THEORY OF QUADRATIC EQUATIONS

Unit 20

• Weightage = 12%

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define discriminant (b^2 - $4ac$) of the quadratic expression $ax^2 + bx + c$.
- Find the discriminant of a given quadratic equation.
- Discuss the nature of the roots of a quadratic equation through discriminant.
- Determine the nature of the roots of a given quadratic equation and verify the result by solvingthe equation.
- Determine the values of an unknown involved in agiven quadratic equation when the nature of its roots is given.
- Find cube roots of unity.
- Recognize complex cube roots of unity as ω and ω^2.
- Verify the properties of cube roots of unity.
- Use properties of cube roots of unity to solve alliedproblems.
- Find the relation between the roots and the coefficients of a quadratic equation.
- Find the sum and the product of the roots of a given quadratic equation without solving it.
- Find the value(s) of the unknown involved in a given quadratic equation when
  - Sum of roots is equal to a multiple of the product of the roots,
  - Sum of the squares of the roots is equal to a given number,
  - Roots differ by a given number,
  - Roots satisfying a given relation (e.g., the relation $2\alpha + 5\beta = 7$, where α and β are the roots of the given equations),
  - Both sum and product of the roots are equal to a given number.
- Define symmetric functions of the roots of a quadratic equation.
- Represent a symmetric function graphically
- Evaluate a symmetric function of the roots of a quadratic equation in terms of its coefficients.
- Establish the formula, x^2 – (sum of the roots) x + (product of the roots) = 0, to find the quadratic equation of the given roots.
- form the quadratic equation whose roots, for examples are of the type:
  - $2\alpha+1, 2\beta+1$
  - α^2, β^2
  - $\frac{1}{\alpha}, \frac{1}{\beta}$
  - $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$
  - $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta}$

  where, α , β are the roots of a quadratic equation
- Find the values of α , β , where the roots of an equation are $\frac{1}{\alpha}, \frac{1}{\beta}$
- Solve the cubic equation if one root of the equation is given,
- Solve a biquadratic (quartic) equation if two of the real roots of the equation are given.
- Solve a system of two equations in two variables, when
  - One equation is linear and the other is quadratic,
  - Both the equations are quadratic.
- Solve the real life problems leading to quadratic equations.

20.1 Nature of the Roots of a Quadratic Equation:

20.1.(i) Define discriminant $\left(b^2-4ac\right)$ of the quadratic expression $ax^2+bx+c, a\neq 0$.

A polynomial of second degree is called Quadratic Expression. General form of quadratic expression is $ax^2+bx+c, (a\neq 0)$ where a and b are coefficients of x^2, and x respectively, where c is a constant term.

The discriminant of a quadratic expression

For quadratic expression $ax^2+bx+c, (a\neq 0)$ the expression b^2-4ac is called its discriminant and is denoted by Δ or D i.e. $\Delta = b^2-4ac$

20.1.(ii) Find the discriminant of a given quadratic equation

We already know that solution of the quadratic equation $ax^2+bx+c=0, a\neq 0$ is $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. Here b^2-4ac enables us to know the nature of the roots either they are real or complex. The expression b^2-4ac is the discriminant of quadratic equation.

$$ax^2+bx+c=0, a\neq 0$$

Example: Find the discriminant of $2x^2+5x-4=0$

Solution: $2x^2+5x-4=0$

Here, $a=2, b=5$ and $c=-4$

$$\Delta = b^2-4ac = (5)^2-4(2)(-4)$$
$$= 25+32 = 57$$

20.1.(iii) Discuss the nature of the roots of a quadratic equation through discriminant.

We can find the nature of the roots of the quadratic equation $ax^2+bx+c=0, a\neq 0$, using discriminant i.e. $\Delta = b^2-4ac$.

The roots of the quadratic equation $ax^2+bx+c=0$ are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$. The nature of the roots depends on the value of b^2-4ac, where a, b and c being real numbers.

i. If $\Delta = b^2-4ac>0$, then the roots are real and unequal.
ii. If $\Delta = b^2-4ac<0$, then the roots are non-real (complex or imaginary) and conjugate of each other.
iii. If $\Delta = b^2-4ac=0$, then the roots are real and equal, each being equal to $-\dfrac{b}{2a}$.
iv. If a,b,c are rationals and $\Delta=b^2-4ac>0$ and perfect square, then roots are rational and unequal otherwise irrational and unequal.

Note: Imaginary roots only arise in case of pure quadratic equation i.e $ax^2+c=0$, $\forall\, a, c\in R^+$.

20.1.(iv) Determine the nature of the roots of a given quadratic equation and verify the result by solving the equation.

Example 1:

Use the discriminant to find the nature of the roots of the following equations and verify by solving the equation:

(i) $x^2+5x-14=0$ (ii) $5x^2+3x+1=0$

(iii) $x^2-9x+5=0$ (iv) $9x^2=6x-1$

Solution:

(i) $x^2+5x-14=0$

Here $a=1, b=5$ and $c=-14$

$$\Delta=b^2-4ac$$

$\therefore$ $\Delta=(5)^2-4(1)(-14)$

$\Rightarrow$ $\Delta=25+56$

$\Rightarrow$ $\Delta=81=(9)^2>0,$

and perfect square

Hence roots are real, rational and unequal

Verification:

$x^2+5x-14=0$

$x^2+7x-2x-14=0$

$x(x+7)-2(x+7)=0$

$x+7=0$ or $(x-2)=0$

$\Rightarrow$ $x=-7$ or $x=2$

Here, roots are real, rational and unequal

Hence verified.

Solution:

(ii) $5x^2+3x+1=0,$

Here $a=5, b=3$ and $c=1$

$$\Delta=b^2-4ac$$

$\therefore$ $\Delta=(3)^2-4(5)(1)$

$\Rightarrow$ $\Delta=9-20$

$\Rightarrow$ $\Delta=-11<0,$ i.e negative.

Hence roots are complex and conjugate of each other

Verification:

By using Quadratic formula

$$x=\frac{-3\pm\sqrt{9-20}}{2}$$

$$x=\frac{-3\pm\sqrt{11}\,i}{2} \qquad (\because i^2=-1)$$

Here, roots are complex and conjugate of each other

Hence verified

Solution:

(iii) $x^2-9x+5=0,$

Here $a=1, b=-9$ and $c=5$

$$\Delta=b^2-4ac$$

$\therefore$ $\Delta=(-9)^2-4(1)(5)$

$\Rightarrow$ $\Delta=81-20$

$\Rightarrow$ $\Delta=61>0,$ not a perfect square.

Hence the roots are real, irrational and unequal.

Verification:

$x^2-9x+5=0,$

By using Quadratic formula

$$x=\frac{9\pm\sqrt{81-20}}{2}$$

$$x=\frac{-3\pm\sqrt{61}}{2}$$

Here the roots are real, irrational and unequal.

Hence verified.

Solution:

(iv) $9x^2 = 6x - 1$

$\Rightarrow 9x^2 - 6x + 1 = 0$

Here $a = 9, b = -6$ and $c = 1$

$\because \quad \Delta = b^2 - 4ac$

$\therefore \quad \Delta = (6)^2 - 4(9)(1)$

$\Rightarrow \quad \Delta = 36 - 36$

$\Rightarrow \quad \Delta = 0$

Hence roots are real, rational and equal.

Verification:

$9x^2 - 6x + 1 = 0$

$(3x)^2 - 2(3x)(1) + (1)^2 = 0$

$(3x - 1)^2 = 0$

$3x - 1 = 0 \quad \text{or} \quad 3x - 1 = 0$

$\Rightarrow \quad x = \dfrac{1}{3} \quad \text{or} \quad x = \dfrac{1}{3}$

Hence roots are real and equal.

Hence verified.

Example 2:

Show that the roots of the equation $abc^2x^2 + c(3a^2 + b^2)x + 3a^2 + b^2 - ab = 0$, $\forall\, a, b, c \in Q$ are rational.

Solution: $abc^2x^2 + c(3a^2 + b^2)x + 3a^2 + b^2 - ab = 0$

Here $\text{A} = abc^2, \text{B} = c(3a^2 + b^2)$ and $\text{C} = 3a^2 + b^2 - ab$

$\Delta = \text{B}^2 - 4\text{AC}$

$\Delta = \left[c(3a^2 + b^2)\right]^2 - 4(abc^2)(3a^2 + b^2 - ab)$

$= c^2(3a^2 + b^2)^2 - 4abc^2(3a^2 + b^2 - ab)$

$= c^2\left[(3a^2 + b^2)^2 - 4ab(3a^2 + b^2) + 4a^2b^2\right]$

$= c^2\left[(3a^2 + b^2)^2 - 2(3a^2 + b^2)(2ab) + (2ab)^2\right] \qquad \because \ 4ab = 2(2ab)$

$= c^2\left[(3a^2 + b^2 - 2ab)^2\right]$

$= \left[c(3a^2 + b^2 - 2ab)\right]^2 \geq 0$ and perfect square. Hence roots are rational.

