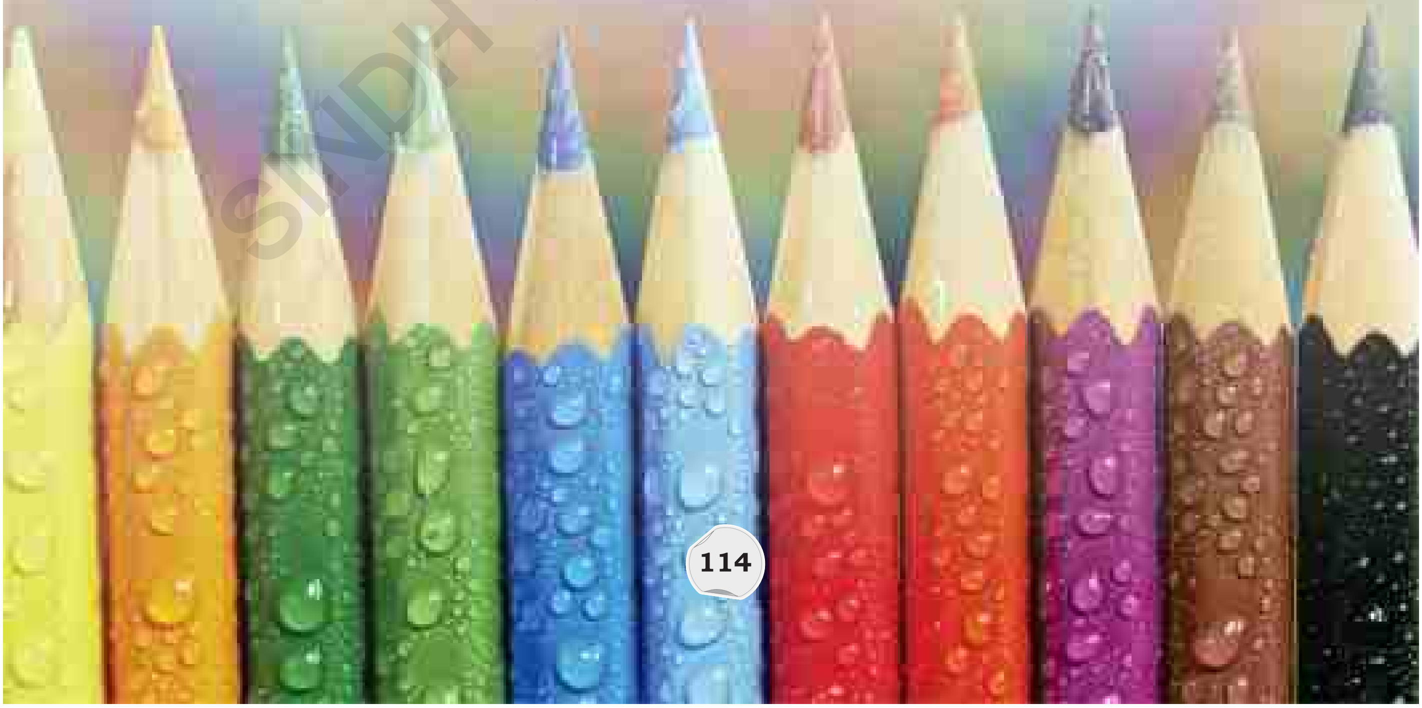
Unit 21

- Weightage = 4%

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define proper, improper and rational fractions.
- Resolve an algebraic fraction into its partial fractionswhen its denominator consist of
  - Non- repeated linear factors,
  - Repeated linear factors,
  - Non- repeated quadratic factors,
  - Repeated quadratic factors.



Introduction:

To split a rational fraction into the sum or difference of two or more fractions, such fractions are known as partial fractions. Partial fractions can only be found if the degree of the polynomial in numerator is strictly less than the degree of the polynomial in denominator. For instance,

(i) $$\frac{2x+3}{(x-1)(x+4)}=\frac{1}{x-1}+\frac{1}{x+4}$$

(ii) $$\frac{-(4x^2+x+11)}{(x^2+1)(x-3)}=\frac{x+2}{x^2+1}-\frac{5}{x-3}$$

21.1 Define Proper, Improper and Rational Fractions

Rational Fraction:

We know that, a number of the form $\frac{p}{q}$, where $p,q\in Z$ and $q\neq 0$ is called a rational number. Similarly, the quotient of two polynomials $\frac{P(x)}{Q(x)}$, $Q(x)\neq 0$ is called a rational algebraic expression. It is commonly known as rational fraction.

