ANGLES IN A SEGMENT OF A CIRCLE

Unit 28

• Weightage = 5%

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with theircorollaries and apply them to solve allied problems.
- The measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segments of a circleare equal.
- The angle
  - In a semi-circle is a right angle,
  - In a segment greater than the semi-circle is less than a right angle, (i.e., an acute angle)
  - In a segment less than a semi-circle is greater than a right angle, (i.e., an obtuse angle)
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

28.1 Angle in a segment of a circle

We have already studied most of the terms related to circle like chord, minor arc and major arc etc.

Now, we have to define some more terms in order to understand the theorems related to the angle in a segment of a circle.

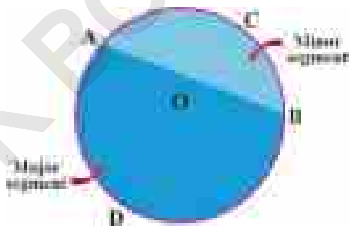
Definitions:

i. Sector of a circle:

A sector of a circle is the part of circular region bounded by two radial segments and the arc which they intercept.
In the given figure, AOB is a sector of the given circle, with centre O.

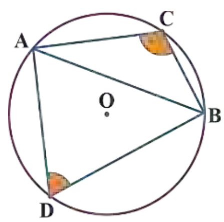
ii. Segment of a circle:

A segment of a circle is the part of circular region bounded by an arc and its chord.
A segment is called major segment if its arc is major arc. Similarly, a segment is called minor segment if its arc is minor arc. In the adjacent figure, ACB is a minor segment and ADB is a major segment.



iii. Angle in a segment of circle:

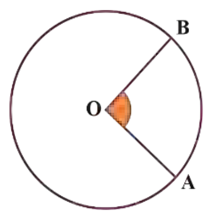
An angle in a segment of a circle is the angle which is subtended by the chord of the segment at a point other than the end points on the arc of the segment.
In the adjacent figure, $\angle ACB$ is an angle in the minor segment ACB.
Whereas $\angle ADB$ is an angle in the major segment ADB.



Note:
Angle in a segment of a circle is also known as inscribed angle of the arc of the segment or simply angle subtended by the arc

iv. Central angle of an arc:

A central angle of an arc of a circle is the angle subtended by the arc at the centre of the circle.
In the figure, $\angle AOB$ is the central angle of the arc AB.



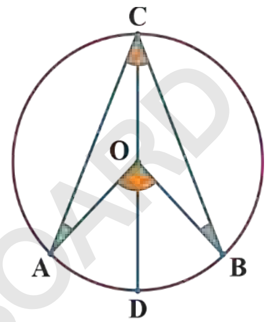
Understand the following theorems along with their corollaries and apply them to solve allied problems.

Theorem 28.1

The measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc.

OR

The central angle of a minor arc is double in measure of the inscribed angle of the corresponding major arc.



Given:

In a circle with centre O, $\angle AOB$ is the central angle of minor arc AB.

$\angle ACB$ is the angle subtended by corresponding major arc ACB.

To prove:

$m\angle AOB = 2m\angle ACB$

Construction:

Draw $\overline{CO}$ and produce it to point D of the circle.

Proof:

Statements	Reasons
In ΔAOC, $\overline{OA} \cong \overline{OC}$	Radial segments of same circle
$m\angle OAC = m\angle OCA$ ...(i)	Angles opposite to congruent sides
Now $m\angle AOD = m\angle OAC + m\angle OCA$	Exterior angle is sum of opposite interior angles of a triangle.
$= 2m\angle OCA$ ...(ii)	Using eq:(i)
Similarly $m\angle BOD = 2m\angle OCB$ ...(iii)	By same process
Now $m\angle AOB = m\angle AOD + m\angle BOD$	Angle addition postulate
$= 2m\angle OCA + 2\,m\angle OCB$	Using (ii) and (iii)
$= 2(m\angle OCA + m\angle OCB)$	Taking 2 common
$m\angle AOB = 2\,m\angle ACB$	Angle addition postulate

Q.E.D

Corollary 1:

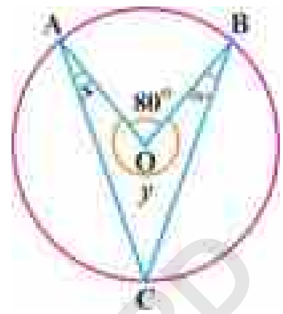
Measure of a central angle of a major arc is double that of the inscribed angle of the corresponding minor arc.

