UNIT

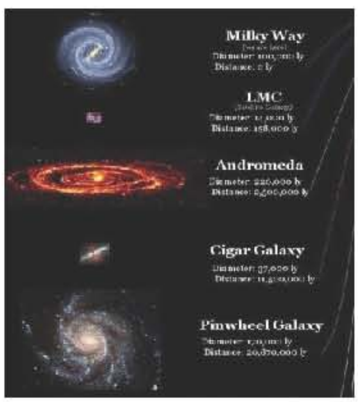
03

SETS AND RELATIONS

In this unit the students will be able to:

- Describe mathematics as a study of patterns, structures and their relationships.
- Identify sets and apply operations on three sets (subset, overlapping and disjoint cases) using Venn diagrams.
- Solve problems on classification and cataloging by using Venn diagram for scenarios involving two sets and three sets.
- Verify and apply laws of union and intersection of three sets through analytical and Venn diagram method.
- Apply concepts from set theory to real world problems (such as in demographic classification, categorizing products in shopping malls and music playlist).
- Explain product, binary relations and identify domain and range of binary relations.
- Recognize that a relation can be represented by tables, ordered pairs and graphs.

ALLAH سبحانہ وتعالی has created this huge universe and a part of it is exposed to humans but a big part of it is still unknown to humans. It is estimated that there are 200 billions to 2 trillion galaxies in the observable universe. The adjoining figure shows a set of 10 naked eye galaxies.



SET

Definition:

A set is a well-defined collection of distinct objects.

The term well-defined, means that the objects follow a given discipline with which presence or absence of some object in the set is checked.

For instance, if we say that we have a collection of lighter stones, then this collection is not well defined. Instead of this, if we say that we have a collection of stones weighing less than 1kg, this collection is well defined.

A set may consist of objects of different types.

e.g. A set of luminous objects may contain a star, a moon, a tube light or a candle.

Similarly, a set of objects present in a library may include books, tables, chairs, newspapers, keys, locks, stock registers etc.

Example 1:

(a) A set of Prime numbers which are also even i.e. {2}

(b) A set of Pakistani currency i.e.
{5, 10, 20, 50, 100, 500, 1000, 5000}

(c) A set of flowers in my garden. i.e.
{Pansy, Lilly, Daisy, Jasmine, Tulip, Rose, Hibiscus}

(d) A set of Natural numbers less than 1 i.e. { }

(e) A set of all letters present in the word 'set' {s, e, t}

(f) A set of Natural numbers among 3, – 9, 2, 3, 4, 2 i.e. {2, 3, 4}.

The repeating elements are taken once only, since repetition is not allowed in a set.

> **Key Fact**
> A set is like a box with some stuff in it which is well defined. When you look inside the box, you should be able to tell if something's in it or not.

Mathematics as a Study of Patterns, Structures and Relationships

In mathematics, patterns are more than a beautiful design. Patterns follow a predictable rule and that rule allows us to predict what will come next.

For example:

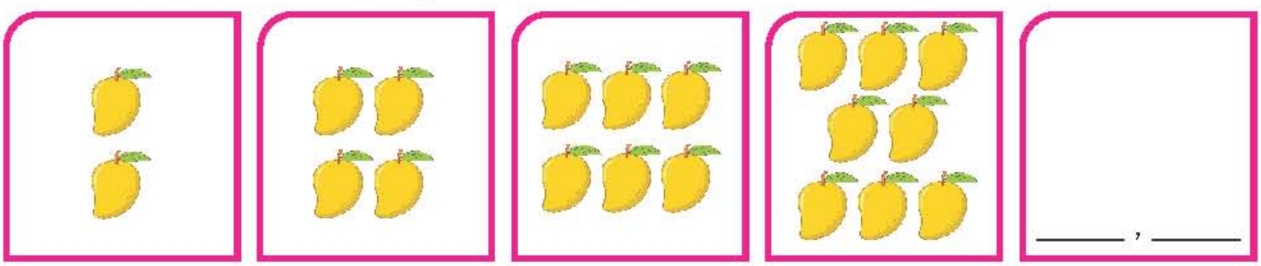
In the set of even numbers,

{2, 4, 6, 8, ...}

a pattern exists and one can determine the next number(s) in the set.

If we want to relate the above pattern with some real situation, we can ask the following question to the students:

What will be the number of mangoes in next two boxes?

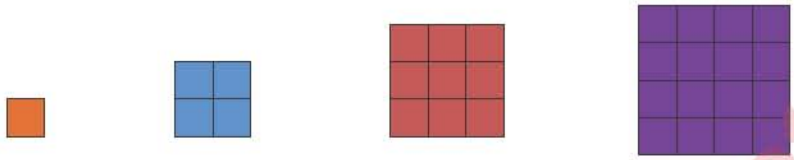


Students can easily predict the number of mangoes in the next two boxes.
Obviously, they will say:

Number of box	Number of mangoes
5	10
6	12

In the same way, we can relate the set of square of natural numbers with structural geometry as:

$$\{1^2, 2^2, 3^2, 4^2, \ldots\} = \{1, 4, 9, 16, \ldots\}$$



$1^2 = 1$ $\quad 2^2 = 4$ $\quad 3^2 = 9$ $\quad 4^2 = 16$

This example relates the patterns in numbers and geometry in the best way where the square numbers represent area of various geometrical shapes.

Key Fact

Mathematicians say that mathematics is the study of patterns in numbers and structure in geometry, and their relationships.

Check Point

Search number patterns in the set of first 100 natural numbers and relate patterns with some kind of geometrical shape or represent pattern in pictorial form.

Set Builder Form (Rule Method)

In set builder form, all the elements of a set are not listed, however we write the set by its defining rule. While writing a set in this method, some variable say x is chosen which represents all the elements of that set according to the defining rule.

e.g. A = Set of all integers, can be written in set builder form as

$A = \{x \mid x \in Z\}$ and read as

"A is the set of all elements x such that x belongs to Z"

Example 2: Write the following sets in the set builder form.

(i) B = Set of Prime numbers less than 17.

$B = \{x \mid x \in P \wedge x < 17\}$.

(ii) C = Set of multiples of 4 greater than or equal to 40.

$C = \{4x \mid x \in N \wedge x \geq 10\}$.

(iii) $D = \{1, 2, 3, 6\}$.

$D = \{x \mid x \text{ is a factor of } 6\}$.

(iv) E = Set of squares of 1st three natural multiples of 10.

$E = \{(10x)^2 \mid x \in N \wedge 1 \leq x \leq 3\}$.

Example 3:

(i) The set $G = \{15x \mid x \in Z \wedge x \geq 1\}$, in descriptive form is written as

G = Set of integral multiples of 15, greater than or equal to 15.

(ii) The set $H = \{y \mid y \in W \wedge y^3 - 1 = 7\}$, in tabular form is $H = \{2\}$.

(iii) The set $I = \{x \mid x \in W \wedge -3 > x > -5\}$, in tabular form is $I = \{\ \}$.

(iv) The set $J = \left\{\frac{x}{2} \mid x \in E \wedge x > 0\right\}$, in tabular form is $J = \{1, 2, 3, 4, \ldots\}$.

(v) The set $K = \{y \mid y \in Z^- \wedge y^2 = 9\}$, in descriptive method is

K = Set of negative integers containing '– 3' only.

