CHORDS AND ARCS

Unit 27

• Weightage = 5%

Students Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
  - If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
  - If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor,major, semi-circular) are congruent.
  - Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
  - If the angles subtended by the two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

Introduction:

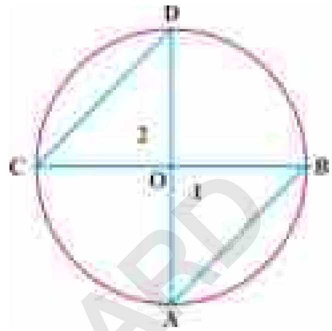
In this unit, we discuss the theorems related to chords and arcs of a circle.

Theorem 27. 1

If two arcs of a circle (or congruent circles) are congruent then the corresponding chords arc equal.

We shall prove the theorem:

i. For one circle
ii. For two congruent circles



i. For one circle

Given: A circle with center O whose $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ arc congruent arcs i.e. $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

$\overline{AB}$ and $\overline{CD}$ are the corresponding chords of the given congruent arcs.

To prove: $\overline{AB} \cong \overline{CD}$

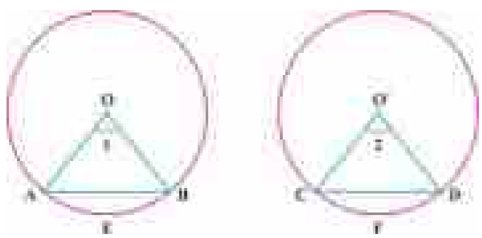
Construction: Join the points O with A,B,C and D.

Proof:

Statement	Reason
In $\Delta OAB \leftrightarrow OCD$	
$\overline{OA} \cong \overline{OC}$	Radii of same circle
$\overline{OB} \cong \overline{OD}$	Radii of same circle
$m\angle 1 = m\angle 2$	Central angles of two congruent arcs
$\Delta OAB \cong OCD$	S.A.S postulate
$\overline{AB} \cong \overline{CD}$	Corresponding sides of congruent triangles

Q.E.D

ii. For two congruent circles



Given: Two congruent circles with centers O and O' respectively. $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent arcs of these circles where $\overline{AB}$ and $\overline{CD}$ are the corresponding chords.

To prove: $\overline{AB} \cong \overline{CD}$

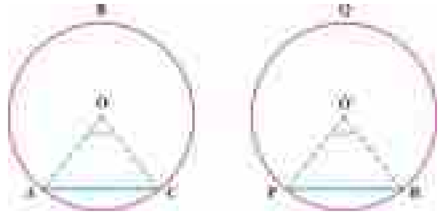
Construction: Join O to A and B. Join O' to C and D.

Proof:

Statement	Reason
In $\Delta OAB \leftrightarrow \Delta O'CD$	
$\overline{OA} \cong \overline{O'C}$	Radii of same circle
$\overline{OB} \cong \overline{O'D}$	Radii of same circle
$m\angle 1 = m\angle 2$	Central angles of two congruent arcs
$\therefore$ $\Delta OAB \cong \Delta O'CD$	S.A.S postulate
$\therefore$ $\overline{AB} \cong \overline{CD}$	Corresponding sides of two congruent triangle

Theorem 27.2 (Converse of theorem 1)

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi-circle) are congruent.



Given: Two congruent circles with center O and O' respectively having two congruent arcs i.e., $\overline{AC} \cong \overline{PR}$.

To prove: $\widehat{AC} \cong \widehat{PR}$

Construction: Join O with A and C, O' with P and R.

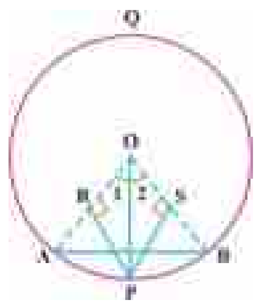
Proof:

Statement	Reason
In $\Delta AOC \leftrightarrow \Delta PO'R$	
$\overline{OA} \cong \overline{O'P}$	Radii of congruent circles
$\overline{OC} \cong \overline{O'R}$	Radii of congruent circles
$\overline{AC} \cong m\overline{PR}$	Given
$\Delta OAC \cong \Delta PO'R$	S.S.S ≅ S.S.S
$m\angle AOC = m\angle PO'R$ (i)	Corresponding angles of congruent Δs.
Hence	
$\widehat{AC} \cong \widehat{PR}$	From eq: (i)

Q.E.D

Example 1:

A point P on the circumference of a circle is equidistant from the radial segments $\overline{OA}$ and $\overline{OB}$. Prove that $m\widehat{AP} = m\widehat{BP}$



Solution:

Given: $\overline{AB}$ is the chord of a circle with center O. Point P on the circumference of the circle is equidistant from the radial segment $\overline{OA}$ and $\overline{OB}$. i.e. $m\overline{PR} = m\overline{PS}$

To prove: $m\widehat{AP} = m\widehat{BP}$

Construction: Join O with P. Write $m\angle 1$ and $m\angle 2$ as shown in the figure.

