UNIT
02

LOGARITHMS

In this unit the students will be able to:

- Express a number in standard form of scientific notation and vice versa.
- Define logarithm of a number to the base *a*.
- Define a common logarithm, characteristic and mantissa of log of a number.
- Use tables to find the log of a number.
- Give concept of antilog and use tables to find the antilog of a number.
- Differentiate between common and natural logarithm.
- Prove the four basic laws of logarithm.
- Apply laws of logarithm to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

October 8, 2005 is an unforgettable day in the history of Pakistan, when the earth started shaking violently and in few minutes the worst disaster had ruined many of the towns and villages from the face of the earth. This earthquake measured 7.6 on Richter scale but what is a Richter scale? Answer to this question will be explained in this unit.

INTRODUCTION

Exponents provide an efficient way of writing very large as well as very small numbers. For example:
approximate mass of Uranus is 87 trillion trillion kg
i.e. 87 followed by 24 zeros or
87, 000000000000, 000000000000.
This style of expressing a number is called *standard form* which is not useful for such a large number, since some error may occur while writing or telling it. There is another method of writing such numbers, to make them handy.
This method involves integral exponents of 10. In this method the mass of Uranus is $8.7\times10{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$ kg $= 8.7\times10^{25}$ kg. This method is called Scientific notation.

> **Historya Mystery**
>
> Al-Khawarizmi did pioneering work on logarithms and the word **logarithm** is also derived from his name.

2.1 Scientific Notation

> *A number 'c' is in scientific notation if it is written as* $c = d\times10^n$, where $1\le d<10$ and $n\in Z$.

For example: 5.3×10^7, 7.412×10^{-2}, 1.592×10^0.

How to Write in Scientific Notation

- Place the decimal point after first left hand nonzero digit, the resulting number is d. (Position after first left hand nonzero digit is called **reference position**.)
- Count the number of digits moved by the decimal point. This is absolute value of n.
- If decimal point is moved to left, value of n is positive.
- If decimal point is moved to right, the value of n is negative.

e.g. $0.05\,432 = 05.432\times10^{-2}$ or 5.432×10^{-2}

and $5\,43.2 = 5.432\times10^2$

> **Key Fact**
>
> Decimal point is at the right of last digit in an integer.

Example 1: Convert the following into scientific notation:

(a) One light year: 5880,000,000,000 miles

$5\;880000000000.0 = 5.88\times10^{12}$

> The number is greater than 10 so exponent must be positive.

(b) Mass of the smallest insect = 0.00000492 g

$0\,.000004\;92 = 4.92\times10^{-6}$ g

> The number is smaller than 1 so exponent must be negative.

Standard Notation: The number already written in scientific notation, can be converted to standard notation by the multiplication of its two factors.

Example 2: Convert the followings into standard notation.

Exponent is negative so the number is smaller than 1.

Exponent is positive so the number will be greater than 10.

(a) Density of hydrogen = 8.99×10^{-5} g/cm^3

$8.99 \times 10^{-5} = 0\ 00008\,.99 \times 10^{-5} = 0.0000899$

(b) Number of air sacs in lungs = 3.5×10^8

$3.5 \times 10^8 = 3\ .50000000\ 0 \times 10^8 = 350000000.0$ or 350000000

Example 3: The closest star to the Earth (other than Sun) is Alpha Centauri, 4.35 light years from Earth. How many kilometers from Earth is Alpha Centauri?

If one light year = 9460920 million km. Write the answer in scientific notation.

Solution: One light year = 9460920×10^6 km

Distance between Earth and Alpha Centauri

= 4.35 light years = $4.35 \times 9460920 \times 10^6$ km

$= 41155002 \times 10^6 = 4.1155002 \times 10^6 \times 10^7$

$= 4.1155002 \times 10^{13}$ km

Calculator Site

Most of the calculators have a key **E** or **EXP**, For entering a number in scientific notation.

Example 4: The speed of light is approximately 3×10^5 km/s and distance between earth and sun is approximately 1.5×10^8 km. If the sun is suddenly to burn out, how long would it take for earthlings to know about it? Write the answer in standard notation.

Solution: Formula for finding time, if the speed and distance are given, is

Time = Distance/Speed

Here, speed of light = 3×10^5 km/s and distance between Earth and Sun =1.5×10^8km.

$$\text{Time} = \frac{1.5\times10^8\,km}{3\times10^5\,km/s} = 0.5 \times 10^{8-5} \text{ sec} = 0.5 \times {}^3 10 \text{ sec} = 500 \text{ sec or } 8 \text{ min } 20 \text{ sec}$$

EXERCISE 2.1

1. Write the following in scientific notation.
   (i) 0.00053407 (ii) 53400000 (iii) 0.000000000012 (iv) 2.5326
2. Write the following in standard notation.
   (i) 9.067×10^{-5} (ii) 5.64×10^0 (iii) 6.53×10^{-6} (iv) 3.1415×10^9
3. Simplify the following by converting into the form indicated.
   (i) $563.71 \times 10^{-3} \times 2.54 \times 10^4$ ⟶ scientific notation
   (ii) $\dfrac{0.023\times10^5}{10^{-3}}$ ⟶ standard notation
   (iii) $\dfrac{2.549\times5067\times10^{-3}}{10^3}$ ⟶ scientific notation
   (iv) 0.0009988×10^{10} ⟶ standard notation

4. If it takes 5 seconds to recite 'Kalma Pak' once, how many hours will it take to recite 'Kalma Pak' one million times? Convert hours into days and write the answer in standard form. Round off the answer, discarding the decimal part.
5. Distance between Earth and Sun is 9.3225600×10^7 miles. If speed of light is approximately $1.86,000 \times 10^5$ miles per second, how long does it take for light to reach the Earth. Convert the answer in minutes writing in standard form.

