MATRICES AND DETERMINANTS

Unit 19

- Weightage = 9%

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define
  - A matrix with real entries and relate its rectangular layout (formation) with representation in real life aswell.
  - Know about the rows and columns of a matrix,
  - The order/size of a matrix,
  - Equality of two matrices.
- Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity/unit matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric (upto three by three, 3 x 3) and skew-symmetric matrices.
- Know whether the given matrices are conformable for addition and subtraction.
- Scalar multiplication of a matrix by a real number.
- Add and subtract matrices
- Verify commutative and associative laws w.r.t.addition.
- Define additive identity of a matrix.
- Find additive inverse of a matrix.
- Know whether the given matrices are conformable for multiplication.
- Multiply two (or three) matrices.
- Verify associative law under multiplication.
- Verify distributive laws.
- Verify, with the help of an example that commutative law w.r.t. multiplication does not hold, in general. (i.e., $AB \neq BA$),
- Define multiplicative identity of a matrix.
- Verify the result $(AB)^t = B^t A^t$.
- A matrix with real entries and relate its rectangular layout (formation) with
- Define the determinant of a square matrix.
- Finding the value of determinant of a matrix.
- Define singular and non-singular matrices.
- Define minors and co-factors
- Define adjoint of a matrix.
- Find the multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$, where, I isthe identity matrix.
- Use the adjoint method to calculate inverse of a non- singular matrix.
- Verify the result $(AB)^{-1} = B^{-1}A^{-1}$.
- Solve the system of two linear equations, related to real life problems, in two unknowns using
  - Matrix inversion method.
  - Cramer's rule.

19.1 Introduction to Matrices

The matrices belong to a field of mathematics, that is, linear algebra. It is mainly used in business, engineering, physics and computer science. **Arthur Cayley** (1821-1895) was the first mathematician who developed the theory of matrices.

19.1 (i) Define a matrix with real entries and relate its rectangular layout (formation) with representation in real life as well.

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix is usually denoted by capital letter of English alphabet and each entry (or "element") is shown by a lower-case letter. Elements of a matrix are enclosed within [ ] or ( ).
For example:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ or } A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (1)$$

Here a_{11}, a_{22} and a_{33} are the elements of main diagonal and are called main diagonal elements, and diagonal itself is called main diagonal, principal diagonal, leading diagonal or primary diagonal.

Let us take an example of matrix in real life, following table shows the number of residents, televisions and computers in three houses as

	Residents	**Televisions**	**Computers**
House A	4	2	1
House B	6	2	3
House C	2	1	0

The above data can be written in matrix form as

$$\begin{bmatrix} 4 & 2 & 1 \\ 6 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

Note: The plural of word "Matrix" is "Matrices".

19.1 (ii) Know about the rows and columns of a matrix

The horizontal line of entries in a matrix is called row of a matrix and the vertical line of entries in a matrix is called column of a matrix.

For Example, matrix $A=\begin{bmatrix} 2 & 5 & 1 & 4 \\ 6 & 3 & -2 & 0 \end{bmatrix}$ has two horizontal lines of entries so it consists of two rows. It has four vertical lines of entries so it consists of four columns.

19.1.(iii) The order/size of a matrix

The order or size of a matrix is defined by the number of rows and columns it contains. The order of matrix is represented by $m\times n$ (read as m by n) where m is the number of rows, and n is the number of columns in the given matrix. For example,

If $A=\begin{bmatrix} 3 & 4 & 9 \\ 12 & 11 & 35 \end{bmatrix}$ then order of A is 2×3

19.1.(iv) Equality of two matrices

Two matrices A and B are called equal matrices if they have the same order and their corresponding elements are equal. Symbolically, we write as A = B. For example,

$A = \begin{pmatrix} 2 & 6 & 1 \\ 0 & 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & 1 \\ 0 & 5 & 3 \end{pmatrix}$ are equal matrices

$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ are not equal matrices because their orders are not same.

19.2 Types of Matrices

19.2.(i) Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity/unit matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric and skew-symmetric matrices (up to three by three, 3×3).

Row matrix

A matrix having one row is called row matrix.

For example, $[a,\ b]$, $[1 \quad 4 \quad 7]$ and $[4 \quad 3 \quad 8 \quad 6]$ are row matrices.

Note: The order of row matrix is $1 \times n$.

Column matrix

A matrix having one column is called column matrix.

For example, $\begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{bmatrix} 7 \\ 5 \\ 9 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \end{bmatrix}$ are column matrices.

Note: The order of column matrix is $m \times 1$.

Rectangular matrix

A matrix in which number of rows is not equal to number of columns is called rectangular matrix. i.e. $m \neq n$. For example,

$A = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 3 & 9 \\ 1 & -7 & 10 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 1 \\ 7 & 2 \\ -1 & 1 \end{bmatrix}$ are rectangular matrices.

Square matrix

A matrix in which number of rows is equal to the number of columns, is called square matrix. i.e. $m = n$. For example

$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 5 & 2 \\ 1 & 8 & 5 \\ 3 & 1 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} x & y & z & 1 \\ a & b & c & 1 \\ p & q & r & 1 \\ m & n & o & 1 \end{bmatrix}$.

Zero/null matrix

A zero/null matrix is a matrix in which each element is equal to zero. It is generally denoted by capital letter "O". Following are few examples of zero matrix.

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \text{ and } \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Diagonal matrix

A square matrix is called diagonal matrix if all the elements of matrix are zero except at least one element of main diagonal.

For example, $A=\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7\end{bmatrix}$ and $C=\begin{bmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$ are diagonal matrices.

Scalar matrix

A diagonal matrix in which main diagonal elements are same. For example,

$A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$ and $C=\begin{bmatrix}k & 0 & 0\\ 0 & k & 0\\ 0 & 0 & k\end{bmatrix}$ are scalar matrices.

