VARIATIONS

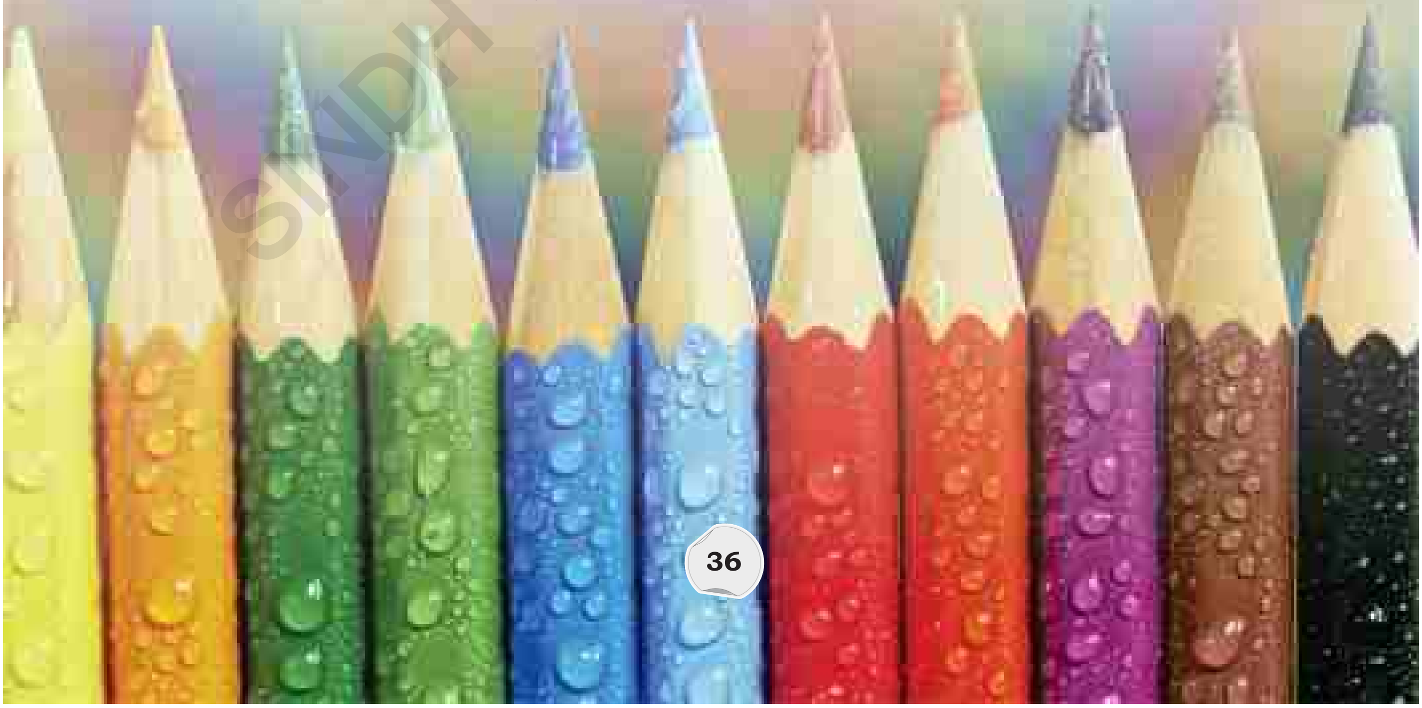
Unit 18

- Weightage = 6%

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Define ratios, proportions and variations (direct and inverse).
- Find 3rd, 4th proportionals in proportion and mean proportional in a continued proportion.
- Apply theorems of:
  - invertendo,
  - alternando,
  - componendo,
  - dividendo and componendo
  - dividendo to find proportions.
- Define joint variations.
- Solve problems related to joint variations.
- Use *k*- Method to prove conditional equalities involving proportions.
- Solve real life problems based on variations.



18.1: Ratio, Proportions and variations

18.1.1 Define ratio, proportion and variations (direct and inverse)

(a) Ratio:

Ratio is a comparison of two quantities having same units. It is the relation between two quantities of the same kind. In other words, ratio means what part of one quantity is of the other. If a and b are two quantities of the same kind and b is not zero, then the ratio of a and b is written as a $a:b$ or $\frac{a}{b}$.

For example: If in a class there are 13 boys and 8 girls, then the ratio of the number of boys to the number of girls can be expressed as $13:8$ or in fraction $\frac{13}{8}$.

Notes:
(i) The order of the elements in a ratio is important.
(ii) A ratio has no unit.
(iii) In a : b the first term a is called antecedent and the second term b called consequent.

Example 1: Find the ratio

(i) 400m to 900m (ii) 700gm to 2kg
(iii) 30sec to 2min (iv) Rs 200 to 300gm

Solution:

(i): Ratio of 400m to 900m

$$400:\ 900 = \frac{400}{900} = \frac{4}{9} = 4:9$$

$4:9$ is the simplest (lowest) form of the ratio $400:900$

(ii) Ratio of 700gm to 2 kg

Since, 1kg = 1000gm
2kg = 2000gm

therefore,

$$700:\ 2000 = \frac{700}{2000} = \frac{7}{20} = 7:20$$

(iii) Ratio of 30 sec to 2 min

Since, 1 min = 60sec
2 min = 120sec

therefore,

$$30:\ 120 = \frac{30}{120} = \frac{1}{4} = 1:4$$

(iv) Ratio of Rs 200 to 300gm

Since, the quantities are not of same kind, so ratio between Rs 200 to 300gm cannot be found.