Most Commonly used Sets of Numbers

i. N = Set of Natural numbers $= \{1, 2, 3, 4, \ldots\ldots\}$

ii. W = Set of Whole numbers $= \{0, 1, 2, 3, 4, \ldots\ldots\}$

iii. Z = Set of Integers $= \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

iv. E = Set of Even numbers $= \{0, \pm 2, \pm 4, \pm 6, \ldots\}$

v. O = Set of Odd numbers $= \{\pm 1, \pm 3, \pm 5, \pm, \ldots\}$

vi. P = Set of Prime numbers $= \{2, 3, 5, 7, 11, 13, \ldots\}$

vii. Q = Set of Rational numbers $= \left\{\frac{p}{q} / p, q \in Z \wedge q \neq 0\right\}$

viii. Q' = Set of Irrational numbers $= \left\{x / x \neq \frac{p}{q}, p, q \in Z \wedge q \neq 0\right\}$

ix. R = Set of Real numbers $= \{x / x \in Q \vee x \in Q'\}$

Check Point

Can we write set of Real numbers in tabular form? Justify!

The above mentioned sets from (i-vi) are first written in descriptive method and then in tabular form (Roster method).

However, the sets from (vii-ix) are first written in descriptive method then in set builder form.

Set Operations

Union of Sets

Let A and B be two given sets. Then union of A and B is the set of all those elements which are taken either from A or from B or from both.

The union of A and B is denoted by '$A \cup B$' and:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

e.g. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 1, 7, 6\}$, then:

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

Intersection of Sets

Let A and B be given sets then the intersection of A and B is the set of elements which belong to both A and B.

The intersection of A and B is denoted by '$A \cap B$' and:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

e.g. If A = {1, 2, 3, 4, 5} and B = {3, 4, 1, 7, 6}, then:

$$A \cap B = \{1, 3, 4\}$$

Difference of Sets

Difference of two sets A and B, denoted by A – B is a collection of those elements of A which are not present in B. For the two sets A and B:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Let A = {1, 2, 3, 4, 5} and B = {3, 4, 6, 7, 8}, then:

A – B = {1, 2, 3, 4, 5} – {3, 4, 6, 7, 8} = {1, 2, 5}

In general, $A - B \neq B - A$

Complement of a Set

Let $A \subset U$ (i.e. A is proper subset of the universal set). Then the set of all elements of U, which are not in A, is called complement of A.

Complement of A is denoted by A^c or $A' = U - A$ and is defined as:

$$\{x \mid x \in U \wedge x \notin A\}$$

e.g. If U = {1, 2, 3, 4, 5, 6, 7}, A = {2, 4, 6}, then:

$$A^c = U - A = \{1, 3, 5, 7\}$$

Key Fact

- The common elements of A and B are written once.
- Two sets A and B are said to be disjoint, if $A \cap B = \phi$
- The elements of A are never present in A^c and vice versa.
- Two sets are said to be overlapping, if neither set is subset of the other and their intersection is non-empty.

Union and Intersection of Three Sets

Disjoint Sets

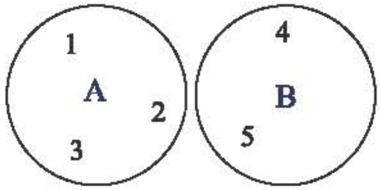
Two sets A and B are said to be disjoint if they have no common elements. i.e. $A \cap B = \phi$.

For example,

A = {1, 2, 3} and B = {4, 5} are disjoint sets as both sets have no common element. i.e.

$$A \cap B = \{1, 2, 3\} \cap \{4, 5\} = \phi$$

Venn diagram representing above disjoint sets is:



Overlapping Sets

Two sets A and B are called overlapping if:

(i) There is at least one element common in both the sets.

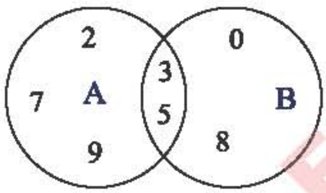
(ii) Neither of the sets is a subset of other set.

For example, the sets

$A = \{2, 3, 5, 7, 9\}$ and $B = \{0, 3, 5, 8\}$ are overlapping as

$A \cap B = \{3, 5\} \neq \phi$ and A and B are not subsets of each other.

Venn diagram representing above overlapping sets is:



Key Fact

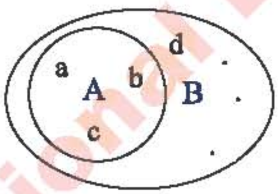
Venn diagrams are used to explain the whole set theory in a very simple way.

Subset

Set A is called subset of a set B if every element of set A is also an element of B.

For example, the set $A = \{a, b, c\}$ is a subset of $B = \{a, b, c, d, \ldots\}$ as all elements of set A are also elements of set B.

Venn diagram representing above subset case is:



The pictorial representation of the relationship $A \subset U$ is called a Venn diagram. In a Venn diagram, universal set is represented by the interior of the rectangle, however inside the rectangle, the subsets are represented by the interior of any other closed shape like circles or ovals etc.

Union and Intersection of Three Sets using Venn Diagrams

To find the union and intersection of three sets say $A \cup B \cap C$, first we find $A \cup B$ or $B \cap C$ and then find the union or intersection with remaining set.

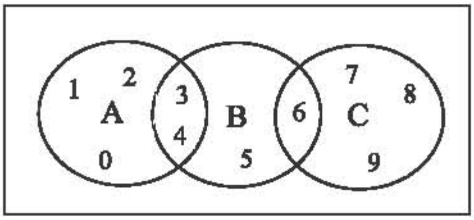
We use brackets to separate two sets from the third one because these represent different sets.

i.e. $(A \cup B) \cap C$ or $A \cup (B \cap C)$

Example 4:

Represent sets $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{6, 7, 8, 9\}$ through Venn diagram.

Solution:

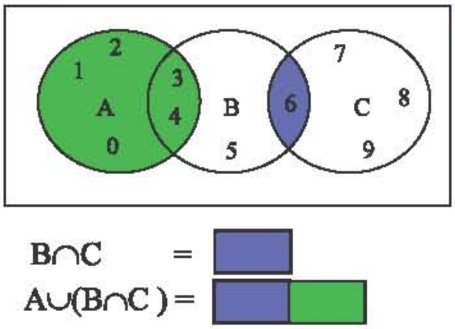


Example 5:

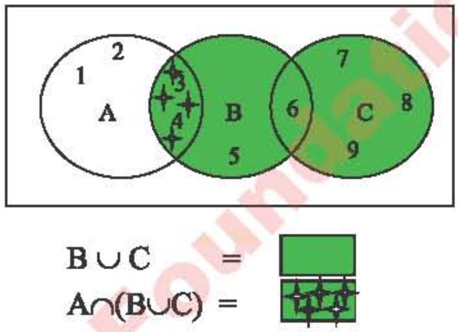
If A = {0, 1, 2, 3, 4}, B = {3, 4, 5, 6} and C = {6, 7, 8, 9}, find A ∪ (B ∩ C) and A ∩ (B ∪ C) through Venn diagram.

Solution:

(i)



(ii)



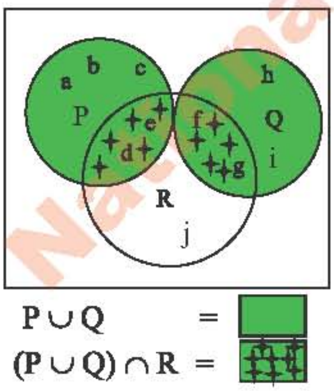
Example 6:

If P = {a, b, c, d, e}, Q = {f, g, h, i}, R = {d, e, f, g, j}

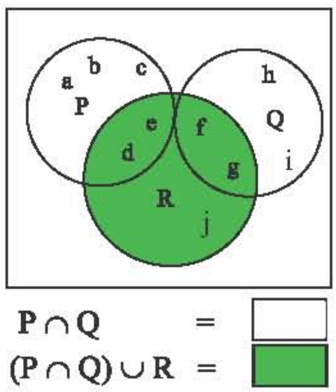
Find (P ∪ Q) ∩ R and (P ∩ Q) ∪ R through Venn diagram.