Identity/unit matrix

A diagonal matrix in which each element of main diagonal is 1. It is generally denoted by capital letter "I". For example,

$$I_1=\begin{bmatrix}1\end{bmatrix}, I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Transpose of a matrix

A matrix that is obtained by interchanging the rows into columns or the columns into rows, is called transpose of the matrix. It is denoted by A^t.

For example, transpose of $\begin{bmatrix}1 & 5\\ 4 & 6\\ 0 & 7\end{bmatrix}$ is $\begin{bmatrix}1 & 4 & 0\\ 5 & 6 & 7\end{bmatrix}$.

Symmetric matrix

A square matrix A is said to be symmetric matrix if its transpose is equal to itself. i.e. $A^t = A$.

For example, if $A=\begin{bmatrix}1 & 2 & 17\\ 2 & 5 & -11\\ 17 & -11 & 9\end{bmatrix}$ then $A^t=\begin{bmatrix}1 & 2 & 17\\ 2 & 5 & -11\\ 17 & -11 & 9\end{bmatrix}$.

As $A^t = A$, so A is a symmetric matrix.

Skew-symmetric matrix

A square matrix is said to be skew-symmetric matrix if its transpose is equal to negative of itself i.e., $A^t = -A$.

For example, if $A = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix} = -A$.

As $A^t = -A$, so A is skew-symmertic matrix.

19.3 Addition and Subtraction of Matrices

19.3.(i) Know whether the given matrices are conformable for addition and subtraction

If two matrices A and B have the same order, then they are conformable for addition and subtraction. For example, $\begin{bmatrix} 1 & 4 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 6 \\ 5 & 9 \end{bmatrix}$ are conformable for addition and subtraction.

19.3.(ii) Scalar multiplication of a matrix by a real number

The scalar multiplication of a matrix refers to the product of a real number and the matrix. In scalar multiplication, each entry in the matrix is multiplied by the given scalar. Here, if c is a scalar and A is a matrix then scalar multiplication of a matrix is denoted by cA. For example,

If $A = \begin{pmatrix} 0 & -1 & 5 \\ -3 & 2 & 1 \\ 2 & 0 & -4 \end{pmatrix}$ and $c = -5$ then $cA = -5\begin{pmatrix} 0 & -1 & 5 \\ -3 & 2 & 1 \\ 2 & 0 & -4 \end{pmatrix}$

i.e. $cA = \begin{pmatrix} 0 & 5 & -25 \\ 15 & -10 & -5 \\ -10 & 0 & 20 \end{pmatrix}$

19.3.(iii) Add and Subtract Matrices

The addition and subtraction of two matrices A and B are obtained by adding or subtracting of the corresponding elements of both matrices.

For example, If $A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 3 & 7 \end{pmatrix}$ then $A+B = \begin{pmatrix} 2+1 & 5+4 \\ -1+3 & 3+7 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 2 & 10 \end{pmatrix}$.

For subtraction,

if $P = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 6 & 1 \\ -2 & 9 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 10 & 4 & 2 \\ 2 & 7 & 1 \\ 1 & 9 & 4 \end{pmatrix}$ then $P-Q = \begin{pmatrix} 1-10 & 0-4 & 2-2 \\ 4-2 & 6-7 & 1-1 \\ -2-1 & 9-9 & 0-4 \end{pmatrix} = \begin{pmatrix} -9 & -4 & 0 \\ 2 & -1 & 0 \\ -3 & 0 & -4 \end{pmatrix}$.

19.3.(iv) Verify commutative and associative laws w.r.t addition

Commutative law w.r.t addition

If A and B are two matrices of the same order, then the commutative law of addition is defined as A + B = B + A.

Example:

Verify commutative law w.r.t addition for $A=\begin{pmatrix}1&0&2\\4&6&1\\-2&9&0\end{pmatrix}$ and $B=\begin{pmatrix}10&4&2\\2&7&1\\1&9&4\end{pmatrix}$.

Here,

$$A+B=\begin{pmatrix}1+10&0+4&2+2\\4+2&6+7&1+1\\-2+1&9+9&0+4\end{pmatrix}=\begin{pmatrix}11&4&4\\6&13&2\\-1&18&4\end{pmatrix}$$

Now,

$$B+A=\begin{pmatrix}10+1&4+0&2+2\\2+4&7+6&1+1\\1-2&9+9&4+0\end{pmatrix}=\begin{pmatrix}11&4&4\\6&13&2\\-1&18&4\end{pmatrix}.$$

Since $A+B=B+A$. Therefore, commutative law w.r.t addition is verified.

Associative law w.r.t addition

If A, B and C are three matrices of the same order, then the associative law of addition is defined as (A + B) +C = A+ (B + C).

Example:

Verify associative law w.r.t addition for $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}, B=\begin{pmatrix}5&6\\7&8\end{pmatrix}$ and $C=\begin{pmatrix}9&10\\11&12\end{pmatrix}$.

Here,

$$(A+B)+C=\left[\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}5&6\\7&8\end{pmatrix}\right]+\begin{pmatrix}9&10\\11&12\end{pmatrix}=\begin{pmatrix}6&8\\10&12\end{pmatrix}+\begin{pmatrix}9&10\\11&12\end{pmatrix}$$

$$(A+B)+C=\begin{pmatrix}15&18\\21&24\end{pmatrix}$$

Now

$$A+(B+C)=\begin{pmatrix}1&2\\3&4\end{pmatrix}+\left[\begin{pmatrix}5&6\\7&8\end{pmatrix}+\begin{pmatrix}9&10\\11&12\end{pmatrix}\right]=\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}14&16\\18&20\end{pmatrix}$$

$$A+(B+C)=\begin{pmatrix}15&18\\21&24\end{pmatrix}$$

Since $(A+B)+C=A+(B+C)$

Therefore, Associative law w.r.t addition is verified.