Example 2: Find the ratio $5a + 2b : 4a + 3b$ if $a : b = 4 : 5$

Solution: Given that $a : b = 4 : 5$ or $\frac{a}{b} = \frac{4}{5}$

Now $5a + 2b : 4a + 3b = \frac{5a + 2b}{4a + 3b}$

$$= \frac{\frac{5a+2b}{b}}{\frac{4a+3b}{b}} = \frac{5(\frac{a}{b})+2(\frac{b}{b})}{4(\frac{a}{b})+3(\frac{b}{b})}$$ (Dividing numerator and denominator by "b")

$$= \frac{5(\frac{4}{5})+2}{4(\frac{4}{5})+3} = \frac{4+2}{\frac{16}{5}+3} = \frac{6}{\frac{31}{5}} = \frac{30}{31} \qquad \because \left(\frac{a}{b}\right) = \left(\frac{4}{5}\right)$$

Hence, $5a + 2b : 4a + 3b = 30 : 31$

Example 3: Find the value of m, when the ratios $3m + 5 : 4m + 3$ and $2 : 3$ are equal.

Solution: According to the given condition

$\Rightarrow \quad \frac{3m+5}{4m+3} = \frac{2}{3}$

$\Rightarrow \quad 3(3m+5) = 2(4m+3)$

$\Rightarrow \quad 9m + 15 = 8m + 6$

$\Rightarrow \quad 9m - 8m = 6 - 15$

$\Rightarrow \quad m = -9$

Thus the required value of m is -9.

Example 4: What number must be added to each term of the ratio $4:15$ to make it equal to $\frac{2}{3}$.

Solution: Let the required number be a.

According to the given condition

$\Rightarrow \quad \frac{4+a}{15+a} = \frac{2}{3}$

$\Rightarrow \quad 3(4+a) = 2(15+a)$

$\Rightarrow \quad 12 + 3a = 30 + 2a$

$\Rightarrow \quad 3a - 2a = 30 - 12$

$\Rightarrow \quad a = 18$

Thus the required number is 18.

(b) Proportion:

The equality of two ratios is called proportion or a proportion is a statement, which is expressed as an equivalence of two ratios. If $\frac{a}{b} = \frac{c}{d}$ then a, b, c and d are in proportion and we can write it as $a : b :: c : d$ where quantities a and d are called extremes where b and c are called means. The proportion of a, b, c and d is written as

$a : b :: c : d$

or $\quad a : b = c : d$

or $\quad \frac{a}{b} = \frac{c}{d}$

$ad = bc$

Note: Product of Extremes = Product of Means

Example 5: Find the value of x, if $40:60=50:x$

Solution:

Given that $40:60=50:x$

Product of extremes = product of means

$$40x = 60\times 50$$

$$x=\frac{60\times 50}{40}=75$$

i.e., $x = 75$

Exercise 18.1

1. Find the ratio of the following.

(i) 70kg and 28kg (ii) 60cm and 1m (iii) 40sec, 3min

(iv) 200 *ml* and 2*l* (v) 135° and 360° (vi) 3.5kg, 5kg 200*gm*

2. In a factory, there are 120 workers in which 45 are women and remaining are men. Find the ratio of

(i) men to women (ii) women to men

(iii) women to total worker (iv) men to total workers

3. If $5(4x-2y)=3x-4y$, find $x:y$

4. Find the value of 'a' if the ratios $3a+4:2a+5$ and $4:3$ are equal

5. What number must be added to antecedent and consequent of the ratio 5:27 to make it equal to 1:3?

6. If $a:b=5:8$, find the value of $3a+4b:5a+7b$

7. Find the value of x in the following.

(i) $2x+5:5::3x-2:7$ (ii) $\frac{4x-3}{5}:\frac{3}{4}::\frac{4x}{3}:\frac{7}{2}$

(iii) $\frac{x-3}{2}:\frac{5}{x-1}::\frac{x-1}{3}:\frac{4}{x+4}$ (iv) $(a^2-ab+b^2):x::\frac{a^3+b^3}{a-b}:(a+b)^2$

(v) $11-x:8-x::25-x:16-x$

(e) Variation:

Variation is defined as the change in one quantity due to change in other quantity. There are two types of variations.

- Direct variation
- Inverse variation

Direct Variation:

If two quantities A and B are so related that when A increases (or decreases) in given ratio, B also increases (or decreases) in the same ratio then this variation is called direct variation. If a quantity y varies directly with regard to a quantity x, we say that y is directly proportional to x and is written as $y\propto x$

or $y=k\,x$ where $k\neq 0$

The sign $\propto$ is called the sign of proportionality or variation and k is called constant of proportionality or variation.

For example:

(i) Length of side of square and its area are in direct proportion.

(ii) The speed of cycle and covered distance are in direct variation.

Example 1: If y varies directly as x, find

(a) the equation connecting x and y.

(b) the relation between x and y when $x=3$ and $y=7$

(c) the value of y when $x=24$ (d) the value of x when $y=21$

Solution: (a) $\because$ y is directly proportional to x

$\therefore$ $y \propto x$

i.e. $y = kx$ ... (i)

where k is constant of variation

(b) By putting $x=3$ and $y=7$ in equation (i),

we get 7=3k $\Rightarrow$ $k=\frac{7}{3}$

By putting the value of k in equation (i), we get $y=\frac{7}{3}x$ ...(ii)

(c) By putting $x=24$ in equation (ii), we get $y=\frac{7}{3}(24)$ =56

(d) By putting $y=21$, in equation (ii),

we get, $21=\frac{7}{3}x \Rightarrow x=\frac{3\times 21}{7}=9$

Example 2: If y varies directly to the square root of x, $y=10$ when $x=16$, then find y when $x=36$

Solution: Given that y varies directly as square root of x

i.e. $y \propto \sqrt{x}$

or $y=k\sqrt{x}$ ... (i)

where k is constant of variation

By putting $x=16$ and $y=10$ in equation (i)

we get $10=k\sqrt{16} \Rightarrow k=\frac{10}{4}=\frac{5}{2}$

By putting $k=\frac{5}{2}$ in eq (i)

$y=\frac{5}{2}\sqrt{x}$ ...(ii)

By putting $x=36$ in eq (ii) we get

$y=\frac{5}{2}\sqrt{36}=$ $y=\frac{5}{2}(6)$ = 15

Example 3: Given that V varies directly as a cube of r and $V = \frac{792}{7}$ when $r = 3$. Find the value of V when $r = 7$.

