INTRODUCTION TO TRIGONOMETRY

Unit 30

• Weightage = 10%

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Measure an angle in sexagesimal system (degrees, minutes and seconds).
- Convert an angle given in $D^{\circ}\ M'\ S''$ form into decimal form (upto two decimal places) and vice- versa.
- Define a radian (measure of an angle in circular system) and prove the relationship between radian and degree measures
- Establish the rule $l = r\Theta$, where r is the radius of the circle, l the length of the circular arc and Θ is the central angle measured in radians.
- Prove that the area of the sector of a circle is $\frac{1}{2}r^2\theta$ or $\frac{1}{2}l\,\theta$
- Define and identify:
    - General angle (coterminal angles)
    - Angle in standard position.
- Recognize quadrants and quadrantal angles.
- Define trigonometric ratios and their reciprocalswith the help of a unit circle.
- Recall the values of the trigonometric ratios for45º, 30º, 60º.
- Recognize signs of trigonometric ratios in different quadrants.
- Find the values of remaining trigonometric ratios ifone trigonometric ratio is given.
- Calculate the values of trigonometric ratios for0º, 90º, 180º, 270º, 360º.
- Prove the trigonometric identities and apply them to showdifferent trigonometric relations.
- Find angles of elevation and depression.
- Solve real life problems involving angles ofelevation and depression.

Introduction:

The word trigonometry has been derived from Greek word, "tri"(Three), gono (angles) and "metron" (measurement). Literally, trigonometry means measurement of the triangle. This branch of mathematics was developed by Greeks in 330 B.C. It deals with solution of triangle. It has vast application in Physics, navigation astronomy surveying etc.

30.1 Measurement of an Angle:

An angle is defined as union of two rays which have a common point (called vertex), one of the rays is called "initial side" and other is called "terminal side".

Measuring of angle depends upon the direction of the rotation from the initial side to the terminal side. Angle measured in the anti clock-wise direction is taken to be "positive angle" as shown in figure (a) and angle measured in the clock-wise direction is to be negative angle, as shown in figure (b).

Note: The rays are also called arms of the angle and their common end point is also known as vertex of the angle.

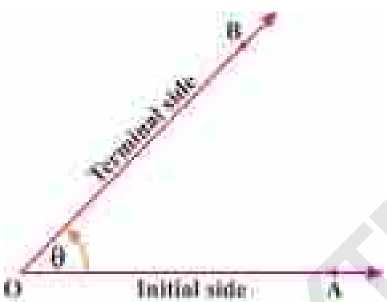

Figure (a)

Figure (b)

An angle is said to be in standard position if its initial side is on the positive of *x*-axis and its vertex is at origin.

30.1.(i) Measure an angle in Sexagesimal system (Degrees, minutes and seconds)

If the initial ray (arm) $\overrightarrow{OA}$ completes one rotation in anticlock-wise direction, then the angle formed is said to be 360 degree or 360°. Hence, the circumference of a circle is divided into 360 equal arcs. The angle subtended at the centre of the circle by such an arc is called one degree and is denoted by 1°. Each degree is divided into 60 equal parts, each part is called 1 minute is denoted as $1'$. Furthermore, each minute is divided into further 60 parts and each part is 1 second and is denoted by $1''$. Hence 1 minute is the 60^{th} part of 1 degree and 1 second is $360^{\circ\, th}$ part of 1 degree. The system in which angle measures in degree, minutes and seconds is called sexagesimal system.

Thus one degree is defined as the measure $\frac{1}{360}$th of a complete rotation and it is denoted by 1°, it is further sub-divided as under

$$1^\circ = 60 \text{ minutes} = 60'$$

$$1' = 60 \text{ seconds} = 60''$$

e.g, 30 degrees, 20 minutes and 10 seconds are written symbolically as $30^\circ 20' 10''$

30.1.(ii) Convert an angle given in $D^\circ M' S''$ into decimal form (up to two decimal places) and vice versa

In this section we convert minutes and seconds into degrees by dividing minute with 60 and second with 3600. After simplifying we get required decimal form.

Example 1

Convert $30^\circ 30' 10''$ into degrees and write in decimal form.

Solution:

$$\text{Since} \quad 1' = \left(\frac{1}{60}\right)^\circ$$

$$\therefore \quad 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2}\right)^\circ$$

$$\text{and} \quad 1'' = \left(\frac{1}{3600}\right)^\circ$$

$$10'' = \left(\frac{10}{3600}\right)^\circ = \left(\frac{1}{360}\right)^\circ$$

$$\therefore \quad 30^\circ 30' 10'' = \left(30 + \frac{1}{2} + \frac{1}{360}\right)^\circ$$

$$= \left[30 + 0.5 + 0.003\right]^\circ$$

$$= 30.503^\circ$$

$$= 30.50^\circ \qquad (\text{round off upto two decimal places})$$

Example 2

Convert 12.51° into $D^\circ M' S''$ form.

Solution:

$$12.51^\circ = 12^\circ + (0.51)^\circ$$

$$= 12^\circ + (0.51 \times 60)' \qquad (\because 1^\circ = 60')$$

$$= 12^\circ + (30.6)'$$

$$= 12^\circ + 30' + (0.6)'$$

$$= 12^\circ 30' + (0.6 \times 60)'' \qquad (\because 1' = 60'')$$

$$= 12^\circ 30' 36''$$

30.1.(iii) Define a radian (measure of an angle in circular system) and prove the relationship between radian and degree measures.

Radian is another unit to measure an angle which is equal to ratio of arc length to the radius of the circle. i.e., $\theta = \frac{l}{r}$, where θ is central angle in radians, l is the arc length and r is the radius of the circle. The system where angle is measured in radian is called circular system. Thus, the angle measure in circular system is said to be of 1 radian if it is subtended by an arc equal to radius of circle. (as shown in figure 30.1).

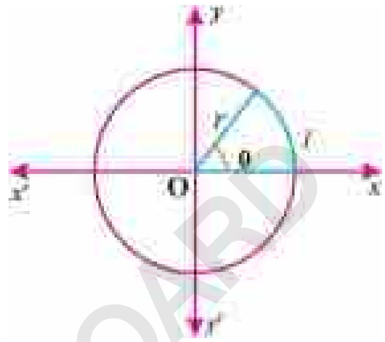

Fig: 30.1

we have $\theta = \frac{l}{r}$,

when $l = r$

then $\theta = \frac{l}{l}$

$$\boxed{\theta = 1 \quad \text{radian}}$$

For a complete rotation the arc length is the circumference of the circle i.e., $l = 2\pi r$.

Now,

$$\theta = \frac{l}{r}$$

$$\theta = \frac{2\pi r}{r}$$

$$\boxed{\theta = 2\pi \ \text{radian}}$$

Hence for a complete rotation, the angle measured in radians is 2π.

Relationship between degree and radian

We know that, for a complete rotation the angle measured in degree is $360°$, and angle measured in radians is 2π.

Therefore

$360° = 2\pi$ radians

$\Rightarrow \quad 180° = \pi$ radians

$\Rightarrow \quad 1° = \frac{\pi}{180}$ radians $\approx$ 0.01745 radians

or, $\quad 1 \text{ radians} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3°$

Some standard angles in degrees and radians are as under

degree	30°	45°	60°	90°	180°	270°	360°
radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Example 1

Convert the following into radian measure.