19.3.(v) Define additive identity of a matrix

If A and O are two matrices of the same order and A + O = A = O + A.Here O is null matrix and called additive identity of the matrix A.

19.3.(vi) Find additive inverse of a matrix

If A and B are two matrices of the same order such that A + B = O = B + A then A and B are called additive inverses of each other. The additive inverse of matrix A is denoted by –A For example,

$$\text{If } A=\begin{pmatrix}1 & 0 & 2\\ 4 & 6 & 1\\ -2 & 9 & 0\end{pmatrix}$$

$$\text{then } -A=-\begin{pmatrix}1 & 0 & 2\\ 4 & 6 & 1\\ -2 & 9 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0 & -2\\ -4 & -6 & -1\\ 2 & -9 & 0\end{pmatrix}$$

A and – A are additive inverse of each other.

19.4 Multiplication of Matrices (up to 2 by 2)

19.4.(i) Know whether the given matrices are conformable for multiplication

Two matrices A and B are said to be conformable for multiplication, if the number of columns of the first matrix is equal the number of rows of the second matrix. For example, $\begin{bmatrix}1 & -4\\ 8 & 2\end{bmatrix}$ and $\begin{bmatrix}4 & -7\\ 2 & 7\end{bmatrix}$ are conformable for multiplication.

19.4.(ii) Multiply two (or three) matrices

If two matrices are conformable for multiplication then the element a_{ij} of the product AB is obtained by multiplying the ith row of A by the jth column of B.

Example:1

If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 6 & 7\end{bmatrix}$ then compute AB.

Solution:

Here, AB is possible because the number of columns of A is 2 and the number of rows of B.

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0 & -1\\ 6 & 7\end{bmatrix}$$

$$AB=\begin{bmatrix}1(0)+2(6) & 1(-1)+2(7)\\ 3(0)+4(6) & 3(-1)+4(7)\end{bmatrix}$$

$$AB=\begin{bmatrix}0+12 & -1+14\\ 0+24 & -3+28\end{bmatrix}=\begin{bmatrix}12 & 13\\ 24 & 25\end{bmatrix}$$

$$AB=\begin{bmatrix}12 & 13\\ 24 & 25\end{bmatrix}$$

Example 2:

Compute the product: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+12 & 5+0 \\ 8+16 & 10+0 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 5 \\ 24 & 10 \end{pmatrix} \cdot \begin{pmatrix} 6 & 8 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 96+25 & 128+5 \\ 144+50 & 192+10 \end{pmatrix} = \begin{pmatrix} 121 & 133 \\ 194 & 202 \end{pmatrix}$$

19.4.(iii) Verify associative law under multiplication

Let A, B and C are three matrices then associative law under multiplication is defined as (AB)C=A(BC).

Example 1: Verify associative law under multiplication when

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} -2 & 3 \\ 4 & 2 \end{pmatrix} \text{and } C = \begin{pmatrix} -1 & 5 \\ 1 & 2 \end{pmatrix}.$$

Verification:

Taking L.H.S $(AB)C$

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix} \right) \begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 13 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 36 \\ -5 & 4 \end{bmatrix}$$

Now taking R.H.S: $A(BC)$

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -2 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 36 \\ -5 & 4 \end{bmatrix}$$

Hence, associative law under multiplication is verified.

19.4.(iv) Verify distributive laws

Let A, B and C are three matrices then the distributive laws are

(i) A(B+C) = (AB)+(AC) (Left distributive law)

(ii) (B+C) A = (BA)+(CA) (Right distributive law)

Example: If $A=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}, B=\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$ then verify the following:

(i) A(B+C) = (AB)+(AC) (ii) (B+C) A = (BA)+(CA)

Verification:

(i) A(B+C) = (AB)+(AC)

Taking $L.H.S = A(B+C)$

$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\left(\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}+\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2 & -1\\ 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3\\ -1 & -2\end{bmatrix}$$

Now, $R.H.S = AB + AC$

$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}+\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}+\begin{bmatrix}-2 & 2\\ 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3\\ -1 & -2\end{bmatrix}$$

Hence, A(B+C) = (AB)+(AC) is verified. While, applying similar procedure, (B+C) A =(BA)+(CA) can be verified.

19.4.(v) Verify with the help of example that commutative law w.r.t. multiplication does not hold in general (i.e., AB ≠ BA)

Example: Let $A=\begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}$ then verify that $AB \neq BA$.

Verification:

$$AB=\begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}\cdot\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}$$

$$AB=\begin{bmatrix}(-5+6) & (10-4)\\ (2-3) & (-4+2)\end{bmatrix}=\begin{bmatrix}1 & 6\\ -1 & -2\end{bmatrix}$$

Now,

$$BA=\begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}\cdot\begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}$$

$$BA=\begin{bmatrix}(-5-4) & (-2-2)\\ (15+4) & (6+2)\end{bmatrix}=\begin{bmatrix}-9 & -4\\ 19 & 8\end{bmatrix}$$

$\because AB \neq BA$ $\therefore$ the commutative law w.r.t multiplication does not hold, in general.

19.4.(vi) Define multiplicative identity of a matrix.

Let A be a square matrix of order n and I_n is unit matrix such that $AI_n = A$ or $I_nA = A$ here I_n is called the multiplicative identity of A.

Example:

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, verify that $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is the multiplicative identity.

Solution, given that

$$AI_2=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=A$$

$$I_2A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=A.$$

19.4(vii) Verify the result $(AB)^t = B^tA^t$

Example: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$ verify that $(AB)^t=B^tA^t$

Verification:

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}=\begin{bmatrix}5 & -2\\ 11 & -4\end{bmatrix}$$

and the transpose of AB is:

$$(AB)^t=\begin{bmatrix}5 & 11\\ -2 & -4\end{bmatrix}.$$

Now,

$$B^tA^t=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=\begin{bmatrix}5 & 11\\ -2 & -4\end{bmatrix}$$

$$(AB)^t=B^tA^t.$$

Hence verified.