Solution: Since V varies directly as cube of r.

$\therefore \quad V \propto r^3$

$V = kr^3 \quad$ (i) $\qquad$ (where k is constant of variation)

By putting $r = 3$ and $V = \frac{792}{7}$ in eq (i)

we get $\quad \frac{792}{7} = k(3)^3 \Rightarrow k = \frac{792}{7 \times 27} = \frac{88}{21}$

Hence $\quad V = \frac{88}{21} r^3 \quad$ (ii)

By putting $r = 7$ in eq: (ii)

we get $\quad V = \frac{88}{21}(7)^3 = \frac{4312}{3}$

Inverse Variation:

If two variables (quantities) are related to each other in such a way that increase in one variable causes decrease in another variable and vice versa then the variables are in inverse variation to each other. If a quantity y varies inversely with regard to quantity x, we say that y is inversely proportional to x and we write it as

$y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ $\quad$ where $k \neq 0$ (constant of variation)

$\Rightarrow \quad yx = k$

Example 1: If y varies inversely as x and $y = 7$ when $x = 2$. Find y when $x = 126$

Solution: Here y varies inversely as x

i.e. $\quad y \propto \frac{1}{x}$

$\Rightarrow \quad y = k(\frac{1}{x}) = \frac{k}{x}$

$\Rightarrow \quad k = xy \quad$ ... (i)

By puting $y = 7$ and $x = 2$ in eq (i)

we get

$k = (2)(7) = 14$

equation (i) becomes

$xy = 14 \quad$ ... (ii)

By putting $x = 126$ in eq (ii)

we get

$(126)y = 14$

$y = \frac{14}{126}$

$y = \frac{1}{9}$

Example 2: If y varies inversely as the cube root of x and $y = 27$ when $x = 8$. Find x when $y = 36$

Solution: Here y varies inversely as $\sqrt[3]{x}$

i.e. $\quad y \propto \frac{1}{\sqrt[3]{x}}$

or $\quad y = \frac{k}{\sqrt[3]{x}} \Rightarrow y\left(\sqrt[3]{x}\right) = k \quad$ ... (i)

By putting $x = 8$ and $y = 27$ in eq (i)

we get $k = (27)(\sqrt[3]{8}) = 54$

Now by putting $k = 54$ in eq (i)

we get $y\sqrt[3]{x} = 54$ ... (ii)

Now, by putting $y = 36$ in eq (ii)

we get $(36)\left(\sqrt[3]{x}\right) = 54$

$\sqrt[3]{x} = \frac{54}{36} = \frac{3}{2}$

$\Rightarrow \quad x = \frac{27}{8}$

Exercise 18.2

1. If y varies directly as x, and $y=10$ when $x=3$, find
   (i) y in terms of x (ii) y when $x=6$ (iii) x when $y=15$
2. If $V \propto T$ and $V=15$ when $T=24$, find
   (i) The equation connecting V and T.   (ii) V when $T=30$
   (iii) T when $V=10$
3. If $u \propto \sqrt[3]{v}$ and $u=4$ when $v=64$, find the value of u when $v=216$ and the value of v when $u=5$
4. If F varies directly as m^3 when $m=3$ and $F=81$, find F when $m=5$.
5. If y varies inversely as x and $y=10$ when $x=3$, find y when $x=10$.
6. The volume V of a gas varies inversely as square root of the pressure P of the gas, if V=12 when P=9, find P when V = 4
7. If $F \propto \frac{1}{r^2}$ and F=8 when $r=2$ then find:
   (i) F when $r=5$ (ii) r when F=24
8. The cube root of x varies as the square of y if $x=8$ when $y=3$ find x when $y=\frac{3}{2}$
9. The force F between two bodies is inversely proportional to the square of the distance between their centers. If F=2 and d=3 then find d when F=72
10. If y varies inversely as $(x-5)$ when $y=6$ and $x=8$, find y when $x=10$

18.1(ii) Find 3rd, 4th, proportionals of proportion and mean proportional in a continued proportion

We are already familiar with proportion that if a, b, c and d are in proportion then

$$a:b::c:d$$

Here a, b, c, and d are called first, second, third and fourth proportional respectively

(a) Third proportional

Example 1:
If 4,7 and 14 are the first, second and fourth proportional in a proportion respectively then find its third proportional.

Solution: let x be the third proportional
then,

$$4:7::x:14$$
$$\Rightarrow \quad 7x=56$$
$$\Rightarrow \quad x=8$$

Example 2:
If $(a-b), (a^2+ab+b^2)$ and (a^3-b^3) are the first, second and fourth proportional respectively then find the third proportional.

Solution: let x be the third proportional
then,

$$(a-b):(a^2+ab+b^2)::x:a^3-b^3$$
$$\Rightarrow \quad (a^2+ab+b^2)x=(a-b)(a^3-b^3)$$
$$(a^2+ab+b^2)\,x=(a-b)^2(a^2+ab+b^2)$$
$$\Rightarrow \quad x=\frac{(a-b)^2(a^2+ab+b^2)}{(a^2+ab+b^2)}$$
$$\Rightarrow \quad x=(a-b)^2$$

(b) fourth proportional

Example 1: Find the fourth proportional if first three proportional are $a^3+b^3, a-b$ and a^2-ab+b^2

Solution: let x be the fourth proportional

then $a^3+b^3 : a-b :: (a^2-ab+b^2) : x$

i.e. $x\,(a^3+b^3) = (a-b)(a^2-ab+b^2)$

$\Rightarrow \quad x = \dfrac{(a-b)(a^2-ab+b^2)}{(a+b)(a^2-ab+b^2)}$

$\Rightarrow \quad x = \dfrac{a-b}{a+b}$

Continued Proportion and mean proportional

The quantities a, b, c, are said to be in continued proportion if $a:b=b:c$

or $\quad a:b::b:c \quad \Rightarrow b^2 = ac$

Here b is called mean proportional

Example 1: Find a mean proportional of x^2-y^2 *and* $\dfrac{x-y}{x+y}$

Solution: Let z be the mean proportional

then

$x^2-y^2 : z :: z : \dfrac{x-y}{x+y}$

i.e., $z^2 = (x^2-y^2)\dfrac{(x-y)}{x+y} = (x-y)^2$

$\Rightarrow \quad z = (x-y)$

Example 2: Find the value of a if $7, a-3$ and 28 are in continued proportion

Solution: Since $7, a-3$ and 28 are in continued proportion.