(a) $120°$ (b) $10° 30'$ (c) $24° 32' 30''$

Solution (a):

we know that

$\because \quad 1^\circ = \frac{\pi}{180}$ radians

$\therefore \quad 120^\circ = 120 \times \frac{\pi}{180}$ radians

$\Rightarrow \quad 120^\circ = \frac{2\pi}{3}$ radians

Solution (b):

we know that

$\because \quad 1' = \left(\frac{1}{60}\right)^\circ$

$\therefore \quad 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2}\right)^\circ$

Now $\quad 10^\circ 30' = \left(10 + \frac{30}{60}\right)^\circ = \left(10 + \frac{1}{2}\right)^\circ = \left(\frac{21}{2}\right)^\circ$

$\therefore \quad = \frac{21}{2} \times \frac{\pi}{180}$ radians $\quad \left(\because 1^\circ = \frac{\pi}{180} \text{ radians}\right)$

$\Rightarrow \quad = \frac{7\pi}{120}$ radians

Solution (c): $24^\circ 32' 30''$

$= 24^\circ + \left(\frac{32}{60}\right)^\circ + \left(\frac{30}{3600}\right)^\circ \qquad 1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{3600}\right)^\circ$

$= \left(24 + \frac{32}{60} + \frac{30}{3600}\right)^\circ = \left(24 + \frac{8}{15} + \frac{1}{120}\right)^\circ$

$= \left(\frac{24 \times 120 + 64 + 1}{120}\right)^\circ$

$= \left(\frac{2945}{120}\right)^\circ \qquad \left(\because 1^\circ \cong \frac{\pi}{180} \text{ radian}\right)$

Now $\quad \left(\frac{2945}{120}\right)^\circ = \frac{2945}{120} \times \frac{\pi}{180}$ radians

$\therefore \quad = \frac{589}{4320} \pi$ radians

So, $24^\circ 32' 30'' = \frac{589}{4320} \pi$ radians

EXERCISE 30.1

1. Convert the following into degree, and write in decimal form.

(i)	$32° 15'$	(ii)	$10° 30'$	(iii)	$8° 15' 30''$
(iv)	$45° 21' 36''$	(v)	$25° 30'$	(vi)	$18° 6' 21''$

2. Convert the following angles into $D° M' S''$ form.

(i)	$32.25°$	(ii)	$47.36°$	(iii)	$57.325°$
(iv)	$-67.58°$	(v)	$22.5°$	(vi)	$225.60°$

3. Express the following angles into degree measures.

(i)	$\frac{\pi}{4}$ radians	(ii)	$\frac{\pi}{3}$ radians	(iii)	$-\frac{3\pi}{4}$ radians
(iv)	3 radian	(v)	$\frac{3}{\pi}$ radians	(vi)	4.5 radians

4. Convert the following angles to radian measures.

(i)	$30°$	(ii)	$45°$	(iii)	$60°$
(iv)	$\left(22\frac{1}{2}\right)^{\circ}$	(v)	$-225°$	(vi)	$60° 35' 48''$

30.2 Sector of a circle:

Definitions:

(i) A part of the circumference of a circle is called an arc.

(ii) A part of the circle bounded by the two radii and an arc is called sector of the circle.

The following figures help to understand the above mentioned definitions.

Fig: 30.2 (i)

Fig: 30.2 (ii)

In fig: 30.2 (i) PQ ia an arc and in fig: 30.2 (ii) POQ is a sector.

30.2.(i) Establish the rule $l = r\,\theta$, where r is the radius of the circle, l the length of the circular arc and θ is the central angle measured in radian.

In a circle of radius r, the arc length l is directly proportional to the central angle θ measured in radians. (as shown in figure 30.3).

i.e., $l \propto \theta$

$\Rightarrow \quad l = c\theta \qquad \text{...(i)}$

For complete rotation

$$l = 2\pi r$$

and $\theta = 2\pi$ π

By using equation (i), we have,

$$2\pi r = c(2\pi)$$

$\Rightarrow$ $c = r$

so equation (i) becomes $l = r\theta$.

Note: (i) l and r are measured in same units

(ii) θ is measured in radian

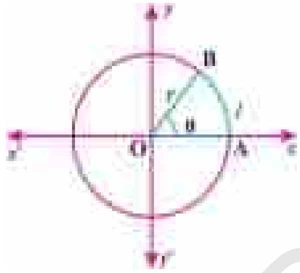

Fig: 30.3

Example 1: Find the length of an arc of a circle of radius 5cm which subtends an angle of $\frac{2\pi}{5}$ radian.

Solution: Given that

$$r = 5\text{cm},\ \theta = \frac{2\pi}{5} \text{ radian and } l = ?$$

Since, $l = r\theta$

$$\therefore \quad l = 5 \times \frac{2\pi}{5}$$

$$\Rightarrow \quad l = 2\pi \cong 2\left(\frac{22}{7}\right) \approx 6.28\text{cm}.$$

Example 2:

How far does a boy on a bicycle travel in 10 revolutions if the diameters of the wheels of his bicycle each equal to 56cm?

Solution:

We know that

one revolution $= 2\pi$ radian

$\therefore$ 10 revolutions $= 20\pi$ radians

i.e. $\theta = 20\pi$ radian

also, diameter of the wheel, d = 56cm

$$\therefore \quad r = \frac{d}{2} = \frac{56}{2}\text{cm} = \frac{56}{2 \times 100}\text{m} \qquad (\because \ 1\text{m}=100\text{cm})$$

$$l = r\theta$$

$$\therefore \quad l = \frac{56}{2 \times 100} \times 20\pi$$

$$\Rightarrow \quad l \approx \frac{56}{2 \times 100} \times 20 \times \frac{22}{7} \qquad (\because \ \pi \cong \frac{22}{7})$$

$$\Rightarrow \quad l \approx 17.6\text{m}.$$

Thus, the boy travels 17.6m in 10 revolutions.

Example 3:

Find r, when $l = 4\text{cm}$ and $\theta = \frac{1}{4}$ radians

Solution: Given that:

$$l = 4\text{cm}$$

$$\theta = \frac{1}{4} \text{ radians}$$

$$\text{and } r = ?$$

$$l = r\theta$$

$$\therefore \quad r = \frac{l}{\theta}$$

$$\Rightarrow \quad r = \left(4 \div \frac{1}{4}\right) = 4 \times \frac{4}{1} = 16\text{cm.}$$

30.2.(ii) Prove that the area of the sector of a circle is $\frac{1}{2}r^2\theta$ or $\frac{1}{2}lr$.

Proof:

Consider a circle of radius r with centre O, AB is an arc which subtends an angle θ radians at the centre as shown in the figure 30.4.

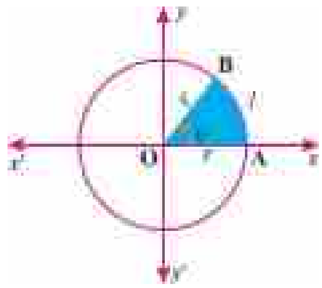

Fig: 30.4

We know that

$$\text{Area of circle} = \pi r^2$$

$$\text{Angle of one revolution} = 2\pi \text{ radian}$$

$$\text{Angle of the sector} = \theta \text{ radian}$$

Then by elementary geometry, using law of proportion, we have

$$\frac{\text{area of sector AOB}}{\text{area of a circle}} = \frac{\text{angle of sector}}{\text{angle of a circle}}$$

$$\therefore \quad \frac{\text{Area of sector AOB}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector AOB} = \frac{\pi r^2 \times \theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector AOB} = \frac{1}{2}r^2\theta$$

$$or \quad \text{Area of sector AOB} = \frac{1}{2}r\theta \times r$$

$$\Rightarrow \quad \text{Area of sector AOB} = \frac{1}{2}rl \qquad (\quad r\theta = l)$$

Hence proved.

Example 1:

Find the area of sector of a circle of radius 5cm with central angle of 60°.

Solution: Given that:

$$r = 5\text{cm}$$

$$\theta = 60^\circ = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3}\text{radians}$$

and area of sector = ?

$\because$ Area of sector $= \frac{1}{2}r^2\theta$

$\therefore$ Area of sector $= \frac{1}{2}(5)^2 \times \frac{\pi}{3}$

$\Rightarrow$ Area of sector $\approx \frac{1}{2} \times 25 \times \frac{22}{7\times 3}$ $\quad \because \pi \approx \frac{22}{7}$

$\Rightarrow$ Area of sector $= 13.09\text{cm}^2$

Example 2:

Find the area of sector whose radius is 4cm, with central angle of 12 radians.

Solution: Given that:

$r = 4\text{cm}$

$\theta = 12$ radian

and Area of sector = ?