20.1.(v) Determine the values of an unknown involved in a given quadratic equation when the nature of its root is given.

The following examples will help us to find the unknown involved in the quadratic equation when nature of roots is given.

Example 1:

For what value of p will $3x^2 + 5x + p = 0$ have

(i) Equal roots (ii) Rational roots (iii) Complex roots.

Solutions (i):

$3x^2 + 5x + p = 0$

Here $a = 3, b = 5$ and $c = p$

$\Delta = b^2 - 4ac$

$\therefore \quad \Delta = (5)^2 - 4(3)(p) = 25 - 12p,$

For equal roots

$$\Delta = 0$$

i.e. $25 - 12p = 0$

$\Rightarrow$ $-12p = -25,$

$\Rightarrow$ $p = \frac{25}{12},$

Thus, at $p = \frac{25}{12}$, given equation has equal roots.

(ii) For rational roots,

Δ should be perfect square

i.e. $25 - 12p$ must be perfect square.

when $p = 0, 2$ and -2 etc.

Thus, at p = 0,2,-2 and some others values, the given equation has rational roots. make the discriminant perfect square.

(iii) For complex roots,

i.e. $25 - 12p < 0$

$\Rightarrow$ $12p > 25$

$\Rightarrow$ $p > \frac{25}{12}$

Thus, when $p > \frac{25}{12}$, given equation has complex roots.

Example 2:

If the equation $x^2 - 15 - k(2x - 8) = 0$ has equal roots, find the values of k.

Solution:

$x^2 - 15 - k(2x - 8) = 0$, re-write it in the standard form of the quadratic equation, we get

i.e. $x^2 - 2kx + 8k - 15 = 0$

Here $a = 1, b = -2k$ and $c = 8k - 15$

$$\Delta = b^2 - 4ac$$

$\therefore$ $\Delta = (-2k)^2 - 4(1)(8k - 15)$

$\Rightarrow$ $\Delta = 4k^2 - 4(8k - 15)$

For equal roots

$$\Delta = 0$$

$\therefore$ $4k^2 - 4(8k - 15) = 0$

$\Rightarrow$ $k^2 - 8k + 15 = 0$ (dividing both sides by 4)

$\Rightarrow$ $k^2 - 5k - 3k + 15 = 0$

$\Rightarrow$ $k(k - 5) - 3(k - 5) = 0$

$\Rightarrow$ $(k - 5)(k - 3) = 0$

i.e. $k - 5 = 0$ or $k - 3 = 0$

$\Rightarrow$ $k = 5$ or $k = 3$

Thus, $k = 3$ and 5 are the required values.

Example 3:

For what value of p, the equation $2x^2 - px + 18 = 0$, has real and unequal roots.

Solution:

$2x^2 - px + 18 = 0,$

Here $a = 2, b = -p$ and $c = 18$

$\Delta = b^2 - 4ac$

$\therefore \quad \Delta = (-p)^2 - 4(2)(18) = p^2 - 144$

For real and unequal roots,

$\Delta > 0$

$\therefore \quad p^2 - 144 > 0$

$\Rightarrow \quad p^2 - 144 > 0 \Rightarrow \sqrt{p^2} > \sqrt{144} \Rightarrow |p| > 12 \Rightarrow \pm p > 12 \Rightarrow p > 12$ or $-p > 12$

$\Rightarrow \quad p > 12$ or $p < -12$

$\Rightarrow$ Thus, for $p > 12$ or $p < -12$ given equation has real and unequal roots.

Exercise 20.1

1. Use discriminant find the nature of the roots of the following quadratic equations:

(i)	$x^2 + 5x - 6 = 0$	(ii)	$x^2 = 6x$	(iii)	$2x^2 + 8 = 6x$
(iv)	$1 + 6x + 9x^2 = 0$	(v)	$24x^2 + 12x + 36 = 0$	(vi)	$x = x^2 + 1$
(vii)	$3x^2 + 6 = 5x$	(viii)	$3x^2 + 9 = 0$		

2. Find the value(s) of k that ensure that following quadratic equations have
   (a) Same solution (b) Different real solutions
   (Hint: for same solution $\Delta = 0$ and for different real solutions $\Delta > 0$).

(i)	$x^2 - 3x + k = 0$	(ii)	$x^2 + k = 4$
(iii)	$x^2 + kx + 2 = 0$	(iv)	$(k-1)x^2 - 4x + 2 = 0$
(v)	$x^2 + kx + 4 = 0$	(vi)	$9x^2 + kx = -16$
(vii)	$(k-2)x^2 = 4x + (k+2)$	(viii)	$x^2 + 1 = kx$

3. Determine the value of m in each of the following quadratic equations that will make the roots equal.
   (i) $(m+1)x^2 + 2(m+3)x + (2m+3) = 0,$ provided $m \neq -1$
   (ii) $9x^2 + mx + 16 = 0$

4. Show that the roots of the following quadratic equations are real.
   (i) $x^2 - 2x\left(k + \frac{1}{k}\right)x + 3 = 0, \ \forall k \in \mathbb{R} - \{0\}$
   (ii) $2nx^2 + 2(l + m + n)x + (l + m) = 0, \ \forall l, m, n \in \mathbb{R}$ and $n \neq 0$

5. Show that the roots of the following quadratic equations are rational.
   (i) $(l - m)x^2 + (m + n - l)x - n = 0, \ \forall l, m, n \in \mathbb{R}$ and $l \neq m$
   (ii) $(a + c - b)x^2 + 2cx + (b + c - a) = 0, \ \forall a, b, c \in \mathbb{R}$

6. For what values of p and q the roots of quadratic equation $x^2+(2p-4)x-(3q+5)=0$ vanish?

7. Show that the roots of the quadratic equation given by $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are real and they can not be equal unless $a=b=c$.

20.2 Cube Roots of Unity and their properties

20.2.(i) Find the cube roots of unity

Let x be the cube root of unity i.e. $x=\sqrt[3]{1}=(1)^{1/3}$.

Cubing on both sides, we have,

$$x^3=1$$

$\Rightarrow$ $(x)^3-(1)^3=0$

$\Rightarrow$ $(x-1)(x^2+x+1)=0,$ $\quad [\because a^3-b^3=(a-b)(a^2+ab+b^2)]$

i.e. $x-1=0$

$\Rightarrow$ $x=1$

or $x^2+x+1=0,$

Here $a=b=c=1,$

Using quadratic formula, we have,

$$\therefore \quad x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{-1\times 3}}{2}=\frac{-1\pm i\sqrt{3}}{2}, \quad (\because i^2=-1)$$

$$\Rightarrow \quad x=\frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}$$

Thus cube root of unity are

$$1, \frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}.$$

20.2.(ii) Recognize complex cube roots of unity as ω and ω^2.

Two complex cube roots are:

$\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$, for these two complex roots we use a small Greek letter "ω" and read as omega, and let us assume $\omega=\frac{-1+i\sqrt{3}}{2}$ then $\omega^2=\frac{-1-i\sqrt{3}}{2}$.

Hence cube roots of unity now are $1, \omega$ and ω^2.

20.2.(iii) Verify the properties of cube roots of unity.

Properties of the cube roots of unity are:

(i) Each of the complex cube root of unity is the square of other.

Verification:

If $\omega=\frac{-1+i\sqrt{3}}{2}$ is one complex cube root of unity,

$\therefore \quad \omega^2 = \left(\dfrac{-1+i\sqrt{3}}{2}\right)^2 = \dfrac{1-2i\sqrt{3}+3i^2}{4} = \dfrac{1-2i\sqrt{3}+3(-1)}{4}$

$\Rightarrow \quad \omega^2 = \dfrac{-2-2i\sqrt{3}}{4} = \dfrac{2(-1-i\sqrt{3})}{4} \qquad (\because i^2=-1)$

$\Rightarrow \quad \omega^2 = \dfrac{-1-i\sqrt{3}}{2}$

Now $\left(\omega^2\right)^2 = \left(\dfrac{-1-i\sqrt{3}}{2}\right)^2 = \dfrac{1+2i\sqrt{3}+3i^2}{4} = \dfrac{1+2i\sqrt{3}+3(-1)}{4} \qquad (\because i^2=-1)$

$\Rightarrow \quad \left(\omega^2\right)^2 = \dfrac{-2+2i\sqrt{3}}{4} = \dfrac{2(-1+i\sqrt{3})}{4} = \dfrac{-1+i\sqrt{3}}{2}$

$\Rightarrow \quad \left(\omega^2\right)^2 = \omega.$

Hence each complex cube root of unity is square of the other
Verified.