$7 : a-3 :: a-3 : 28$

i.e. $(a-3)^2 = 7\times 28$

$\Rightarrow (a-3)^2 = 196$

$\Rightarrow a-3 = 14$

$\Rightarrow a = 14+3$

$\Rightarrow a = 17$

Exercise 18.3

1. Find the third proportional if first, second and fourth proportional are

(i) 6, 18 and 54 (ii) $a^2-b^2, a+b$ and $a-b$

(iii) $(x+y)^2, x^3+y^3$ and $x+y$ (iv) $\dfrac{a^3+b^3}{a^2-b^2}, \dfrac{a^2-ab+b^2}{a-b}$ and $a+b$

2. Find the fourth proportional to

(i) 8, 4, 2 (ii) $a^3+b^3,\ a^2-b^2, a^2-ab+b^2$

(iii) $a^2-8a+12,\ a-2,\ 2a^3-12a^2$

(iv) $(a^2-b^2)(a^2-ab+b^2), a^3+b^3, a^3-b^3$

3. Find the mean proportional to

(i) 8, 18 (ii) $5ab^2, 20a^3b^2$

(iii) $a^4-b^4, \dfrac{a^2-b^2}{a^2+b^2}$ (iv) $a^3-b^3, \dfrac{a-b}{a^2+ab+b^2}$

4. Find the value of x in the following continued proportions.

(i) 45, x, 5 (ii) 16, x, 9

(iii) 12, 3x-6, 27 (iv) 7, x-3, 112

18.2 Theorems on proportions

18.2.1: Apply theorems of invertendo, alternando, componendo, dividendo to solve the problem of proportions.

(i). Theorem of invertendo:

If $a:b=c:d$ then $b:a=d:c$

The above statement is called theorem of invertendo.

Proof: Since $a:b=c:d$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{b}{a}=\frac{d}{c} \qquad \text{(Taking reciprocal of both sides)}$$

i.e. $b:a=d:c$ **Hence proved.**

For example: (i) If 2:3 = 4:6 then by theorem of invertendo

$3:2=6:4$

(ii) If $4p:5q=2r:5s$ then by theorem of invertendo

$5q:4p=5s:2r$

(ii). Theorem of alternando:

If $a:b=c:d$ then $a:c=b:d$

This statement is called theorem of alternando.

Proof: Since $a:b=c:d$

or $\dfrac{a}{b}=\dfrac{c}{d}$

$\Rightarrow$ $ad=bc$

Dividing both sides by cd

we get

we get $\dfrac{a}{c}=\dfrac{b}{d}$

i.e. $a:c=b:d$ **Hence proved.**

Example:

(i) If $2:5=6:15$ then by theorem of alternando

$2:6=5:15$

(ii) If $4p-1:2-3q=5+2r:2s+1$ then by theorem of alternando

$4p-1:5+2r=2-3q:2s+1$

(iii) Theorem of componendo:

If $a:b=c:d$ then according to theorem of componendo

(i) $a+b:b=c+d:d$ (ii) $a:a+b=c:c+d$

Since $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

Adding 1 to both sides

$\frac{a}{b}+1=\frac{c}{d}+1$

$\Rightarrow$ $\frac{a+b}{b}=\frac{c+d}{d}$

or $a+b:b=c+d:d$

Hence proved

(ii) $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow$ $\frac{b}{a}=\frac{d}{c}$ (By invertendo theorem)

By adding 1 to both sides

$\frac{b}{a}+1=\frac{d}{c}+1$

$\frac{b+a}{a}=\frac{d+c}{c}$

$\Rightarrow$ $\frac{a}{b+a}=\frac{c}{c+d}$

or $a:a+b=c:c+d$

Hence proved

If $p+2:q=r:s-3$ then by theorem of componendo (i)

$p+2+q:q=r+s-3:s-3$

Similarly by theorem of componendo (ii)

$p+2:p+2+q=r:r+s-3$

(iv). Theorem of Dividendo:

If $a:b=c:d$ then according to theorem of dividendo

(i) $a-b:b=c-d:d$ (ii) $a:a-b=c:c-d$

Proof: (i) Since $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

Subtracting 1 from both sides.

$$\frac{a}{b}-1=\frac{c}{d}-1$$

$$\frac{a-b}{b}=\frac{c-d}{d}$$

or $a-b:b=c-d:d$

Hence proved.

(ii) Since $a:b=c:d$

or $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow \frac{b}{a}=\frac{d}{c}$ (By invertendo theorem)

Subtracting 1 from both sides.

$$\frac{b}{a}-1=\frac{d}{c}-1$$

$$\frac{b-a}{a}=\frac{d-c}{c}$$

$\Rightarrow \frac{a}{b-a}=\frac{c}{d-c}$ (By invertendo theorem)

or $a:b-a=c:d-c$

Hence proved.

Example:

If $m+5:n-3=4p+7:3q+2$ then show that $m-n+8:n-3=4p-3q+5:3q+2$

Solution: Since

$$m+5:n-3=4p+7:3q+2$$

or $\frac{m+5}{n-3}=\frac{4p+7}{3q+2}$

By dividendo theorem

$$\frac{(m+5)-(n-3)}{n-3}=\frac{(4p+7)-(3q+2)}{3q+2}$$

$$\frac{m-n+8}{n-3}=\frac{4p-3q+5}{3q+2}$$

i.e. $m-n+8:n-3=4p-3q+5:3q+2$

Hence shown.