$\because$ area of sector $= \frac{1}{2}r^2\theta$

$\therefore$ area of sector $= \frac{1}{2}(4)^2 \times 12$

$\Rightarrow$ area of sector $= 16\times 6 = 96\text{cm}^2$

EXERCISE 30.2

1. Find θ, if

(i) $l = 20\text{cm}$ and $r = 5\text{cm}$ (ii) $l = 30.2\text{cm}$ and $r = 2\text{cm}$

(iii) $l = 6\text{cm}$ and $r = 2.87\text{cm}$ (iv) $l = 4.5\text{cm}$ and $r = 2.5\text{cm}$

2. Find l, if

(i) $r = 1.01\text{cm}$ and $\theta = 2.1$ radian

(ii) $r = 5.1\text{cm}$ and $\theta = 2$ radian

(iii) $r = 6\text{cm}$ and $\theta = 30^\circ$

(iv) $r = 15\text{cm}$ and $\theta = 60^\circ\ 30'$

3. Find r, if

(i) $\theta = 60^\circ$ and $l = 2\text{m}$ (ii) $\theta = 3.5$ radians and $l = \frac{7}{4}\text{m}$

(iii) $\theta = \frac{1}{4}$ radians and $l = 4\text{cm}$ (iv) $\theta = 180^\circ$ and $l = 15.4\text{cm}$

4. Find the arc length of a unit circle, corresponding to the central angle measuring.
   (i) 30° (ii) 45° (iii) 60° (iv) 90°

5. The arc of a circle subtends an angle of $\frac{\pi}{6}$ radians at the centre. The radius of a circle is 5cm, find;
   (i) length of an arc (ii) area of a circular sector

6. A point is moving on the circle of radius 10cm. If it makes 3.5 revolutions, find the distance travelled by the point.

7. Find the area of the sector with central angle of $\frac{\pi}{4}$ radian in circle of radius 4cm.

8. If a point on the rim of a 21cm diameter fly wheel travels 5040 meters per minute through, how many radian does the wheel turn in a second?

9. A car is running on a circular path of radius equal to double the arc of the circle travelled by the car. Find the angle subtended by the arc at the centre of the circular path.

10. In a circle of radius 12cm, an arc subtends a central angle of $84°$. Find its arc length and also calculate area of the sector.

30.3 Trigonometric Ratios

30.3.(i) Define and identify:

(a) General angle (Coterminal angles)

(b) Angle in standard position

30.3.1(a) Angles having the same initial and terminal sides are called coterminal angles and they differ by multiple of 2π radians. They are also called general angles.

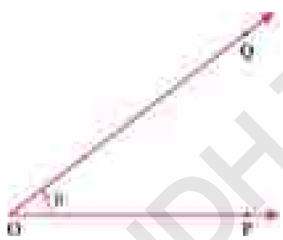

Fig: 30.5 (i)

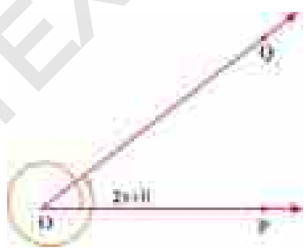

Fig: 30.5 (ii)

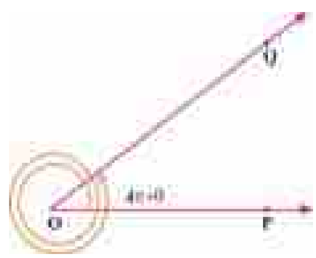

Fig: 30.5 (iii)

$m\angle POQ = \theta$ radian, where $0 \le \theta < 2\pi$.

(i) θ radian , (Zero revolution)

(ii) $(2\pi + \theta)$ radians , (After one revolution)

(iii) $(4\pi + \theta)$ radians , (After two revolution)

Hence $\theta, 2\pi + \theta$ and $4\pi + \theta$ and coterminal angles as shown in figure 30.5.

However, if the rotation are made in the clock-wise directions as mentioned in the below figures

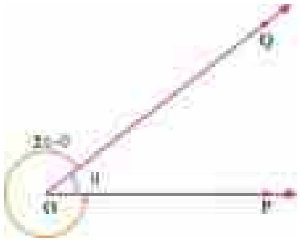

Fig: 30.6 (i)

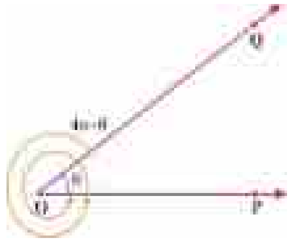

Fig: 30.6 (ii)

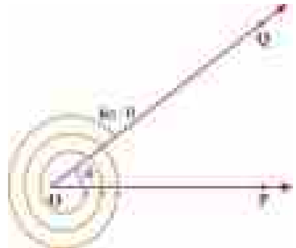

Fig: 30.6 (iii)

(i) $2\pi-\theta$ radians $\quad$ (no revolution)

(ii) $4\pi-\theta$ radians $\quad$ (after one clockwise revolution)

(iii) $6\pi-\theta$ radians $\quad$ (after two clock wise revolution)

$m\angle POQ=\theta$ radian, where $0\le\theta<2\pi$.

Hence $2\pi-\theta$, $4\pi-\theta$ and $6\pi-\theta$ and coterminal angles as shown in figure 30.6.

Examples:

Which of the following angles are coterminal with $120°$?

$-240°, 480°, \frac{14\pi}{3}$ and $-\frac{14\pi}{3}$.

Solution:

$120°-(-240°)=120°+240°=360°$, yes $-240°$ is coterminal angle of $120°$

$480°-120°=360°$, yes $480°$ is coterminal angle of $120°$

$\frac{14\pi}{3}-120°=\frac{14\pi}{3}-\frac{2\pi}{3}=\frac{12\pi}{3}=4\pi$, yes $\frac{14\pi}{3}$ is coterminal angle of $120°$

$120°-\left(\frac{14\pi}{3}\right)=\frac{2\pi}{3}+\frac{14\pi}{3}=\frac{16\pi}{3}$, it is not multiple of 2π,

so, $-\frac{14\pi}{3}$ is not coterminal angle of $120°$.

30.3. (i)(b) An angle is said to in standard position if its vertex is at origin and its initial side on positive *x*-axis as shown in figure 30.7.

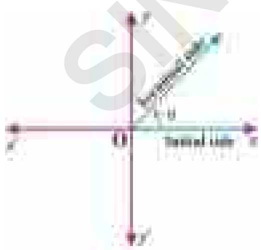

Fig: 30.7 (i)

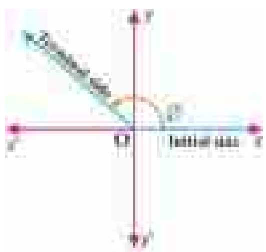

Fig: 30.7 (ii)

Fig: 30.7 (iii)

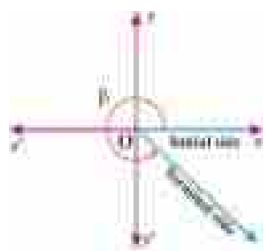

Fig: 30.7 (iv)

30.3.(ii) Recognize Quadrants and quadrantal angles:

30.3.(ii)(a) In rectangular coordinate system two axes divide the plane into equal parts, each part is called quadrant. Hence, four quadrants are formed as shown in figure 30.8.

Angles between 0° and 90° are in Quadrant I

i.e., $0 < \theta < \frac{\pi}{2}$

Angles between 90° and 180° are in Quadrant II

i.e., $90° < \theta < 180°$

Angles between 180° and 270° are in Quadrant III

i.e., $180° < \theta < 270°$

Angles between 270° and 360° are in Quadrant IV

i.e., $270° < \theta < 360°$

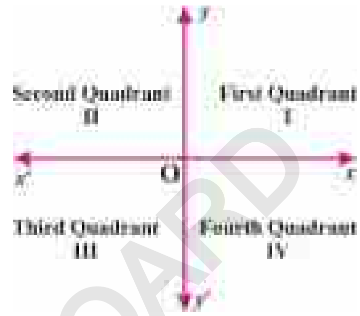

Fig: 30.8

An angle in standard position is said to be in a quadrant, if its terminal side lies in that quadrant.

30.3.(ii) (b) Quadrantal angles.

If the terminal side of the standard angle coincides with x-axis or y-axis, then it is called a quadrantal angle, for example $90°, 180°, 270°$ and $360°$ which are shown in the following figure.