Proof:

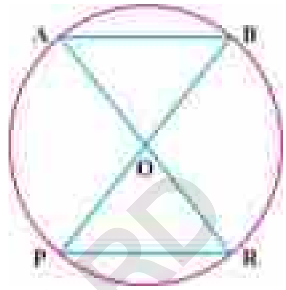
Statement	Reason
In right angle $\Delta OPR \leftrightarrow \Delta OPS$	
$m\overline{OP} = m\overline{OP}$	Common
$m\overline{PR} = m\overline{PS}$	
$\therefore$ $\Delta OPR \cong \Delta OPS$	H.S ≅ H.S
So $m\angle 1 = m\angle 2$ (i)	Corresponding angles of congruent Δs
$\Rightarrow$ $\widehat{AP} \cong \widehat{BP}$	From (i)
Hence $m\widehat{AP} = m\widehat{BP}$	By definition of congruent arcs

Theorem 27.3

Equal chords of a circle (or of congruent circles) subtend equal angles at the center (at the corresponding centers).

We shall prove theorem:

i) For one circle ii) For two circles



i) For one circle

Given: Circle with centre O, having two congruent chords i.e., $\overline{AB} \cong \overline{PR}$.

To prove: $m\angle AOB = m\angle POR$

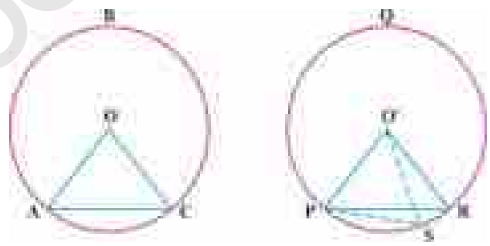
Construction: Join O with A and B and also O with P and R.

Proof:

Statement	Reason
$\overline{AB} \cong \overline{PR}$ (i)	Given
$\overline{OB} \cong \overline{OR}$ (ii)	$\overline{OB}, \overline{OR}, \overline{OA}$ and $\overline{OP}$ are radii of same circle
and $\overline{OA} \cong \overline{OP}$ (iii)	
$\Delta AOB \cong \Delta POR$	S.S.S $\cong$ S.S.S
$m\angle AOB = m\angle POB$	Corresponding angles of congruent triangles

Q.E.D

ii) For two congruent circles



Given:

Two congruent circles with center O and O′ respectively so that $m\overline{AC} = m\overline{PR}$. Where $\overline{AC}$ and $\overline{PR}$ are the chords of circles.

To prove:

$m\angle AOC = m\angle PO'R$

Construction:

Take a point S on arc PR and join S to P and O′ such that $\angle AOC \cong \angle PO'S$.

Proof:

Statement	Reason
Let $m\angle AOC = m\angle PO'R$	
But $m\angle AOC \cong m\angle PO'S$	Construction
$\therefore$ $\overset{\frown}{AC} \cong \overset{\frown}{PS}$ (i)	Arcs subtended by equal central angles of congruent circles
$\therefore$ $m\overline{AC} = m\overline{PS}$ (ii)	From theorem (1)
But $m\overline{AC} = m\overline{PR}$ (iii)	Given
$\therefore$ $m\overline{PR} = m\overline{PS}$	Using (ii) and (iii)
Which is only possible, if R concides with S.	Because P is common point
$\Rightarrow$ $m\angle AOC = m\angle POR$	Our supposition is working

Corollary 1:

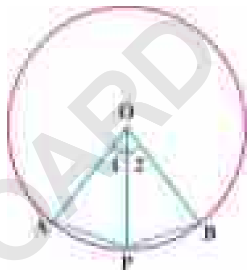
In congruent circles or in same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2:

In congruent circles or in same circle, arcs will subtend unequal central angles for unequal chords.

Example 1:

Prove that the internal bisector of a central angle in a circle bisects the corresponding arc.



Given: A circle with center O. $\overline{OP}$ is an internal bisector of central angle AOB. i.e. $\angle 1 \cong \angle 2$

To prove: $\overset{\frown}{AP} \cong \overset{\frown}{PB}$

Construction: Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle 1$ and $\angle 2$ are corresponding angles of $\overset{\frown}{AP}$ and $\overset{\frown}{BP}$ respectively.