Example:

(i) $$\frac{2x^3-5x^2-3x-10}{x^2-1}$$ (ii) $$\frac{5x+8}{3x^2-2x-1}$$

Proper fraction:

A rational fraction $\frac{P(x)}{Q(x)}$ is said to be proper fraction if the degree of numerator $P(x)$ is less than the degree of denominator $Q(x)$.

Examples:

(i) $$\frac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$$ (ii) $$\frac{6x+27}{3x^3-9x}$$

Improper fraction:

A rational fraction $\frac{P(x)}{Q(x)}$ is said to be improper fraction if the degree of numerator $P(x)$ is equal or greater than the degree of denominator $Q(x)$.

Examples:

(i) $$\frac{2x^3-5x^2-3x-10}{x^2-1}$$ (ii) $$\frac{6x^3-5x^2-7}{3x^2-2x-1}$$

(iii) $$\frac{x^2+1}{x^2-1}$$

Additionally, any improper fraction can be resolved into the sum of polynomial and proper fraction.

e.g. $\dfrac{3x^2-2x+1}{x+2}=3x-8+\dfrac{17}{x+2}$.

Here, $\dfrac{3x^2-2x+1}{x+2}$ is improper fraction , $3x-8$ is polynomial and $\dfrac{17}{x+2}$ is proper fraction. To resolve the improper fraction into the sum of polynomial and proper fraction, one needs to divide the numerator with denominator.

21.2 Resolution of fraction into its partial fractions

To resolve the rational fraction into partial fractions, it is necessary that the rational fraction must be a proper fraction. If it is not, then it must be converted into proper fraction by division.

21.2.(i) Resolve an algebraic fraction into its partial fractions when its denominator consist of

- **Non- repeated linear factors,**
- **Repeated linear factors,**
- **Non- repeated quadratic factors,**
- **Repeated quadratic factors.**

Case-I: Denominator Consists of Non-repeated linear factors

Let $R(x)=\dfrac{\text{P}(x)}{\text{Q}(x)}$ is a rational fraction, where, its denominator $\text{Q}(x)$ is the product of non-repeated linear factors which can be written as

$\text{Q}(x)=(x-a_1)(x-a_2)(x-a_2).....(x-a_n)$.

Now $R(x)=\dfrac{\text{P}(x)}{\text{Q}(x)}$ is resolved as under:

$$\frac{\text{P}(x)}{\text{Q}(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\frac{A_3}{x-a_3}+...+\frac{A_n}{x-a_n}.$$

Here, the constants $A_1, A_2, A_3, ..., A_n$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\dfrac{11-3x}{(x-1)(x+3)}$ into partial fractions.

Solution:

Let $\dfrac{11-3x}{(x-1)(x+3)}=\dfrac{\text{A}_1}{(x-1)}+\dfrac{\text{A}_2}{(x+3)}$ (i)

Here, A_1 and A_2 are unknown constants to be determined.

From equation (i),

$$\frac{11-3x}{(x-1)(x+3)}=\frac{A_1(x+3)+A_2(x-1)}{(x-1)(x+3)}$$

Multiplying both sides by $(x-1)(x+3)$

$$\Rightarrow 11-3x=A_1(x+3)+A_2(x-1) \qquad \text{(ii)}$$

To determine constants A_1 and A_2, values of x are chosen. To get A_1, put $x=1$, on both sides in equation (ii),

we get $11-3(1)=A_1(1+3)+A_2(0)$

$\Rightarrow \quad 8=4A_1$

$\Rightarrow \quad A_1=2$

To get A_2, put $x=-3$, in equation (ii),

we get $11-3(-3)=A_1(0)+A_2(-3-1)$

$\Rightarrow \quad 20=-4A_2$

$\Rightarrow \quad A_2=-5$

Finally, by putting the values of constants A_1 and A_2 in equation (ii),

we get $\dfrac{11-3x}{(x-1)(x+3)}=\dfrac{2}{x-1}-\dfrac{5}{x+3}$.

Example 2:

Resolve $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$ into partial fractions.

Solution:

The given rational fraction is an improper fraction; hence it is converted in the proper fraction by dividing numerator with denominator.

$$\frac{6x^3+5x^2-7}{2x^2-x-1}=\frac{6x^3+5x^2-7}{(x-1)(2x+1)}=3x+4+\frac{7x-3}{(x-1)(2x+1)} \qquad \text{(i)}$$

Consider the expression $\dfrac{7x-3}{(x-1)(2x+1)}$ for resolving into partial fraction.