Example 2: Verify the result: $(AB)'=B'A'$, If $A=\begin{pmatrix}6 & 5 & 2\\ 4 & 6 & 3\\ 0 & 1 & 7\end{pmatrix}$ and $B=\begin{pmatrix}6 & 8 & 0\\ 5 & 1 & 2\\ 3 & 6 & 4\end{pmatrix}$.

Verification:

$$AB=\begin{pmatrix}6 & 5 & 2\\ 4 & 6 & 3\\ 0 & 1 & 7\end{pmatrix}\cdot\begin{pmatrix}6 & 8 & 0\\ 5 & 1 & 2\\ 3 & 6 & 4\end{pmatrix}$$

$$=\begin{pmatrix}36+25+6 & 48+5+12 & 0+10+8\\ 24+30+9 & 32+6+18 & 0+12+12\\ 0+5+21 & 0+1+42 & 0+2+28\end{pmatrix}=\begin{pmatrix}67 & 65 & 18\\ 63 & 56 & 24\\ 26 & 43 & 30\end{pmatrix}$$

$$(AB)^t = \begin{pmatrix} 67 & 65 & 18 \\ 63 & 56 & 24 \\ 26 & 43 & 30 \end{pmatrix}^t = \begin{pmatrix} 67 & 63 & 26 \\ 65 & 56 & 43 \\ 18 & 24 & 30 \end{pmatrix}$$

$$\text{Now, } B^tA^t = \begin{pmatrix} 6 & 8 & 0 \\ 5 & 1 & 2 \\ 3 & 6 & 4 \end{pmatrix}^t \cdot \begin{pmatrix} 6 & 5 & 2 \\ 4 & 6 & 3 \\ 0 & 1 & 7 \end{pmatrix}^t$$

$$= \begin{pmatrix} 6 & 5 & 3 \\ 8 & 1 & 6 \\ 0 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 & 4 & 0 \\ 5 & 6 & 1 \\ 2 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 36+25+6 & 24+30+9 & 0+5+21 \\ 48+5+12 & 32+6+18 & 0+1+42 \\ 0+10+8 & 0+12+12 & 0+2+28 \end{pmatrix}$$

$$= \begin{pmatrix} 67 & 63 & 26 \\ 65 & 56 & 43 \\ 18 & 24 & 30 \end{pmatrix}$$

$(AB)^t = B^tA^t.$

Hence verified.

Exercise 19.1

1. Specify the type of each of the following matrices.

(i) $\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ (v) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

2. What is the order of each of the following matrix?

(i) $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \\ -1 & -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ (v) $[0]$

3. Check whether the following matrices or symmetric matrix or skew symmetric matrix

(i) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & -6 & 4 \\ -6 & 0 & 7 \\ 4 & 7 & 0 \end{bmatrix}$

4. If $A=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}, B=\begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix}, C=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}, D=\begin{bmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 1 & 2 & 3\end{bmatrix}$ and $E=\begin{bmatrix}1 & 2 & -1\\ -1 & 0 & 2\\ 2 & 1 & 0\end{bmatrix}$ then compute the following (if possible)

(i) $A+C$ (ii) $A+E$ (iii) $B-D$ (iv) $2B+3A$

(v) $B-C$ (vi) A^2 (vii) B^2 (viii) $D+E$

5. For the matrices $A=\begin{bmatrix}-1 & 0 & 1\\ 2 & 1 & 0\\ 3 & 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 2 & 3\\ -2 & 4 & 1\\ 3 & 2 & 1\end{bmatrix}$. Find

(i) $A+B$ (ii) $A-B$ (iii) $3A+2B$

(iv) AB (v) BA (vi) A^2

6. Verify that: (i) $A+B=B+A$ and (ii) $C-D\neq D-C$ for the following matrices

$A=\begin{pmatrix}2 & -1\\ -7 & 4\end{pmatrix}, B=\begin{pmatrix}-3 & 0\\ 7 & -4\end{pmatrix}, C=\begin{pmatrix}3 & 1 & -4\\ 4 & 3 & 1\\ 1 & 4 & -3\end{pmatrix}$ and $D=\begin{pmatrix}2 & 7 & -5\\ -2 & 1 & 0\\ 6 & 3 & 4\end{pmatrix}$.

7. Show that $AB\neq BA$ when $A=\begin{pmatrix}8 & 2 & -1\\ 2 & 1 & 2\\ 2 & 3 & 0\end{pmatrix}$ and $B=\begin{pmatrix}1 & 4 & -4\\ 2 & 5 & 7\\ 0 & 4 & -3\end{pmatrix}$.

8. If $A=\begin{pmatrix}-3 & 0\\ 7 & -4\end{pmatrix}, B=\begin{pmatrix}2 & -1\\ -7 & 4\end{pmatrix}$ and $C=\begin{pmatrix}1 & 0\\ -2 & -4\end{pmatrix}$, find $2A-3B+4C$.

9. If $\begin{bmatrix}a & b\\ c & d\end{bmatrix}+\begin{bmatrix}3 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then find the values of a, b, c and d.

10. If $\begin{bmatrix}x & -1 & y\\ 2 & 0 & 3\\ z & 3 & 2\end{bmatrix}=\begin{bmatrix}3 & a & x\\ b & 0 & c\\ 2 & 3 & d\end{bmatrix}$, then find the values of a, b, c, d, x, y and z.

11. Evaluate possible products of the following matrices,

$A=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}, B=\begin{bmatrix}2 & 1 & 0\\ 1 & 0 & -1\end{bmatrix}$ and $C=\begin{bmatrix}0 & -1 & 2\\ -1 & 0 & -4\\ 3 & 1 & 2\end{bmatrix}$.