Solution:

(i)

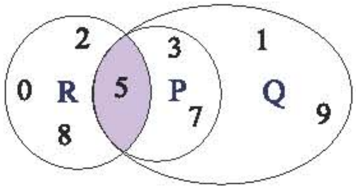


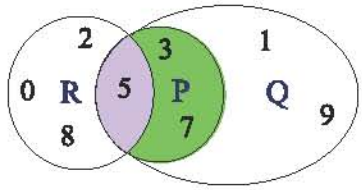
(ii)



Example 7: If P = {3, 5, 7}, Q = {1, 3, 5, 7, 9}, R = {0, 2, 5, 8}, then find P ∪ (Q ∩ R) using Venn diagram.

Solution: P = {3, 5, 7}, Q = {1, 3, 5, 7, 9}, R = {0, 2, 5, 8}





$Q \cap R = \{5\} =$ ▭ $\qquad P \cup (Q \cap R) = \{3, 5, 7\} =$ ▭▭

Verification of Associative Laws Using Venn Diagram

We illustrate the concept with the help of following examples.

Associative Property of Union

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof:

Let $x \in (A \cup B) \cup C$

$\Rightarrow \quad x \in (A \cup B) \quad \text{or} \quad x \in C$

$\Rightarrow \quad (x \in A \quad \text{or} \quad x \in B) \quad \text{or} \quad x \in C$

$\Rightarrow \quad x \in A \quad \text{or} \quad (x \in B \quad \text{or} \quad x \in C)$

$\Rightarrow \quad x \in A \quad \text{or} \quad x \in (B \cup C)$

$\Rightarrow \quad x \in A \cup (B \cup C)$

$\Rightarrow \quad (A \cup B) \cup C \subseteq A \cup (B \cup C) \qquad \text{(a)}$

Similarly, we can prove that:

$A \cup B \cup C) \subseteq (A \cup B) \cup C \qquad \text{(b)}$

From (a) and (b), we have:

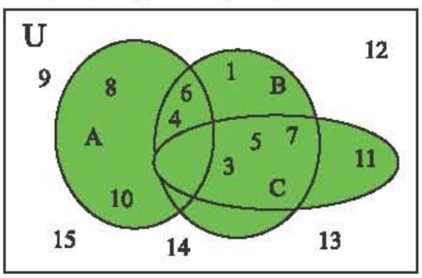
$$(A \cup B) \cup C = A \cup (B \cup C)$$

Example 8:

If $U = \{1, 2, 3, 4, 5, \ldots, 15\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $C = \{2, 3, 5, 7, 11\}$, then verify the associative property of union using Venn diagram.

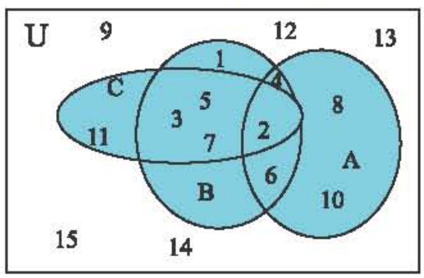
Solution:

LHS = $(A \cup B) \cup C$



$(A \cup B) \cup C =$ ▭

RHS = $A \cup (B \cup C)$



$A \cup (B \cup C) =$ ▭

From both the figures, it is observed that same region is shaded.

i.e. $(A \cup B) \cup C = A \cup (B \cup C)$

$\therefore$ Associative property of Union is verified.

Associative Property of Intersection

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Proof:

Let $y \in (A \cap B) \cap C$

$\Rightarrow$ $y \in A \cap B$ and $y \in C$

$\Rightarrow$ $(y \in A$ and $y \in B)$ and $y \in C$

$\Rightarrow$ $y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $y \in A$ and $y \in (B \cap C)$

$\Rightarrow$ $y \in A \cap (B \cap C)$

$\Rightarrow$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ (a)

Similarly, we can prove that:

$A \cap (B \cap C) \subseteq (A \cap B) \cap C$ (b)

From (a) and (b), we have:

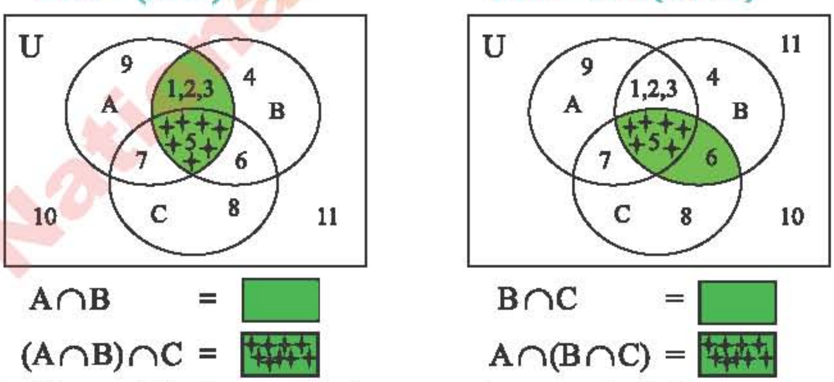
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Example 9:

If $U = \{1, 2, 3, \ldots, 11\}$, $A = \{1, 2, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$ then verify the associative property of intersection.

Solution:

LHS = $(A \cap B) \cap C$ RHS = $A \cap (B \cap C)$



From both the figures it is observed that same regions are shaded.

i.e. $(A \cap B) \cap C = A \cap (B \cap C)$

$\therefore$ Associative property of intersection is verified.

Verification of Distributive Laws Using Venn Diagram

(a) Distributive Property of Union over Intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

Let $x \in A \cup (B \cap C)$

$\Rightarrow \quad x \in A \text{ or } x \in (B \cap C)$

$\Rightarrow \quad x \in A \text{ or } (x \in B \text{ and } x \in C)$

$\Rightarrow \quad (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow \quad x \in (A \cup B) \text{ and } x \in (A \cup C)$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow \quad A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \qquad \text{(a)}$

Again, let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow \quad y \in (A \cup B) \text{ and } y \in (A \cup C)$

$\Rightarrow \quad (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C)$

$\Rightarrow \quad y \in A \text{ or } (y \in B \text{ and } y \in C)$

$\Rightarrow \quad y \in A \text{ or } y \in (B \cap C)$

$\Rightarrow \quad y \in A \cup (B \cap C)$

$\Rightarrow \quad (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \qquad \text{(b)}$

From (a) and (b), we have:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(b) Distributive Property of Intersection over Union

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:

Let $x \in A \cap (B \cup C)$

$\Rightarrow \quad x \in A \text{ and } x \in (B \cup C)$

$\Rightarrow \quad x \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Rightarrow \quad x \in (A \cap B) \text{ or } x \in (A \cap C)$

$\Rightarrow \quad x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow \quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \qquad \text{(a)}$

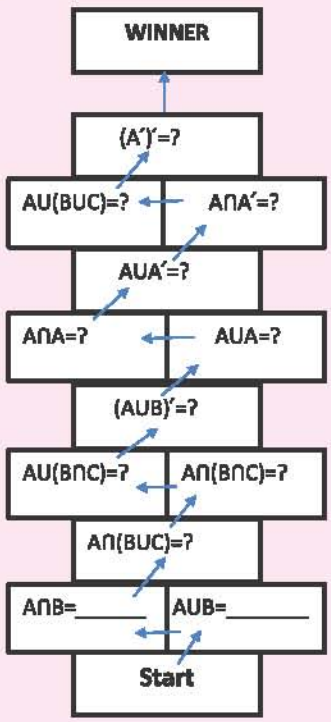
Again, let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow \quad y \in (A \cap B) \text{ or } y \in (A \cap C)$

$\Rightarrow \quad (y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C)$

Math Play Ground

1. Take students to the playground and make a hopscotch as shown:
2. Ask a student to start hoping and filling the blanks.