(ii) Sum of three cube roots of unity is zero, i.e. $1+\omega+\omega^2=0$

Verification: L.H.S

$= 1+\omega+\omega^2$

$= 1+\left(\dfrac{-1+i\sqrt{3}}{2}\right)+\left(\dfrac{-1-i\sqrt{3}}{2}\right)$, where $\omega = \dfrac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$

$= \dfrac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}$

$= \dfrac{2-2}{2} = \dfrac{0}{2} = 0 = \text{R.H.S.}$

Verified.

(iii) Product of three cube roots of unity is 1, i.e, $\omega\cdot\omega^2\cdot 1 = 1$ **or** $\omega^3 = 1$

Verification: L.H.S $= \omega\cdot\omega^2\cdot 1$

$= 1\left(\dfrac{-1+i\sqrt{3}}{2}\right)\left(\dfrac{-1-i\sqrt{3}}{2}\right)$

$= \dfrac{1-i^2 3}{4}$

$= \dfrac{1-(-1)(3)}{4}, \qquad (\because i^2=-1)$

$= \dfrac{1+3}{4} = \dfrac{4}{4} = 1$

$= \text{R.H.S} \qquad$ i.e. $1\cdot\omega\cdot\omega^2 = 1$ or $\omega^3 = 1$

Verified.

(iv) Each complex cube root of unity is reciprocal of the other.

Verification One complex cube root is ω,

$\Rightarrow \quad \omega^3 = 1,$

$\Rightarrow \quad \omega \cdot \omega^2 = 1,$

$\Rightarrow \quad \omega^2 = \frac{1}{\omega}$ or $\omega = \frac{1}{\omega^2}.$

Verified

(v) Every integral power of ω^3 is unity.

Verification

$\because \quad \omega^3 = 1,$

$\therefore \quad \left(\omega^3\right)^m = 1, \qquad \forall m \in \mathbb{Z}.$

$\Rightarrow \quad \omega^{3m} = 1.$

Verified

20.2.(iv) Use properties of cube roots of unity to solve allied problems.

Following are the allied problem related to the cube root of unity.

Example 1: Find all the cube roots of –27.

Solution:

Let x be the cube root of –27,

$\therefore \quad x = (-27)^{\frac{1}{3}}$

Cubing on both the sides, we have,

$x^3 = -27$

$\Rightarrow \quad x^3 + 27 = 0$

$\Rightarrow \quad (x)^3 + (3)^3 = 0$

$\Rightarrow \quad (x+3)\left(x^2 - 3x + 9\right) = 0, \qquad \left[\because a^3 + b^3 = (a+b)\left(a^2 - ab + b^3\right)\right]$

i.e, $\quad x + 3 = 0 \Rightarrow x = -3,$

Now $\quad x^2 - 3x + 9 = 0$

Here, $a = 1, b = -3$ and $c = 9$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{9 - 36}}{2}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{-27}}{2}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{-1 \times 27}}{2}$$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{i^2 \times 27}}{2} \qquad \left(\because i^2 = -1\right)$$

$\Rightarrow \quad x=\dfrac{-(-3)\pm i3\sqrt{3}}{2}$

$\Rightarrow \quad x=\dfrac{-3\left(-1\pm i\sqrt{3}\right)}{2}$

$\Rightarrow \quad x=\dfrac{-3\left(-1+i\sqrt{3}\right)}{2}$ and $\dfrac{-3\left(-1-i\sqrt{3}\right)}{2}$

$\Rightarrow \quad x=-3\omega,-3\omega^2$

Hence, three cube roots of -27 are $-3,-3\omega$ and $-3\omega^2$.

Note: The complex cube roots of a^3 are $a\omega$ and $a\omega^2, \forall a\in R-\{0\}$
e.g the complex cube roots of 8 are 2ω and $2\omega^2$.

Example 2: Show that:

(a) $2+\omega=\dfrac{3}{2+\omega^2}$

(b) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$... to $2n$ factors $=1$, where $n\in\mathbb{N}$

(a) Solution: L.H.S $=2+\omega$

$=\dfrac{(2+\omega)(2+\omega^2)}{(2+\omega^2)}$ Multiply and divide by $(2+\omega^2)$

$=\dfrac{4+2\omega+2\omega^2+\omega^3}{2+\omega^2}$

$=\dfrac{4+2(\omega+\omega^2)+1}{2+\omega^2}$ $(\because \omega^3=1)$

$=\dfrac{5+2\ (-1)}{2+\omega^2}$ $(\because \omega+\omega^2=-1)$

$=\dfrac{5-2}{2+\omega^2}=\dfrac{3}{2+\omega^2}=$ RHS

Hence shown.

(b) Solution: L.H.S $=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$... to $2n$ factors $\forall n\in\mathbb{N}$

$=[(1+\omega)(1+\omega^2)]\,[(1+\omega^4)(1+\omega^8)]$...to n factors

$=\left[(-\omega^2)(-\omega)\right]\left[(1+\omega^3\cdot\omega)(1+\omega^6\cdot\omega^2)\right]$... to n factors, $\left(\begin{array}{l}\because 1+\omega=-\omega^2\\ 1+\omega^2=-\omega\\ \omega^3=\omega^6=1\end{array}\right)$

$=\left[(-1)^2\cdot(\omega^3)\right]\left[(1+\omega)(1+\omega^2)\right]$... to n factors,

$=\left[(-1)^2\cdot 1\right]\left[(-\omega^2)(-\omega)\right]$... to n factors,

$=\left[(-1)^2\right]\left[(-1^2)\cdot\omega^3\right]$... to n factors,

$=(-1)^2\cdot(-1)^2$ ... to n factors,

$=(1)(1)$... to n factors,

$=(1)^n=1$. **Hence Shown.**

Example 3: Prove that

$$\left(x+y+z\right)\left(x+y\omega+z\omega^2\right)\left(x+y\omega^2+z\omega\right)=x^3+y^3+z^3-3xyz,\text{where}$$

ω and ω^2 are complex cube roots of unity.

Proof: **L.H.S** $=\left(x+y+z\right)\left(x+y\omega+z\omega^2\right)\left(x+y\omega^2+z\omega\right)$

$$=\left(x+y+z\right)\left[\left(x+y\omega+z\omega^2\right)\left(x+y\omega^2+z\omega\right)\right]$$

$$=\left(x+y+z\right)\left[x^2+xy\omega^2+xz\omega+xy\omega+y^2\omega^3+yz\omega^2+xz\omega^2+yz\omega^4+z^2\omega^3\right]$$

$$=\left(x+y+z\right)\left[x^2+y^2\left(1\right)+z^2\left(1\right)+xy\left(\omega^2+\omega\right)+yz\left(\omega^2+\omega^4\right)+zx\left(\omega+\omega^2\right)\right]$$

$$=\left(x+y+z\right)\left[x^2+y^2+z^2+xy\left(-1\right)+yz\left(-1\right)+zx\left(-1\right)\right],$$

$$=\left(x+y+z\right)\left[x^2+y^2+z^2-xy-yz-zx\right] \qquad \left(\begin{array}{l}\because\ \omega^3=1,\omega^2+\omega=-1\\ \text{and } \omega^2+\omega^4=\omega+\omega^2=-1\end{array}\right)$$

$$=x^3+y^3+z^3-3xyz.$$

= R.H.S

Hence Proved.

Exercise 20.2

1. Find all the cube roots of:

(i) 64 (ii) -125 (iii) 216 (iv) $-b^3$

2. Evaluate the following:

(i) $\left(1+\omega^2\right)^4$ (ii) $\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)$

(iii) $\left(2+5\omega+2\omega^2\right)^6$ (iv) $\left(-1+\sqrt{-3}\right)^4+\left(-1-\sqrt{-3}\right)^4$

3. Show that:

(i) $\left(1+\omega\right)\left(1+\omega^2\right)\left(1+\omega^4\right)\left(1+\omega^8\right)=\left(\omega+\omega^2\right)^4$

(ii) $\left(a+b\right)\left(a\omega+b\omega^2\right)\left(a\omega^2+b\omega\right)=a^3+b^3$

(iii) $\left(a+\omega b+\omega^2c\right)\left(a+\omega^2b+\omega c\right)=a^2+b^2+c^2-ab-ba-ca$

(iv) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^9+\left(\dfrac{-1-i\sqrt{3}}{2}\right)^9-2=0$

(v) $\left(1+\omega-\omega^2\right)^3-\left(1-\omega+\omega^2\right)^3=0$

20.3 Roots and Coefficient of a Quadratic Equation

20.3.(i) Find the relation between the roots and the coefficient of a quadratic equation

Let the two roots of $ax^2+bx+c=0, a\neq 0$ be denoted by α and β, then

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad \alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad \alpha+\beta = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a} \qquad (1)$$

and $$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$\Rightarrow \quad \alpha\beta = \frac{(-b)^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \qquad (2)$$

Therefore (1) represents the sum of the roots $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

and (2) represents the product of the roots $= \dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

Hence, we have the important result:

If α, β be the roots of the equation $ax^2+bx+c=0, a \neq 0$, then

$\alpha+\beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

20.3.(ii) Find the sum and product of the roots of a given quadratic equation without solving it.