Let $\dfrac{7x-3}{(x-1)(2x+1)}=\dfrac{A_1}{x-1}+\dfrac{A_2}{2x+1}$ (ii)

Here, A_1 and A_2 are unknown constants to be determined.

From equation (ii),

$$\frac{7x-3}{(x-1)(2x+1)}=\frac{A_1(2x+1)+A_2(x-1)}{(x-1)(2x+1)}$$

By multiplying $(x-1)(2x+1)$ to both sides

we get

$$\Rightarrow \quad 7x-3 = A_1(2x+1)+A_2(x-1) \qquad \text{(iii)}$$

To determine constants A_1 and A_2, values of x are chosen. To get A_1, put $x=1$, in equation (iii),

we get

$$7(1)-3 = A_1(2+1)+A_2(0)$$

$$\Rightarrow \quad 4 = 3A_1$$

$$\Rightarrow \quad A_1 = \frac{4}{3}$$

To get A_2, put $x=-\frac{1}{2}$, in equation (iii), we get

$$7\left(-\frac{1}{2}\right)-3 = A_1(0)+A_2\left(-\frac{1}{2}-1\right)$$

$$\Rightarrow \quad -\frac{13}{2} = -\frac{3}{2}A_2$$

$$\Rightarrow \quad A_2 = \frac{13}{3}$$

By putting the values of constants A_1 and A_2 in equation (ii), we get

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{4}{3(x-1)}+\frac{13}{3(2x+1)}.$$

Finally, equation (i) becomes

$$\frac{6x^3+5x^2-7}{2x^2-x-1} = 3x+4+\frac{4}{3(x-1)}+\frac{13}{3(2x+1)}.$$

Exercise 21.1

Resolve the following into partial fractions

1. $\dfrac{12}{x^2-9}$
2. $\dfrac{4(x-4)}{x^2-2x-3}$
3. $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$
4. $\dfrac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}$
5. $\dfrac{x^2+9x+8}{x^2+x-6}$
6. $\dfrac{x^2-x-14}{x^2-2x-3}$
7. $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$

Case-II: Denominator consists of repeated linear factors

Let $R(x)=\frac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ is the product of repeated linear factors which can be written as $Q(x)=(x-a)^n$.

Now, $R(x)=\frac{P(x)}{Q(x)}$ is resolved as:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)^1}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+...+\frac{A_n}{(x-a)^n}.$$

Here, the constants $A_1, A_2, A_3, ..., A_n$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\frac{2x+3}{(x-2)^2}$ into partial fractions.

Solution:

Let $\frac{2x+3}{(x-2)^2}=\frac{A_1}{(x-2)}+\frac{A_2}{(x-2)^2}$ (i)

Here, A_1 and A_2 are unknown constants to be determined.

From of equation (i)

$$\frac{2x+3}{(x-2)^2}=\frac{A_1(x-2)+A_2}{(x-2)^2}$$

By multiplying $(x-2)^2$ both sides

we get $2x+3=A_1(x-2)+A_2$

$\Rightarrow$ $2x+3=A_1x-2A_1+A_2$ (ii)

To determine constants A_1 and A_2, by equating the like-terms on both sides of equation (ii). Now, by equating the coefficients of x in equation (ii), we get,

$$2=A_1$$

Again, by equating the constants in equation (ii), we get

$3=-2A_1+A_2$

$3=-2(2)+A_2$ $(\because A_1=2)$

$7=A_2$

Finally, by putting the values of constants A_1 or A_2 in equation (i), we get

$$\frac{2x+3}{(x-2)^2}=\frac{2}{(x-2)}+\frac{7}{(x-2)^2}.$$

Example 2:

Resolve $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ into partial fractions.

Solution:

Let $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}=\dfrac{A_1}{(x+3)}+\dfrac{A_2}{(x-1)}+\dfrac{A_3}{(x-1)^2}$ (i)

Here, A_1, A_2 and A_3 are unknown constants to be determined.