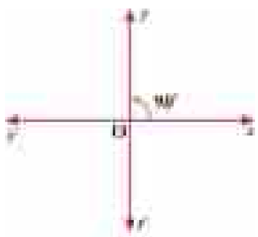

Fig: 30.9 (i)

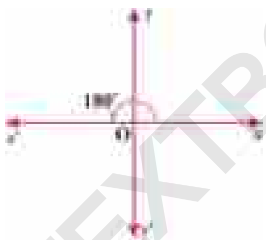

Fig: 30.9 (ii)

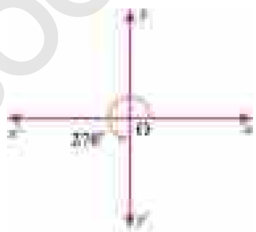

Fig: 30.9 (iii)

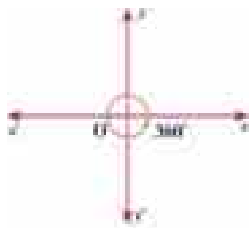

Fig: 30.9 (iv)

30.3.(iii) Define trigonometric ratios and their reciprocals with the help of a unit circle.

We have already learnt about trigonometric ratios for any acute angle θ of a right-angled triangle ABC in previous classes, which are as under.

1. $\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}} = \dfrac{a}{c}$

2. $\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \dfrac{b}{c}$

3. $\tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}} = \dfrac{a}{b}$

and their reciprocals are respectively

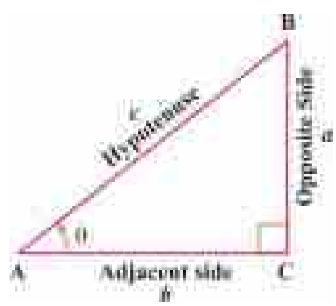

Fig: 30.10

4. $\text{cosec}\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}} = \dfrac{1}{\sin\theta} = \dfrac{c}{a}$

5. $\sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}} = \dfrac{1}{\cos\theta} = \dfrac{c}{b}$

6. $\cot\theta = \dfrac{\text{length of adjacent side}}{\text{length of opposite side}} = \dfrac{1}{\tan\theta} = \dfrac{b}{a}$

For each point of the circle, there always exist an angle corresponds to it.

Suppose that $P(x, y)$ be any point on the unit circle lying in the 1st quadrant and θ is the corresponding angle a shown in figure 30.11.

We define the trigonometric ratios as under

$$\sin\theta = \frac{|\overline{BP}|}{|\overline{OP}|} = \frac{y}{1} = y,$$

$$\text{and } \cos\theta = \frac{|\overline{OB}|}{|\overline{OP}|} = \frac{x}{1} = x,$$

$$\text{Now, } \tan\theta = \frac{\overline{BP}}{\overline{OB}} = \frac{y}{x}, \text{ provided } x \neq 0$$

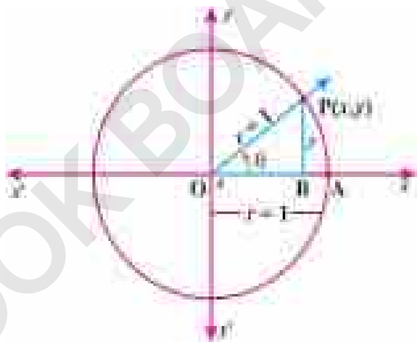

Fig: 30.11

similarly,

$\cot\theta = \dfrac{x}{y}$ provided $y \neq 0$

$\sec\theta = \dfrac{1}{x}$ provided $x \neq 0$

$\text{cosec}\theta = \dfrac{1}{y}$ provided $y \neq 0$

Reciprocal ratios, listed below:

$\sin\theta = \dfrac{1}{\text{cosec}\theta}$ or $\text{cosec}\theta = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{1}{\cot\theta}$ or $\cot\theta = \dfrac{1}{\tan\theta}$

Notes:

1. All trigonometric ratios in terms of x and y are called trigonometric function.
2. The functions $\cos\theta$ and $\sin\theta$ are also called circular functions.
3. The functions $\sin\theta$ and $\cos\theta$ are called primary functions whereas other functions are called secondary functions.

Example:

Find the value of trigonometric ratios at angle θ , if point P (6, 8) is on the terminal side of θ

Solution:

Here $\quad P(x, y) = (6,8)$

$\Rightarrow \quad x = 6$ and $y = 8$ as shown in figure 30.12.

By using Pythogras theorem,

$$r^2 = x^2 + y^2$$

$$\therefore \quad r^2 = (6)^2 + (8)^2$$

$$\Rightarrow \quad r^2 = 36 + 64 = 100$$

$$\Rightarrow \quad r = \sqrt{100} = 10, \text{ where } r = |\overline{OP}|$$

Thus, values of all six trigonometric ratios are

$$\sin\theta = \frac{y}{r} = \frac{8}{10} = \frac{4}{5}, \qquad \text{cosec}\theta = \frac{5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{6}{10} = \frac{3}{5}, \qquad \sec\theta = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{8}{6} = \frac{4}{3} \quad \text{and} \quad \cot\theta = \frac{3}{4}$$

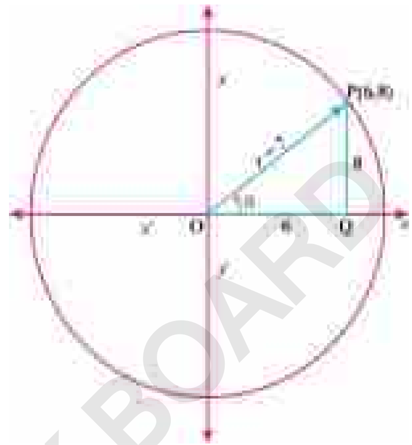

Fig: 30.12

30.3.(iv) Recall the values of trigonometric ratios for 45°,30° and 60°

We have already learnt that values of trigonometric ratios for $30°, 45°$ and $60°$ which are given in the table below

θ	$30°$	$45°$	$60°$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Now we can find the values of reciprocals trigonometric ratios with the help of above table.

$$\text{cosec}30° = \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2$$

$$\sec30° = \frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{1}{\tan 30^\circ} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3} \qquad \operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2} \qquad \cot 45^\circ = \frac{1}{\tan 45^\circ} = \frac{1}{1} = 1$$

$$\operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{\frac{1}{2}} = 2$$

and $\cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

30.3.(v) Recognize signs of trigonometric rations in different quadrants.

If θ is not a quadrantal angles, then signs of trigonometric ratios can be determined according to point $P(x, y)$ on its terminal side.

1. If θ lies in first quadrant, then point (x, y) on its terminal side has x and y coordinates positive.
   i.e. $x > 0$ and $y > 0$.

$\therefore$ All trigonometric functions/ratios are positive in the first quadrant.

2. In 2nd quadrant $x < 0$ and $y > 0$. Therefore $\sin\theta$ and $\operatorname{cosec}\theta$ are positive in second quadrant and remaining ratios are negative.
3. In 3rd quadrant $x < 0$ and $y < 0$. Therefore $\tan\theta$ and $\cot\theta$ are positive in third quadrant and remaining ratios are negative.
4. In 4th quadrant $x > 0$ and $y < 0$. Therefore $\cos\theta$ and $\sec\theta$ are positive in fourth quadrant and remaining ratios are negative.

Example 1: Find the signs of the following trigonometric functions/ratios

(i) $\cos 120^\circ$ (ii) $\sin 1030^\circ$ (iii) $\tan 229^\circ$ (iv) $\tan\left(-\frac{\pi}{4}\right)$ (v) $\operatorname{cosec}\left(-\frac{\pi}{3}\right)$

Solution:

(i) $\cos 120^\circ$

$\because$ 120° lies in Quadrant II, and $\cos\theta$ is negative in II Quadrant

$\therefore$ $\cos 120^\circ$ is negative.

(ii) $\sin 1030^\circ = \sin(720^0 + 310^0)$

$\because$ 1030° lies in Quadrant IV and Sin θ is negative in IV Quadrant,

$\therefore$ Sine 1030° is negative.

(iii) $\tan 229^\circ$

$\because$ 229° lies in Quadrant III and $\tan\theta$ is positive in Quadrant III,

$\therefore$ $\tan 229^\circ$ is positive.