Example: Without solving, find the sum and product of roots in each of the following equations.

(i) $4x^2+6x+1=0$ (ii) $3(5x^2+1)=17x$

Solutions (i):

$4x^2+6x+1=0$

Here $a=4, b=6$ and $c=1$

$\because \quad \alpha+\beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

$\therefore \quad \alpha+\beta = -\dfrac{6}{4}$ and $\alpha\beta = \dfrac{1}{4}$

$\Rightarrow \quad \alpha+\beta = -\dfrac{3}{2}$ and $\alpha\beta = \dfrac{1}{4}$

Solution (ii): $3(5x^2+1)=17x$

$\Rightarrow \quad 15x^2-17x+3=0$

Here $a=15, b=-17$ and $c=3$

$\because \quad \alpha+\beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

$\therefore \quad \alpha+\beta = -\dfrac{(-17)}{15}$ and $\alpha\beta = \dfrac{3}{15} = \dfrac{1}{5}$

$\Rightarrow \quad \alpha+\beta = \dfrac{17}{15}$ and $\alpha\beta = \dfrac{1}{5}$

20.3 (iii) Find the value (s) of the unknown involved in a given quadratic equation when:

(a) Sum of the roots is equal to a multiple of the product of the roots.
(b) Sum of the squares of the roots is equal to a given number.
(c) Roots differ by a given number.

(d) Roots satisfying a given relation. (e.g, the relation $2\alpha + 5\beta = 7$, where α and β are the roots of the equation)

(e) Both sum and product of the roots are equal to a given number.

The all above mentioned conditions can be explained through examples.

(a) Sum of the roots is equal to a multiple of the product of the roots.

Example 1: Find the values of k, if the sum of the roots of $6x^2 - 3kx + 5 = 0$ is equal to the product of its roots.

Solution: Let α, β be the roots of the equation $6x^2 - 3kx + 5 = 0$

Here, $a = 6$, $b = -3k$ and $c = 5$

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-3k)}{6} = \frac{3k}{6} \text{ and } \alpha\beta = \frac{c}{a} = \frac{5}{6}$$

It is given that

Sum of the roots = product of the roots

i.e, $\alpha + \beta = \alpha\beta$

$\therefore \quad \frac{3k}{6} = \frac{5}{6} \quad \Rightarrow \quad k = \frac{5}{3}$.

Example 2: Find the value of p, if the sum of the roots is equal to two times the product of the roots of the equation $2x^2 + (8 - 4p)x + 3p = 0$.

Solution: Let α, β be the roots of the equation $2x^2 + (8 - 4p)x + 3p = 0$

Here $a = 2$, $b = 8 - 4p$ and $c = 3p$

$$\alpha + \beta = -\frac{b}{a} = -\frac{(8-4p)}{2} = 2p - 4 \text{ and } \alpha\beta = \frac{c}{a} = \frac{3p}{2}$$

As per condition in the problem, we have

$\alpha + \beta = 2(\alpha\beta)$

$\therefore \quad 2p - 4 = 2\left(\frac{3p}{2}\right) = 3p$

$\Rightarrow \quad p = -4$.

(b) Sum of the squares of the roots is equal to a given number.

Example 3: Find k, if the sum of the squares of the roots of the equation $2x^2 + 3kx + k^2 = 0$ is 5.

Solution: Let α, β be the roots of the equation

$2x^2 + 3kx + k^2 = 0$

Here, $a = 2$, $b = 3k$ and $c = k^2$

$$\alpha + \beta = -\frac{b}{a} = -\frac{3k}{2} \text{ and } \alpha\beta = \frac{c}{a} = \frac{k^2}{2}$$

According to given condition

$\alpha^2 + \beta^2 = 5$

$\Rightarrow \quad (\alpha+\beta)^2 - 2\alpha\beta = 5 \qquad \because [a^2+b^2=(a+b)^2-2ab]$

$\Rightarrow \quad \left(\frac{-3k}{2}\right)^2 - 2\left(\frac{k^2}{2}\right) = 5$

$\Rightarrow \quad \frac{9k^2}{4} - k^2 = 5$

$\Rightarrow \quad \frac{9k^2-4k^2}{4} = 5$

$\Rightarrow \quad 5k^2 = 5\times 4$

$\Rightarrow \quad k^2 = 4$

$\Rightarrow \quad k = \pm 2$

(c) Roots differ by a given number.

Example 4: Find p, if the roots of the equation $x^2 - px + 8 = 0$ differ by 2.

Let α, β be roots of the equation $x^2 - px + 8 = 0$

Here, $a = 1, b = -p$ *and* $c = 8$

$\alpha+\beta = -\frac{b}{a} = -\frac{(-p)}{1} = p$ and $\alpha\beta = \frac{c}{a} = \frac{8}{1} = 8$

According to given condition,

$\alpha - \beta = 2$

Squarring on both sides, we have

$(\alpha-\beta)^2 = 4$

$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta = 4 \qquad \left[\because (a-b)^2 = (a+b)^2 - 4ab\right]$

$\Rightarrow \quad p^2 - 4(8) = 4$

$\Rightarrow \quad p^2 - 32 = 4$

$\Rightarrow \quad p^2 = 36$

$\Rightarrow \quad p = \pm 6$

(d) Roots satisfying a given relation (e.g, the relation $2\alpha+5\beta=7$, where α and β are the roots of the equation)

Example 5: Find k, if the roots α and β of the equation $x^2 - 5x + k = 0$ satisfy the condition $2\alpha + 5\beta = 7$

Solution: In the given equation $x^2 - 5x + k = 0$,

Here $a = 1, b = -5$ and $c = k$ and let α, β be the roots of given equation.

$\because \quad \alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$\therefore \quad \alpha+\beta = -\frac{(-5)}{1} = 5$

$\Rightarrow \quad \alpha = 5 - \beta \qquad$ (i)

and $\quad \alpha\beta = \frac{k}{1} = k$

$\Rightarrow \quad \alpha\beta = k \qquad \text{(ii)}$

Given that

$\because \quad 2\alpha + 5\beta = 7$

$\therefore \quad 2(5-\beta)+5\beta = 7 \qquad [\because \text{using (i)}]$

$\Rightarrow \quad 10 - 2\beta + 5\beta = 7$

$\Rightarrow \quad 3\beta = -3$

$\Rightarrow \quad \beta = -1$

$\therefore$ From equation (i), we have,

$\alpha = 5-(-1) = 5+1 = 6,$

To find the value of k, substitute the values of α and β in equation (ii), we get

$k = 6(-1) = -6$

$\Rightarrow \quad k = -6$

(e) Both sum and product of the roots are equal to a given number:

Example 6: Find k, if sum and product of the roots of the equation $6x^2 - 3kx + 5 = 0$ is equal to $\frac{5}{6}$.

Solution: Let α, β be the roots of the equation $6x^2 - 3kx + 5 = 0$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-3k)}{6} = \frac{k}{2} \qquad \text{(i)}$$

and $$\alpha\beta = \frac{c}{a} = \frac{5}{6} \qquad \text{(ii)}$$

According to the given condition that sum and product of the roots equal to $\frac{5}{6}$.

i.e., $\alpha + \beta = \alpha\beta = \frac{5}{6}$

i.e., $\frac{k}{2} = \frac{5}{6}$

$\Rightarrow \quad k = \frac{5}{3},$

Hence, at $k = \frac{5}{3}$, given equation has sum of the roots equal to its product of the roots equal to $\frac{5}{6}$.

Exercise 20.3

1. Without solving, find the sum and product of the roots in each of the following quadratic equations

   (i) $x^2 - x - 1 = 0$ (ii) $2x^2 - 3x - 4 = 0$

   (iii) $x^2 - \frac{3}{4}a^2 = ax$ (iv) $7x^2 - 5kx + 7k = 0$

2. Find the value of m, if,

   (i) Sum of the roots of the equation $x^2 + (3m-7)x + 5m = 0$ is $\frac{3}{2}$ times the product of its roots.