(v). Theorem of componendo and dividendo:

If $a:b=c:d$ then according to theorem of componendo and dividendo

(i) $a+b:a-b=c+d:c-d$

(ii) $a-b:a+b=c-d:c+d$

Example 1: If $m:n=p:q$ then show that $3m-2n:3m+2n=3p-2q:3p+2q$

Solution: Since $m:n=p:q$

or $\frac{m}{n}=\frac{p}{q}$

Multiplying both sides by $\frac{3}{2}$

$$\frac{3m}{2n}=\frac{3p}{2q}$$

By using componendo and dividendo theorem (ii)

$$\frac{3m-2n}{3m+2n}=\frac{3p-2q}{3p+2q}$$

or $3m-2n:3m+2n=3p-2q:3p+2q$

Hence shown.

Example 2: If $3p+4q:3p-4q=3r+4s:3r-4s$ then show that $p:q=r:s$

Solution: Since $3p+4q:3p-4q=3r+4s:3r-4s$

By using componendo and dividendo theorem (i)

$$\therefore \quad \frac{(3p+4q)+(3p-4q)}{(3p+4q)-(3p-4q)}=\frac{(3r+4s)+3r-4s}{(3r+4s)-(3r-4s)}$$

$$\frac{3p+4q+3p-4q}{3p+4q-3p+4q}=\frac{3r+4s+3r-4s}{3r+4s-3r+4s}$$

$$\frac{6p}{8q}=\frac{6r}{8s} \qquad \text{(by cancelling } \frac{6}{8} \text{ from both side)}$$

$$\Rightarrow \quad \frac{p}{q}=\frac{r}{s}$$

or $p:q=r:s$

Hence shown.

Example 3: Using theorem of componendo and dividend theorem find the value of x

if $\dfrac{\sqrt{x+6}-\sqrt{x-6}}{\sqrt{x+6}+\sqrt{x-6}}=\dfrac{2}{5}$.

Solution: We have $\dfrac{\sqrt{x+6}-\sqrt{x-6}}{\sqrt{x+6}+\sqrt{x-6}}=\dfrac{2}{5}$

By componendo and dividendo theorem

$$\frac{(\sqrt{x+6}-\sqrt{x-6})+(\sqrt{x+6}+\sqrt{x-6})}{(\sqrt{x+6}-\sqrt{x-6})-(\sqrt{x+6}+\sqrt{x-6})}=\frac{2+5}{2-5}$$

$$\frac{\sqrt{x+6}-\sqrt{x-6}+\sqrt{x+6}+\sqrt{x-6}}{\sqrt{x+6}-\sqrt{x-6}-\sqrt{x+6}-\sqrt{x-6}}=\frac{7}{-3}$$

$$\frac{2\sqrt{x+6}}{-2\sqrt{x-6}}=\frac{7}{-3} \Rightarrow \frac{\sqrt{x+6}}{\sqrt{x-6}}=\frac{7}{3} \qquad \text{....(1)}$$

squaring **(1)** both sides

$$\frac{x+6}{x-6}=\frac{49}{9}$$

$$9x+54=49x-294$$

$$9x-49x=-294-54$$

$$-40x=-348$$

$$x=\frac{348}{40}=\frac{87}{10}$$

As (1) is a radical equation, so verification of root is necessary. On verification it is found that $x=\dfrac{87}{10}$ satisfies the original equation. Therefore, $\text{S.S}=\left\{\dfrac{87}{10}\right\}$

Example 4: Solve the equation: $\dfrac{(x+3)^2+(x-1)^2}{(x+3)^2-(x-1)^2}=\dfrac{5}{4}$ by using componendo and dividendo theorem.

Solution: We have $\dfrac{(x+3)^2+(x-1)^2}{(x+3)^2-(x-1)^2}=\dfrac{5}{4}$

By componendo and dividendo theorem

$$\frac{(x+3)^2+(x-1)^2+(x+3)^2-(x-1)^2}{(x+3)^2+(x-1)^2-(x+3)^2+(x-1)^2}=\frac{5+4}{5-4}$$

$$\Rightarrow \quad \frac{2(x+3)^2}{2(x-1)^2}=\frac{9}{1}$$

$$\Rightarrow \quad \left(\frac{x+3}{x-1}\right)^2=9$$

$$\Rightarrow \quad \frac{x+3}{x-1}=\pm 3$$

$$\frac{x+3}{x-1}=3 \qquad \text{or} \qquad \frac{x+3}{x-1}=-3$$

$$x+3=3x-3 \qquad\qquad x+3=-3x+3$$

$$-2x=-6 \qquad\qquad 4x=0$$

$$x=3 \qquad\qquad x=0$$

$\therefore$ The solution set is $\{0,3\}$

Example 5: Prove that $a:b=c:d$ if $\dfrac{ac^2-bd^2}{ac^2+bd^2}:\dfrac{c^3-d^3}{c^3+d^3}$

Solution:

Since $\dfrac{ac^2-bd^2}{ac^2+bd^2}=\dfrac{c^3-d^3}{c^3+d^3}$

By componendo and dividendo theorem

$$\frac{ac^2-bd^2+ac^2+bd^2}{ac^2-bd^2-ac^2-bd^2}=\frac{c^3-d^3+c^3+d^3}{c^3-d^3-c^3-d^3}$$

$$\frac{2ac^2}{-2bd^2}=\frac{2c^3}{-2d^3}$$

$$\Rightarrow \quad \frac{a}{b}=\frac{c}{d}$$

or $\quad a:b=c:d$

Hence Proved.