(iv) $\tan\left(-\frac{\pi}{4}\right)$

$\because$ $\frac{-\pi}{4}$ lies in Quadrant IV and $\tan\theta$ is negative in Quadrant IV,

$\therefore$ $\tan\frac{-\pi}{4}$ is negative.

(v) $\operatorname{cosec}\left(-\frac{\pi}{3}\right)$

$\because$ $\frac{-\pi}{3}$ lies in IV Quadrant and $\operatorname{cosec}\theta$ is negative in IV Quadrant,

$\therefore$ $\operatorname{cosec}\left(\frac{-\pi}{3}\right)$ is of negative "–" sign.

30.3.(vi) Find the values of remaining trigonometric ratios if one trigonometric ratio is given.

The method is illustrated by the following examples:

Example 1: If $\sin\theta = \frac{3}{5}$ then find values of remaining trigonometric function/ratios.

Solution: Given that:

Since $\sin\theta = \frac{y}{r}$

By using figure 30.13

$\frac{3}{5} = \frac{y}{r}$

$\Rightarrow \quad y = 3$ and $r = 5$

Since $x^2 + y^2 = r^2$

$x^2 + (3)^2 = (5)^2$

$x^2 + 9 = 25$

$x^2 = 25 - 9$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = 4$

Now

$\cos\theta = \frac{x}{r}$

$\cos\theta = \frac{4}{5}$

$\sec\theta = \frac{r}{x} = \frac{5}{4}$

$\operatorname{cosec}\theta = \frac{r}{y} = \frac{5}{3}$

$\tan\theta = \frac{y}{x} = \frac{3}{4}$

$\cot\theta = \frac{x}{y} = \frac{4}{3}$

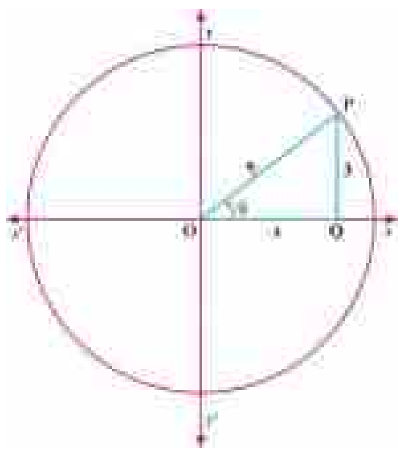

Fig: 30.13

Example 2: If $\tan\theta = 1$ and θ lies in the 3^{rd} quadrant. Find values of remaining trigonometric functions.

Solution: Given that:

By using figure 30.14

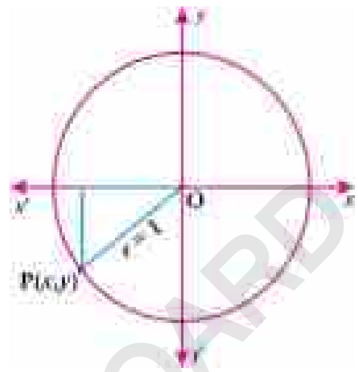

Fig: 30.14

$$\tan\theta = 1 \qquad \therefore\ \cot\theta = \frac{1}{\tan\theta} = \frac{1}{1} = 1,$$

$$\therefore \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} = 1,$$

$$\Rightarrow \qquad y = x$$

$$\because \qquad x^2 + y^2 = 1$$

$$\therefore \qquad x^2 + x^2 = 1 \qquad \because\ y = x$$

$$\Rightarrow \qquad 2x^2 = 1$$

$$\Rightarrow \qquad x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Here we take negative value, because

θ lies in 3rd quadrant

$$x = \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } y = \sin\theta = -\frac{1}{\sqrt{2}}$$

Therefore remaining trigonometric functions are

$$\cot\theta = 1,$$

$$\sin\theta = -\frac{1}{\sqrt{2}},$$

$$\text{cosec}\theta = \frac{1}{\sin\theta} = -\sqrt{2},$$

$$\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\text{and} \qquad \sec\theta = \frac{1}{\cos\theta} = -\sqrt{2}$$

Example 3: Find the remaining trigonometric functions/ratios, using the unit circle. If

(i) $\cos\theta = \frac{2}{3}$ and θ is in 1st quadrant.

(ii) $\sin\theta = 0.6$ and $\tan\theta$ is negative.

Solution (i): Given that:

$$\cos\theta = x = \frac{2}{3}, \qquad \therefore\ \sec\theta = \frac{1}{\cos\theta} = \frac{3}{2},$$

θ lies in 1st quadrant $\therefore\ p(x, y) \in$ Quadrant I and $\quad x^2 + y^2 = 1 \quad$ (Given)

Here $\quad x = \cos\theta \quad$ and $\quad y = \sin\theta$

$\therefore \quad \left(\frac{2}{3}\right)^2 + y^2 = 1$

$\Rightarrow \quad y^2 = 1 - \frac{4}{9} = \frac{5}{9}$

$\Rightarrow \quad y = \pm\frac{\sqrt{5}}{3}$

Here, we take +ve value, because θ lies in 1st quadrant

$\therefore \quad y = \frac{\sqrt{5}}{3} = \sin\theta \quad ; \qquad \therefore \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{3}{\sqrt{5}},$

and $\quad \tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \frac{\sqrt{5}}{2}; \qquad \therefore \quad \cot\theta = \frac{1}{\tan\theta} = \frac{2}{\sqrt{5}},$

Therefore required remaining trigonometric ratios are

1. $\sec\theta = \frac{3}{2}$ $\qquad$ 2. $\sin\theta = \frac{\sqrt{5}}{3}$ $\qquad$ 3. $\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$

4. $\tan\theta = \frac{\sqrt{5}}{2}$ $\quad$ and $\quad$ 5. $\cot\theta = \frac{2}{\sqrt{5}}$

Solution (ii): Given that:

$\sin y = 0.6 = \frac{6}{10} = \frac{3}{5}, \qquad \therefore \quad \operatorname{cosec}\theta = \frac{5}{3} = 1.6,$

since $\quad \sin\theta > 0$ and $\tan\theta < 0$

$x^2 + y^2 = 1$

$\therefore \quad x^2 + \left(\frac{3}{5}\right)^2 = 1$

$\Rightarrow \quad x^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\Rightarrow \quad x^2 = \pm\frac{4}{5}$

Here, we take negative value because $\rho(\theta)$ is in 2nd quadrant

$\therefore \quad x = \cos\theta = -\frac{4}{5} = -0.8$ and $\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4} = -1.25$

Now, $\quad \tan\theta = \frac{y}{x} = \frac{\frac{3}{5}}{\frac{-4}{5}} = -\frac{3}{4} = -0.75$ and $\cot\theta = \frac{1}{\tan\theta} = -\frac{4}{3} = 1.3$

Therefore required remaining trigonometric ratios are

1. $\text{cosec}\,\theta = 1.6$ 2. $\cos\theta = -0.8$ 3. $\sec\theta = -1.25$
4. $\tan\theta = -0.75$ 5. $\cot\theta = -1.3$

30.3.(vii) Calculate the values of trigonometric ratios of $0°, 90°, 180°, 270°$ and $360°$.

We have already discussed quadrantal angles in section 30.3.(ii)(b). Here we calculate trigonometric ratios of the quadrantal angle

When $\theta = 0°$

In a unit circle, the point $P(1,0)$ lies on terminal side of an angle $\theta°$.

$\therefore$ Here $x = 1$, $y = 0$ and $r = 1$. Where r is the radius of unit circle.

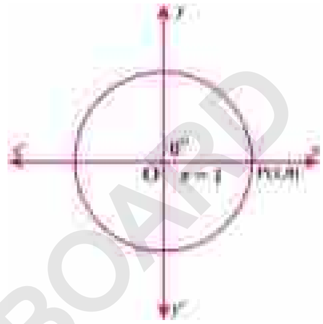

Fig: 30.15

$\therefore \quad \sin 0° = \frac{y}{r} = \frac{0}{1} = 0$; $\quad \therefore \quad \text{cosec}\,0° = \frac{r}{y} = \frac{1}{0}$ (undefined)

$\therefore \quad \cos 0° = \frac{x}{r} = \frac{1}{1} = 1$; $\quad \therefore \quad \sec 0° = \frac{r}{x} = \frac{1}{1} = 1$

$\therefore \quad \tan 0° = \frac{y}{x} = \frac{0}{1} = 0$; $\quad \therefore \quad \cot 0° = \frac{x}{y} = \frac{1}{0}$ (undefined)

When $\theta = 90°$

In a unit circle, the point $P(0,1)$ lies on terminal sides of an angle $90°$ and coincide with the (+ve) y-axis.