   (ii) Sum of the roots of the equation $2x^2 - 3x + 4m = 0$ is 6 times the product of its roots.

3. Find the value of p, if

   (i) Sum of the squares of the roots of the equation of $x^2 - px + 6 = 0$ is 13.

   (ii) Sum of the squares of the roots of the equation of $x^2 - 2px + (2p-3) = 0$ is 6.

4. Find the value of m, if

   (i) the roots of the equation $x^2 - 5x + 2m = 0$ differ by 1.

   (ii) the roots of the equation $x^2 - 8x + m + 2 = 0$ differ by 2.

5. Find the value of k, if

   (i) The roots of the equation on $5x^2 - 7x + k - 2 = 0$ satisfy the relation $2\alpha + 5\beta = 1$

   (ii) The roots of the equation on $3x^2 - 2x + 7k + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$.

6. Find the value of p given that sum and product of the roots of the following quadratic equation are equal.

   (i) $(2p+3)x^2 + (7p-5)x + (3p-10) = 0$ (ii) $4x^2 - (5p+3)x + 17 - 9p = 0$

20.4 Symmetric Function of Roots of a Quadratic Equation.

20.4.(i) Define symmetric function of the roots of a Quadratic Equation.

Let α and β be the roots of the quadratic equation, then a function in α and β is said to be symmetric function if the function remains the same when α and β are interchanged. i.e. $f(\alpha,\beta) = f(\beta,\alpha)$.

$\alpha + \beta$ and $\alpha\beta$ are known as the primary symmetric functions of α and β.

It is noted that $\alpha - \beta \neq \beta - \alpha$, so that $\alpha - \beta$ is not symmetric function of roots.

The value of a symmetric function of roots in terms of the coefficients of the quadratic equation can be obtained using $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ by expressing in terms of $\alpha + \beta$ and $\alpha\beta$.

Examples of symmetric functions are as under:

- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
- $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
- $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=\left[(\alpha+\beta)^2-2\alpha\beta\right]-2(\alpha\beta)^2$ etc.

Example: Find the value of $\alpha^2+\beta^2+\alpha\beta$, when $\alpha=2$ and $\beta=3$ and show that it is symmetric function of α and β, where α and β are the roots of quadratic equation.

Solution: Let $f(\alpha,\beta)=\alpha^2+\beta^2+\alpha\beta$,

Given that $\alpha=2$ and $\beta=3$

$\therefore \quad f(2,3)=2^2+3^2+(2)(3)=4+9+6=19,$

Now,

$f(\beta,\alpha)=\beta^2+\alpha^2+\beta\alpha=\alpha^2+\beta^2+\alpha\beta=f(\alpha,\beta)$

$\because \quad f(\alpha,\beta)=f(\beta,\alpha)$

$\therefore \quad f$ is symmetric function.

Given expression is symmetric function of roots of α and β.

Hence Shown.

20.4.(ii) Represent a symmetric function graphically.

In article 20.4.1 we have already defined symmetric functions of roots of quadratic equations, e.g, $\alpha+\beta, \alpha\beta, \alpha^2+\beta^2, \alpha^2+\beta^2+2\alpha\beta, \alpha^3+\beta^3$ etc.

When a symmetric function (expression) is equated to constant $c\in\mathbb{R}$, we obtain a symmetric equation, so that symmetric equation of the roots can be written as $f(\alpha,\beta)=c$. Every equation can be expressed graphically as the set of all points denoted by $G(f)$ defined as

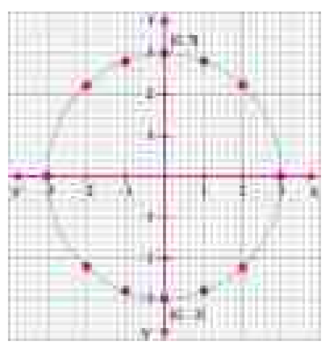
$$G(f)=\{(\alpha,\beta)\mid f(\alpha,\beta)=c\wedge\alpha,\beta\in\mathbb{R}\}$$

Example: Represent symmetric equation $\alpha^2+\beta^2=9$ graphically and sketch the graph.

Solution: Given that

$\therefore \quad \alpha^2+\beta^2=9$

$\Rightarrow \quad \beta=\pm\sqrt{9-\alpha^2}$



To plot the graph, we take some points. By assigning different values of α and get corresponding values of β.

α	0	1	2	3	-1	-2	-3
β	±3	±2.828	±2.2360	0	±2.828	±2.2360	0

Thus the symmetric function $\alpha^2+\beta^2$ represents a circle when equated to constant $c=9$.

20.4.(iii) Evaluate a Symmetric functions of roots of a quadratic equation in terms of its coefficients.

Let α and β be the roots of the quadratic equation $ax^2+bx+c=0, a\neq 0$ (i)

Then $\alpha+\beta=-\frac{b}{a}$ (ii)

and $\alpha\beta=\frac{c}{a}$ (iii)

The above equations (ii) and (iii) represent the primary symmetric functions of the quadratic equation (i).

Example 1: If α and β are the roots of $x^2-px+q=0$, find the value of :

(i) $\alpha^2+\beta^2$ (ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

Solution: Given that α and β are the roots of $x^2-px+q=0$

Here $a=1$, $b=-p$ and $c=q$

$\therefore \quad \alpha+\beta=-\frac{b}{a}=-\frac{(-p)}{1}=p$ and $\alpha\beta=\frac{c}{a}=\frac{q}{1}=q$

Now

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=(p)^2-2q \qquad (\because \alpha+\beta=p \text{ and } \alpha\beta=q)$$
$$=p^2-2q$$

(ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{p^2-2q}{q} \qquad (\because \alpha+\beta=p \text{ and } \alpha\beta=q)$$

Exercise 20.4

1. If α,β are the roots of a quadratic equation. Express the following symmetric functions in terms of $\alpha+\beta$ and $\alpha\beta$.

   (i) $(\alpha-\beta)^2$ (ii) $(\alpha+\beta)^3$ (iii) $\alpha^2\beta^{-1}+\beta^2\alpha^{-1}$

   (iv) $\alpha^3\beta+\alpha\beta^3$ (v) $\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3}$ (vi) $\left(\dfrac{1}{\alpha}-\dfrac{1}{\beta}\right)^2$

2. If α,β are the roots of the equation $2x^2-3x+7=0$, find the value of following symmetric functions.

   (i) $\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)^2$ (ii) $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}$ (iii) $\dfrac{1}{a\alpha+1}+\dfrac{1}{a\beta+1}$

3. If α,β are the roots of the equation $px^2+qx+q=0, p\neq 0$, find the value of $\sqrt{\dfrac{\alpha}{\beta}}+\sqrt{\dfrac{\beta}{\alpha}}$

4. Represent symmetric equation $\alpha+\beta=8$ graphically when α and β are the root of a quadratic equation and also plot the graph.

20.5 Formation of Quadratic Equations

20.5.(i) Establish the formula $x^2-(\text{sum of the roots})\,x+(\text{product of the roots})=0$, to find the quadratic equation of the given roots.

Let α,β be the roots of the quadratic equation $ax^2+bx+c=0$, $a\neq 0$, ...(i)

then $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

Now, rewrite the equation (i) as under

$$\Rightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0, \qquad \text{provided } a\neq 0$$

$$\Rightarrow \quad x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0$$

or $\quad x^2-(\alpha+\beta)\,x+\alpha\beta=0$

In words, $x^2-(\text{sum of the roots})\,x+(\text{product of the roots})=0$

$x^2-Sx+P=0$, where S and P respectively denote the sum and product of the roots of the given quadratic equation.

Example 1: Form the quadratic equation whose roots are:

(i) $-\dfrac{4}{5},\dfrac{3}{7}$ (ii) $7\pm 2\sqrt{5}$

Solution (i): Let $\alpha = -\frac{4}{5}$ and $\beta = \frac{3}{7}$

$\therefore$ $S = \alpha + \beta = -\frac{4}{5} + \frac{3}{7} = \frac{-28+15}{35} = \frac{-13}{35}$

and $P = \alpha\beta = \left(-\frac{4}{5}\right)\left(\frac{3}{7}\right) = \frac{-12}{35}$

When roots are known then equation is

$x^2 - Sx + P = 0$

i.e, $x^2 - \left(\frac{-13}{35}\right)x + \left(\frac{-12}{35}\right) = 0$

$\Rightarrow$ $35x^2 + 13x - 12 = 0.$

is the required equation

Solution (ii): Let $\alpha = 7 + 2\sqrt{5}$ and $\beta = 7 - 2\sqrt{5}$

$\therefore$ $S = \alpha + \beta = 7 + 2\sqrt{5} + 7 - 2\sqrt{5} = 14$

and $P = \alpha\beta = \left(7 + 2\sqrt{5}\right)\left(7 - 2\sqrt{5}\right) = 49 - 20 = 29$

When roots are known, then equation is

$x^2 - Sx + P = 0$

$\Rightarrow$ $x^2 - 14x + 29 = 0,$ is the required equation.