Exercise 18.4

1. If $x:y=z:w$ then prove that

(i) $\frac{4x+3y}{4x-3y}=\frac{4z+3w}{4z-3w}$ (ii) $\frac{5x-3y}{5x+3y}=\frac{5z-3w}{5z+3w}$

(iii) $\frac{x^3+y^3}{x^3-y^3}=\frac{z^3+w^3}{z^3-w^3}$ (iv) $\frac{3x+2y}{3x-2y}=\frac{3z+2w}{3z-2w}$

(v) $\frac{2x+3y+2z+3w}{2x+3y-2z-3w}=\frac{2x-3y+2z-3w}{2x-3y-2z+3w}$

2. Prove that $a:b=c:d$ if

(i) $a^2-b^2:a^2+b^2=ac-bd:ac+bd$

(ii) $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$

3. Solve the following equation by using componendo – dividendo theorem

(i) $\frac{(x+3)^2-(x-5)^2}{(x+3)^2+(x-5)^2}=\frac{4}{5}$ (ii) $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{1}{2}$

(iii) $\frac{\sqrt{x+5}-\sqrt{x-5}}{\sqrt{x+5}+\sqrt{x-5}}=\frac{1}{10}$ (iv) $\frac{(x-4)^3-(x-3)^3}{(x-4)^3+(x-3)^3}=\frac{63}{65}$

18.3 Joint variation

18.3(i) Define joint variation

When a variable depends on the product or quotient of two or more variables is called the joint variation. In other words, joint variation occurs when a variable varies directly or inversely with multiple variables.

For example

(i) If x varies directly with both y and z we have $x \propto yz \Rightarrow x=kyz$

(ii) If x varies directly with y and inversely with z we have $x \propto \frac{y}{z} \Rightarrow x=\frac{ky}{z}$

18.3 (ii) Solve problem related to joint variation.

Example 1:

If y varies directly as xz^2 and $y=4$ when $x=6$ and $z=3$. Find the value of y when $x=-81$ and $z=5$

Solution Since y varies directly as xz^2

$\therefore$ $y \propto xz^2$

i.e. $y=k\,xz^2$ (i)

By putting, $y=4$, $x=6$ and $z=3$ in eq (i)

$4=k\,(6)\,(3)^2 \Rightarrow k=\frac{4}{54}=\frac{2}{27}$

By putting, $k = \frac{2}{27}$ in eq (i)

we get $y = \frac{2}{27} x z^2$ ...(ii)

Now by putting $x = -81$ and $z = 5$ in eq (ii)

$$y = \frac{2}{27}(-81)(5)^2 = 2(-3)(25) = -150$$

Example 2. If s varies directly as v^3 and inversely as $\sqrt{t}$ u and s = 16 when v = 4, t = 9 and u = 24; find the value of s when u = 25, t = 36 and v = 5.

Solution. Since s varies directly as v^3 and inversely as $\sqrt{t}$ u.

$$\therefore \quad s \propto \frac{v^3}{u\sqrt{t}} \quad \Rightarrow \quad s = k\frac{v^3}{u\sqrt{t}} \quad \text{... (i)}$$

By putting s = 16, v = 4, t = 9 and u = 24 in eq (i)

$$16 = \frac{k(4)^3}{\sqrt{9}(24)}$$

$$16 = \frac{k(64)}{72} \Rightarrow k = 16 \times \frac{72}{64} = 18$$

By putting k = 18 in eq (i)

$$s = \frac{18\,v^3}{u\sqrt{t}} \quad \text{... (ii)}$$

By putting u = 25, t = 36 and v = 5 in eq (ii) we get

$$s = \frac{18\,(5)^3}{\sqrt{36}\,(25)} = \frac{18 \times 125}{6 \times 25} = 15$$

Exercise 18.5

1. If y varies directly as x^2 and z and y=6 when $x = 4, z = 9$. Write y as a function of x and z and determine the value of y when $x = -8$ and $z = 12$
2. If y varies directly as x and u^2 and inversely as v and t. y=40 when $x = 8$, u = 5, v = 3 and t = 2 find y in terms of x, u, v and t. Also find the value of y when $x = -2$, u = 4, v = 3, t = −1.
3. If w varies directly as u^2 and inversely as cubic root of v and w = 216 when u = 6 and v=27. Find the value of w when u =10 and v =125.
4. If the time period T of simple pendulum is directly proportional to the square root of its length L and inversely proportional to the square root of acceleration due to gravity ‘g’ if T =2sec L=100cm and g =9.8m/s. Find time period of simple pendulum when L =200m and g =7.6 at a particular height.
5. The volume V of a particular gas is directly proportional to the temperature T and inversely proportional to the square root of pressure. If V=100, T =30 and P =64. Find the value of V when T=60 and P=81.

18.4 k – Method

When two or more ratios are equal many useful proportion may be proved by introducing a single symbol 'k' to denote each of the equal ratios. This method of using symbol k to each ratio is called k-method, the k method is a very useful method in solving many problems.

18.4 (i) Use k-method to prove conditional equalities involving proportions

If $a:b::c:d$ is a proportion.

Then by k – method

i.e $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow \frac{a}{b} = k$ and $\frac{c}{d} = k$

$a = b\,k$ ... (i) and $c = d\,k$ ... (ii)

By using these equations, we can solve problem relating to proportion.

Example1. If $a:b::c:d$ then show that $\frac{8a-5b}{8a+5b} = \frac{8c-5d}{8c+5d}$

Solution: Since $a : b = c : d$

Let $\frac{a}{b} = \frac{c}{d} = k$

Then $a = bk$ and $c = dk$.

Now L H.S $= \frac{8a-5b}{8a+5b} = \frac{8bk-5b}{8bk+5b} = \frac{b(8k-5)}{b(8k+5)} = \frac{8k-5}{8k+5}$

R.H.S $= \frac{8c-5d}{8c+5d} = \frac{8dk-5d}{8dk+5d} = \frac{d(8k-5)}{d(8k+5)} = \frac{8k-5}{8k+5}$

$\because$ L.H.S = R.H.S

$\therefore \frac{8a-5b}{8a+5b} = \frac{8c-5d}{8c+5d}$

Hence shown.

Example 2: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then prove then $\frac{(a+c+e)^3}{(b+d+f)^3} = \frac{ace}{bdf}$.