Corollary 2:

Measure of the central angle of a semi-circle is double that of the inscribed angle of the corresponding semi-circle.

Example 1:

In the adjacent figure, point O is the centre of the circle. Find the value of x if $m\angle AOB = 80^\circ$ and $m\angle OBC = 25^\circ$



Solution: In the figure, $\angle AOB$ is the central angle of $\overset{\frown}{AB}$ and $\angle ACB$ is inscribed angle of corresponding $\overset{\frown}{ACB}$

Central angle of $\overset{\frown}{AB}$ is double of the inscribed angle of the corresponding ACB.

$\therefore \quad m\angle AOB = 2m\angle ACB$

i.e. $\quad 80^\circ = 2m\angle ACB$

$\Rightarrow \quad m\angle ACB = 40^\circ$

As we know that the sum of all angles around a point is 360°.

So,

$80^\circ + y = 360^\circ$

$\Rightarrow \quad y = 280^\circ$

In quadrilateral AOBC

$x + y + 25^\circ + m\angle ACB = 360^\circ$

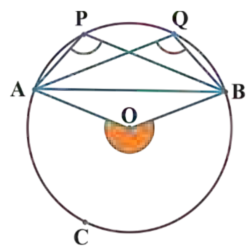
$\Rightarrow \quad x + 280^\circ + 25^\circ + 40^\circ = 360^\circ$

or $\quad x + 345^\circ = 360^\circ$

$\Rightarrow \quad x = 15^\circ$

Theorem 28.2

Any two angles in the same segment of a circle are equal.



Given: $\angle APB$ and $\angle AQB$ are the two angles in the same segment APQB of circle with centre O.

To prove: $m\angle APB = m\angle AQB$

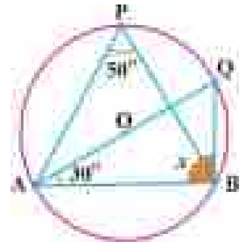
Construction: Join O with A and B so that $\angle AOB$ is the central angle of $\overset{\frown}{ACB}$.

Proof:

Statements	Reasons
$\angle AOB$ is the central angle of $\overset{\frown}{ACB}$.	Construction
$\angle APB$ and $\angle AQB$ are the angles in the same segment	Given
Now $m\angle AOB = 2m\angle APB$	Central angle of major arc is double of inscribed angle of corresponding minor arc.
Also $m\angle AOB = 2m\angle AQB$	By same process
$2m\angle APB = 2m\angle AQB$	By transitive property
$m\angle APB = m\angle AQB$	Dividing both sides by 2

Corollary 1: Any two angles in a major segment are equal.

Example:



In the adjacent figure, $\angle P$ and $\angle Q$ are the two angles in the same segment APQB of circle with centre O. Find the value of x or $m\angle ABQ$ where the measures of angles are indicated in the figure.

Solution:

$\angle P$ and $\angle Q$ are the angles on the same segment

$\therefore \quad m\angle Q = m\angle P$

i.e. $\quad m\angle Q = 50^\circ \qquad (\because m\angle P = 50^\circ)$

In ΔABQ,

$m\angle A + m\angle Q + x = 180^\circ$

i.e. $\quad 30^\circ + 50^\circ + x = 180^\circ$

$\Rightarrow \quad x = 100^\circ$

EXERCISE 28.1

1. In figure 1, point O is the centre of the circle. Find x where $m\angle C = 30^\circ$.

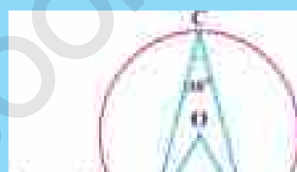


Figure 1

2. In figure 2, point O is the centre of the circle. Find x where $m\angle Q = 50^\circ$.

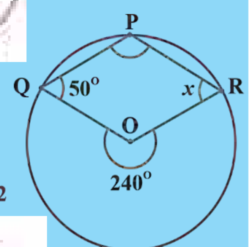


Figure 2

3. In figure 3, point O is the centre of the circle. Find $x + y$ when $m\angle D = 30^\circ$.

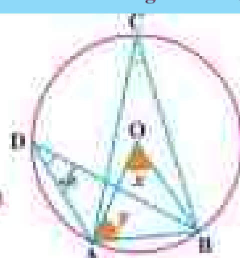


Figure 3

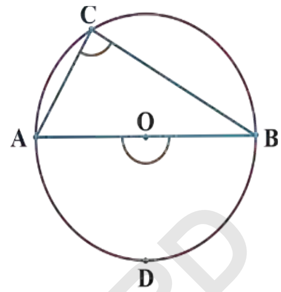
4. The inscribed angles of two congruent major arcs of two congruent circles are congruent. Prove it.
5. Prove that the inscribed angles of major arc and its corresponding minor arc in a circle are supplementary.

Theorem 28.3 (a)

The angle in a semi-circle is a right angle.

OR

The angle inscribed in a semi-circle is right angle.



Given:

In a circle with centre O, $\angle AOB$ is the central angle of semi-circle ADB.

$\angle ACB$ is the inscribed angle of corresponding semi-circle.

To prove: $\angle ACB$ is a right angle.

Proof:

Statements	Reasons
$m\angle AOB = 180^\circ$ ...(i)	Central angle of semi-circle
Now $m\angle AOB = 2m\angle ACB$	Central angle of semi circle is double of inscribed angle of corresponding semi-circle
$\Rightarrow$ $2m\angle ACB = 180^\circ$	Using eq: (i)
or $m\angle ACB = 90^\circ$	Dividing both sides by 2
i.e. $\angle ACB$ is right angle	By definition of right angle.

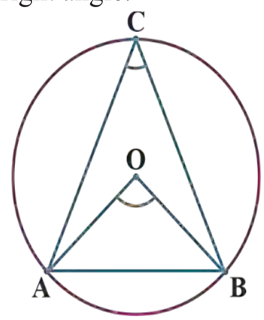
Q.E.D

Theorem 28.3 (b)

The angle in a segment greater than the semi-circle is less than a right angle.

OR

The angle inscribed in a major arc is acute.



Given:

In a circle with centre O, $\angle ACB$ is the angle in the segment ACB greater than semi-circle.

$\angle AOB$ is the central angle of corresponding minor segment AB.

To prove: $\angle ACB$ is an acute angle.

Proof:

Statements	Reasons
$m\angle AOB < 180^\circ$ ...(i)	Central angle of minor arc
Now $m\angle AOB = 2m\angle ACB$	Central angle of minor arc is double of inscribed angle of corresponding major arc.
$\Rightarrow$ $2m\angle ACB < 180^\circ$	Using eq: (i)
or $m\angle ACB < 90^\circ$	Dividing both sides by 2
i.e. $\angle ACB$ is an acute angle	By definition of acute angle.

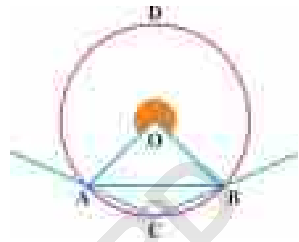
Q.E.D

Theorem 28.3 (c)

The angle in a segment less than a semi-circle is greater than a right angle.

OR

The angle inscribed in a minor arc is obtuse.



Given:

In a circle with centre O, $\angle ACB$ is the inscribed angle of minor arc AB.

$\angle AOB$ is the central angle of corresponding major arc ADB.

To prove: $\angle ACB$ is obtuse.

Proof:

Statements	Reasons
$m\angle AOB > 180^\circ$ ...(i)	Central angle of major arc
Now	
$m\angle AOB = 2m\angle ACB$	Central angle of major arc is double of inscribed angle of corresponding minor arc.
$\Rightarrow$ $2m\angle ACB > 180^\circ$	Using eq: (i)
or $m\angle ACB > 90^\circ$	Dividing both sides by 2
i.e. $\angle ACB$ is obtuse	By the definition of obtuse angle.