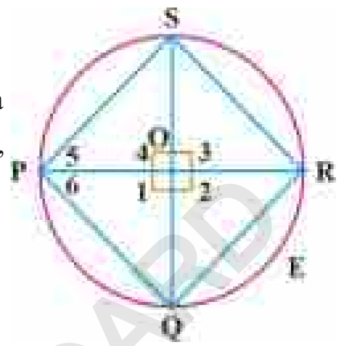
Proof:

Statement	Reason
In $\Delta OAP \leftrightarrow \Delta OBP$	
$m\overline{OA} = m\overline{OB}$	Radii of same circle
$m\angle 1 = m\angle 2$	Given
and $m\overline{OP} = m\overline{OP}$	Common
$\therefore$ $\Delta OAP \cong \Delta OBP$	S.A.S postulate
Hence $\overline{AP} \cong \overline{BP}$	Corresponding sides of congruent Δs
$\Rightarrow$ $\overset{\frown}{AP} \cong \overset{\frown}{BP}$	Arcs corresponding to equal chords in a circle.

Q.E.D

Example 2:

Prove that in a circle, if two diameters are perpendicular to each other then the lines joining their ends form a square.



Given:

Given $\overline{PR}$ and $\overline{QS}$ are two perpendicular diameters of a circle with center O. So PQRS is a quadrilateral. Naming $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in figure.

To prove:

PQRS is a square.

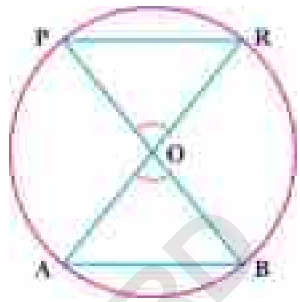
Proof:

Statement	Reason
$\overline{PR}$ and $\overline{QS}$ are two perpendicular diameters of a circle with center O.	Given
$m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = 90^\circ$	Diameters are perpendicular to each other
$m\text{PQ} = m\text{QR} = m\text{RS} = m\text{SP}$ (i)	Arcs opposite to the equal central angles in a circle
$m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{SP}$ (ii)	Chords corresponding to equal arcs.
In right Δ POS,	
$m\overline{PO} = m\overline{OS}$ (iii)	$\overline{PO}$ and $\overline{OS}$ are radii of circle
ΔPOS is right isosceles, triangle	From (iii)
$m\angle 5 = 45^\circ$ (iv)	ΔPOS is right isosceles, triangle
Similarly, ΔPOQ is right isosceles triangle	By the above process
$m\angle 6 = 45^\circ$ (v)	
Moreover	
$m\angle P = m\angle 5 + m\angle 6$	Angle addition postulate
$m\angle P = 45^\circ + 45^\circ$	By using (iv) and (v)
$m\angle P = 90^\circ$ (vi)	
Similarly, $m\angle Q = m\angle R = m\angle S = 90^\circ$ (vii)	By the above process
PQRS is a square	Using eq (ii), (vi) and (vii)

Q.E.D

Theorem 27.4

If the angles subtended by two chords of a circle (or congruent circles) at the centres (corresponding centres) are equal, the chord are equal.



(a) For one circle

Given:

Circle with centre O, having chords $\overline{PR}$ and $\overline{AB}$. The angles subtended by both arcs are congruent i.e. $\angle POR \cong \angle AOB$.

To prove: $m\overline{AB} \cong m\overline{PR}$

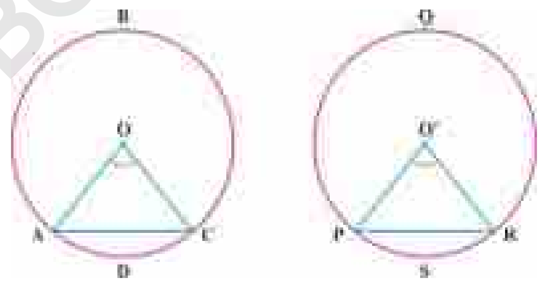
Construction: Join O with P and R, and also with A and B.