12. Determine: $\begin{pmatrix}1 & 0 & 3\\ 2 & 1 & 2\\ 1 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 2 & 0\\ 1 & 3 & 2\\ 3 & 2 & 0\end{pmatrix}$.

13. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}, B=\begin{bmatrix}2 & -1 & 0\\ 3 & 1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}1 & 1 & 2\\ -2 & 2 & 3\end{bmatrix}$

then verify that (i). $A(B+C)=AB+AC$ (ii). $(B+C)A=BA+CA$.

14. If $A=\begin{pmatrix}6 & 5 & 2\\ 4 & 6 & 3\\ 0 & 1 & 7\end{pmatrix}, B=\begin{pmatrix}6 & 8 & 0\\ 5 & 1 & 2\\ 3 & 6 & 4\end{pmatrix}$ and $C=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$ then verify the following:

(a) $A+B=B+A$ (b) $(A+B)+C=A+(B+C)$

(c) $(A+B)C=AC+BC$ (d) $A(B+C)=AB+AC$

15. If $A=\begin{bmatrix}2 & 3 & 1\\ 1 & -1 & 1\\ 0 & 2 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 1\\ -1 & 2 & -1\\ 2 & 0 & 2\end{bmatrix}$ and $C=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & 2\end{bmatrix}$ then

show that $A(BC)=(AB)C$.

19.5 Determinant of a Matrix

19.5.(i) Define the determinant of a square matrix

We can associate with every square matrix A over R, a number $|A|$, known as the determinant of the matrix A.

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then determinant of the matrix A is denoted by $|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}$.

19.5.(ii) Finding the value of determinant of a matrix

For a 2×2 matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$ is called value of its determinant.

For example, $\begin{vmatrix}3 & 7\\ 1 & -4\end{vmatrix}=3\times(-4)-7\times 1=-19$.

Now, The Leibniz formula or laplacian expression for the determinant of 3×3 matrix

$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ is the following

$$|A|=a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}.$$

Notes: 1. The above evaluation of the determinant is said to be the expansion by row one (R_1).

2. If $A=\begin{bmatrix}a_{11}\end{bmatrix}$ then $|A|=|a_{11}|=a_{11}$, i.e just the element itself of a matrix

e.g $A=\begin{bmatrix}-2\end{bmatrix}$ then $|A|=|-2|=-2$ and $B=\begin{bmatrix}3\end{bmatrix}$ then $|B|=|3|=3$

Example: Find $|A|$ if $A=\begin{bmatrix}4 & -3 & 5\\ 1 & 0 & 3\\ -1 & 5 & 2\end{bmatrix}$

Solution: To find $|A|$, let us expand the determinant by R_1.

$$|A|=4\begin{vmatrix}0 & 3\\ 5 & 2\end{vmatrix}-(-3)\begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix}+5\begin{vmatrix}1 & 0\\ -1 & 5\end{vmatrix}$$

$$|A|=4(0-15)+3(2+3)+5(5+0)=-20.$$

Hence, the determinant a matrix has a unique value.

19.5.(iii) Define singular and non-singular matrices

A square matrix A is said to be singular if its determinant is equal to zero. i.e., $|A|=0$.

A square matrix A is said to be non-singular if its determinant is not equal to zero. i.e, $|A|\neq 0$.

Note: A singular matrix is also called non invertible matrix, and a non-singular matrix is also called invertible matrix.

Example: Determine whether $A=\begin{bmatrix}2 & 4 & 6\\ 2 & 0 & 2\\ 6 & 8 & 14\end{bmatrix}$ is a singular matrix or not.

Solution:

$$|A|=\begin{vmatrix}2 & 4 & 6\\ 2 & 0 & 2\\ 6 & 8 & 14\end{vmatrix}=2(0-16)-4(28-12)+6(16-0)=-32-64+96=0.$$

$\because$ $|A|=0$

$\therefore$ A is singular martix.

19.5.(iv) Define minors and co-factors

Let us consider a square matrix A of order 3×3 then the minor of an element a_{ij}, denoted by M_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and the j^{th} column of A. For example, if

$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ then the matrix obtained by deleting the first row and second column of A

is $\begin{bmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix}$ and its determinant is the minor as defined below:

$M_{12}=\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$ while, cofactor of an element a_{ij}, is denoted A_{ij} and can be computed by

$A_{ij}=(-1)^{i+j}M_{ij}$ Here, M_{ij} is the minor of an element a_{ij} of the square matrix A.

Example: Let $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ then compute M_{12} and A_{12}.

Solution: Here $M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = 6(-7) - 4(1) = -46$

and $A_{ij} = (-1)^{i+j} M_{ij}$,

so $A_{12} = (-1)^{1+2} M_{12}$

$A_{12} = (-1)^3(-46)$

$A_{12} = (-1)(-46) = 46.$

19.6 Multiplicative Inverse of a Matrix

19.6.(i) Define adjoint of a matrix

Adjoint of a matrix with order 2 × 2

The adjoint of a matrix A is the transpose of the matrix of cofactors of A.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the adjoint of matrix A is denoted by $\text{Adj}(A)$ or $\text{adj}(A)$.

and $\text{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{Adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t.$$

Where $A_{11}, A_{12}, ... A_{33}$ are the cofactors of A.

Example: Find the adjoint of the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Solution: First we, find all the cofactors of matrix A

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}3 & -1\\ 1 & -1\end{vmatrix} = -2$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix} = -5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}1 & -1\\ -2 & 0\end{vmatrix} = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}3 & -1\\ 2 & 0\end{vmatrix} = -2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}3 & 1\\ 2 & -2\end{vmatrix} = -8$$

Now, $\text{Adj}(A) = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^t$.

$$\text{Adj}(A) = \begin{bmatrix}2 & 2 & 6\\ -1 & -2 & -5\\ -2 & -2 & -8\end{bmatrix}^t = \begin{bmatrix}2 & -1 & -2\\ 2 & -2 & -2\\ 6 & -5 & -8\end{bmatrix}.$$

19.6.(ii) Multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$, where, I is the identity matrix.