Proof: we have $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

Then $\frac{a}{b} = k\ ,\ \frac{c}{d} = k\ ,\ \frac{e}{f} = k$

$\Rightarrow a = bk \quad c = dk$ and $e = fk$

L.H.S $= \frac{(a+c+e)^3}{(b+d+f)^3} = \frac{(bk+dk+fk)^3}{(b+d+f)^3} = \frac{[k(b+d+f)]^3}{(b+d+f)^3} = k^3$

$$\text{R.H.S} = \frac{ace}{bdf} = \frac{(bk)(dk)(fk)}{bdf} = \frac{k^3 bdf}{bdf} = k^3$$

L.H.S = R.H.S

$$\frac{(a+c+e)^3}{(b+d+f)^3} = \frac{ace}{bdf}$$

Hence proved.

Example 3:

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then show that $(a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2$

Proof: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$\Rightarrow \quad a = bk, \quad c = dk$ and $e = fk$

$$\begin{aligned}\text{L.H.S} &= (a^2+c^2+e^2)(b^2+d^2+f^2)\\ &= (b^2k^2+d^2k^2+f^2k^2)(b^2+d^2+f^2)\\ &= k^2(b^2+d^2+f^2)(b^2+d^2+f^2)\\ &= k^2(b^2+d^2+f^2)^2\end{aligned}$$

$$\begin{aligned}\text{R.H.S} &= (ab+cd+ef)^2\\ &= \left[(bk)b+(dk)d+(fk)f\right]^2\\ &= \left[kb^2+kd^2+kf^2\right]^2\\ &= \left[k(b^2+d^2+f^2)\right]^2\\ &= k^2(b^2+d^2+f^2)^2\end{aligned}$$

L.H.S = R.H.S

$\therefore \quad (a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2$

Hence shown.

Exercise 18.6

1. If $p : q = r : s$ then show that

(i) $\frac{8p-3q}{8p+3q} = \frac{8r-3s}{8r+3s}$

(ii) $\sqrt[3]{\frac{p^3+r^3}{q^3+s^3}} = \frac{p}{q}$

(iii) $(p^2+q^2) : \frac{p^3}{p+q} = (r^2+s^2) : \frac{r^3}{r+s}$

(iv) $p^5+r^5 : q^5+s^5 = p^3 r^2 : q^3 s^2$

(v) $\frac{p-q}{p} : \frac{q}{p+q} = \frac{r-s}{r} : \frac{s}{r+s}$

2. If $a : b = c : d = e : f$ then show that

(i) $\frac{a^4 b^2 + a^2 e^2 - e^4 f}{b^6 + b^2 f^2 - f^5} = \frac{a^4}{b^4}$

(ii) $\frac{a^2 b + c^2 d + e^2 f}{ab^2 + cd^2 + ef^2} = \frac{a+c+e}{b+d+f}$

(iii) $\frac{ac}{bd} + \frac{ce}{df} + \frac{ea}{fb} = \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2}$

18.4 (ii) Solve real life problems based on variations:

Example 1.

The potential energy of a body varies jointly as the mass "m" of the body and the height "h" If potential energy is 1960 joules when m=2kg and h=100 then determine potential energy of the body having mass 5kg and height 300m.

Solution: Since potential energy "p" varies jointly as the mass m and the height h

i.e. $p \propto mh$

$p = kmh$ … (i) when k is constant of proportionality

By putting m = 2, h = 100 and p = 1960 in equation (i)

We get $1960 = k(2)(100) \Rightarrow k = \frac{1960}{200} = 9.8$

By putting the value of k in equal (i),

$p = 9.8mh$ … (ii)

Now by putting m=5 and h = 300 in eq: (ii)

We get $p = 9.8(5)(300) = 14700$ joules.

Example 2.

According to Hook's law the force F applied to stretch a spring varies directly as the amount of elongation S when F=64 *lb* and S=3.2 inches.

Find. (i) S when F=100 *lb*

(ii) F when S=0.4 inches.

Solution: Since $F \propto S$

$F = kS$ … (i)

By Putting F=64 & S=3.2

we get

$64 = 3.2k$

$\Rightarrow \quad k = \frac{64}{3.2} = 20$

By putting k = 20 in eq (i)

we get $F = 20S$ … (ii)

(i) By putting F=100 in eq (ii)

we get $100 = 20S$

$\Rightarrow \quad S = 5$ inch

(ii) By putting S=0.4 in eq (ii)

We get $F = 20(0.4) = 8$ *lb*

Example 3:

Labour cost C varies jointly as the number of workers "*n*" and the average number of days *d*, if the cost of 200 workers for 10 days is Rs.450,000. Find the labour cost of 300 workers for 16 days.

Solution: Since cost varies directly as the number of workers and the days.

i.e., $C \propto nd$

$C = knd \qquad$ (i)

By putting $C = 450,000, n = 200$ and $d = 10$ in eq (i)

$$450,000 = k(200)(10)$$

$$\Rightarrow \quad k = \frac{450,000}{2000} = 225$$

By putting $k = 225$ in eq (i)

we get $C = 225 \cdot nd \qquad$ (ii)

By putting $n = 300$ and $d = 16$ in eq (ii)

we get $C = 225 \times 300 \times 16$

$C = 1080000$

Hence the cost for 300 workers for 16 days is Rs.1080000.

Example 4:

The expenses of a hostel are partly constant and partly vary according to the number of students. When the number of students are 140 the expenditure is Rs 19300 and for 200 students the expenditure is Rs 26500. Find the expenses when number of students are 300.

Solution: Let the constant expenses be denoted as C, Q denotes partly expense, x is the number of students and P denotes the total expense.

Since the expenses varies directly to number of students.

i.e., $Q \propto x$

$Q = nx$ where n is constant

$P = C + Q$

$P = C + nx$ where n is constant

According to given condition

Hence $19300 = C + 140n$ ...(i)

and $26500 = C + 200n$ ...(ii)

also $P = C + 300n$ ...(iii)

From (i) and (ii), we get $n = 120$ and $C = 2500$

Now, equation (iii) becomes

$$P = 2500 + 300(120) = 38500$$

The total expense for 300 students is 38500.