2.2 Logarithms

2.2.1 Why We Use Logarithms

Population of the world is growing and the radioactive wastes are decaying continuously. The mathematical tool used to predict the future and explore the past of such rates of growth and decay over time, is an exponential relation.

Key Fact

In late 1500s, John Napier extended the work of Al–Khawarizmi and started developing log tables.

i.e. an equation of the form $x = b^y$ where b, x and y are real numbers, $b > 0$, $x > 0$ and $b \neq 1$. This relation is widely used by archaeologists, scientists and business people. The inverse relation of this exponential relation is called logarithmic relation.

Definition of Logarithm

If $b^y = x$ where $x, y, b \in R$; $b > 0, x > 0$ and $b \neq 1$, then y is the **logarithm** of x with base b, written as $y = \log_b x \Leftrightarrow b^y = x$.

While evaluating logarithms, remember that a logarithm is an exponent, e.g. if $\log_9 81 = 2$, then 2 is the logarithm of 81 with base 9, since 9 raised to power 2 gives 81.

Example 5: Convert the following exponential equations to logarithmic equations and the logarithmic equations to exponential equations.

(a) $2^7 = 128$, here base = 2, exponent = 7 and $x = 128$

exponent

x

$$2^7 = 128 \Leftrightarrow \log_2 128 = 7$$

Base

Enlighten Yourself

Exponential equations are used by
a. Archaeologists, for finding the age of very old bones, fossils etc.
b. Scientists for finding the life time of radioactive elements etc.

(b) $7^{-3} = \dfrac{1}{343} \Leftrightarrow \log_7 \dfrac{1}{343} = -3$

(c) $\sqrt[3]{125} = 5$ or $(125)^{\frac{1}{3}} = 5$

$125^{\frac{1}{3}} = 5 \Leftrightarrow \log_{125} 5 = \dfrac{1}{3}$

(d) $\log_5 625 = 4 \quad \Leftrightarrow \quad 5^4 = 625$

(e) $\log_2 \frac{1}{64} = -6 \quad \Leftrightarrow \quad 2^{-6} = \frac{1}{64}$

(f) $\log_{81} \frac{1}{3} = -\frac{1}{4} \quad \Leftrightarrow \quad (81)^{-\frac{1}{4}} = \frac{1}{3}$

Check Point

$Log_t 5^0 = ?$
$Log_3 (\log_2 2) = ?$

Key Fact

- $\log_b x$ is defined only for positive x.
- $\log_b 1 = 0 \quad \because b^0 = 1$
- $\log_b b = 1 \quad \because b^1 = b$
- $\log_a a^x = x \quad \because a^x = a^x$
- $\log_b x_1 - \log_b x_2 \Rightarrow x_1 = x_2$

Example 6: Check whether these logs are defined or not?

(a) $\log_1 2 = y \Rightarrow 1^y = 2$

None of the exponents of 1 can give answer 2, so $\log_1 2$ is undefined.

(b) $\log_5 (-1) = y \Rightarrow 5^y = -1$

None of the exponents of 5 can give answer '–1', so $\log_5 (-1)$ is undefined i.e. log of negative number is not defined.

(c) $\log_2 0 = y \Rightarrow 2^y = 0$

None of the exponent of 2, can give answer 0, so log of 0 is not defined.

(d) Is $\log_2 (4 - 2x) = y$ true or not for $x = 0, 1, 2$.

$\log_2 (4 - 2x) = y \Rightarrow 2^y = 4 - 2x$

If $x = 0$, then $2^y = 4$ is true for $y = 2$.
If $x = 1$, then $2^y = 2$ is true for $y = 1$.
If $x = 2$, then $2^y = 0$ is not true for any value of y.

Example 7: Find the value of unknowns by converting logarithmic form to exponential form.

(a) $\log_2 x = 4 \Rightarrow 2^4 = x \Rightarrow x = 2 \times 2 \times 2 \times 2 = 16$

(b) $\log_{64} x = -\frac{4}{3} \Rightarrow (64)^{-\frac{4}{3}} = x \Rightarrow x = (4^3)^{-\frac{4}{3}} = 4^{3 \times -\frac{4}{3}} = \frac{1}{4^4} = \frac{1}{256}$

(c) $\log_b \frac{1}{128} = -7 \Rightarrow b^{-7} = \frac{1}{128} = \frac{1}{2^7}$ or $b^{-7} = 2^{-7}$

As exponents are same, so bases must be same i.e $b = 2$

(d) $\log_{27} 3 = y \Rightarrow 27^y = 3$ or $(3^3)^y = 3 \Rightarrow 3^{3y} = 3^1$

As bases are same so exponents must be same. i.e. $3y = 1$ or $y = \frac{1}{3}$

Check Point

Decide which log is defined:

Log 1	Log 0
Log–2	$Log_{-10}2$

Example 8: Find y if $\log_b (y^3 + 1) = \log_b 28$

Solution: $\log_b (y^3 + 1) = \log_b 28$

$\Rightarrow y^3 + 1 = 28$ or $y^3 = 28 - 1 = 27$

$y = \sqrt[3]{27} = 3$

EXERCISE 2.2

1. Check whether $\log_x(7-x)$ is defined for
   (i) $x=0$ (ii) $x=1$ (iii) $x=6$ (iv) $x \geq 7$
2. Convert the form of following equations i.e. from exponential form to logarithmic form and vice versa
   (i) $\log_6 216 = 3$ (ii) $7^4 = 2401$ (iii) $\log_5 x = 5$ (iv) $b^{-\frac{3}{4}} = \frac{1}{27}$
   (v) $125^{\frac{x}{3}} = 25$ (vi) $\log_{10} 10^{12} = y$ (vii) $(256)^{\frac{x}{4}} = \frac{1}{64}$
   (viii) $\log_3(x^3+1) = 2$ (ix) $\log_5(2x-3) = 1$ (x) $2x+1 = 2^3$
3. Find the value of x in the following questions.
   (i) $\log_x 3 = 1$ (ii) $\log_{x+1} 9 = 2$ (iii) $\log_3 81 = x$
   (iv) $\log_2 64 = x+1$ (v) $\log_2 x = 4$ (vi) $\log_2(x^2-1) = 3$
4. Find the unknowns appeared in the question 2.

2.2.2 Common Logarithm

There are two most commonly used bases for logarithm i.e.'10' and 'e ≈ 2.71828'(an irrational number). Base 10 was used by Henry Briggs.

*If $10^y = x$, for $x > 0$, then y is called **common log** of x i.e. $10^y = x \Leftrightarrow \log_{10} x = y$*

These logarithms are also called Briggs's logarithms, denoted by $\log_{10} x$ or simply log x. If none of the base is mentioned with log then it is obviously a common logarithm e.g. $\log_{10} 36$ can be simply written as log 36.

Logarithm of a number = Characteristic + Mantissa

2.2.3 Characteristic

*Integral part of the logarithm is called **characteristic.***

Characteristic is an integer. It is infact the integral power of 10, when the number is written in scientific notation. e.g characteristic of $\log 3.3 \times 10^2$ is '2' and in $\log 5.632 \times 10^{-4}$, characteristic is negative 4. This negative characteristic is usually written as $\bar{4}$.

Characteristic of the log of some number can also be found using **reference position**. In 0.00532, the reference position is between 5 and 3. By counting the number of digits between the decimal point and the reference position we get the numerical value of the characteristic however, the sign is taken negative if the reference position is on right side of the decimal point and it is taken positive otherwise.

Memory Plus

If $\log x_1 = \log x_2 \Rightarrow x_1 = x_2$
- $\log 10^0 = 0$
  $\log 1 = 0$
- $\log 10^1 = 1$
- $\log 10^2 = 2$
- $\log 10^3 = 3$
- $\log 10^{12} = 12$

So log of an integral power of 10 is that integral whole number,
- lso If $1 < x < 10$, then $0 < \log x < 1$.

2.2.4 Mantissa

Decimal part or the fractional part of a logarithm is called mantissa.

Mantissa is always a nonnegative number less than 1, i.e. it can be either zero or positive but never negative. Mantissa is found from the log table.

A small part of log table:

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 13	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	19 17	22 20	25 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 6	9 8	11 11	14 14	16 17	20 19	23 22	26 24

Example 9: Find **(a)** log 156.3 **(b)** log 0.0123

Solution: (a) $\log 156.3 = \log 1.563 \times 10^2$ Convert the number into scientific notation

characteristic = 2

Explanation: Llook at the log table in the extreme left column for the number 15. The next digit in 156.3 is 6 From the top row, look at the digit 6. Move vertically downward from 6 and horizontally rightwards from 15. The number present at the intersection of row of 15 and the column of 6 is 1931. Go ahead horizontally and see the number present at the intersection of row of 15 and column of 3 (in the difference tables) i.e. 8. Add 1931 and 8 to get 1939. Since mantissa is less than 1, so mark the decimal point before first digit so mantissa is '.1939'.

mantissa = .1939 or 0.1939

log 156.3 = characteristics + mantissa = 2 + 0.1939 = 2.1939

(b) $\log 0.0123 = \log 1.230 \times 10^{-2}$

characteristic = –2 or $\bar{2}$

For mantissa, use the log table to see the number present at the intersection of row of 12 and the column of 3 i.e. 0899, as there is no difference table for '0' so mark the decimal point before the first digit i.e. mantissa is .0899.

mantissa = .0899

log 0.0123 = $\bar{2}$ + .0899 = $\bar{2}$.0899 (never write '–2.0899')

Example 10: Find log 1009

Solution:

log 1009 = $\log 1.009 \times 10^3$
characteristic = 3
mantissa = .0038 (≠ .38)
∴ log 1009 = 3.0038

10 | 0 | 9
.0000
+ .0038
.0038

Memory Plus

Number of digits in a whole number = characteristics +1

i.e If characteristic of log of some whole number is 3, then the number of digits in that number will be 3 +1=4

see Example 10 for confirmation.

EXERCISE 2.3

Find the logarithms of following numbers if possible.