If A and B are $n \times n$ non singular matrices such that $AB = BA = I_n$, then A and B are multiplicative inverses of each other. The multiplicative inverse of A is denoted by A^{-1}

Example: Check whether A and B are multiplicative inverses of each other.

If $A = \begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B = \begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$

$$AB = \begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix} = \begin{bmatrix}15-14 & -21+21\\ 10-10 & -14+15\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Similarly,

$$BA = \begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix} = \begin{bmatrix}15-14 & 35-35\\ -6+6 & -14+15\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

as $AB = BA = I_2$

Hence A and B are multiplicative inverses of each other.

19.6.(iii) Use the adjoint method to calculate inverse of a non- singular matrix.

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a non-singular matrix then A^{-1} can be found as, $A^{-1}=\dfrac{\text{Adj}(A)}{|A|}$ where $|A|\neq 0$.

The method of finding inverse in this way is called adjoint method

Example 1: Find the inverse of the matrix $A=\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ by adjoint method.

Solution: Here $|A|=\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}=2$. Since $|A|\neq 0$ therefore, inverse of A exists.

Now $\text{Adj}(A)=\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ and

$$A^{-1}=\frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}.$$

Example 2: Find the inverse of the matrix $A=\begin{bmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{bmatrix}$ by adjoint method.

Solution:

Here

$$|A|=\begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix}$$

$$=9(2-0)-2(-10-24)+1(0+4)$$

$$=18+68+4$$

$$=90.$$

Since $|A|\neq 0$, Here A^{-1} inverse exists. Now, we find all cofactors of A

$$A_{11}=(-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 0 & -2 \end{vmatrix}=2-0=2$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} 5 & 6 \\ 4 & -2 \end{vmatrix}=-(-10-24)=34$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & -1 \\ 4 & 0 \end{vmatrix} = 0 + 4 = 4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} = -(-4-0) = 4$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 9 & 1 \\ 4 & -2 \end{vmatrix} = (-18-4) = -22$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 9 & 2 \\ 4 & 0 \end{vmatrix} = -(0-8) = 8$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix} = 12 + 1 = 13$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 9 & 1 \\ 5 & 6 \end{vmatrix} = -(54-5) = -49$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 9 & 2 \\ 5 & -1 \end{vmatrix} = -9 - 10 = -19$$

$$\text{Adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t. \text{ Thus, Adj}(A) = \begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix}$$

Now $\quad A^{-1} = \dfrac{\text{Adj}(A)}{|A|}$

$$A^{-1} = \frac{1}{90}\begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix} = \begin{bmatrix} \frac{1}{45} & \frac{2}{45} & \frac{13}{90} \\ \frac{17}{45} & \frac{-11}{45} & \frac{-49}{90} \\ \frac{2}{45} & \frac{4}{45} & \frac{-19}{90} \end{bmatrix}$$

19.6.(iv) Verify the result $(AB)^{-1} = B^{-1}A^{-1}$.

Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution:

Given $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$.

Inverse of AB:

$$AB=\begin{bmatrix}3&7\\2&5\end{bmatrix}\begin{bmatrix}6&8\\7&9\end{bmatrix}$$

$$AB=\begin{bmatrix}18+49&24+63\\12+35&16+45\end{bmatrix}=\begin{bmatrix}67&87\\47&61\end{bmatrix}.$$

Now, $|AB|=4087-4089=-2$

Since, $|AB|\neq 0$

Therefore, $(AB)^{-1}$ exists.

Now

$$(AB)^{-1}=\frac{\text{adj }(AB)}{|AB|}$$

$$=\frac{1}{-2}\begin{bmatrix}61&-87\\-47&67\end{bmatrix}\text{...(i)}$$

Now, Inverse of A

We have $A=\begin{bmatrix}3&7\\2&5\end{bmatrix}$

$|A|=15-14=1$

Since, $|A|\neq 0$

Hence, A^{-1} exists.

$$A^{-1}=\frac{\text{adj A}}{|A|}=\begin{bmatrix}5&-7\\-2&3\end{bmatrix}$$

Inverse of B:

We have $B=\begin{bmatrix}6&8\\7&9\end{bmatrix}$

Now

$|B|=54-56=-2$

Since, $|B|\neq 0$

Therefore, B^{-1} exists.

$$B^{-1}=\frac{\text{adj B}}{|B|}=\frac{1}{-2}\begin{bmatrix}9&-8\\-7&6\end{bmatrix}$$

Now

$$B^{-1}A^{-1}=\frac{1}{-2}\begin{bmatrix}9&-8\\-7&6\end{bmatrix}\begin{bmatrix}5&-7\\-2&3\end{bmatrix}$$

$$=\frac{1}{-2}\begin{bmatrix}45+16&-63-24\\-35-12&49+18\end{bmatrix}$$

$$B^{-1}A^{-1}=\frac{1}{-2}\begin{bmatrix}61&-87\\-47&67\end{bmatrix}\text{...(ii)}$$

From equation (i) and (ii),

$(AB)^{-1}=B^{-1}A^{-1}$

Hence verified.

19.7 Solution of Simultaneous Linear Equations

Consider a system of two linear equations in two variables:

$$\left.\begin{aligned}a_1x+b_1y=c_1\\a_2x+b_2y=c_2\end{aligned}\right\}\quad\text{(i)}$$

The system can be written in matrix form as

$$AX=B\quad\text{(ii)}$$

Where, $A=\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}c_1\\c_2\end{bmatrix}$.

The matrix A is called the matrix of the coefficients of system i, matrix X is the column matrix of unknown and B is the column matrix of constants. An ordered pair (x,y) is called the solution of given system satisfied by these two values. The set of all solutions of the given system is called the solution set of simultaneous linear equation.