$\Rightarrow \quad y \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow \quad y \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad y \in A \cap (B \cup C)$

$\Rightarrow \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \qquad$ (b)

From (a) and (b), we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 10: Verify through Venn diagram

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

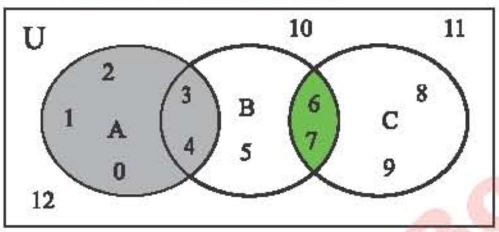
When $U = \{x : x \in W \wedge x \leq 12\}$, $A = \{x : x \in W \wedge x \leq 4\}$

$B = \{y : y \in N \wedge 3 \leq y \leq 7\}$ and $C = \{z : z \in N \wedge 3 \leq z \leq 7\}$

Solution:

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

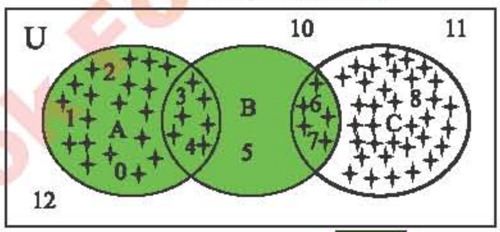
LHS = $A \cup (B \cap C)$



$B \cap C$ =

$A \cup (B \cap C)$ =

RHS = $(A \cup B) \cap (A \cup C)$



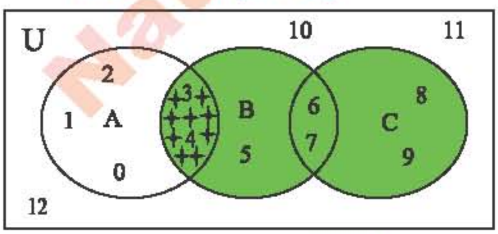
$A \cup B$ =

$A \cup C$ =

$(A \cup B) \cap (A \cup C)$ =

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C$

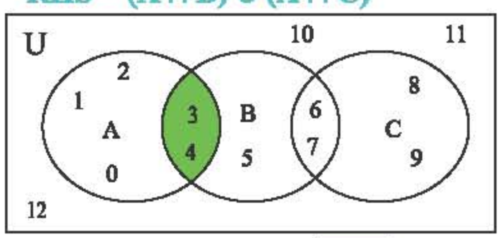
LHS = $A \cap (B \cup C)$



$B \cup C$ =

$A \cap (B \cup C)$ =

RHS = $(A \cap B) \cup (A \cap C)$



$A \cap B$ =

$A \cap C$ =

$(A \cap B) \cup (A \cap C)$ =

Key Fact

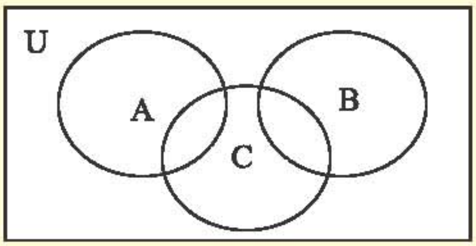
(a) If A = {0, 1, 3, 5, 7}, B = { x | $x \in N \wedge x \leq 5$} and C = {1, 2, 3, 4, 6, 12}, then find $(A \cup B) \cap C$ using Venn diagrams.

(b) If A = {0, 2, 4, 6}, B = {1, 3, 5, 7} and C = {1, 2, 3, 6}, then find $(A \cap B) \cap C$, $A \cap (B \cap C)$ and $(A \cup B) \cap C$ using Venn diagram.

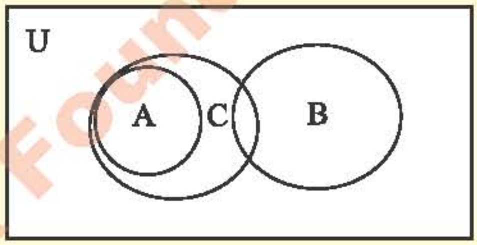
EXERCISE 3.1

1. Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $(A \cup B) \cup C$ and $A \cap (B \cap C)$ using following Venn diagrams.

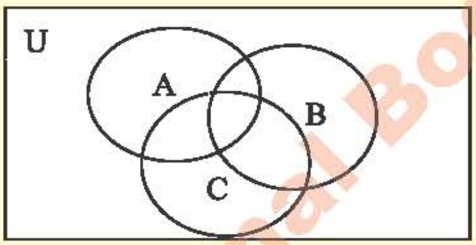
(i)



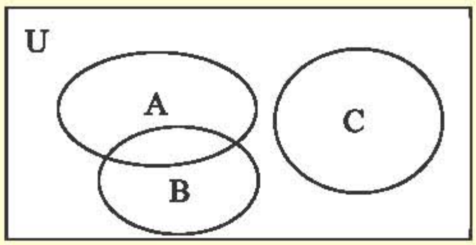
(ii)



(iii)



(iv)



2. If X = {a, b, c, d, e}, Y = {a, c, e}, Z = {g, h, i, j} then, find the following using Venn diagram.

   (i) $(X \cup Y) \cup Z$ (ii) $X \cup (Y \cup Z)$ (iii) $(X \cap Y) \cap Z$

   (iv) $X \cap (Y \cap Z)$ (v) $(X \cup Y) \cap Z$ (vi) $(X \cap Y) \cup Z$

3. Verify associative law of union and intersection by using diagrams of question 1.

4. Verify:

   (i) distributive property of union over intersection,

   (ii) distributive property of intersection over union,

   by using diagrams of question 1.

5. Prove by using Venn diagram:

   (a) $(P \cup Q) \cup R = P \cup (Q \cup R)$ (b) $(P \cap Q) \cap R = P \cap (Q \cap R)$

   when (i) P = {0,1 ,2, 3}, Q = {2, 3, 4, 5, 6}, R = {5, 6, 7, 8, 9}

(ii) P = {m, n, o, p, q}, Q = {r, s, t, u}, R = {t, u, v, w}

6. Verify $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ using Venn diagram for the following sets.
$X = \{-1, -2, -3\}$, $Y = \{0, 1, 2, 3\}$, $Z = \{0, \pm 1, \pm 2, \pm 3\}$

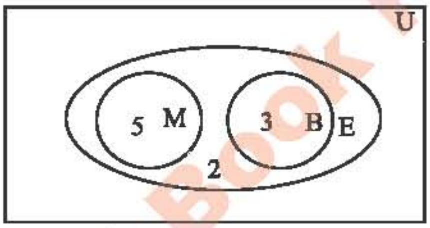
7. Verify $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
X = Set of first three Vowels, Y = Set of letters of the word " energy",
Z = Set of letters of the word "algebra"

Application of Venn Diagrams

Venn diagram is a practical mathematical tool for solving real world problems of set theory. Few of the examples depict the vital role of the Venn diagrams in problem solving.