1. 5313
2. 4580
3. 9.613
4. 110.9
5. 52.39
6. 0.01207
7. 0.0093
8. 1 Trillion
9. 0 .00004
10. 4
11. 4000
12. 54
13. 1.009
14. 0.1009
15. -4

2.2.5 Antilogarithm

The inverse operation of taking log is called antilog. An antilog is used to cancel the effect of log.

If $\log x = y$ *then x is called the* ***antilog*** *of y* or $x = \text{antilog } y$.

Antilogarithm tables are used for finding it. Before finding antilog, recall that log x consists of two parts, characteristic and mantissa. For finding antilog, mantissa is used for looking in the antilog table. Characteristic is not used in table, however characteristic is used for locating the decimal point in the number obtained from antilog table.

A small part of Antilog of Table:

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

Example 11(a): Find antilog 2.4541

Solution: 2.4541 is obviously log of some number so,

characteristic = 2 (integral part) , mantissa = .4541 (fractional part)

Explanation: Look at the antilog table for .45 in extreme left column and 4 (the digit next to .45) in the top row. The number present at the intersection of row of .45 and column of 4 is 2844. Now go ahead horizontally, the number present at the intersection of row of .45 and difference column of 1 is 1. Add 2844 and 1 to get 2845 . Antilog table is no more needed, however the antilog of 3.4541 is not yet completely found. Locate the reference position in the number obtained from antilog table i.e. $2 \wedge 845$, now characteristic will locate the decimal point . As characteristic is +2, so mark the decimal point moving two digits rightwards from reference position i.e. $2 \wedge 84 . 5$

$\therefore$ antilog 2.4541 = 284.5

(b) If $\log x = \bar{2}.0000$, then find x.

Solution: $\log x = \bar{2}.0000$

$\text{antilog}(\log x) = \text{antilog}(\bar{2}.0000)$ (taking antilog on both sides.)

$x = \text{antilog}(\bar{2}.0000)$

here, characteristic = $\bar{2}$ and mantissa = .0000 (see log table at the back of this book.)

$\therefore\ x = \text{antilog}\,(\bar{2}.0000)$

$= .01000$

Example 12: Find antilog of $\bar{2}.4900$

Solution: antilog $\bar{2}.4900$

characteristic = $\bar{2}$ mantissa = .4900

antilog $(\bar{2}.49)$ = $.03_{\wedge}090 = 0.03090$

Rough work for antilog

.49↓0↓0

3090

EXERCISE 2.4

Find the antilog of following numbers.

1. 2.4324	2. 1.5890	3. 0.2425	4. 3.5636
5. 4	6. 0.0038	7. $\bar{1}.2429$	8. $\bar{2}.9281$
9. $\bar{3}.5219$	10. 0.0000	11. –3	12. 5.9990
13. 2.4900	14. 0.49000	15. 2.34	

2.3 Common and Natural Logarithm

John Napier started developing log tables with base *e*, so the logarithms with base *e* are called natural logarithms or Naperian logarithms, represented by '*ln x*'.

2.3.1

*If $e^y = x$, for positive values of x then y is called **natural log** of x i.e. $e^y = x \Leftrightarrow y = \ln x$.*

Napier spent last 20 years of his life working with log tables of base *e*, which he never finished and died. Henry Briggs, then completed these tables. The difference between common and natural logarithms is depicted below.

Log Type	Representation	Base	Nature of base	Properties
Common (Briggs)	*log x*	10	Rational	$\log 1 = 0$ $\log 10 = 1$ $\log 10^x = x$
Natural (Naperian)	*ln x*	$e \approx 2.71828$	Irrational	$\ln 1 = 0$ $\ln e = 1$ $\ln e^x = x$

2.4 Laws of Logarithms

Laws of logarithms are closely related to the laws of exponents, since logarithms in nature, are exponents. The laws of exponents are given in unit 1. In this section the laws of exponents are used to develop some laws of logarithms, for solving complicated equations involving exponents and logarithms. Also the complicated questions of multiplication, division and root extraction are converted to handy sums of addition and subtraction.

2.4.1 Product Law of Logarithms

For any real numbers m, n and b where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$

Proof: Let $\log_b m = x$ .... (i) and $\log_b n = y$ ...... (ii)

Their respective exponential equations are

$m = b^x$ ...... (iii) and $n = b^y$ ...... (iv)

Now product of equations (iii) and (iv) is

$m \times n = b^x . b^y$

$mn = b^{x+y}$ ← using product rule of exponents

$\Rightarrow \log_b mn = x + y$ ←logarithmic form of above equation

or $\log_b mn = \log_b m + \log_b n$ ← substituting the values of x and y

The Product law of logarithms states that:

Logarithm of a product of two (or more) numbers is equal to the sum of their logarithms, provided that all logarithms are defined.

Key Fact

$\log_{10}(3 \times 2) = \log_{10}3 + \log_{10}2$

Example 13 (a): Use the product rule to expand $\log_e(x^2 y^3 z)$.

Solution $\log_e(x^2 y^3 z) = \log_e x^2 + \log_e y^3 + \log_e z$

(b) Use product rule to combine $ln\, a + ln\sqrt{b} + ln\, c^3$ in a single logarithmic term.

Solution $ln\, a + ln\sqrt{b} + ln\, c^3 = ln(a\sqrt{b}\, c^3)$

2.4.2 The Quotient Law of Logarithm

For any positive numbers m, n and b, where $b \neq 1$.