By multiplying $(x+3)(x-1)^2$ to both sides of eq (i)

$\Rightarrow \quad 5x^2-2x-19=A_1(x-1)^2+A_2(x+3)(x-1)+A_3(x+3)$ (ii)

$5x^2-2x-19=A_1x^2+2A_1x+A_1+A_2x^2+2A_2x-3A_2+A_3x+3A_3$

$\Rightarrow \quad 5x^2-2x-19=(A_1+A_2)x^2+(A_3+2A_2-2A_1)x+(A_1-3A_2+3A_3)$ (iii)

To determine constants A_1, A_2 and A_3, values of x are chosen. To get A_1, put $x=-3$, in equation (ii), we get.

$$5(-3)^2-2(-3)-19=A_1(-4)^2+A_2(0)(-4)+A_3(0)$$

$$\Rightarrow \quad 32=16A_1$$

$$\Rightarrow \quad A_1=2$$

By equating the like-terms on both sides of equation (iii). Now, by equating the coefficients of x^2 in equation (iii), we get,

$$5=A_1+A_2$$

$$\Rightarrow \quad 5=(2)+A_2 \qquad (\quad A_1=2)$$

$$\Rightarrow \quad A_2=3$$

Again, by equating the coefficients of x in equation (iii)

$$-2=A_3+2(3)-2(2)$$

$$-2=A_3+2$$

$$A_3=-4$$

Finally, by putting the values of constants A_1, A_2 and A_3 in equation (i), we get

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{2}{(x+3)}+\frac{3}{(x-1)}-\frac{4}{(x-1)^2}.$$

Exercise 21.2

Resolve the following into partial fractions

1. $\dfrac{4x-3}{(x+1)^2}$
2. $\dfrac{x^2+7x+3}{x^2(x+3)}$
3. $\dfrac{5x^2-30x+44}{(x-2)^3}$
4. $\dfrac{18+21x-x^2}{(x-5)(x+2)^2}$
5. $\dfrac{x^2-x+3}{(x-1)^3}$

Case-III: Denominator consists of non-repeated quadratic factors

Let $R(x)=\frac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ is the product of non-repeated irreducible quadratic factors.

Now $R(x)=\frac{P(x)}{Q(x)}$ is resolved as:

$$\frac{P(x)}{Q(x)}=\frac{A_1x+A_2}{a_1x^2+b_1x+c_1}+\frac{A_3x+A_4}{a_2x^2+b_2x+c_2}+\ldots+\frac{A_{2n-1}x+A_{2n}}{a_nx^2+b_nx+c_n}$$

Here, the constants $A_1, A_2, A_3, \ldots, A_{2n}$ are to be found. The method is explained by following examples.

Example 1:

Resolve $\frac{7x^2+5x+13}{(x+1)(x^2+2)}$ into partial fractions.

Solution:

Let $\frac{7x^2+5x+13}{(x+1)(x^2+2)}=\frac{A_1}{x+1}+\frac{A_2x+A_3}{x^2+2}$ (i)

Here, A_1, A_2 and A_3 are unknown constants to be determined.

By multiplying $(x+1)(x^2+2)$ to both sides of eq: (i)

$7x^2+5x+13=A_1(x^2+2)+(A_2x+A_3)(x+1)$ (ii)

$7x^2+5x+13=A_1x^2+2A_1+A_2x^2+A_2x+A_3x+A_3$

$7x^2+5x+13=(A_1+A_2)x^2+(A_2+A_3)x+(2A_1+A_3)$ (iii)

To determine constants A_1, A_2 and A_3, values of x are chosen. To get A_1, put $x=-1$, in equation (ii), we get

$7(-1)^2+5(-1)+13=A_1((-1)^2+2)+(A_2(-1)+A_3)(-1+1)$

$\Rightarrow$ $15=3A_1$

$\Rightarrow$ $A_1=5$

By equating the like-terms on both sides of equation (iii), we get,

$7=(A_1+A_2)$ $5=(A_2+A_3)$

$\Rightarrow$ $7=(5+A_2)$ $\Rightarrow$ $5=(2+A_3)$

$\Rightarrow$ $2=A_2$ $\Rightarrow$ $3=A_3$

Finally, by putting the values of constants A_1, A_2 and A_3 in equation (i), we get

$$\frac{7x^2+5x+13}{(x+1)(x^2+2)}=\frac{5}{(x+1)}+\frac{2x+3}{(x^2+2)}.$$

Example 2:

Resolve $\dfrac{2x+7}{(x^2+3)(x^2+6)}$ into partial fractions.

Solution:

Let $\dfrac{2x+7}{(x^2+3)(x^2+6)} = \dfrac{A_1x+A_2}{x^2+3} + \dfrac{A_3x+A_4}{x^2+6}$ (i)

Here, A_1, A_2, A_3 and A_4 are unknown constants to be determined.