Example 11: Among the ten teachers of a secondary school, five teach Mathematics and three teach Biology. However, all these teachers also teach English? Show the data by Venn diagram. Also find how many teachers teach only English.

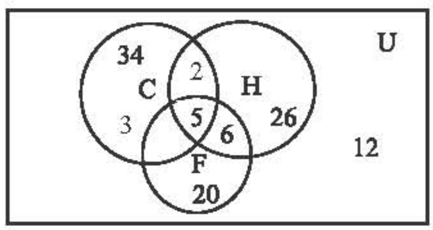
Solution: If E represents set of English Teachers, M represents set of Mathematics Teachers, B represents the set of Biology Teachers and then both M and B are the subsets of E, as shown in the figure below.



From the Venn diagram it is observed that two of the teachers only teach English.

Example 12: In a survey, people were asked whether they like cricket, hockey or football. Using the Venn diagram find the number of people playing:

(i) only cricket,
(ii) hockey and football,
(iii) all three games,
(iv) either of the three games,
(v) neither of the three games,
Also find number of people surveyed.



Solution: From Venn diagram, it is clear that:

(i) Number of people who play only cricket = 34
Which shows $n(C - C \cap H \cap F)$

(ii) Number of people who play both hockey and football = 6 + 5 = 11
Which shows $n(H \cap F)$

(iii) Number of people who play all three games = 5
Which shows $n(C \cap H \cap F)$

(iv) Number of people who play either of the games = 34 + 5 + 6 + 26 + 20 = 91
Which shows $n(C \cup H \cup F)$

(v) Number of people who do not play any game = 12
Which shows $n(U - C \cup H \cup F)$

Total number of people = n(U) = 34 + 5 + 6 + 26 + 20 + 12 = 103

Application of Set Theory

Applications of set theory are most frequently used in science and mathematics fields like biology, chemistry, and physics as well as in computer and electrical engineering. These applications range from forming logical foundations for all branches of mathematics. Therefore, understanding set theory is crucial for learning many subjects.

Following formulae are helpful in the set theory.

(i) For any two overlapping sets A and B:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A \cup B) - n(B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

Key Fact

For any sets A and B:

$A \cup A = A, \quad A \cap A = A, \quad A \cap \emptyset = \emptyset$

$A \cup \emptyset = A, \quad A \cap B \subseteq A, \quad A \subseteq A \cup B$

(ii) For any two sets A and B that are disjoint:

- $n(A \cup B) = n(A) + n(B)$
- $n(A - B) = n(A)$

(iii) For any three sets A, B and C:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Example 13:

In a class of 80 students, 40 like English, 34 like Mathematics and 9 like both. How many students like either of both subjects and how many like neither?

Solution:

Total number of students = n(T) = 80

Number of students that like English = n(E) = 40

Number of students that like Mathematics = n(M) = 34

Therefore, total number of students that like both subjects is:

$$n(E \cup M) = n(E) + n(M) - n(E \cap M)$$
$$= 40 + 34 - 9 = 65$$

Number of students that do not like both subjects is:

$$n(T) - n(E \cup M) = 80 - 65 = 15$$

EXERCISE 3.2

1 Let A and B be two finite sets such that $n(A) = 24$, $n(B) = 18$ and $n(A \cup B) = 31$. Find $n(A \cap B)$.

2. If $n(A - B) = 23$, $n(A \cup B) = 44$ and $n(A \cap B) = 2$, then find $n(B - A)$. Also find $n(B)$. (Hint: $n(B) = n(A \cap B) + n(B - A)$)

3. In a group of 30 Mathematics students, 20 like Algebra and 15 like both Geometry and Algebra. Show the data by Venn diagram. Also find how many students like Geometry.

4. In a street with 50 houses, 25 houses have lawns, 32 houses have car porch and 15 houses have both lawn and car porch. Show the data by Venn diagram. Also find how many houses have neither lawn nor porch.

5. In a survey of 940 children, 400 students were found studying at primary level, 240 students at elementary and 175 at secondary level. Create a Venn diagram to illustrate this information. How many children were found out of school?

6. ABC Dairy polls its customers on their favorite flavor: chocolate, vanilla or mango? 100 customers said they like mango flavor, 90 customers said they like vanilla, 40 polled for chocolate, 20 customers liked both mango and vanilla while 14 liked both chocolate and vanilla. How many customers said they like:
(i) only mango? (ii) only vanilla (iii) only chocolate

7. In a survey of university 200 students were interviewed. It was found that:
42 students have laptops, 80 students have cell phones, 100 students have iPods,
23 students have both a laptop and a cell phone, 10 students have both a laptop and iPod,
14 students have both a cell phone and iPod and 8 students have all three items.
(a) How many students have only cell phone?
(b) How many students have none of the three items?
(c) How many students have both iPod and laptop but not cellphone?

8. In a girl college, every student plays either badminton or table tennis or both. If 350 students play badminton, 280 play table tennis and 150 play both. Find how many students are there in the college?

9. Among 50 students, 8 are learning both English and Chinese languages. A total of 26 students are learning English. If every student is learning at least one language, how many students are learning Chinese?

10. Out of 70 people, 48 like tea and 40 like coffee and each person likes at least one of the two drinks. How many like both tea and coffee?

11. There are 46 students in science group and 50 students in arts group. Find the number of students who are either in science or arts group.

12. In a group of people, 52 people can speak Arabic and 112 can speak French. How many can speak Arabic only? How many can speak French only if 12 of them can speak both languages? How many people were in the group?

13. In a high school, 360 students like reading story books, 170 like practical activities and 150 like both. Find
   (i) The number of students who like reading story books only.
   (ii) The number of students who like only practical activities.
   (iii) The total number of students in the school.

14. In a survey of 60 people, it was found that 25 people watch channel A, 16 watch channel B, 13 watch channel C, 4 watch both A and B, 7 watch both B and C, 8 watch both A and C, 3 watch all three channels. Find the number of people who watch at least one of the channels? Also find number of people who do not watch these channels.

Binary Relations

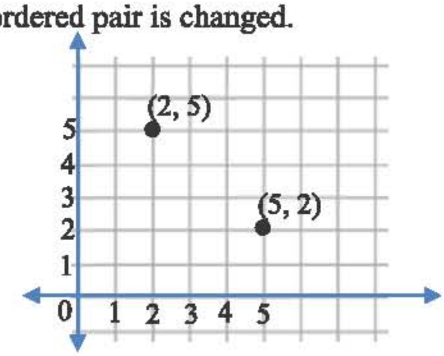
Ordered Pair

Pairs of two numbers in which order of numbers is not invertible, is called an ordered pair. The numbers in an ordered pair are written within small brackets (parenthesis) and are separated by comma.

For example, (a, b) is an ordered pair in which a is called first element and b is called second element. By interchanging the positions of elements, the ordered pair is changed.

As in geometry, position of a point is determined by ordered pair, therefore (2, 5) and (5, 2) represent two different points.