$\log_b \frac{m}{n} = \log_b m - \log_b n$

Proof: Let $\log_b m = x$ ...... (i) and $\log_b n = y$...... (ii)

Respective exponential forms of above equations are

$m = b^x$ ...... (iii) and $n = b^y$ ...... (iv)

Dividing (iii) by (iv) i.e. $\frac{m}{n} = \frac{b^x}{b^y}$

$\frac{m}{n} = b^{x-y}$ ←quotient rule of exponents with same base

$\Rightarrow \log_b \frac{m}{n} = x - y$ ←conversion to logarithmic form

$\log_b \frac{m}{n} = \log_b m - \log_b n$ ←Substituting the value of x and y

The Quotient law of logarithms states that:

Logarithms of quotient of two numbers is equal to the difference of their logarithms provided that all the logarithms are defined.

Example 14 (a): Use quotient law of logarithms to expand $\log \dfrac{13}{7b}$.

(b) Use quotient law of logarithms to combine log 39 – log t – log b into single logarithmic term.

Solution (a): $\log \dfrac{13}{7b} = \log 13 - (\log 7 + \log b) = \log 13 - \log 7 - \log b$

Solution (b): $\log 39 - \log t - \log b = \log 39 - (\log t + \log b) = \log 39 - \log tb = \log \dfrac{39}{tb}$

Key Fact

$\log(50-5) \neq \log 50 - \log 5$

2.4.3 The Power Law of Logarithm

For any real number m, n and b, where $m > 0$, $b > 0$, $b \neq 1$, $\log_b m^n = n \log_b m$

Proof: Let $\log_b m = x$

Converting into exponential equation i.e $m = b^x$

$m^n = (b^x)^n$ ← taking n^{th} power on both sides.

or $m^n = b^{nx}$ ← using power of power rule.

$\Rightarrow \log_b m^n = nx$ ← converting into logarithmic form.

$\log_b m^n = n \log_b m$ ← substituting the value of x.

The Power Law of logarithms states that:

Logarithm of power of a number is equal to the power times the logarithm of the number, provided that all logarithms are defined.

Example 15(a): Use power law of logarithms to expand $\log_2 100^{-3}$.

Solution: $\log_2 100^{-3} = -3 \log_2 100$

(b) Use power law of logarithms to combine $-\dfrac{4}{3}\log_{\sqrt{3}} 7$

Solution: $-\dfrac{4}{3}\log_{\sqrt{3}} 7 = \log_{\sqrt{3}} 7^{-\frac{4}{3}}$

2.4.4 Change of Base Law of Logarithms

Although only common logarithms and natural logarithm are programmed into a calculator still the logarithms for other positive real bases can be found by changing that base into some frequently used base that is 'e' or '10'. The law which enables us to change the base is called change of base law which is given below.

If a, b and m are positive real numbers and, $a \neq 1$, $b \neq 1$, then

$\log_a m = \log_b m \cdot \log_a b$

Proof: Let $\log_b m = x$

$\Rightarrow \quad m = b^x$ ←converting above equation into exponential equation.

$\log_a m = \log_a b^x$ ←taking log on both sides with base 'a'.

$\log_a m = x.\log_a b$ ←applying power law of logarithm.

$\log_a m = \log_b m.\log_a b$ or $\log_b m = \dfrac{\log_a m}{\log_a b}$

Slide Rule

based upon laws of logarithms were used for complicated calculations, before the invention of calculators.

Example 16: Convert the base of $\log_b 536$ into 10.

Solution: $\log_b 536 = \dfrac{\log_{10} 536}{\log_{10} b}$ or $\dfrac{\log 536}{\log b}$

Key Fact

The change of base law and the quotient law of logarithms are often confused. Remember the difference between $\log_b(\dfrac{m}{n})$ and $\dfrac{\log_a m}{\log_a n}$. These two expressions look alike, however they are totally different.

Example 17(a): Use laws of logarithms to expand $\log \dfrac{5p^2q^{\frac{1}{2}}}{4\sqrt{s}t^3}$.

Solution: $\log \dfrac{5p^2q^{\frac{1}{2}}}{4\sqrt{s}t^3} = \log(5p^2q^{\frac{1}{2}}) - \log(4\sqrt{s}\ t^3)$ ←using quotient law of log

$= \log 5 + \log p^2 + \log q^{\frac{1}{2}} - [\log 4 + \log\sqrt{s} + \log t^3]$ ←using product law of log

$= \log 5 + 2\log p + \dfrac{1}{2}\log q - \log 4 - \dfrac{1}{2}\log s - 3\log t$ ←using power law of log

(b) Use laws of logarithms to evaluate

(i) $\log_2 5\sqrt{3}$ **(ii)** $\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

Solution: **(i)** $\log_2 5\sqrt{3} = \dfrac{\log_{10} 5\sqrt{3}}{\log_{10} 2}$ ← change of base law

$= \dfrac{\left[\log 5 + \log 3^{\frac{1}{2}}\right]}{\log 2} = (\log 5 + \dfrac{1}{2}\log 3) \div \log 2$