$\therefore$ Here $x = 0, y = 1$ and $r = 1$.

$\therefore \quad \sin 90° = \frac{y}{r} = \frac{1}{1} = 1$;

$\text{cosec}\,90° = \frac{r}{y} = \frac{1}{1} = 1$

$\cos 90° = \frac{x}{r} = \frac{0}{1} = 0$;

$\sec 90° = \frac{r}{x} = \frac{1}{0}$ (undefined)

$\tan 90° = \frac{y}{x} = \frac{1}{0}$ (undefined)

and $\quad \cot 90° = \frac{x}{y} = 0$

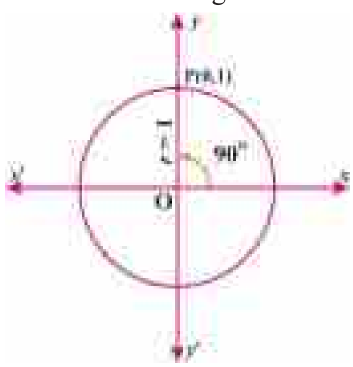

Fig: 30.16

When $\theta = 180°$

In a unit circle, the point $P(-1,0)$ lies on terminal sides of an angle $180°$ and coincide with the –ve x-axis. In this case

$P(-1,0) \Rightarrow x=-1, y=0$ and $r=1$.

$$\sin 180° = \frac{y}{r} = \frac{0}{1} = 0;$$

$$\therefore \quad \text{cosec}\,180° = \frac{r}{y} \text{(undefined)}$$

$$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1;$$

$$\sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0;$$

and $$\cot 180° = \frac{x}{y} = \frac{1}{0} \text{(undefined)}$$

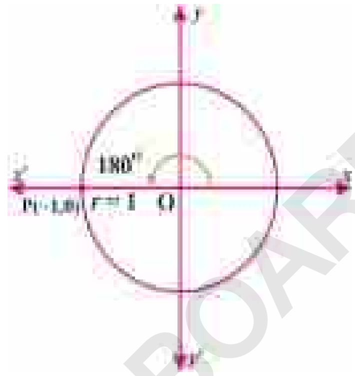

Fig: 30.17

When $\theta = 270°$

In a unit circle, the point $P(0,-1)$ lies on terminal sides of an angle $180°$ and coincide with the –ve y-axis. In this case

$\therefore \quad P(0,-1) \Rightarrow x=0, y=-1$ and $r=1$.

$$\therefore \quad \sin 270° = \frac{y}{r} = \frac{-1}{1} = -1$$

$$\text{cosec}\,270° = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos 270° = \frac{y}{r} = \frac{0}{1} = 0$$

$$\sec 270° = \frac{r}{x} = \frac{1}{0} \text{(undefined)}$$

$$\tan 270° = \frac{y}{x} = \frac{-1}{0} = \text{undefined}$$

and $$\cot 270° = \frac{0}{-1} = 0$$

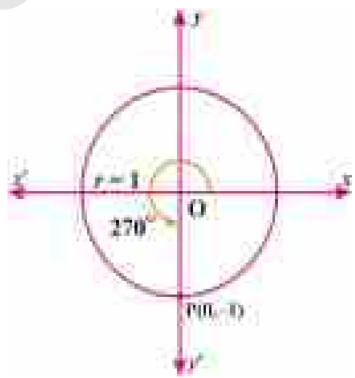

Fig: 30.18

When $\theta = 360°$

In a unit circle, the point $P(1,0)$ lies on terminal sides of an angle $360°$ and coincide with the +ve x-axis. In this case,

$P(1,0) \Rightarrow x=1, y=0$ and $r=1$.

$$\therefore \qquad \sin 360^\circ = \frac{y}{r} = \frac{0}{1} = 0$$

$$\operatorname{cosec}360^\circ = \frac{1}{\sin 360^\circ} = \frac{1}{0}\text{(undefined)}$$

$$\cos 360^\circ = \frac{x}{r} = \frac{1}{1} = 1$$

$$\sec 360^\circ = \frac{1}{\cos 360^\circ} = \frac{1}{1} = 1$$

$$\tan 360^\circ = \frac{y}{x} = \frac{0}{1} = 0$$

and $$\cot 360^\circ = \frac{1}{\tan 360^\circ} = \frac{1}{0}\text{(undefined)}$$

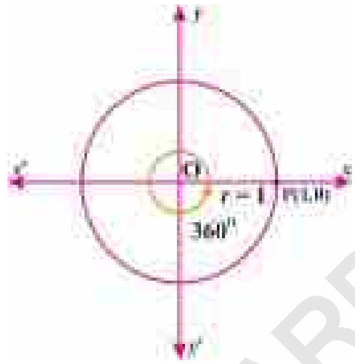

Fig: 30.19

Summarizing all the results, we have a table

θ	0°	30°	45°	60°	90°	180°	270°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	−∞	0
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	−1	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	−1	∞	1
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

EXERCISE 30.3

1. Find coterminal angles of the following angles.

(i) 55° (ii) $\frac{\pi}{6}$ (iii) $-45°$ (iv) $\frac{-3\pi}{4}$

2. Identify the quadrants in which of the following angles lie.

(i) $\frac{8\pi}{5}$ (ii) 75° (iii) $-818°$ (iv) 1090° (v) $\frac{7\pi}{6}$ (vi) $-\frac{5\pi}{4}$

3. Find the signs of the following.

(i) $\cos 120^\circ$ (ii) $\sin 340^\circ$ (iii) $\sec 200^\circ$

(iv) $\operatorname{cosec} 198^\circ$ (v) $\tan\left(\frac{-\pi}{3}\right)$ (vi) $\cot\left(\frac{-2}{3}\pi\right)$

4. In which quadrant does terminal side of angle θ lie if:

(i) $\cos\theta < 0$ and $\sin\theta > 0$ (ii) $\tan\theta > 0$ and $\cos\theta < 0$

(iii) $\sin\theta < 0$ and $\sec\theta < 0$ (iv) $\cot\theta > 0$ and $\sec\theta < 0$

(v) $\cos\theta < 0$ and $\tan\theta < 0$ (vi) $0 < \cot\theta < 1$

5. If $\cos\theta = \frac{-3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then find the remaining trigonometric ratios.

6. Find remaining trigonometric functions/ratios, if:

(i) $\sin\theta = \frac{\sqrt{3}}{2}$ and θ lies in second quadrant

(ii) $\cos\theta = \frac{2}{3}$ and θ lies in fourth quadrant

(iii) $\tan\theta = -\frac{1}{2}$ and θ lies in second quadrant

(iv) $\sec\theta = \operatorname{cosec}\theta = \sqrt{2}$ and θ lies in first quadrant

(v) $\cos\theta = \frac{1}{2}$ and $\tan\theta$ is positive.

7. Find the values of:

(i) $\tan 30^\circ \tan 60^\circ + \tan 45^\circ \cot 45^\circ$ (ii) $\dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

(iii) $\dfrac{\sin 45^\circ}{\sin 45^\circ + \cot 45^\circ}$ (iv) $\cos\frac{\pi}{6}\cdot\cos\frac{\pi}{3} + \sin\frac{\pi}{6}\cdot\cos\frac{\pi}{3}$

(v) $\dfrac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}$ (vi) $\dfrac{\tan 45^\circ + \cot 45^\circ}{\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ}$

30.4 Trigonometric Identities:

An identity is an equation which is true for all values of the variable (except where value is not defined).

30.4.1 Prove the trigonometric identities and apply them to show different trigonometric relations.

For any real number θ related to the right triangle in a unit circle, we have the following fundamental trigonometric identities

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta = 1 + \tan^2\theta$

(iii) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(i) Proof:

In a unit circle, consider a right ΔOAP, in which $\angle AOP = \theta$ radians is in standard position. Let $P(x, y)$ be on the terminal side of the angle.