20.5.(ii) Form the quadratic equation whose roots are of the types:

(a) $2\alpha + 1, 2\beta + 1$ (b) α^2, β^2 (c) $\frac{1}{\alpha}, \frac{1}{\beta}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta},$

where α, β are the root of a quadratic equation.

Example 1: If α, β are the roots of the equation $x^2 - 5x + 6 = 0$, form the equation whose roots are

(i) $2\alpha + 1, 2\beta + 1$ (ii) α^2, β^2 (iii) $\frac{1}{\alpha}, \frac{1}{\beta}$

(iv) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (v) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution (i): Since α, β are the roots of the equation $x^2 - 5x + 6 = 0$

Here $a = 1, b = -5$ and c=6

$\therefore$ $\alpha + \beta = -\frac{b}{a} = -\frac{(-5)}{1} = 5$ and $\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6,$

The roots of required equation are $2\alpha + 1$ and $2\beta + 1$.

Now find the sum and product of these roots.

$S = (2\alpha + 1) + (2\beta + 1)$
$S = 2(\alpha + \beta) + 2$
$S = 2(5) + 2 \qquad (\alpha + \beta = 5)$
$S = 10 + 2 = 12$
$P = (2\alpha + 1)(2\beta + 1)$
$P = 4\alpha\beta + 2(\alpha + \beta) + 1$
$P = 4(6) + 2(5) + 1 \qquad (\alpha + \beta = 5 \text{ and } \alpha\beta = 6)$
$P = 24 + 10 + 1 = 35$

The required equation is $x^2 + 12x + 35 = 0$

Solution (ii): The roots of required equation are α^2, β^2.

$S = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (5)^2 - 2(6) = 25 - 12 = 13, (\because \alpha+\beta=5 \text{ and } \alpha\beta = 6)$

and $P = \alpha^2\beta^2 = (\alpha\beta)^2 = (6)^2 = 36$

$x^2 - Sx + P = 0$

$\therefore \quad x^2 - 13x + 36 = 0,$ is the requied equation.

Solution (iii): The roots of required equation are $\frac{1}{\alpha}, \frac{1}{\beta}$.

$S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{5}{6}, \qquad (\alpha+\beta=5 \text{ and } \alpha\beta = 6)$

and $P = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{6}$

$x^2 - Sx + P = 0$

$\therefore \quad x^2 - \frac{5}{6}x + \frac{1}{6} = 0$

$\Rightarrow \quad 6x^2 - 5x + 1 = 0,$ is the requied equation.

Solution (iv): The roots of required equation are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.

$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \qquad (\because \alpha+\beta=5 \text{ and } \alpha\beta = 6)$

$= \frac{25 - 2(6)}{6} = \frac{25 - 12}{6} = \frac{13}{6},$

and $P = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = \frac{\alpha\beta}{\alpha\beta} = \frac{6}{6} = 1$

$x^2 - Sx + P = 0$

$\therefore \quad x^2 - \frac{13}{6}x + 1 = 0$

$\Rightarrow \quad 6x^2 - 13x + 6 = 0,$ is the required equation.

Solution (v): The roots of required equation are $\alpha+\beta$ and $\frac{1}{\alpha}+\frac{1}{\beta}$

$$S=(\alpha+\beta)+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$
$$=(\alpha+\beta)+\left(\frac{\alpha+\beta}{\alpha\beta}\right)$$
$$=(\alpha+\beta)\left(1+\frac{1}{\alpha\beta}\right)$$
$$=5\left(1+\frac{1}{6}\right)=5\left(\frac{7}{6}\right)=\frac{35}{6} \qquad (\because \alpha+\beta=5 \text{ and } \alpha\beta=6)$$
$$P=(\alpha+\beta)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$
$$=(\alpha+\beta)\left(\frac{\alpha+\beta}{\alpha\beta}\right)=\frac{(\alpha+\beta)^2}{\alpha\beta}=\frac{25}{6} \qquad (\because \alpha+\beta=5 \text{ and } \alpha\beta=6)$$
$$x^2-Sx+P=0$$
$$\therefore \quad x^2-\frac{35}{6}x+\frac{25}{6}=0$$
$\Rightarrow \quad 6x^2-35x+25=0,$ is the required equation.

20.5.(iii) Find the values of $\alpha,\beta,$ where the roots of an equation are $\frac{1}{\alpha},\frac{1}{\beta}$.

Example: If $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the $x^2-6x+8=0,$ find the value(s) of α and β.

Solution: Since $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $x^2-6x+8=0.$

Here $a=1,\ b=-6$ and $c=8$

$$\therefore \quad \frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{a}=-\left(\frac{-6}{1}\right)=6,$$
$$\Rightarrow \quad \frac{\alpha+\beta}{\alpha\beta}=6$$
$$\Rightarrow \quad \alpha+\beta=6\alpha\beta \qquad \ldots \qquad \text{(i)}$$
$$\text{and} \quad \frac{1}{\alpha}\times\frac{1}{\beta}=\frac{c}{a}=\frac{8}{1}=8$$
$$\Rightarrow \quad \alpha\beta=\frac{1}{8} \qquad \text{(ii)}$$

From equation (i), $\beta=6\left(\frac{1}{8}\right)-\alpha=\frac{3-4\alpha}{4}$ (iii)

By substituting the value of β in equation (ii), we have,

$$\alpha\left(\frac{3-4\alpha}{4}\right)=\frac{1}{8}$$

$\Rightarrow \quad \frac{8\alpha}{4}(3-4\alpha)-1=0$

$\Rightarrow \quad 2\alpha(3-4\alpha)-1=0$

$\Rightarrow \quad 6\alpha-8\alpha^2-1=0$

$\Rightarrow \quad 8\alpha^2-6\alpha+1=0$

$\Rightarrow \quad 8\alpha^2-4\alpha-2\alpha+1=0$

$\Rightarrow \quad 4\alpha(2\alpha-1)-1(2\alpha-1)=0$

$\Rightarrow \quad (2\alpha-1)(4\alpha-1)=0$

i.e, $\quad 2\alpha-1=0 \quad$ or $\quad 4\alpha-1=0$

$\Rightarrow \quad \alpha=\frac{1}{2} \quad$ or $\quad \alpha=\frac{1}{4},$

then from equation (iii), we have,

$$\beta=\frac{3-4\left(\frac{1}{2}\right)}{4}=\frac{1}{4} \quad \text{or} \quad \beta=\frac{3-4\left(\frac{1}{4}\right)}{4}=\frac{2}{4}=\frac{1}{2},$$

Hence, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{4}$ or $\alpha=\frac{1}{4}$ and $\beta=\frac{1}{2}$.

Exercise 20.5

1. Form the equations whose roots are:

(i) $-2,-3$ (ii) ω,ω^2 (iii) $2+i, 2-i$ (iv) $2\sqrt{2},-2\sqrt{2}$

2. If α and β are the roots of the equation $6x^2-3x+1=0$. Form the equations whose roots are:

(i) $2\alpha+1, 2\beta+1$ (ii) α^2,β^2 (iii) $\frac{1}{\alpha},\frac{1}{\beta}$

(iv) $\frac{\alpha}{\beta},\frac{\beta}{\alpha}$ (v) $\alpha+\beta,\frac{1}{\alpha}+\frac{1}{\beta}$ (vi) $-\frac{1}{\alpha},-\frac{1}{\beta}$

3. Find the equation whose roots are the reciprocals of the roots of $px^2-qx+r=0, p\neq 0$.

4. Find the equation whose roots are double the roots of $x^2-px+q=0$.

5. Find the equation whose roots exceed by 2 the roots of $px^2+qx+r=0$.

6. Find the condition that one root of $ax^2+bx+c=0, a\neq 0,$ may be:

(i) 3 times the other (ii) square of the other
(iii) additive inverse of the other (iv) multiplicative inverse of the other.

20.6 Higher Dgree Equations reducible to Quadratic Form.

20.6.(a) Solve the cubic equation if one root of the equation is given

Let $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, be the cubic equation and its one root be α.

Suppose that

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0$$

and $f(x) = 0$ has a root say α, then,

$f(x) = (x-\alpha)\cdot f_1(x)$, where $f_1(x)$ is the quadratic expression.