$= [0.6990 + \dfrac{1}{2}(0.4771)] \div (0.3010)$

$= \dfrac{0.9376}{3.010} = 3.1148$

(ii) Evaluate $\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

$\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

$$=\log_2\left(\log_2\left(2^3\right)^2-\log_{\sqrt{3}}\left(\left(\sqrt{3}\right)^2\right)^3+\log_{\sqrt{10}}\left(\sqrt{10}\right)^2\right)$$

$$=\log_2\left(\log_2 2^6-\log_{\sqrt{3}}\left(\sqrt{3}\right)^6+\log_{\sqrt{10}}\left(\sqrt{10}\right)^2\right)$$

$$=\log_2\left(6\log_2 2-6\log_{\sqrt{3}}\sqrt{3}+2\log_{\sqrt{10}}\sqrt{10}\right) \qquad (\because \log_b b^x = x)$$

$$=\log_2(6-6+2)$$

$$=\log_2 2 = 1 \qquad (\because \log_b b = 1)$$

c. If $\log_b 2 = 0.3010$, $\log_b 3 = 0.4771$ and $\log_b 5 = 0.6990$, then evaluate $\log_b 0.0036$, applying laws of logarithms.

Math Play Ground

Maths play ground

1. Take students to the play ground.
2. Give each student a strip of paper with a simple logarithms sum written on it.
3. Spread answers of all questions in the play ground and ask students to find their respective answers.
4. Specimen questions may include: (i) log10 = ? (ii) lne = ? etc.

Solution: $\log_b 0.0036 = \log_b\left(\frac{36}{10000}\right)$

$$=\log_b\left(\frac{2^2\times 3^2}{2^4\times 5^4}\right)$$

$$=\log_b\left(\frac{3^2}{2^2\times 5^4}\right)$$

$$=2\log_b 3-2\log_b 2-4\log_b 5$$

$$=2(0.4771)-2(0.3010)-4(0.6990)$$

$$=-2.4438$$

(which is negative number having both characteristic and mantissa as negative.)

Since mantissa can never be negative, to make the mantissa positive check the next positive integer to the magnitude of the answer. The next integer is '+3' as shown on the number line

Now 'add 3 to' and 'subtract 3 from' the answer

i.e. $\log_b 0.0036 = -2.4438 + 3 - 3 = 0.5562 - 3$.

The positive term is mantissa and negative term is characteristic.

So, $\log_b 0.0036 = \bar{3}.5562$

Important results deduced from Laws of Logarithms

(i) $\log_b a \times \log_a b = 1$

(ii) $\log_c a = \dfrac{1}{\log_a c}$

(iii) $\log_b a \,.\, \log_c b = \log_c a$

(iv) $\log_s r^t \times \log_t s^r \times \log_r t^s = rst$

(v) $\log_x z \times \log_y x \times \dfrac{1}{\log_y z} = 1$

(vi) $\log_b a \,.\, \log_c b \times \log_a c = 1$

EXERCISE 2.5

1. Use laws of logarithms to expand the followings.

(i) $\log 9t$ (ii) $\log \frac{59}{s}$ (iii) $\log \frac{(5pq^2)}{(xy^3)}$ (iv) $\log \sqrt{\frac{53.3}{46.4}}$

(v) $\log \left(\frac{5^2 t^5 a^{\frac{1}{3}}}{\sqrt[3]{4.4tb^3}}\right)$ (vi) $\log \sqrt[5]{\frac{7^2 t^3 p}{d^6 b^2}}$ (vii) $\log \left(\frac{\sqrt[3]{5.512}pm^{\frac{1}{2}}}{\sqrt[4]{5.91}a^2 b}\right)^t$

2. Use laws of logarithms to combine the followings into single logarithmic terms.

(i) $3\log x - 5\log y$

(ii) $\frac{1}{2}\log t + \frac{1}{3}\log r - \frac{1}{5}\log s$

(iii) $\frac{1}{7}[\log 57.7 - 3\log 9.24 + 4\log 36.6 - 2\log 23.3]$

(iv) $5\log 6 - 7\log 9.42 + \frac{1}{3}\log t - \frac{1}{2}\log 32.2 + \frac{2}{3}\log a$

(v) $\frac{5}{4}\log 37.74 - \frac{1}{4}\log 53.71 + \frac{1}{4}\log 28.83$

3. Use laws of logarithms to evaluate the followings.

(i) $\log_2 15$ (ii) $\log_9 \sqrt[3]{9}$ (iii) $\log_3 65$ (iv) $\log_{\sqrt{3}} 72.34$ (v) $\log_{\sqrt{7}} 343$

4. If $\log_b 2 = 0.3010$, $\log_b 3 = 0.4771$, $\log_b 5 = 0.6990$
then evaluate the followings with laws of logarithms.

(i) $\log_b \frac{6}{5}$ (ii) $\log_b \frac{100}{9}$ (iii) $\log_b \frac{\sqrt[3]{450}}{\sqrt{27}}$ (iv) $\log_b 0.024$ (v) $\log_b \sqrt[7]{5\frac{2}{5}}$

2.5 Applications of Logarithms

Common logarithms appear in many scientific formulae.

Richter Scale

Opening of this unit depicts the 2005 earthquake in Pakistan, when thousands of people lost their lives. The Richter Scale used for measuring the magnitude (M) of earthquake is a logarithmic scale. If I is intensity of its shock waves and I_0 is a constant then

$$M = \log\left(\frac{I}{I_0}\right)$$

Example 18: An earthquake that occurred in Pakistan in 2005, measured 7.6 on the Richter scale. In 1978, an earthquake in China measured 8.2 on the Richter scale. How many times, the China's earthquake was stronger than the Pakistan's earthquake.