Exercise 18.7

1. The current in a wire is directly proportional to the potential difference and inversely proportional to the resistance R. If I=6 amperes, when V=220 volts and R=5 ohms. Find I when V=180 volts and R=8 ohms
2. The intensity of light measured in foot candles varies inversly with the square of the distance from the light source. Suppose the intensity of a high bulb is 0.08 foot candles at a distance of 3 meters .find the intensity level at 8 meters.
3. The strength S of a rectangular beam can varies directly as the breadth b and the square of the depth d. if a bean 9 cm wide a 12 cm deep with support 1200 pound. What weight a bean of 12 cm wide and 9cm deep will support.
4. Labor cots C varies jointly as the number of worker n and the average number of days d, if the cost of 100 workers for 15 days is 9000, then find the labour cost of 300 workers for 20 days.
5. The sales tax on the purchase of a new car varies directly as the price of a car. If a new car is purchased in Rs 2000000, then the sale tax is Rs 40000. How much sale tax is charged if the new car is priced at Rs 2800000.

Review Exercise 18

1. Multiple Choice Question

Select the correct option in the following

(i) In a proportion $p : q :: r : s$, p is called

(a) third proportional (b) mean (c) fourth proportional (d) first proportional

(ii) In a ratio $l : m$, l is called

(a) consequent (b) antecedent (c) relation (d) none of these

(iii) In a ratio $u:v$, v is called

(a) antecedent (b) consequent (c) relation (d) none of these

(iv) If a,b,c are in continued proportion then b is called ____________ proportion between a & c

(a) 1st (b) mean (c) 3rd (d) None of these

(v) The mean proportional between a^2 and b^2 is ______________

(a) $\sqrt{ab}$ (b) ab (c) $\frac{a}{b}$ (d) $-ab$

(vi) If $x+5:x+7=5:7$ then x is equal to ____________

(a) 2 (b) -1 (c) 0 (d) 1

(vii) If 1,9, x and 45 are in proportion, then $x=$

(a) 27 (b) $\frac{1}{5}$ (c) 5 (d) 405

(viii) If $p:q=r:s$ then $p:r=q:s$ this property is called

(a) compodendo (b) invertendo (c) dividendo (d) alternando

(ix) If $x:y=z:w$ then according to componendo

(a) $\frac{x}{x+y}=\frac{z}{z+w}$ (b) $\frac{x}{x-y}=\frac{z}{z-w}$

(c) $\frac{x-y}{x+y}=\frac{z-w}{z+w}$ (d) $\frac{x-y}{y}=\frac{z-w}{w}$

(x) If $a:b=c:d$ then according to alternando property

(a) $\frac{a}{b}=\frac{c}{d}$ (b) $\frac{a+b}{b}=\frac{c+d}{c}$ (c) $\frac{b}{a}=\frac{d}{c}$ (d) $\frac{a}{c}=\frac{b}{d}$

(xi) The fourth proportional to 3,5,12 is

(a) 20 (b) 15 (c) 60 (d) 36

(xii) If 2x, 3y and 6z are in continued proportion then

(a) $y^2=12xy$ (b) $9y^2=xz$ (c) $9y^2=12xz$ (d) $3y^2=4xz$

(xiii) If $\frac{x}{y}=\frac{w}{z}$ then according to dividendo property is

(a) $\frac{x-y}{y}=\frac{w-z}{z}$ (b) $\frac{x+y}{y}=\frac{w+z}{z}$

(c) $\frac{x}{x+y}=\frac{w}{w+z}$ (d) None of these

(xiv) Force and acceleration are in

(a) direct proportion (b) joint proportion

(c) inverse proportion (d) None of these

(xv) If $a:4::15:5$ then a =______________
(a) 20 (b) 15 (c) 12 (d) 10

2. Find the ratios of the
(i) 100m and 500cm (ii) 50kg and 300g

3. Find the value of x in the following
(i) $5:x-3=x+11:3$ (ii) $9:x-10=x+13:12$

4. If y varies directly as x and $y = 25$ when $x = 75$ then find y when $x = 144$.

5. If y varies inversely as x and $y = 100$ when $x = \frac{1}{2}$ then find y when $x = 4$.

6. If $x:y=z:w$ then prove that $\frac{7x+5y}{7x-5y}=\frac{7z+5w}{7z+5w}$.

7. Solve $\frac{(x-3)(x-5)}{(x-7)(x-2)}=\frac{(x-6)(x-2)}{(x-1)(x-8)}$ by componendo – dividendo theorem.

8. If x varies directly as y and inversely as z. If $x = 30$ when $y = 15$ and $z = 2$. Find x if $y = 30$ when $z = 12$.

9. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 ampere, the resistance in 10ohms. Find the current if resistance is 12 ohm.

Summary

- The comparison between two quantities of the same kind is called ratio
- If two ratio $a:b$ and $c:d$ are equal then we can write.
  $a:b::c:d$
- Equality of two ratios is called proportion.
- If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) respectively in the other quantity is called direct variation.
- If two quantities are related in such a way that when one quantity increase, then the other decreases and vice versa is called the quantities are in direct variation.
- Theorems on proportion
  (i) Theorems invertendo If $a:b=c:d$ then $b:a=d:c$
  (ii) Theorems of alternendo If $a:b=c:d$ then $a:c=b:d$
  (iii) Theorems of componendo
  If $a:b=c:d$ then (a) $a+b:b=c+d:d$ (b) $a:a+b=c:c+d$
  (iv) Theorem of dividend
  If $a:b=c:d$ then $a-b:b=c-d:d$ and $a:a-b=c:c-d$
  (v) Theorem componendo and dividendo
  If $a:b=c:d$ then
  $a+b:a-b=c-d:c-d$
- If one variable varies directly or inversely with two or more than two variables then it is called joint variation