Q.E.D

Example:

In figure 1, Point O is the centre of the circle. Find x and y where $m\angle BCO = 30^\circ$. $\overline{AC}$ and $\overline{BD}$ are diameters.

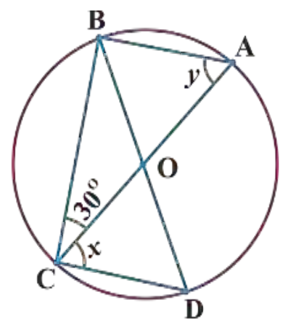


Figure 1

Solution:

$\overline{BD}$ is diameter

$\therefore$ $\angle BCD$ is inscribed angle of semi-circle

Hence $m\angle BCD = 90^\circ$

$\Rightarrow$ $x + 30^\circ = 90^\circ$

$\Rightarrow$ $x = 60^\circ$

$\overline{AC}$ is diameter

$m\angle CBA$ is inscribed angle of semi-circle

So $m\angle CBA = 90^\circ$

In ΔABC,

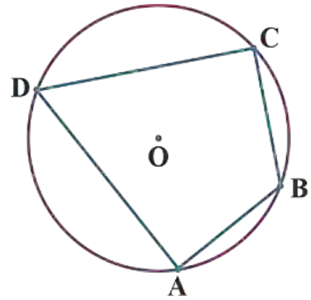
$30^\circ + m\angle CBA + y = 180^\circ$

$30^\circ + 90^\circ + y = 180^\circ$

$\Rightarrow$ $y = 60^\circ$

Inscribed Quadrilateral or Cyclic Quadrilateral:

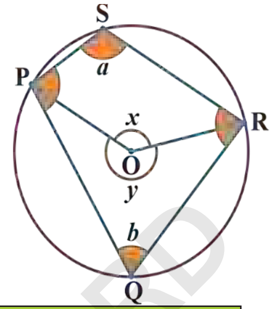
A quadrilateral is called inscribed quadrilateral or cyclic quadrilateral if all of its vertices lie on the same circle. In the figure, ABCD is a cyclic quadrilateral.



Theorem 28.4

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Given: PQRS is a quadrilateral inscribed in a circle with center O.



To prove:

$m\angle Q + m\angle S = 180^\circ$

and $m\angle P + m\angle R = 180^\circ$

Construction:

Draw $\overline{OP}$ and $\overline{OR}$ so that $\angle POR$ or $\angle x$ is central angle of $\overparen{PR}$ and $\angle y$ is central angle of corresponding $\overparen{PQR}$.

Proof:

Statements	Reasons
$\angle S$ or $\angle a$ is inscribed angle of $\overparen{PR}$	By definition
$\angle y$ is central angle of corresponding $\overparen{PQR}$.	Construction
$m\angle y = 2m\angle a$	Central angle of major arc is double of inscribed angle of corresponding minor arc.
$\Rightarrow \quad m\angle a = \frac{1}{2}m\angle y \ldots(\text{i})$	Dividing both sides by 2
Now	
$\angle Q$ or $\angle b$ is inscribed angle of $\overparen{PQR}$.	By definition
$\angle x$ is central angle of corresponding $\overparen{PR}$.	Construction
So, $m\angle x = 2m\angle b$	Central angle of minor arc is double of inscribed angle of corresponding major arc.
i.e. $\quad m\angle b = \frac{1}{2}m\angle x \ldots(\text{ii})$	Dividing both sides by 2
Now	
$m\angle a + m\angle b = \frac{1}{2}m\angle x + \frac{1}{2}m\angle y$	Adding eq: (i) and eq: (ii)
$= \frac{1}{2}(m\angle x + m\angle y)$	Taking $\frac{1}{2}$ common
$= \frac{1}{2}(360^\circ)$	Sum of all angles around a point is 360°.
i.e. $\quad m\angle a + m\angle b = 180^\circ$	
or $\quad m\angle Q + m\angle S = 180^\circ$	$\angle a \cong \angle S$ and $\angle b \cong \angle Q$
Similarly	
$m\angle P + m\angle R = 180^\circ$	By same process

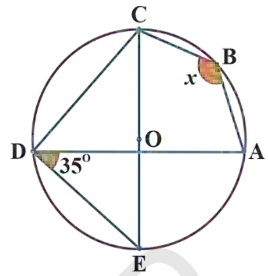
Q.E.D

Corollary:

A parallelogram inscribed in a circle is rectangle.