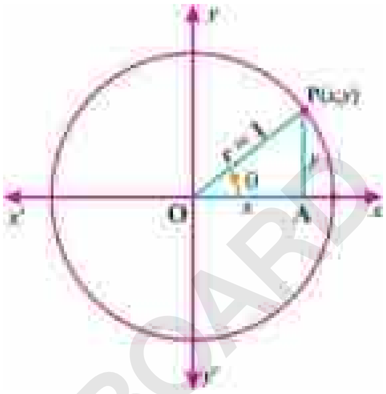

Fig: 30.20

By Pythagoras theorem, we have

$x^2 + y^2 = 1$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 1 \qquad [\because \ x = \cos\theta \text{ and } y = \sin\theta]$

Hence proved.

(ii) Proof:

By Pythagoras theorem

$1 = x^2 + y^2$

dividing both sides by x^2, we have,

$\Rightarrow \quad \dfrac{1}{x^2} = 1 + \dfrac{y^2}{x^2}$ or $\dfrac{1}{x^2} = 1 + \left(\dfrac{y}{x}\right)^2$

$\Rightarrow \quad \dfrac{1}{\cos^2\theta} = 1 + \tan^2\theta, \qquad$ (Provided $x = \cos\theta \neq 0$)

$\Rightarrow \quad \boxed{\sec^2\theta = 1 + \tan^2\theta}$

Hence proved.

(iii) Proof:

By Pythagoras theorem

$1 = x^2 + y^2$

Dividing both sides by y^2, we have,

$\dfrac{1}{y^2} = \dfrac{x^2}{y^2} + 1,$ or $\left(\dfrac{1}{y}\right)^2 = \left(\dfrac{x}{y}\right)^2 + 1$

$\Rightarrow \quad \left(\dfrac{1}{\sin\theta}\right)^2 = (\cot\theta)^2 + 1, \qquad$ (Provided $\sin\theta \neq 0$)

$\Rightarrow \quad \boxed{\operatorname{cosec}^2\theta = 1 + \cot^2\theta} \qquad \left(\because \ \dfrac{\cos\theta}{\sin\theta} = \cot\theta\right)$

Hence proved.

Example 1:

Prove that $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Proof:

$$\begin{aligned} \text{L.H.S} &= (\sin\theta + \cos\theta)^2 \\ &= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \\ &= 1 + 2\sin\theta\cos\theta \qquad [\because \ \sin^2\theta + \cos^2\theta = 1] \\ &= \text{R.H.S} \end{aligned}$$

$$\text{L.H.S} = \text{R.H.S}$$

$\therefore \quad (\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Hence proved.

Example 2:

Prove that: $\sin\theta = \sqrt{1-\cos^2\theta}$

Proof:

$$\text{L.H.S} = \sin\theta$$

$$= \sqrt{(\sin\theta)^2}$$

$$= \sqrt{\sin^2\theta},$$

$$= \sqrt{1-\cos^2\theta}, \qquad \left[\because \ \sin^2\theta + \cos^2\theta = 1\right]$$

$$= \text{R.H.S}$$

$\because \quad \text{L.H.S} = \text{R.H.S}$

$\therefore \quad \sin\theta = \sqrt{1-\cos^2\theta}$

Hence proved.

Example 3:

Prove that $\dfrac{1+\sec\theta}{1-\sec\theta} = \dfrac{\sin\theta + \tan\theta}{\sin\theta - \tan\theta}$ (Provided $\cos\theta \neq 0$)

Proof:

$$\text{L.H.S} = \frac{1+\sec\theta}{1-\sec\theta}$$

$$= \frac{1+\dfrac{1}{\cos\theta}}{1-\dfrac{1}{\cos\theta}}$$

$$= \frac{\cos\theta + 1}{\cos\theta - 1}$$

$$= \frac{\sin\theta(\cos\theta + 1)}{\sin\theta(\cos\theta - 1)} \qquad \text{(Provided } \sin\theta \neq 0)$$

$$= \frac{\sin\theta\cos\theta + \sin\theta}{\sin\theta\cos\theta - \sin\theta}$$

$$= \frac{\dfrac{\sin\theta\cos\theta}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\sin\theta\cos\theta}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}, \text{ Dividing numerator and denominator by } \cos\theta$$

$$= \frac{\sin\theta + \tan\theta}{\sin\theta - \tan\theta}$$

$= \text{R.H.S}$

$\because \quad \text{L.H.S} = \text{R.H.S}$

$$\therefore \quad \frac{1+\sec\theta}{1-\sec\theta} = \frac{\sin\theta + \tan\theta}{\sin\theta - \tan\theta}$$

Hence proved.

Example4: Prove that $\sin\theta = \tan\theta \cdot \cos\theta$

Proof:

$\text{R.H.S} = \tan\theta \cdot \cos\theta$

$$= \frac{\sin\theta}{\cos\theta} \cdot \cos\theta \qquad \text{(Provided } \cos\theta \neq 0)$$

$= \tan\theta \cdot \cos\theta$

$= \text{R.H.S}$

$\text{L.H.S} = \text{R.H.S}$

$\therefore \quad \sin\theta = \tan\theta \cdot \cos\theta$

Hence proved.

EXERCISE 30.4

1. Prove the following trigonometric identities:

(i) $\sin^2\theta = \left(\sec^2\theta - 1\right)\cos^2\theta \qquad (\cos\theta \neq 0)$

(ii) $\dfrac{1+\sin\theta}{\cos\theta} = \tan\theta + \sec\theta \qquad (\cos\theta \neq 0)$

(iii) $\tan\theta = \sin\theta\sqrt{1+\tan^2\theta} \qquad (\cos\theta \neq 0)$

(iv) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

(v) $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

(vi) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \dfrac{\sin\theta}{1-\cos\theta} \qquad (\cos\theta \neq 1)$

(vii) $\sqrt{\dfrac{\sec\theta+1}{\sec\theta-1}} = \dfrac{\sec\theta+1}{\tan\theta} \qquad (\tan\theta \neq 0)$

(viii) $\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta \qquad (\tan\theta \neq 0)$

(ix) $\dfrac{\cot\theta+\text{cosec}\theta}{\sin\theta+\tan\theta}=\text{cosec}\theta\cot\theta \qquad (\sin\theta\neq 0)$

(x) $\sin^2\theta=\dfrac{\tan^2\theta}{1+\tan^2\theta} \qquad (\cos\theta\neq 0)$

(xi) $\cos^2\theta=\dfrac{\cot^2\theta}{1+\cot^2\theta} \qquad (\sin\theta\neq 0)$

(xii) $\sin\theta\cos\theta\tan\theta+\sin\theta\cos\theta\cot\theta=1 \qquad (\sin\theta\neq 0 \text{ and } \cos\theta\neq 0)$

2. Transform the first expression into second one:

(i) $(\sec\theta+1)(\sec\theta-1)$ into $\tan^2\theta$

(ii) $\dfrac{\cos\theta}{\sin\theta}-\dfrac{\text{cosec}\,\theta}{\cos\theta}$ into $-\tan\theta$

(iii) $\text{cosec}\,\theta+\cot\theta$ into $\dfrac{1+\cos\theta}{\sin\theta}$

30.5 Angles of Elevation and Depression:

30.5.(i) Find angles of elevation and depression.

Fig: 30.21 (i)

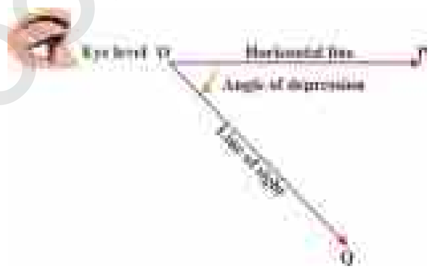

Fig: 30.21 (ii)

When an object is above the eye level of an observer, then the counter clockwise angle from the horizontal line to the line of sight is the angle of elevation as shown in Fig 30.21 (i). When the object is below the eye level, then the clockwise angle from the horizontal line to the line of sight is the angle of depression as shown in Fig 30.21 (ii)

In our daily life, it is used to find distance and angle relationships for object sighted a long distance from observer either above or below the observer's position. This involve uses of angles of elevation or angle of depression.

30.5. (ii) Solve real life problems involving angles of elevation and depression.