Proof:

Statement	Reason
$\overline{OR} \cong \overline{OB}$ (i)	Radius of same circle
and $\overline{OP} \cong \overline{OA}$ (ii)	Radii of same circle
$\angle POR \cong \angle AOB$ (iii)	Given
$\Delta PQR \cong \Delta AOB$	$S.A.S \cong S.A.S$ (Postulate)
$\overline{AB} \cong \overline{PR}$	Corresponding side of congruent triangles

Q.E.D

(b) For two congruent circles



Given:

Two congruent circles with center O and O′ with $\overline{AC}$ and $\overline{PR}$ are chords. Subtending equal angles at centre i.e. $\angle AOC \cong \angle PO'R$.

To prove: $m\overline{AC} = m\overline{PR}$

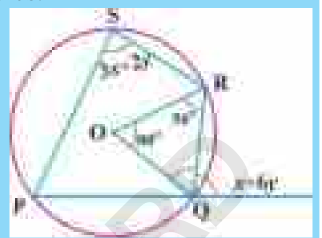
Proof:

Statement	Reason
In $\Delta OAC \leftrightarrow \Delta O'PR$	
$\overline{OA} \cong \overline{O'P}$	Radii of congruent circles
$\angle AOC \cong \angle PO'R$	Given
$\overline{OC} \cong \overline{O'R}$	Radii of congruent circle
$\therefore$ $\Delta OAC \cong \Delta O'PR$	S.A.S postulate
Hence $m\overline{AC} = m\overline{PR}$	Corresponding sides of congruent Δs

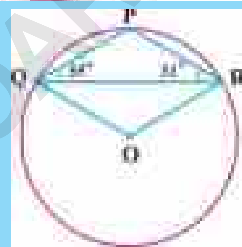
Q.E.D

EXERCISE 27.1

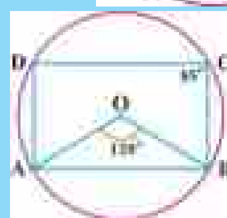
1. In a circle prove that the arcs between two parallel and equal chords or equal.
2. Prove that in equal circles, equal central angles have equal arcs.
3. In the following figure, O is the centre of the circle. Find the value of x and y.



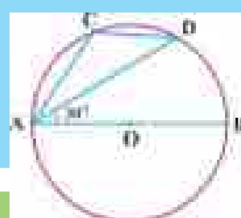
4. In the following figure $m\angle PQR = 35°$ and $m\angle PRQ = 31°$. Find $m\angle QPR$ and $m\angle OQR$.



5. In the figure O is the center of circle, given that $m\angle AOB = 120°$ and $m\angle BCD = 85°$. Find $m\angle OAD$



6. In the following figure, $\overline{AB}$ is diameter of the circle given that $m\angle DAB = 30°$ Find $m\angle ACD$



REVIEW EXERCISE 27

1. Multiple Choice Question:

Tick the correct option.

i. If a chord of a circle subtends, a central angle of $60°$. Then the chord and the radial segment are _____.
   (a) Parallel (b) Perpendicular (c) Congruent (d) Incongruent

ii. An arc subtends a central angle of $45°$ then the corresponding chords will subtend a central angle of ?
   (a) $15°$ (b) $30°$ (c) $45°$ (d) $60°$

iii. A pair of chords of a circle subtending two congruent central angle are.
   (a) Perpemdicular (b) Non congruent (c) Congruent (d) None of these

iv. The arcs opposite to congruent central angles of a circle are always.
(a) Parallel (b) Congruent (c) Perpendicular (d) None of these

v. A 6cm long chord subtends a central angle of 60°. The radial segment of this circle is
(a) 4cm (b) 6cm (c) 5cm (d) 8cm

vi. The chord length of a circle subtending a central angle of 180° is always
(a) equal to the radial segment
(b) less than radial segment
(c) double of radial segment
(d) half of the radial segment.

viii. The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:
(a) 60° (b) 75° (c) 90° (d) 45°

ix. Out of two congruent arcs of a circle, if one arc makes a central angle of 30° then the other arc will subtend the central angle:
(a) 60° (b) 90° (c) 75° (d) 30°

x. Diameter divides the circle into ______ parts.
(a) Two (b) Three (c) Four (d) All of these

SUMMARY

- The circles are congruent of their radii are equal.
- Equal chords of circle subtend equal angles at the centre.
- The straight line joining any two points of the circumference is called a chord of the circle.
- The portion of a circle bounded by an arc and a chord is known as the segment of a circle.
- The boundary traced by a moving point in a circle is called its circumference.
- An portion of the circumference of a circle is called an arc of the circle.
- A straight line, drawn from the centre of a circle bisecting a chord is perpendicular to the chord
- A perpendicular drawn from the centre of a circle on a chord, it bisect the chord.
- If two arcs of a circle(or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, semi circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angle subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.