By multiplying $(x^2+3)(x^2+6)$ to both sides

$\Rightarrow 2x+7 = (x^2+6)(A_1x+A_2)+(x^2+3)(A_3x+A_4)$

$\Rightarrow 2x+7 = (A_1+A_3)x^3+(A_2+A_4)x^2+(6A_1+3A_3)x+(6A_2+3A_4)$ (ii)

By equating the like-terms on both sides of equation (ii), we get,

$A_1+A_3=0$ (iii)

$A_2+A_4=0$ (iv)

$6A_1+3A_3=2$ (v)

$6A_2+3A_4=7$ (vi)

solving equations (i) and (iv) simultaneously, we get $A_1=\dfrac{2}{3}$, and $A_3=-\dfrac{2}{3}$.

similarly, the values are $A_2=\dfrac{7}{3}$ and $A_4=-\dfrac{7}{3}$.

Finally, by putting the values of constants A_1, A_2, A_3 and A_4 in equation (i), we get

$$\frac{2x+7}{(x^2+3)(x^2+6)} = \frac{\frac{2}{3}x+\frac{7}{3}}{x^2+3}+\frac{-\frac{2}{3}x-\frac{7}{3}}{x^2+6} = \frac{2x+7}{3(x^2+3)}+\frac{-2x-7}{3(x^2+6)}.$$

Exercise 21.3

Resolve the following into partial fractions

1. $\dfrac{x^2-x-13}{(x^2+7)(x-2)}$
2. $\dfrac{6x-5}{(x^2+10)(x+1)}$
3. $\dfrac{15+5x+5x^2-4x^3}{x^2(x^2+5)}$
4. $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$
5. $\dfrac{x^2-x+2}{(x+1)(x^2+3)}$

Case-IV: Denominator consists of repeated quadratic factors

Let $R(x)=\dfrac{P(x)}{Q(x)}$ is a rational fraction, where, its denominator $Q(x)$ has the repeated irreducible quadratic factors. For the sake of simplicity, it is taken that $Q(x)=(ax^2+bx+c)^n$.

Now $R(x)=\dfrac{P(x)}{Q(x)}$ is resolved as:

$$\frac{P(x)}{Q(x)}=\frac{A_1x+A_2}{ax^2+bx+c}+\frac{A_3x+A_4}{(ax^2+bx+c)^2}+\dots+\frac{A_{2n-1}x+A_{2n}}{(ax^2+bx+c)^n}.$$

Here, the constants $A_1, A_2, A_3, \dots, A_n$ are to be found. The method is explained by following examples.

Example 1: Resolve $\dfrac{5x^2+2}{\left(x^2+x+1\right)^2}$ into partial fractions.

Solution:

Let $$\frac{5x^2+2}{\left(x^2+x+1\right)^2}=\frac{A_1x+A_2}{x^2+x+1}+\frac{A_3x+A_4}{\left(x^2+x+1\right)^2} \qquad \text{(i)}$$

Here, A_1, A_2, A_3 and A_4 are unknown constants to be determined.

By multiplying $\left(x^2+x+1\right)^2$ to both sides

$5x^2+2=\left(x^2+x+1\right)\left(A_1x+A_2\right)+A_3x+A_4$

$5x^2+2=x^3A_1+x^2A_1+x^2A_2+xA_1+xA_2+xA_3+A_2+A_4$

$0.x^3+5x^2+0.x+2=x^3A_1+x^2\left(A_1+A_2\right)+x\left(A_1+A_2+A_3\right)+\left(A_2+A_4\right) \qquad \text{(ii)}$

By equating the like-terms on both sides of equation (ii), we get,

$A_1=0$ (iii)

$A_1+A_2=5$ (iv)

$A_1+A_2+A_3=0$ (v)

$A_2+A_4=2$ (vi)

Put the value of $A_1=0$, in eq (iv) we get $A_2=5$.

Similarly, by putting the values of A_2 and A_1 in eq (v)

We get

$0+5+A_3=0$

$\Rightarrow \quad A_3=-5$

Again by putting the values of A_2 in eq (vi)

$5+A_4=2$

$\Rightarrow \quad A_4=-3$

$$\frac{5x^2+2}{\left(x^2+x+1\right)^2}=\frac{0.x+5}{x^2+x+1}+\frac{-5x-3}{\left(x^2+x+1\right)^2}.$$

Example: 2

Resolve $\dfrac{x^4+x^3+x^2+x+1}{x\left(x^2+1\right)^2}$ into partial fractions.