19.7.(i) Solve the system of two linear equations related to real life problems in two unknown using

Matrix inversion method:

Consider the matrix equation as discussed in previous section 19.7.

$$AX = B, \qquad (i)$$

where, A is a non-singular matrix. Multiplying both sides of (i) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow \quad (A^{-1}A)X = A^{-1}B \qquad [\because \text{By associative law under } \times]$$

$$\Rightarrow \quad X = A^{-1}B \quad (ii) \qquad [\because A^{-1}.A = I, IX = X]$$

To find X, the matrix of unknown by eq (ii) is called matrix inversion method.

Example: Solve the following system of linear equations by using matrix inversion method.

$5x+2y=3$
$3x+2y=5$

Solution:

Writing the given system in matrix form:

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

First, we find A^{-1}

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}}{10-6} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

Now, by matrix inversion method

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

The solution is $x=-1$ and $y=4$. The solution set is $\{(-1,4)\}$.

Example 2: A civil engineer needs 6000 m^3 of sand and 9000 m^3 of coarse gravel for a building project. There are two pits from which these materials can be obtained. The number of these pits is shown in table:

	Sand	Coarse Gravel
Pit - I	3	8
Pit - II	4	11

By using inverse matrix method to compute volume of materials (sand and coarse gravel) that must be hauled from each pit in order to meet the engineer's need.

Solution:

Let x represents volume and and y represents coarse gravel. The inverse matrix method is

$$X = A^{-1}B \text{.............(i)}$$

Writing the given data in matrix form:

$$AX = B$$

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6000 \\ 9000 \end{bmatrix}$$

First, we need to calculate A^{-1}

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}}{3(11)-8(4)} = \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

so, $$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now, from equation (i),

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 6000 \\ 9000 \end{bmatrix} = \begin{bmatrix} -6000 \\ 3000 \end{bmatrix}$$

By ignoring '–' sign, we have $x = 6000$ and $y = 3000$.

Hence 6000cm^3 can be haulded of sand and 3000 m^3 can be haulded of coarse graved.

19.2.(ii) Cramer's rule

Cramer's rule is one of the methods used to solve a system of linear equations. Let us consider a system of linear equations in two variables x and y written as

$$\begin{aligned} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{aligned} \quad \text{(i)}$$

System (i) is written in the matrix form:

$$AX = B,$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Where $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

System (i) has exactly one solution if $|A| \neq 0$.

Which is obtained by $x=\dfrac{|A_1|}{|A|}=\dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ and $y=\dfrac{|A_2|}{|A|}=\dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$.

Where $A_1=\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}$ and $A_2=\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}$

Example 1: Solve the system using Cramer's Rule:

$$3x+2y=10$$
$$-6x+4y=4$$

Given system can be written in matrix format as:

Here $|A|=\begin{bmatrix} 3 & 2 \\ -6 & 4 \end{bmatrix}$

$12+12=24$

$\therefore \quad |A| \neq 0$

$x=\dfrac{|A_1|}{|A|}=\dfrac{\begin{vmatrix} 10 & 2 \\ 4 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -6 & 4 \end{vmatrix}}$ is by cramers rule (This method is called crammers rule)

$=\dfrac{40-8}{24}$

$=\dfrac{32}{24}$

$x=\dfrac{4}{3}$

$y=\dfrac{|A_2|}{|A|}=\dfrac{\begin{vmatrix} 3 & 10 \\ -6 & 4 \end{vmatrix}}{24}$

$=\dfrac{12+60}{24}$

$=\dfrac{72}{24}$

$y=3$ Hence, The solution set is $\left\{\left(\dfrac{4}{3},3\right)\right\}$

Exercise 19.2

1. Evaluate each of the following determinants:

i) $\begin{vmatrix} -5 & -3 \\ 3 & -4 \end{vmatrix}$ ii) $\begin{vmatrix} -4 & 3 \\ 1 & -3 \end{vmatrix}$ iii) $\begin{vmatrix} -1 & -5 \\ 2 & 3 \end{vmatrix}$

iv) $\begin{vmatrix} 2 & -5 \\ 2 & -1 \end{vmatrix}$ v) $\begin{vmatrix} 3 & -3 & -4 \\ 4 & 1 & -5 \\ 0 & -1 & -4 \end{vmatrix}$ vi) $\begin{vmatrix} -3 & 4 & -5 \\ 2 & -3 & -5 \\ 1 & 3 & 5 \end{vmatrix}$

vii) $\begin{vmatrix} 0 & 5 & -4 \\ -3 & 4 & -5 \\ 1 & 0 & -5 \end{vmatrix}$ viii) $\begin{vmatrix} 4 & 3 & -5 \\ -5 & -1 & -5 \\ 2 & 4 & 4 \end{vmatrix}$ ix) $\begin{vmatrix} 1 & 0 & 4 \\ -2 & 3 & -5 \\ 3 & 5 & 3 \end{vmatrix}$

x) $\begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & x & y \end{vmatrix}$ xi) $\begin{vmatrix} -2 & 0 & 0 \\ -6 & x & 1 \\ -4 & 0 & -1 \end{vmatrix}$ xii) $\begin{vmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{vmatrix}$

2. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ then compute M_{12}, M_{22}, M_{21}, A_{12}, A_{22}, and A_{21}.

3. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?