Solution: Let I_1 be the intensity of Pakistan's earthquake and I_2 be the intensity of China's earthquake.

As, $\quad M = \log\left(\frac{I}{I_0}\right)$

Then, $7.6 = \log\left(\frac{I_1}{I_0}\right) = \log I_1 - \log I_0$ .... (i)

$8.2 = \log\left(\frac{I_2}{I_0}\right) = \log I_2 - \log I_0$ .... (ii)

> **Project**
>
> Search more applications of logarithms in Biology, Chemistry and Physics.

Subtracting (i) from (ii)

$\log I_2 - \log I_1 = 8.2 - 7.6$

$\Rightarrow \quad \log\left(\frac{I_2}{I_1}\right) = 0.6$

$\text{antilog}\log\left(\frac{I_2}{I_1}\right) = \text{antilog}\ 0.6$ (taking antilog on both sides)

$\frac{I_2}{I_1} = 3.98 \approx 4$

$\therefore$ China's earthquake is nearly 4 times stronger than Pakistan's earthquake.

Example 19: Use laws of logarithms to evaluate.

(a) $\frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}}$ **(b)** $\sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$

Solution: **(a)** let $x = \frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}}$

Then, by taking common log on both sides.

$\log x = \log \frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}}$

$= \log(8.59)^{\frac{1}{3}} + \log(55.6)^2 - \log(2.51) - \log(2.12)^{\frac{1}{2}}$

$= \frac{1}{3}\log 8.59 + 2\log 55.6 - \log 2.51 - \frac{1}{2}\log(2.12)$

$\approx \frac{1}{3}(0.9340) + 2(1.7451) - (0.3997) - \frac{1}{2}(0.3263)$

$\approx 0.3113 + 3.4902 - 0.3997 - 0.1632$

$\log x \approx 3.2386$

$\text{antilog}\log x \approx \text{antilog}\ 3.2386$ ← Taking antilog on both sides

$x \approx 1732.21$

$\therefore \frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}} \approx 1732.21$

(b) let $x = \sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$

$\log x = \log \sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$ ← Taking common log on both sides

$$\log x = \log \left(\frac{16}{3}\right)^{\frac{1}{3}} - \log \left(\frac{22}{7}\right)^{\frac{1}{2}} = \frac{1}{3}\log\left(\frac{16}{3}\right) - \frac{1}{2}\log\left(\frac{22}{7}\right)$$

$$= \frac{1}{3}(\log 16 - \log 3) - \frac{1}{2}(\log 22 - \log 7)$$

$$= \frac{1}{3}\log 16 - \frac{1}{3}\log 3 - \frac{1}{2}\log 22 + \frac{1}{2}\log 7$$

$$\approx \frac{1}{3}(1.2041) - \frac{1}{3}(0.4771) - \frac{1}{2}(1.3424) + \frac{1}{2}(0.8451)$$

$\log x \approx 0.4014 - 0.1590 - 0.6712 + 0.4226$

$\log x \approx -0.0062$

$\log x \approx (-0.0062 + 1) - 1$ ← making mantissa positive

$\approx 0.9938 - 1$

$\log x \approx \bar{1}.9938$

Taking antilog on both sides.

antilog $\log x \approx$ antilog $\bar{1}.9938$

$x \approx 0.9858$

Example 20: Find the number of digits in 5^{50}.

Solution:

By finding the log of the given whole number, we can find the relevant characteristic. But the number of digits in a whole number is always one more than the characteristic of the log of that number.

Now $\log 5^{50} = 50 \log 5$

$= 50 \times (.6990) = 34.95$

Characteristic = 34

Number of digits in 5^{50} = characteristic + 1 = 34 + 1 = 35

EXERCISE 2.6

1. Find the number of digits in
   (i) 3^{30} (ii) 100^{100} (iii) 2^{10} (iv) 5^{37} (v) 529^{30} (vi) 23^{15}
2. Evaluate applying laws of logarithms.
   (i) 23.57×5.967 (ii) $\dfrac{65.89}{7.392}$ (iii) $\dfrac{47.27 \times 5.321}{9.712 \times 4.171}$ (iv) $\dfrac{\sqrt[3]{27.98}}{\sqrt[2]{28.73}}$
   (v) $\dfrac{\sqrt[7]{129.4}}{\sqrt[3]{27.37}}$ (vi) $\dfrac{\sqrt{39.24} \times \sqrt[3]{1.931}}{\sqrt[4]{64.4} \times \sqrt{23.91}}$ (vii) $\dfrac{\sqrt{16\frac{3}{4}}}{\sqrt[3]{53}}$
   (viii) $\dfrac{(27.98)^2}{(28.73)^3}$
3. The Kansu, China earthquake of 1920 was measured about 8.5 on Richter Scale and the Tokyo, Japan earthquake of 1923 was measured 7.8 on that scale how many times stronger was the 1920 earthquake than 1923 earthquake?