Solution

Let $\dfrac{x^4+x^3+x^2+x+1}{x\left(x^2+1\right)^2}=\dfrac{A_1}{x}+\dfrac{A_2x+A_3}{\left(x^2+1\right)}+\dfrac{A_4x+A_5}{\left(x^2+1\right)^2}$ (*i*)

Here, A_1, A_2, A_3, A_4 and A_5 are unknown constants to be determined.

By multiplying $x\left(x^2+1\right)^2$ to both sides of eq (i)

$$x^4+x^3+x^2+x+1=A_1\left(x^2+1\right)^2+x\left(A_2x+A_3\right)\left(x^2+1\right)+x\left(A_4x+A_5\right)$$
$$x^4+x^3+x^2+x+1=A_1x^4+2A_1x^2+A_1+A_2x^4+A_2x^2+A_3x^3+A_3x+A_4x^2+A_5x$$
$$x^4+x^3+x^2+x+1=\left(A_1+A_2\right)x^4+A_3x^3+\left(2A_1+A_2+A_4\right)x^2+\left(A_3+A_5\right)x+A_1 \quad \text{(ii)}$$

To determine constants A_1, A_2, A_3, A_4 and A_5, we equate the like-terms of equation (ii), we have

$$1=A_1+A_2 \quad \text{(iii)}$$
$$1=A_3 \quad \text{(iv)}$$
$$1=2A_1+A_2+A_4 \quad \text{(v)}$$
$$1=A_3+A_5 \quad \text{(vi)}$$
$$1=A_1 \quad \text{(vii)}$$

Put the value of A_1 in equation (iii), we get

$$1=A_1+A_2$$
$$\Rightarrow \quad 1=\left(1+A_2\right)$$
$$\Rightarrow \quad 0=A_2$$

Put the value of A_3 in equation (vi), we get

$$1=1+A_5$$
$$0=A_5$$

Put the value of A_1 and A_2 in equation (v), we get

$$1=2(1)+(0)+A_4$$
$$-1=A_4$$

Finally, putting all values of constants A_1, A_2, A_3, A_4 and A_5 in equation (i), we get

$$\frac{x^4+x^3+x^2+x+1}{x\left(x^2+1\right)^2}=\frac{1}{x}+\frac{1}{x^2+1}-\frac{x}{(x^2+1)^2}.$$

Exercise 21.4

Resolve the following into partial fractions

1. $\dfrac{x^2}{(x^2+1)^2(1-x)}$
2. $\dfrac{x^2+x+2}{x^2(x^2+3)^2}$
3. $\dfrac{x^2}{(1+x)(1+x^2)^2}$
4. $\dfrac{7}{(x+1)(2+x^2)^2}$
5. $\dfrac{49}{(x-2)(3+x^2)^2}$

Review Exercise 21

1. Tick the correct option

i. An improper fraction can be reduced into proper fraction by
   (a) addition (b) multiplication
   (c) subtraction (d) division

ii. Partial fractions of $\dfrac{x}{(x-a)(x-b)(x-c)}$ can have a form
   (a) $\dfrac{A}{x+a}+\dfrac{B}{x+b}+\dfrac{C}{x+c}$ (b) $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
   (c) $\dfrac{A}{x+a}+\dfrac{B}{x-b}+\dfrac{C}{x+c}$ (d) None of these

iii. Find the partial fractions of $\dfrac{x-3}{x^3+3x}$ are ______________.
   (a) $\dfrac{-1}{x}-\dfrac{x-1}{x^2+3}$ (b) $\dfrac{1}{x}+\dfrac{x+1}{x^2+3}$
   (c) $\dfrac{1}{x}-\dfrac{x+1}{x^2+3}$ (d) $\dfrac{-1}{x}+\dfrac{x+1}{x^2+3}$

iv. $\dfrac{x^3+1}{(x-1)(x+2)}$ is
   (a) Proper fraction (b) An improper fraction
   (c) An identity (d) A constant term

v. The fraction $\dfrac{2x+5}{x^2+5x+6}$ is known as:
   (a) Proper (b) Improper
   (c) Both proper and improper (d) None of these

2. Define proper, improper and rational fraction.

3. Resolve the following fractions into partial functions.

i) $\dfrac{5x+8}{(x-1)(x+2)}$

ii) $\dfrac{9x^2+5x+7}{x(x+2)(x-5)}$

iii) $\dfrac{x^2+2x+3}{(x^2+1)(x-2)}$

iv) $\dfrac{x^3+8x^2+9}{(x^2+x+1)(x+1)}$

v) $\dfrac{x+5}{(x^2+1)^2(x-3)}$

vi) $\dfrac{7x+3}{(x-1)^2(x+2)}$