Thus, $(2, 5) \neq (5, 2)$



Equality of Two Ordered Pairs

Two ordered pairs (a, b) and (c, d) are equal if:

$$a = b \text{ and } c = d$$

Key Fact
- $(a, b) \neq (b, a)$
- $(a, b) = (c, d)$
  $\Leftrightarrow a = c \;\&\; b = d$

Example 15:

Find the values of x and y when $(x - 3y, 5x + 1) = (4, 6)$

Solution: By the equality of ordered pairs, we have:

$x - 3y = 4$ (i)

$5x + 1 = 6$ (ii)

From equation (ii), $5x = 6 - 1$

$5x = 5 \quad \Rightarrow \quad x = 1$

Substituting $x = 1$ in equation (i), we get:

$$1 - 3y = 4 \Rightarrow -3y = 3$$
$$\Rightarrow y = -1$$

Check Point

Find a and b when:
$(a + 1, 4) = (2, b - 3)$

Cartesian Product of Sets

If A and B are two non-empty sets, then Cartesian product $A \times B$ is the set of ordered pairs (x, y) such that $x \in A$ and $y \in B$. Mathematically:

$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

Key Fact

- $n(A \times B) = n(A) \times n(B)$
- $A \times B = \phi$ if either $A = \phi$ or $B = \phi$

Similarly, $B \times A = \{(y, x) \mid y \in B \wedge x \in A\}$

e.g. If $A = \{0, 1, 2\}$, $B = \{3, 4\}$

then $A \times B = \{0, 1, 2\} \times \{3, 4\}$

$= \{(0, 3), (0, 4), (1, 3), (1, 4), (2, 3), (2, 4)\}$

$A \times B$ can also be represented through table as follows:

A × B	0	1	2
3	(0, 3)	(1, 3)	(2, 3)
4	(0, 4)	(1, 4)	(2, 4)

In the same way, we can find $B \times A$ as:

$B \times A = \{3, 4\} \times \{0, 1, 2\}$

$= \{(3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2)\}$

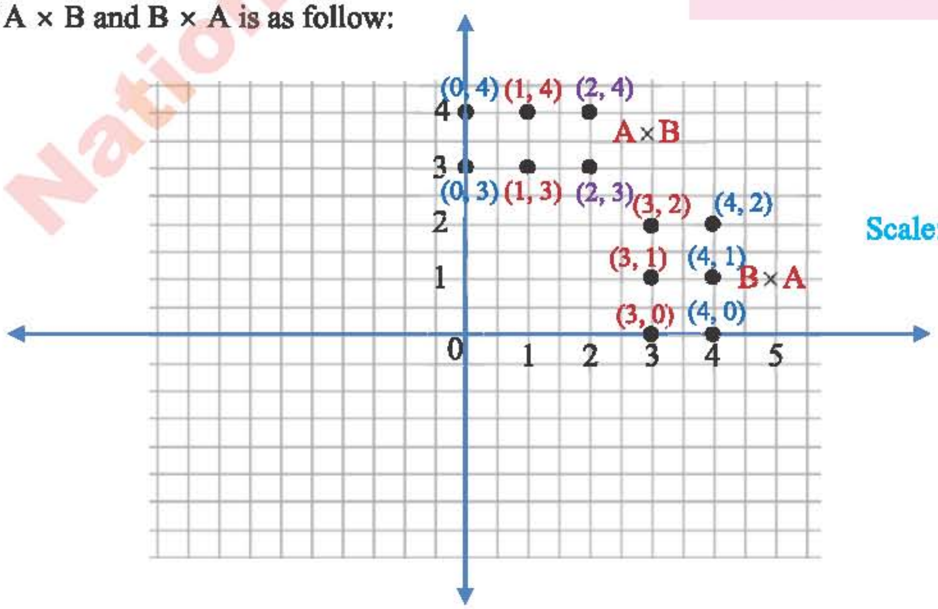
Key Fact

- Each element of the set $A \times B$ is called an ordered pair.
- The ordered pair $(a, 1)$ cannot be written as $(1, a)$.
- The number of subsets of $A \times B = 2^{n(A \times B)}$.

Table for $B \times A$ is:

B × A	3	4
0	(3, 0)	(4, 0)
1	(3, 1)	(4, 1)
2	(3, 2)	(4, 2)

Graph of $A \times B$ and $B \times A$ is as follow:



Scale: 2 squares = 1

From the definition, table and graph, we see that:

$$A \times B \neq B \times A$$

There are many subsets of $A \times B$

e.g. $R_1 = \phi$, $R_2 = \{(0, 3)\}$, $R_3 = \{(1, 3), (0, 4)\}, \ldots$

Key Fact
- In general $A \times B \neq B \times A$
- $A \times B = B \times A$ if and only if $A = B$.
- $n(A \times B) = n(B \times A)$

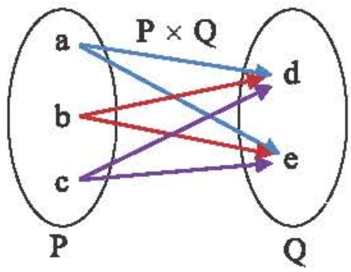
Example 16:

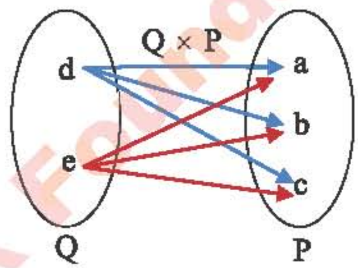
Exhibit $P \times Q$ and $Q \times P$ by arrow diagram when $P = \{a, b, c\}$ and $Q = \{d, e\}$.

Solution:

$P \times Q = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$

$Q \times P = \{(d, a), (d, b), (d, c), (e, a), (e, b), (e, c)\}$





EXERCISE 3.3

1. Find the values of unknowns when:
   (i) $(a, -b) = (7, 1)$ (ii) $(2a, 2b + 3) = (-10, -b)$ (iii) $(2a - 4, 6) = (8, -b + 1)$
   (iv) $(x + 2y, y - 3) = (2, 5)$ (v) $(2x - y, y - 3x) = (4, 2)$
   (vi) $(4x + 6y, x - 12y) = (6, -3)$ (vii) $(5x + y, -x + y) = (6, 1)$
2. Let $A = \{1, 4, 8\}$ and $B = \{1, 0\}$. Find:
   (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$
   How many elements are there in $A \times B$, $B \times A$, $A \times A$ and $B \times B$?
3. Let $E = \{1, 3\}$ and $F = \{4, 6, 8\}$. Express $E \times F$, $F \times E$, $E \times E$, $F \times F$ graphically.
4. If $L \times M = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$, then find sets L, M and $M \times L$.
5. Given that $A = \{1, 3, 5\}$, $B = \{2, 4\}$, $C = \{6, 7\}$.
   (i) Find $A \times (B \cup C)$ (ii) Find $(A \times B) \cup (A \times C)$
   (iii) Verify $A \times (B \cup C) = (A \times B) \cup A \times C)$
6. Given that $D = \{a, e, i\}$, $E = \{a, c\}$, $F = \{b, c\}$.
   (i) Find $D \times (E \cap F)$ (ii) Find $(D \times E) \cap (D \times F)$
   (iii) Verify $D \times (E \cap F) = (D \times E) \cap (D \times F)$

7. Given that $A = \{x / x \in N, x < 3\}$, $B = \{y / y \in W, y < 2\}$, $C = \{0, 2, 4\}$.
   (i) Verify $A \times (B - C) = (A \times B) - (A \times C)$
   (ii) Verify $(A - B) \times C = (A \times C) - (B \times C)$
8. Let $X = \{x / x \in W, x \leq 2\}$ and $Y = \{-1, -2, -3\}$. Exhibit $X \times Y$ and $Y \times X$ by arrow diagram.