Example : Solve $x^3 - 6x^2 + 11x - 6 = 0$, if one root of this equation is 1.

Solution: $x^3 - 6x^2 + 11x - 6 = 0$,

It is given that one root is 1, i.e, $x = 1$, is the multiplier factor.

By synthetic division method

1	1	−6	11	−6
	↓	1	−5	6
	1	−5	6	0 = remainder

Thus, we have quadratic equation as under

$$x^2 - 5x + 6 = 0$$
$$\Rightarrow \quad x^2 - 2x - 3x + 6 = 0$$
$$\Rightarrow \quad x(x-2) - 3(x-2) = 0$$
$$\Rightarrow \quad (x-2)(x-3) = 0$$

i.e, $x - 2 = 0$ or $x - 3 = 0$

$\Rightarrow$ $x = 2$ or $x = 3$

$\therefore$ Solution set $= \{1, 2, 3\}$.

20.6.(b) Solve a biquadratic (quartic) equation if two of the real roots of the equation are given

Let $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, a_0 \neq 0$, be the biquadratic equation and its two real roots be α and β.

If $f(x) = 0$, has two roots say α and β, then

$$f(x) = (x-\alpha)(x-\beta)\cdot f_1(x),$$

where $f_1(x)$ is the quadratic expression.

Example 1: Find the remaining two roots of the biquadratic equation

$x^4 - 9x^3 + 19x^2 + 9x - 20 = 0$, if its two roots are 5 and 1.

Solution: $x^4 - 9x^3 + 19x^2 + 9x - 20 = 0$, is the given quartic equation whose given two roots are 1 and 5 i.e., multipliers 5 and 1.

By synthetic division method, we have,

5	1	−9	19	9	−20
	↓	5	−20	−5	20
1	1	−4	−1	4	0 = remainder
	↓	1	−3	−4	
	1	−3	−4	0 = remainder	

Thus, we have quadratic equation

$$x^2 - 3x - 4 = 0$$
$$\Rightarrow \quad x^2 - 4x + x - 4 = 0$$
$$\Rightarrow \quad x(x-4) + 1(x-4) = 0$$
$$\Rightarrow \quad (x-4)(x+1) = 0$$

i.e., $x - 4 = 0$ or $x + 1 = 0$

$\Rightarrow$ $x = 4$ or $x = -1$

Therefore, remaining two roots are 4 and −1.

Exercise 20.6

1. Find the remaining two roots of the following cubic equations, when one root is given:
   (i) $2x^3 - x^2 - 2x + 1 = 0$, and $x = 1$
   (ii) $x^3 - 4x^2 + x + 6 = 0$, and $x = 3$
   (iii) $x^3 - 28x + 48 = 0$, and $x = 2$
2. Find the remaining two roots of the following biquadratic equations, when its two roots are given:
   (i) $12x^4 - 8x^3 - 7x^2 + 2x + 1 = 0$; and $x = 1, -\frac{1}{2}$
   (ii) $x^4 + 4x^2 - 5 = 0$; and $x = 1, -1$
   (iii) $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$; and $x = 3, -4$
3. Find the value of m and remaining two roots of the equation $2x^3 - 3mx^2 + 9 = 0$, if its one root is 3.
4. If one root of $x^3 - 3ax^2 - x + 6 = 0$ is 1, then find the value of a. Also find its remaining roots.
5. Find the value of a and b, if two roots of the equation $x^4 - ax^2 + bx + 252 = 0$ are 6 and -2. Also find its remaining two roots.

20.7 Simultaneous Equations

Two or more equations taken together is called a system of simultaneous equations. To determine the values of two unknown variables we need a pair of equations.

The set of all ordered pairs (x, y) which satisfies the system of equation is called the solution set of the system.

20.7.(i) (a) Solve a system of two equations in two variables, when one equation is linear and other is quadratic.

The complete method (procedure) to solve the system of equation when one is quadratic other is linear is illustrated by solving the examples given below.

Example 1: Solve the system of equations $2x + y = 10$ and $4x^2 + y^2 = 68$.

Solution:

$2x + y = 10$ (i)

and $4x^2 + y^2 = 68$ (ii)

From equation (i), we have,

$y = 10 - 2x$ (iii)

Substitute this value of y in equaion (ii), we have,

$$4x^2 + (10 - 2x)^2 = 68$$

$$\Rightarrow \quad 4x^2 + 100 - 40x + 4x^2 - 68 = 0$$

$$\Rightarrow \quad 8x^2 - 40x + 32 = 0$$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$x^2 - 4x - x + 4 = 0$

$(x-4)(x-1) = 0$

i.e. $\quad x - 4 = 0 \quad$ or $\quad x - 1 = 0$

$\Rightarrow \quad x = 4 \quad$ or $\quad x = 1$

Putting these values of x in equation (iii)

when $x = 4$, then $y = 10 - 2(4) = 2$

when $x = 1$, then $y = 10 - 2(1) = 8$

Thus, solution set is $\{(4,2),(1,8)\}$.

Example 2: Solve the system of equations $3x - 2y = 7$ and $xy = 20$.

Solution:

$3x - 2y = 7 \quad$ (i)

and $\quad xy = 20 \quad$ (ii)

From equation (i), we have, $\quad x = \dfrac{7+2y}{3} \quad$ (iii)

Substitute this value of x in equaion (ii), we have

$$\left(\frac{7+2y}{3}\right)y = 20$$

$\Rightarrow \quad 7y + 2y^2 = 20 \times 3 = 60$

$\Rightarrow \quad 2y^2 + 7y - 60 = 0$

$\Rightarrow \quad 2y^2 - 8y + 15y - 60 = 0$

$\Rightarrow \quad 2y(y-4) + 15(y-4) = 0$

$\Rightarrow \quad (y-4)(2y+15) = 0$

i.e. $\quad y - 4 = 0 \quad$ or $\quad 2y + 15 = 0$

$\Rightarrow \quad y = 4 \quad$ or $\quad y = -\dfrac{15}{2}$

Putting these values of x in equation (iii)

when $y = 4$, then $x = \dfrac{7+2(4)}{3} = \dfrac{15}{3} = 5$

when $y = -\dfrac{15}{2}$, then $x = \dfrac{7+2\left(\dfrac{-15}{2}\right)}{3} = -\dfrac{8}{3}$

Thus solution set is $\left\{(5,4),\left(-\dfrac{8}{3},-\dfrac{15}{2}\right)\right\}$.

20.7.(i) (b) When both the equations are quadratic

We explain the method of solution of system when both equations are quadratic by the following example.

Example 3: Solve the system of equations $x^2 + y^2 = 4$ and $2x^2 - y^2 = 8$.

Solution:

$x^2 + y^2 = 4$ (i)

and $2x^2 - y^2 = 8$ (ii)

To eliminate y^2 adding equations (i) and (ii), we have

$\therefore$ $3x^2 = 12$

$\Rightarrow$ $x^2 = 4$

$\Rightarrow$ $x = \pm 2,$

$\therefore$ By using $x = \pm 2,$ in equation (i), we get

$(2)^2 + y^2 = 4$	$(-2)^2 + y^2 = 4$
$\Rightarrow$ $4 + y^2 = 4$	$\Rightarrow$ $4 + y^2 = 4$
$\Rightarrow$ $y^2 = 4 - 4 = 0$	$\Rightarrow$ $y^2 = 4 - 4 = 0$
$\Rightarrow$ $y = 0$	$\Rightarrow$ $y = 0$

Thus, solution set is $\{(-2,0),(2,0)\}$.

Example 4: Solve the system of equations $x^2 + y^2 = 13$ and $xy = 6$.

Solution:

$x^2 + y^2 = 13$ ... (i)

and $xy = 6$ ... (ii)

In this type of equations we eliminate constant by multiplying equation (i) by 6 and equation (ii) by 13, we get

$\therefore$ $6x^2 + 6y^2 = 78$ ... (iii)

and $13xy = 78$ ... (iv)

By substracting equation (iv) from equation (iii)

$6x^2 - 13xy + 6y^2 = 0$

$\Rightarrow$ $6x^2 - 9xy - 4xy + 6y^2 = 0$

$\Rightarrow$ $3x(2x - 3y) - 2y(2x - 3y) = 0$

i.e., $2x - 3y = 0$ or $3x - 2y = 0$

$\Rightarrow$ $x = \frac{3y}{2}$ or $x = \frac{2y}{3}$

Thus, we have following two systems

$$\left.\begin{array}{l} xy = 6 \\ x = \dfrac{3y}{2} \end{array}\right\} \text{System (A)}$$

and $$\left.\begin{array}{l} xy = 6 \\ x = \dfrac{2y}{3} \end{array}\right\} \text{System (B)}$$

In system (A),

$$\left(\frac{3y}{2}\right)y = 6, \quad \left(\because x = \frac{3y}{2}\right)$$

$\Rightarrow \quad 3y^2 = 2\times 6$

$\Rightarrow \quad y^2 = \dfrac{2\times 6}{3} = 4$

$y = \pm 2$

when $y = \pm 2$ then $x = \dfrac{3}{2}(\pm 2) = \pm 3$

In system (B),

$$\left(\frac{2y}{3}\right)\cdot y = 6 \qquad \left(\because x = \frac{2y}{3}\right)$$

$\Rightarrow \quad 2y^2 = 3\times 6$

$\Rightarrow \quad y^2 = \dfrac{3\times 6}{2} = 9$

$\Rightarrow \quad y = \pm 3,$

when $y = \pm 3$, then $x = \dfrac{2}{3}(\pm 3) = \pm 2$

Thus, the solution set is

$\{(3,2),(-3,-2),(2,3),(-2,-3)\}$.