KEY POINTS

- A number written as $d \times 10^n$,(where $1 \le d < 10$, $n \in Z$) is said to be in Scientific notation.
- Reference position is the place after first left hand non–zero digit.
- For $x, y, b \in R$; $b > 0, x > 0$ and $b \neq 1$, y is called logarithm of x with base b written as
  $$y = \log_b x$$
- Logarithm of a number consists of two parts,
  - Characteristic: The integral part of logarithm.
  - Mantissa: The fractional part of logarithm, which is never negative.
- If $\log x = y$ then x is called the antilog of y.
- $y = \log_{10} x$, is called common logarithm and $y = \log_e x$ is called natural logarithm.
- $\log_b mn = \log_b m + \log_b n$
- $\log_b \dfrac{m}{n} = \log_b m - \log_b n$
- $\log_b m^n = n \log_b m$
- $\log_n m = \dfrac{\log_b m}{\log_b n}$

MISCELLANEOUS EXERCISE 2

1. Encircle the correct option in the following.

(i) If $a = b \times 10^n$ is written in scientific notation then ,

(a) $0 \le b \le 10$ **(b)** $0 \le b < 10$ **(c)** $1 \le b \le 10$ **(d)** $1 \le b < 10$

(ii) In 0.537, reference position is

(a) after 0 **(b)** after 7 **(c)** after 5 **(d)** before 7

(iii) $\log_1 100$ is

(a) 2 **(b)** –2 **(c)** 0 **(d)** impossible

(iv) If $\log(x+3) = \log(15x-4)$ then x is

(a) 0.5 **(b)** 7 **(c)** 14 **(d)** 2

(v) $\log_7 7^{-3} + \log_2 4^3$ is

(a) 3 **(b)** –3 **(c)** 0 **(d)** ±3

(vi) For the log 0.00327, characteristic is

(a) –2 **(b)** –3 **(c)** 3 **(d)** 0

(vii) $\log_b(M+N)$ is

(a) $\log_b MN$ **(b)** $\log_b M + \log_b N$ **(c)** both a and b **(d)** none of these

(viii) $\log_b g^h$ is

(a) $g \log_b h$ **(b)** $\log_b (gh)$ **(c)** $(\log_b g) \times h$ **(d)** $h \log_g b$

(ix) $\log_b M - \log_b N$ is

(a) $\dfrac{\log_b M}{\log_b N}$ **(b)** $\log_b \dfrac{M}{N}$ **(c)** $\log_N M$ **(d)** $\dfrac{\log_b N}{\log_b M}$

(x) $\log_{\sqrt{10}} 100^2$ is

(a) 2 **(b)** 1 **(c)** 4 **(d)** 8

(xi) log18 is

(a) 3 log2 + log3 **(b)** log 2 + 2log3 **(c)** 3log 3 + 2log2 **(d)** 2 log 3 + 3 log 2

(xii) log 5 – log 8 + log 3 – log 2 is

(a) $\log\dfrac{5\times2}{8\times3}$ **(b)** $\log\dfrac{15}{16}$ **(c)** $\log\dfrac{30}{8}$ **(d)** log –2

(xiii) $\log_{10} 100^0$ is

(a) 2 **(b)** 0 **(c)** 1 **(d)** impossible

(xiv) Scientific notation of 6.25 is

(a) 6.25×10^1 (b) 6.25×10^0 (c) 6.25×10 (d) 0.625×10^2

(xv) Base of natural logarithm is

(a) rational number (b) integer (c) irrational number (d) 10

(xvi) If $\log_{\sqrt{x}} 25 = 4$ then x is

(a) +5 (b) –5 (c) ±5 (d) impossible

(xvii) $\log_{\sqrt{b}} 10^4 \div \log_{\sqrt{b}} 10$ is

(a) $\log_{\sqrt{b}} \frac{10^4}{10}$ (b) $\log_{\sqrt{b}} 10^4 - \log_{\sqrt{b}} 10$ (c) 4 (d) $\log_{\sqrt{b}} (10^4 - 10)$

(xviii) $5 \log 2 - 2 \log 5$ is

(a) $\frac{(\log 2)^5}{(\log 5)^2}$ (b) $\frac{\log 2^5}{\log 5^2}$ (c) $\log \frac{2^5}{5^2}$ (d) $\frac{5}{2} \log \frac{2}{5}$

2. Convert the following into scientific notation.

(i) 53.36 (ii) 0.0000000000000102 (iii) 523.4×10^{-3}

3. Convert the following into standard notation.

(i) 7.232×10^{-2} (ii) $10.53 \times 10^2 \times 20.31$ (iii) 5.6×10^0

4. Evaluate the following.

(i) $\log_5 5^3 - \log_2 2^3$ (ii) $\log_2 4 - \log_3 1$ (iii) $\log_8 (\log_x x - \log_b b^{-7})$

5. Find x if

(i) $\log_3 9 - \log_b 1 = x$ (ii) $\log_2 x - \log_2 16^{1/4} = 3$

(iii) $\log_2 (x^2 - 1) = \log_2 3$ (iv) $\log_7 x = \log_7 (8 \log_y y)$

6. If $\log_b 2 = 0.3010, \log_b 3 = 0.4771$ & $\log_b 5 = 0.6990$, then evaluate the following by applying laws of logarithms.

(i) $\log_b 30$ (ii) $\log_b 0.24$ (iii) $\log_b 360$

7. Simplify with the help of laws of logarithm $\left(\frac{0.5327\times\sqrt[3]{42.97}}{0.0059}\right)^3$.

8. Prove that:

$$\log_v U \times \log_w V \times \log_u W = 1$$