Binary Relation

A binary relation R in the set $A \times B$ is a subset of the Cartesian product $A \times B$.
Symbolically R is the relation in a set $A \times B$ if and only if $R \subseteq A \times B$.
If R is elation from A to B, then:

$$R = \{(a, b) / a \in A, b \in B\}$$

A binary relation can also be taken from only one set after taking Cartesian product of the set with itself e.g. from $A \times A$
If R_1 is elation from A to A, then:

$$R_1 = \{(a, b) / a \in A, b \in A\}$$

Key Fact
- A binary relation is a set of ordered pairs.
- Number of binary relations in $A \times B = 2^{n(A \times B)}$

Example 17:
If $A = \{5, 10, 15, 20, 25\}$ then find the number of binary relations in A.
Solution: Number of elements in $A = n(A) = 5$
Number of elements in $A \times A = n(A \times A) = 5 \times 5 = 25$
Number of binary relations in A (or $A \times A$) $= 2^{25}$

Example 18:

If $P = \{2, 3\}$, $Q = \left\{\frac{1}{2}, \frac{1}{3}\right\}$, then find all possible binary relations in $P \times Q$.

Solution:
Number of elements in $P = n(P) = 2$
Number of elements in $Q = n(Q) = 2$
Number of elements in $P \times Q = n(P \times Q) = n(P) \times n(Q)$
$= 2 \times 2 = 4$
Number of binary relations in $P \times Q = 2^{n(P \times Q)}$
$= 2^{2 \times 2} = 2^4 = 16$

Now $P \times Q = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$. Here,

$R_1 = \phi$, $R_2 = \left\{\left(2, \frac{1}{2}\right)\right\}$, $R_3 = \left\{\left(2, \frac{1}{3}\right)\right\}$, $R_4 = \left\{\left(3, \frac{1}{2}\right)\right\}$, $R_5 = \left\{\left(3, \frac{1}{3}\right)\right\}$,

$R_6 = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right)\right\}$, $R_7 = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{2}\right)\right\}$, $R_8 = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$,

$$R_9=\left\{\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right)\right\},\ R_{10}=\left\{\left(2,\frac{1}{3}\right),\left(3,\frac{1}{3}\right)\right\},\ R_{11}=\left\{\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\},$$

$$R_{12}=\left\{\left(2,\frac{1}{2}\right),\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right)\right\},\ R_{13}=\left\{\left(2,\frac{1}{2}\right),\left(2,\frac{1}{3}\right),\left(3,\frac{1}{3}\right)\right\}$$

$$R_{14}=\left\{\left(2,\frac{1}{2}\right),\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\},\ R_{15}=\left\{\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\}$$

$$R_{16}=\left\{\left(2,\frac{1}{2}\right),\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\}$$

Example 19:

Let A = {1, 2, 3}, B ={0, 1, 3} and R = {(a, b) / $a \in$ A, $b \in$ B} and $a > b$}, then find R and show it by arrow diagram.

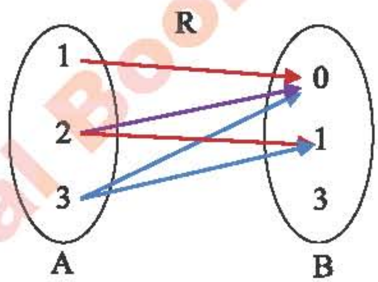
Solution:

A = {1, 2, 3}, B ={0, 1, 3}

A × B = {(1, 0), (1, 1), (1, 3), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 3)}

Now R = {(a, b) / $a \in$ A, $b \in$ B} and $a > b$}

R = {(1, 0), (2, 0), (2, 1), (3, 0), (3, 1)}



Domain of Binary Relation

Set of all the first elements of ordered pairs in a binary relation is called domain of that binary relation.

In example 18, Domain of R_{11} = {3} and Domain of R_{14} = {2, 3}.

Range of Binary Relation

Set of all the second elements of ordered pairs in a binary relation is called range of that binary relation.

In example 18, Range of $R_6=\left\{\frac{1}{2},\frac{1}{3}\right\}$ and Range of $R_7=\left\{\frac{1}{2}\right\}$.

Example 20:

(a) If A = Set of Natural numbers and $R = \{(x, y) \mid x \in A \wedge y \in A\}$ i.e. $R \subseteq A \times A$

Then find the domain and range of R.

(b) If $T = \{0, \pm 1, \pm 2\}$ and $R_1 = \{(x, y) \mid x \in T \wedge y \in T \wedge x + y = 0\}$, then find the Dom R and Range R.

(c) If $E = \{2, 4, 6\}$, $F = \{0, 1, 2\}$ and $R_2 = \{(x, y) \mid x \in E, y \in F \wedge x + y = 6\}$, then

(i) Write $E \times F$ **(ii)** Write R_2 in tabular form **(iii)** Find Dom R_2 and Range R_2.

Solution:

(a) $R = \{(1, 1), (1, 2), (1, 3), \ldots,(2, 1), (2, 2), (2, 3), \ldots(3, 1), (3, 2), (3, 3),\ldots\}$

So, Domain of $R = \{1, 2, 3, \ldots\}$ = Set of Natural numbers

Range of $R = \{1, 2, 3, \ldots\}$ = Set of Natural numbers

$\therefore$ Dom R = Range R = A

(b) $R_1 = \{(0, 0), (-1, 1), (1, -1), (-2, 2), (2, -2)\}$

Dom $R_1 = \{0, \pm 1, \pm 2\}$, Range $R_1 = \{0, \pm 1, \pm 2\}$

(c) $E \times F = \{(2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$

$R_2 = \{(4, 2), (6, 0)\}$

Dom $R_2 = \{4, 6\}$, Range $R_2 = \{0, 2\}$

> **Key Fact**
> $R_1 = \phi$ is called a void relation.

Inverse Relation

Let $R = \{(a, b) / a \in A, b \in B\}$ be a relation from A to B then the inverse of R is defined by:

$R^{-1} = \{(b, a) / b \in B, a \in A\}$

Example 21:

Given that $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 9, 16\}$ and $R = \{(x, y) \mid x \in A \wedge y \in B \wedge x^2 = y\}$.

Verify: Dom R^{-1} = Range R and Range R^{-1} = Dom R

Solution:

Given $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 9, 16\}$ and $R = \{(x, y) / x \in A, y \in B\} \wedge x^2 = y\}$

R in tabular form is:

$R = \{(0, 0), (2, 4), (3, 9)\}$

Dom $R = \{0, 2, 3\}$ and Range $R = \{0, 4, 9\}$

Inverse of R is:

$R^{-1} = \{(0, 0), (4, 2), (9, 3)\}$

Dom $R^{-1} = \{0, 4, 9\}$ and Range $R^{-1} = \{0, 2, 3\}$

Which shows that:

Dom R^{-1} $\{0, 4, 9\}$ = Range R and Range $R^{-1} = \{0, 2, 3\}$ = Dom R

> **Key Fact**
> - Dom R^{-1} = Range R
> - Range R^{-1} = Dom R

EXERCISE 3.4

1. Find the number of binary relations in the following cases.
   (i) $A = \{1, 3\}, B = \{0, 2, 4\}$ (ii) $n(C) = 7$ (iii) $D = \{1, 3, 5\}$
2. Find all possible binary relations in the following cases mentioning the number of binary relations in each case.
   (i) $A = \left\{\sqrt{2}, \sqrt{3}, \sqrt{5}\right\}$, $B = \left\{\sqrt[3]{5}\right\}$ (ii) $C = \{\pi, e\}$ (iii) $D = \{5\}, E = \{1, 10\}$
3. If $P = \{7, 8, 9\}$ then find 2 binary relations from P to P. Also find domain and range of each relation.
4. Let $H = \{5, 6, 7, 8, 9\}$ and $G = \{5, 7, 9, 11\}$. Write the following relations from H to G in tabular form.
   (i) 'is equal to' (ii) 'is less than' (iii) 'is greater than'
   (iv) 'is one less than' (v) 'is one greater than' (vi) 'is two less than'
5. Let $C = \{2, 4, 6\}$, $D = \{4, 6, 8, 9, 12\}$ and $R = \{(x, y) / x \in C, y \in D\} \wedge x$ is factor of $y\}$.
   (i) Write R in tabular form. (ii) Find domain and range of R.
   (iii) Find R^{-1}. (iv) Represent R by arrow diagram.
6. Let $R = \{(2, 0), (4, 2), (6, 4), (8, 6), (10, 8)\}$
   (i) Write R in set builder form. (ii) Find domain and range of R.
   (iii) Write R^{-1} in tabular and set builder from.
   (iv) Represent R and R^{-1} by arrow diagram.
7. Let $A = \{0, 1, 3\}$ and $B = \{1, 2, 3, 5, 7\}$. Write $R = \{(x, y) \mid x \in A, y \in B \wedge y = 2x + 1\}$ in tabular form. Also find R^{-1}.
8. If $S = \{1, 2, 4, 8\}$, $T = \{3^0, 3^1, 3^2\}$, then write the following binary relations in tabular form.
9. Find the domain and range in each case.
   (i) $R_1 = \{(x, y) \mid x \in S, y \in T \wedge x = y\}$ (ii) $R_2 = \{(x, y) \mid x \in S, y \in T \wedge y < x\}$
   (iii) $R_3 = \{(x, y) \mid x \in S, y \in T \wedge x + y \in E\}$ (iv) $R_4 = \{(x, y) \mid x \in S, y \in T \wedge x \times y \in O\}$
   (v) $R_5 = \{(x, y) \mid x \in S, y \in T \wedge y > 2x\}$

KEY POINTS

- If A and B are any two sets, then the set consisting of all the elements of these two sets is called union of these two sets.
- If A and B are any two sets then the set consisting of all the common elements of these two sets is called intersection of these two sets.
- The pictorial representation of any set is called a Venn diagram.
- We can prove associative laws and distributive laws by using Venn diagrams.
- Any subset of the Cartesian product $A \times B$ is a binary relation.

MISCELLANEOUS EXERCISE 3

1. Encircle the correct option.

i. Set builder form of A – B is:

(a) $\{x \mid x \in A\}$ (b) $\{x \mid x \in A \wedge x \notin B\}$ (c) $\{x \mid x \in A \wedge x \in B\}$ (d) $\{x \mid x \in B\}$

ii. If $A \cup B = A$ and $A \cap B = B$, then:

(a) $A \subset B$ (b) $A \not\supseteq B$ (c) $A \supseteq B$ (d) $A \neq B$

iii. If $A \subset B$ then $A - B =$

(a) A (b) B (c) ϕ (d) B – A

iv. If $A - B = B - A = \phi$, then:

(a) A = B (b) $B \subseteq A$ (c) $A \subseteq B$ (d) all a, b & c

v. Set of common elements of A and A^c is _____ set.

(a) infinite (b) null (c) universal (d) singleton

vi. Set of real numbers can be written in:

(a) tabular form. (b) descriptive form.

(c) set builder form. (d) both b and c.

vii. If R = {(2, 1), (4, 3), (2, 2)}, then Dom R =

(a) {2, 4, 2} (b) {2, 4} (c) {1, 3, 2} (d) none of these

viii. If $R = \{(a, b) / a, b \in Z \wedge a + b = 0\}$, then:

(a) Dom R $\subset$ Range R (b) Range R $\subset$ Dom R

(c) Dom R = Range R (d) none of these.

ix. If $R = \{(a, b) / a, b \in N \wedge a \times b = 12\}$ then tabular form of R is:

(a) {(1, 12), (2, 6), (3, 4), (4, 3), (6, 2), (12, 1)} (b) {(1, 12), (2, 6), (3, 4)}

(c) {(12, 1), (6, 2), (4, 3)} (d) {(1, 12), (4, 3), (2, 6)}

x. If n (A) = p, then $n(A \times A) =$

(a) p (b) 2p (c) $2p^2$ (d) p^2

xi. If n (B) = t, then number of binary relations in B × B is:

(a) t (b) t^2 (c) 2^t (d) 2^{t^2}

2. If R = {a, f, h, s} and S = {b, e, j, n}, then

(i) Find the number of binary relations in R × S.

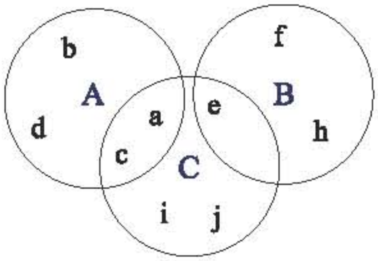
(ii) Write any 3 binary relations from R × S.

(iii) Write a binary relation whose domain is equal to set R.

(iv) Write a binary relation whose range is equal to set S.

(v) Write a binary relation whose domain is equal to set R and range is equal to set S.

3. Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $A - (B \cap C)$ and $B - (A \cup C)$, in the following Venn diagram.



4. Verify associative properties of union and intersection through Venn diagram for
$X = \{2x \mid x \in N \wedge x < 20\}$, Y = Set of first 6 natural multiples of 3, $Z = \{6x \mid x \in W \wedge x < 20\}$

5. Verify distributive law of union over intersection through Venn diagram for the following sets.

$A = \{1, 2, 3, \ldots\}$, $B = \{-1, -2, -3, \ldots\}$, $C = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

6. Verify distributive law of intersection over union through Venn diagram for the following sets.

$P = \{a, b, c, d, e\}$, $Q = \{c, d, e, f\}$, $R = \{g, h, i, j, k\}$

7. Create a Venn diagram to illustrate the following information regrading the subsets A and B in the universal set.
   (i) $n(A) = 60, n(B) = 48, n(A \cap B) = 20, n(U) = 90$
   (ii) $n(A) = 34, n(B) = 52, n(A \cup B) = 60, n(U) = 85$

8. 10 boys participated in a Qiraat competition. Among them, Haani, Zubair and Haider recited in Naafi Qiraat style, Abdullah, Umer, Bilal and Ali recited in Al-Kissai Qiraat style while Hassan, Jaffer and Usman recited in Al-Kufi Qiraat style. Represent boys participated in Naafi, Al-Kissai and Al-Kufi styles by sets A, B and C respectively. Find:
   (i) Find tabular form of A, B and C.
   (ii) Draw Venn diagram of situation.
   (iii) Find $A \cap (B \cap C)^c$, $(A \cup B) \cap C^c$, $A - (B \cap C)$ and $A - (A \cup B)$.

9. 100 candidates were appeared in an examination. Out of which 45 candidates passed in Mathematics, 40 in Science and 50 in Health. If 12 were passed in Mathematics and Science, 15 in Science and Health, 20 in Health and Mathematics and 5 were passed in all three subjects.
   (i) Illustrate the above information by drawing a Venn diagram.
   (ii) How many candidates were passed at least one subject?
   (iii) How many candidates did not pass any subject?