20.7.(ii) Solve the real life problems leading to quadratic equations

Example 1:

Find two consecutive positive integers whose product is 72.

Solution:

Let x and $x+1$ be the two consecutive integers.

According to given condition.

$x(x+1) = 72$

$\Rightarrow \quad x^2 + x - 72 = 0$

$\Rightarrow \quad (x+9)(x-8) = 0$ (By factoriziation)

i.e., $\quad x+9 = 0 \quad$ or $\quad x-8 = 0$

$\Rightarrow \quad x = -9 \quad$ or $\quad x = 8$

$\because \quad$ Neglecting $x = -9$ being negative integer

$\therefore \quad$ Required consecutive integers are 8 and 9.

Example 2:

The area of a rectangular plot is $320m^2$. The width of the plot is less $4m$ than the length of the plot. Find the length and width of the plot.

Solution:

Let x m be the length of the plot, then the width is $(x-4)$ m.

Given that

$\because \quad$ Area of the plot $= 320m^2$

$\therefore \quad x(x-4) = 320 \qquad [\because \text{Area} = \text{Length}\times \text{Width}]$

$\Rightarrow \quad x^2 - 4x - 320 = 0$

$\Rightarrow \quad x^2 - 20x + 16x - 320 = 0$

$\Rightarrow \quad x(x-20) + 16(x-20) = 0$

$\Rightarrow \quad (x-20)(x+16) = 0$

i.e. $\quad x - 20 = 0 \quad$ or $\quad x + 16 = 0$

$\Rightarrow \quad x = 20 \quad$ or $\quad x = -16$

Neglecting $x = -16$ because length is always positive.

Thus, length of the plot is $20m$ and width of the plot $= 20 - 4 = 16m$.

Exercise 20.7

Solve the following systems of equations:

1. $2x - y = 3$ and $x^2 + y^2 = 2$

2. $2x + y = 4$ and $x^2 - 2x + y^2 = 0$

3. $\frac{4}{x} + \frac{3}{y} = 2$ and $4x + 3y = 25$

4. $(x-1)^2 + (y+3)^2 = 25$ and $x^2 + (y+1)^2 = 10$

5. $x^2 + y^2 = 25$ and $(4x - 3y)(x - y - 5) = 0$

6. $x^2 + y^2 = 16$ and $2x^2 - 3xy + y^2 = 0$

7. $x^2 + y^2 = 5$ and $xy = 2$

8. $x^2 + xy = 5$ and $x^2 - 2xy = 2$

9. $x + \frac{4}{y} = 1$ and $y + \frac{4}{x} = 25$

10. Divide 12 into two parts such that the sum of their squares is greater than twice their product by 4.

11. The length of a prayer hall is 5 meters more than its width. If the area of the hall is $36m^2$, find the length and width of hall.

12. The sum of the squares of two positive numbers is 100. One number is 2 more than the other find the numbers.

13. The length of the base of the right triangle exceeds the length of the perpendicular by 3cm. When hypotenuse of the triangle is 15cm. Find the length of base and perpendicular.

14. The perimeter of an isosceles triangle is 36cm. The altitude to unequal side of the triangle is 12cm. Find the length of the three sides of the triangle.

15. The difference of two numbers is 5, and difference of their squares is 275. Find the numbers.

Review Exercise 20

1. Multiple Choice Question MCQs.

Encircle the correct answer.

i. If p, q are the roots of $2x^2+5x+3=0$, then $p+q=$__________

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{5}{2}$ (d) $-\frac{5}{2}$

ii. If $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of the $ax^2+bx+c=0,\ a\neq 0$, then $\alpha+\beta=$_____

(a) $-\frac{b}{a}$ (b) $\frac{b}{c}$ (c) $-\frac{b}{c}$ (d) $-\frac{c}{b}$.

iii. If the sum of the roots of $(p+1)x^2+(2p+3)x+(3p+4)=0$ is -1, then product of the root is

(a) 0 (b) 1 (c) 2 (d) 3

iv. The nature of the roots of $ax^2+bx+c=0,\ a\neq 0$ is determined by

(a) sum of the roots (b) product of the roots

(c) discriminant (c) none of these

v. If sum of the roots of a quadratic equation is $\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$, then equation is

(a) $ax^2+bx+c=0$ (b) $ax^2+bx-c=0$

(c) $ax^2-bx+c=0$ (d) $ax^2-bx-c=0$

vi. The required equation whose roots are the reciprocal of the roots of $ax^2+bx+c=0$ is

(a) $ax^2+bx+c=0$ (b) $cx^2+bx+a=0$

(c) $cx^2+ax+b=0$ (d) $cx^2-bx-a=0$

vii. If α,β are the roots of $x^2-2x-15=0$, then the value of $\alpha^2+\beta^2=$_____

(a) 34 (b) -34 (c) 26 (d) -26

viii. If $\Delta=b^2-4ac$ of a quadratic equation with rational coefficients is perfect square, then roots are

(a) real and equal (b) real, rational and un equal

(c) real, irrational and unequal (d) imaginary or complex and unequal

ix. If one root of quadratic equation with rational coefficients is $2+\sqrt{3}$, then other root will be

(a) 2 (b) $-2+\sqrt{3}$ (c) $2-\sqrt{3}$ (d) $-2-\sqrt{3}$

x. The quadratic equation whose roots are complex cube roots of umity is

(a) $x^2-x-1=0$ (b) $x^2-x+1=0$

(c) $x^2+x+1=0$ (d) $x^2+x-1=0$

2. Discuss the nature of the following quadratic equations.

i) $x^2-7x+12=0$ ii) $x^2-14x+49=0$

iii) $x^2-x+7=0$ iv) $x^2-5=0$

3. For what value of k, the equation $x^2+kx+4=0$

i) has equal roots ii) has complex roots

iii) has real roots iv) has rational roots

4. Find the cube roots of 729.

5. Find the sum and product of roots of the following quadratic equation.

i) $x^2-7x+29=0$ ii) $x^2-px+q=0$

iii) $7x-8=5x^2$ iv) $11x=9x^2-28$

6. Define symmetric function of roots of quadratic equations.

7. Form the quadratic equation whose roots are $1-\sqrt{3}$ and $1+\sqrt{3}$.

8. Find the equation whose roots are the reciprocals of the roots of $x^2-10x+16=0$

9. Solve the following system of equations.

i) $x^2+y^2=13$, $x+y=5$ ii) $x^2+y^2=37$, $2xy=12$

SUMMARY

- Discriminant of the quadratic equation $ax^2+bx+c=0, a\neq 0$ is $\Delta=b^2-4ac$.
  - i. If $\Delta=b^2-4ac>0$, then the roots are real and unequal.
  - ii. If $\Delta=b^2-4ac<0$, then the roots are non-real (complex or imaginary), unequal in pair form and conjugate of each other.
  - iii. If $\Delta=b^2-4ac=0$, then the roots are equal, each being equal to $-\frac{b}{2a}$.
  - iv. If a,b,c are rationals and $\Delta=b^2-4ac>0$ and perfect square, then roots are rational and unequal otherwise irrational and unequal.
- The cube roots of unity are $1, \omega$ and ω^2.
- Properties of the cube roots of unity are:
  - i. Each of the complex cube of unity is the square of other.
  - ii. Sum of three cube roots of unity is zero.
  - iii. Product of three cube roots of unity is 1.
  - iv. Each complex cube root of unity is reciprocal of the other.
- If α and β are the roots of a quadratic equation $ax^2+bx+c=0, a\neq 0$, then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$
- Symmetric functions of the roots of quadratic equation are those functions which remain unchanged when roots are interchanged, i.e. $f(\alpha,\beta)=f(\beta,\alpha)$.