Example 1: The angle of elevation of the top of the electric pole from a point on the ground 18 meter away from its base is $30°$. Find the height of the electric pole.

Solution:

Angle of elevation from a point P (above the eye level) to the top of the tower B is $30°$ and let "h" be the height of the pole. Pole is 18m away from the the top of the tower point P. In ΔPAB, we have $\theta=30°$ and $\left|\overline{PA}\right|=18$ m

$\therefore \quad \tan 30° = \frac{h}{|PA|} = \frac{h}{18}$

$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{18}$

$\Rightarrow \quad h = \frac{18}{\sqrt{3}}$

$\Rightarrow \quad h = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$ meter.

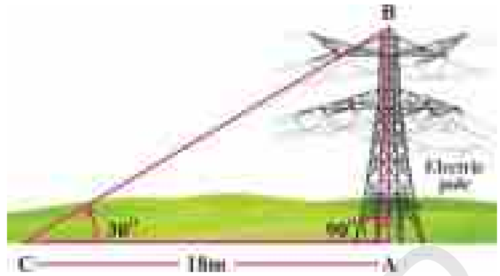

Fig: 30.22

Thus, height of the electric pole is $6\sqrt{3}$ meter.

Example 2: Find the height of an object if the angle of elevation of the sun is $19°$ and the length of the shadow of the object is 1.7 meters.

Solution: Given that angle of elevation of sun i.e. $\theta = 19°$ and the length of shadow $a = 1.7$m as shown in the figure.

Then,

$$\tan 19° = \frac{h}{a} = \frac{h}{1.7}$$

$\Rightarrow \quad h = (1.7)\cdot \tan 19°$

$\Rightarrow \quad h = (1.7)\cdot(0.3443)$

$\Rightarrow \quad h = 0.585$m approx is the required height of the object.

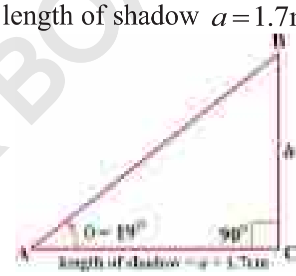

Fig: 30.23

Example 3: From the top of a tower, the angle of depression to the nearest port of a ship at its waterline is $40°$. If the height of tower is 35m, find the distance between the ship and the foot of the tower.

Solution:

Let 'x' be the distance of the ship from point B to the foot of the tower at point A as shown in the figure and "h" be the height of the tower.

Here,

$h = 35$m, height of the tower,

$\theta = 40°$, angle of depression,

and $x = ?$

In $\Delta ABC \quad \tan 40° = \frac{h}{x} = \frac{35}{x}$

$\Rightarrow \quad x = \frac{35}{\tan 40°} = \frac{35}{0.8391} = 41.71$m (approx),

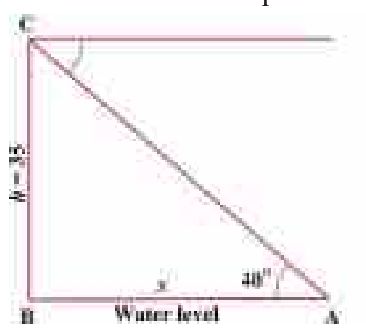

Fig: 30.24

Thus, the distance of ship to the foot of tower is 41.71 meters, (approx).

EXERCISE 30.5

1. From the top of a light house 102 meters high, measure of the angle of depression of a ship is $18°30'$. How far is the ship from the light house.
2. A ladder makes angle $60°$ with the ground and reaches a height of 6m on the wall, find the length of the ladder.
3. Find the angle of elevation when a 6m high bamboo makes a shadow of length $2\sqrt{3}$m.
4. An angle of elevation of the top of cliff is $30°$. Walking 210 m from the point towards the cliff, the angle of elevation is $45°$. Find the height of cliff.
5. An observation balloon is 4280m above the ground and 9613m away from a farm house. Find angle of depression of the farm house as observed from the balloon.

REVIEW EXERCISE 30

1. Multiple Choice Question M.C.Qs.
   Spot the correct option.

i. The system of measurement in which angle is measured in radian is called:
(a) CGS System (b) Sexagesimal
(c) MKS system (d) circular system

ii. The union of two non-collinear rays at common vertex is called
(a) an angle (b) a degree
(c) a minute (d) radian

iii. $10° =$ _________
(a) $600''$ (b) $3600''$ (c) $600'$ (d) 600

iv. $5\frac{\pi}{4}$ radians = _________
(a) $115°$ (b) $225°$ (c) $135°$ (d) $45°$

v. $\frac{1}{2}\sec 45° =$ _________
(a) $\sqrt{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{2}{\sqrt{2}}$ (d) $\frac{1}{2}$

vi. $\operatorname{cosec}\theta \cdot \sin\theta =$ _________
(a) 1 (b) 0 (c) -1 (d) 0.5

vii. $\sec^2\theta - \tan^2\theta =$ _________
(a) 0 (b) 1 (c) -1 (d) $\cos^2\theta$

viii. In unit circle for point (x, y), $\sin\theta =$ _________

(a) x (b) $-x$ (c) y (d) $-y$

ix. If $\sin\theta = -\frac{1}{\sqrt{2}}$ and θ lies in the 3rd quadrant, then $\tan\theta = ?$

(a) -1 (b) 1 (c) ± 1 (d) $\frac{1}{\sqrt{2}}$

x. If an object is above the level of an observer then angle formed between the horizontal line and observer's line of sight is called:

(a) an angle of depression (b) an angle of elevation

(c) an obtuse angle (d) None of these

2. Define an angle and angle in standard position.

3. Convert the $70°30'90''$ into radians.

4. How many degrees are there in $(7200)'$.

5. Covert $\frac{2\pi}{3}$ radians to degree measure.

6. Covert $\frac{3}{\pi}$ radians to $D°M'S''$ form.

7. Find r, when $l = 7\text{cm}$ and $\theta = \frac{\pi}{4}$ radian

8. Find θ, when arc length equal to radius of a circle.

9. Prove that $(1-\cos^2\theta)(1+\cot^2\theta) = 1$

10. Calculate area of the circular sector when $r = 2\text{cm}$ and $\theta = 3$ radians.

SUMMARY

- Division of circumference of a circle into 360 equal parts and angle subtended at centre by each part is called one degree, and is denoted by $1°$.
- The angle subtended at the centre of the circle such that arc length and radius of the circle are equal then angle measured is of one radian.
- Sub-division of radian/degree measures,
  - $1° = \frac{\pi}{180}$ radian ≈ 0.01745 radians and
  - $1° = \left(\frac{180°}{\pi}\right) \approx 57.3°$
- Formula for finding arc length of circle i.e. $l = r\theta$.
- Formula for area of sector $= \frac{1}{2}r^2\theta$ or $= \frac{1}{2}lr$.

- Angles having the same initial and terminal sides are called coterminal angles and they differ by multiple of 2π radians. They are also called general angles.
- An angle is said to in standard position if its vertex is at origin and its initial side on positive x-axis.
- In rectangular coordinate system two axes divide the plane into four equal parts, each part is called quadrant. Hence, four quadrants are formed.

  Angles between $0°$ and $90°$ are in Quadrant I
  Angles between $90°$ and $180°$ are in Quadrant II
  Angles between $180°$ and $270°$ are in Quadrant III
  Angles between $270°$ and $360°$ are in Quadrant IV
- If the terminal side of the standard angle coincides with x-axis or y-axis, then it is called a quadrantal angle.
- There are six basic trigonometric ratios for angle θ. These are $\sin\theta, \cos\theta, \tan\theta, \operatorname{cosec}\theta$ $\sec\theta$ and $\cot\theta$
- Reciprocal ratios, listed below:

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \quad \text{or} \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{or} \quad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{or} \quad \cot\theta = \frac{1}{\tan\theta}$$

- An identity is an equation which is true for all values of the variable (except where value is not defined).
- Three fundamental trigonometric identities are:

  (i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta = 1 + \tan^2\theta$
  (iii) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$
- When an object is above the eye level of an observer, then the counter clockwise angle from the horizontal line to the line of sight is the angle of elevation.
- When the object is below the eye level, then the clockwise angle from the horizontal line to the line of sight is the angle of depression.