Example:

In given figure, point O is the centre of the circle. $\overline{CE}$ is its diameter and ABCD is cyclic quadrilateral. Find x if $m\angle ODE = 35^o$.



Solution:

$\angle CDE$ is inscribed angle of semi-circle

$\therefore \quad m\angle CDE = 90^\circ$

i.e. $\quad 35^\circ + m\angle ADC = 90^\circ$

$\Rightarrow \quad m\angle ADC = 55^\circ$

Now

In cyclic quadrilateral ABCD

$x + m\angle ADC = 180^\circ$

$\Rightarrow \quad x + 55^\circ = 180^\circ$

$\Rightarrow \quad x = 180^\circ - 55^\circ$

$\Rightarrow \quad x = 125^\circ$

EXERCISE 28.2

1. In figure 1, O is the centre of the circle, whereas $\overline{AC}$ is its diameter. Find x.

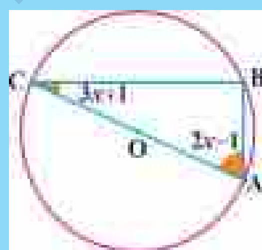


Figure 1

2. In figure 2, $\overset{\frown}{ACB}$ is a major arc of circle with centre O and $\angle C$ is its inscribed angle. Find x where $x \in \mathbb{R}$.

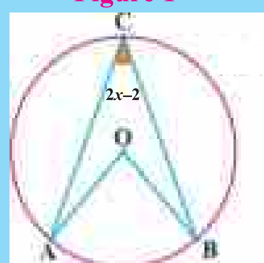


Figure 2

3. In a circle, $7x - 1$ is the measure of inscribed angle of a minor arc of a circle. Find x where $x \in \mathbb{R}$.

4. Prove that a rhombus inscribed in a circle is a square.

5. Prove that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Review Exercise 28

1. Tick the correct option.

i. Inscribed angle of --------- is obtuse.
(a) minor arc (b) major arc (c) semi-circle (d) all of these

ii. -----------is the circular region bounded by arc and its chord
(a) Sector (b) Segment
(c) Major arc (d) Circumference

iii. If $2x$ is the measure of inscribed angle of minor arc then the central angle of corresponding major arc is ---------.
(a) x (b) $2x$ (c) $3x$ (d) $4x$

iv. If $2x$ and 60° are the measures of inscribed angles of same segment then $x =$ -------.
(a) 120° (b) 60° (c) 20° (d) 30°

v. The inscribed angle of minor arc of circle is ------------ angle.
(a) acute (b) obtuse
(c) right (d) reflex

vi. The inscribed angle of major arc of circle is ------------ angle.
(a) acute (b) obtuse
(c) right (d) reflex

vii. Sum of opposite angles of cyclic quadrilateral is -------------.
(a) 90° (b) 180° (c) 270° (d) 360°

viii. A parallelogram inscribed in a circle is --------------.
(a) kite (b) trapezium
(c) rectangle (d) rhombus

ix. The central angle of an arc is ----- than inscribed angle of corresponding arc.
(a) less (b) greater
(c) less or equal (d) greater or equal

x. The sum of central angles of all the arcs of a circle is -----------.
(a) 90° (b) 180° (c) 360° (d) 1000°

SUMMARY

- The circular region bounded by an arc and two radial segments is called sector.
- The circular region bounded by an arc and its chord is called a segment.
- Angle in a segment of circle is also known as inscribed angle of the segment.
- Angle subtended by an arc at the centre of circle is called the central angle.
- The central angle of minor (or major) arc is double than inscribed angle of the corresponding major (or minor) arc respectively.
- Any two angles in the same segment of a circle are equal.
- The angle inscribed in semi-circle is right angle.
- The angle inscribed in major arc is acute.
- The angle inscribed in minor arc is obtuse.
- All the vertices of a cyclic quadrilateral are on the same circle.
- The opposite angles of cyclic quadrilateral are supplementary.