4. Classify the square matrices as singular or non-singular given by

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

5. Find the adjoint of the following:

$$A = \begin{pmatrix} 1 & 6 \\ 4 & 7 \end{pmatrix}, B = \begin{pmatrix} -3 & 2 & -5 \\ -1 & 0 & -2 \\ 3 & -4 & 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

6. Verify $A(\text{adj } A) = |A|I$, where $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$.

7. Find the inverse of the following matrices by adjoint method if exist

i) $A = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$ ii) $B = \begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}$

iii) $C=\begin{bmatrix}2 & 1 & 1\\ 3 & 2 & 1\\ 2 & 1 & 2\end{bmatrix}$ iv) $D=\begin{bmatrix}1 & 0 & 1\\ -4 & 1 & -1\\ 6 & -2 & 1\end{bmatrix}$ v) $E=\begin{bmatrix}1 & -1 & 3\\ 2 & 1 & 2\\ -2 & -2 & 1\end{bmatrix}$.

8. Find the solution by matrix inversion method and cramer's rule:

i) $2x+3y=14$, $-4x+y=28$ ii) $2x-4y=-12$, $2y+3x=0$

Review Exercise 19

1. Tick the correct option

i. If m denotes the number of rows and n denotes the number of columns such that $m=n$, then matrix is called __________ matrix.

(a) Rectangular (b) Equal (c) Square (d) Null

ii. If $A=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ then A^2 is

(a) I_2 (b) I_3 (c) $-I_2$ (d) O

iii. If A is any square matrix such that $A^t=-A$, then A is said to be:

(a) Diagonal matrix (b) Scalar matrix
(c) Symmetric matrix (d) Skew Symmetric matrix

iv. If A , B and C are matrices of same order then $(ABC)^t=$

(a) $A^t.B^t.C^t$ (b) $C^tB^tA^t$
(c) $C^tA^tB^t$ (d) $(B^tA^t)C^t$

v. If $A=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$, $B=\begin{bmatrix}0 & i^2\\ -i^2 & 0\end{bmatrix}$ and $C=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$ then

(a) $A^2=-I_2$ (b) $B^2=-I_2$
(c) $C^2=-I_2$ (d) All of them

vi. For two matrices A and B if $A=\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$ and $B=\begin{bmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{bmatrix}$ then AB =

(a) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$
(c) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ (d) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

vii. If $2\begin{bmatrix}x\\ 3\\ 4\end{bmatrix}+3\begin{bmatrix}-1\\ y\\ 0\end{bmatrix}=4\begin{bmatrix}0\\ 4\\ z\end{bmatrix}$ then the values of x, y and z respectively are______

(a) $2,\frac{3}{2},\frac{10}{3}$ (b) $\frac{3}{2},2,\frac{10}{3}$
(c) $\frac{3}{2},\frac{10}{3},2$ (d) 1,2,3

viii. For matrix A, $\left(A^{-1}\right)^{-1}=$ ———

(a) A^{-2} (b) A (c) A^{-1} (d) A^{2}

ix. Find x if $\begin{vmatrix} 5 & 1 \\ 2 & x \end{vmatrix} = x+4$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 0 (d) None of them

x. If the matrix $\begin{bmatrix} \lambda & -3 & 4 \\ -3 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}$ is invertible then $\lambda \neq$ ———

(a) −15 (b) −17 (c) −16 (d) None of these

2. Define the row and column of a matrix.

3. Find the order of the matrices:

i) $\begin{bmatrix} 1 & 5 & 8 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \end{bmatrix}$ iii) $\begin{bmatrix} 7 & 2 \\ 9 & 3 \\ 8 & 1 \end{bmatrix}$ iv) $\begin{bmatrix} a+b+c \end{bmatrix}$

4. Define the following matrices:

i) Square matrix ii) Rectangular matrix
iii) Diagonal matrix iv) Scalar matrix
v) Symmetric matrix vi) Skew Symmetric matrix

5. If $A = \begin{bmatrix} 1 & 8 & 9 \\ 2 & 1 & 0 \\ -2 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 4 \\ 7 & 2 & 4 \end{bmatrix}$ then find

i) $A+B$ ii) $A-B$ iii) AB iv) BA
v) $5A$ vi) $7B$ vii) $8A-9B$

6. Evaluate $\begin{vmatrix} 1 & 8 & 9 \\ 2 & 0 & -1 \\ -7 & 8 & -10 \end{vmatrix}$

7. Define singular and non-singular matrices with examples.

8. If $A = \begin{bmatrix} 1 & 4 & 2 \\ 7 & 0 & 9 \\ 0 & 2 & -3 \end{bmatrix}$ then find A^{-1}

9. Solve $2x+5y=27$
$7x+y=12$
by i) matrix inversion method ii) Cramer's rule

Summary

- A matrix is a rectangular array of elements.
- The horizontal line of entries in a matrix is called the row of a matrix and vertical line of entries in a matrix is called column of a matrix.
- Two matrices are said to be equal if they have same order and same corresponding elements.
- A matrix having one row is called row matrix, having one column is called column of matrix.
- A matrix in which number of rows is equal to number of columns is known as square matrix otherwise it is rectangular matrix.
- A square matrix is diagonal matrix if all the elements are zero except at least one diagonal element.
- A diagonal matrix in which diagonal elements are same is called scalar matrix.
- A diagonal matrix in which each element of main diagonal is 1 is called identity matrix.
- A square matrix is said to be symmetric matrix if its transpose is equal to itself.
- A square matrix is said to be skew-symmetric matrix if its transpose is equal to negative of itself .
- The addition and subtraction of two matrices are obtained by adding or subtracting of the corresponding elements of both matrices.
- If A and B are two matrices of the same order, then the commutative law of addition is defined as $A+B=B+A$.
- If A, B and C are three matrices of the same order, then the associative law of addition is defined as $(A+B)+C=A+(B+C)$.
- Two matrices A and B are said to be conformable for multiplication, if the number of columns of the first matrix is equal the number of rows of the second matrix.
- For every square matrix A over R, a number $|A|$, known as the determinant of the matrix A.
- If A and B are $n\times n$ non singular matrices such that $AB=BA=I$, then A and B are multiplicative inverse of each other.
- System of linear equations can be solved by matrix inversion method and Cramer's rule.