Based on National
Curriculum of Pakistan 2022-23

Model Textbook of

MATHEMATICS

Grade-9

National Book Foundation
as
Federal Textbook Board
Islamabad

Based on National Curriculum of Pakistan 2022-23

Model Textbook of Mathematics

Science Group

Grade

09

National Curriculum Council
Ministry of Federal Education and Professional Training

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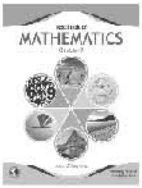
Model Textbook of **Mathematics**
for Grade 9



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Preface

This Model Textbook for Mathematics grade 9 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building upon the foundation of learning from the previous grades. A key emphasis of the present textbook is on creating real life linkages of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they go up the learning curve and for them to fully grasp the conceptual basis that will be built upon in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its books. The present book features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement and the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this book.

May Allah guide and help us (Ameen).

Dr. Raja Mazhar Hameed
Managing Director

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

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UNIT
01

REAL NUMBERS

In this unit the students will be able to:

- Recall the history of numbers.
- Recall the set of real numbers as a union of sets of rational and irrational numbers.
- Depict real numbers on the number line.
- Demonstrate a number with terminating and non-terminating recurring decimals on the number line.
- Give decimal representation of rational and irrational numbers.
- Know the properties of real numbers.
- Explain the concept of radicals and radicands.
- Diferentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.

A number is an abstract idea used in *counting* and *measuring*. A symbol which represents a number is called a numeral, but in common usage the word number is used for both the idea and the symbol. In addition to their use in counting and measuring, numerals are often used for labels (telephone numbers), for ordering (serial numbers), and for codes (ISBN, i.e. International Standard Book Number). In mathematics, the definition of number has been extended over the years to include such numbers as zero, negative numbers, rational numbers, irrational numbers, real numbers and complex numbers.

1.1 History of Real Numbers

Can you imagine a world without numbers?

In our daily conversations, our domestic activities and at our jobs, we cannot spend a single day without using numbers. Our lives will be quite strange without involvement of numbers. In this way one can imagine about the life of a person in the era when numbers were not discovered.

There is an interesting story regarding how humans started using numbers for the first time. The story is about a shepherd boy who used pebbles to count his goats /sheep he sent for grazing each day to avoid missing any. The number discovered in that way were simply the counting numbers which are now called **Natural numbers**. People usually used their fingers or sticks to count objects and were symbolized by tally marks. They counted objects by carving tally marks into cave walls, bones, woods or stones about 30000 BC. In fact, the earliest numerals recorded so far were simple marks for small numbers and special symbol for 10. These symbols appeared in early Egyptians inscriptions around 3500 BC to 3000 BC.

Sumerians Contribution to Numeral System

Some historians believe that Sumerians were first to use symbols for numerals

Around 5000 BC. Sumerian was a Great civilization settled at the Fertile Crescent area near present Iraq. They had a great contribution to the development of number system and basic Mathematics as they are considered to be very advanced in many fields at that era. e.g.

Interesting Information

The first record of existence of tally marks is on a leg bone of a baboon dating prior to 30000 BC. The bone has 29 clear notches in a row. It was discovered in South Africa.

- They were first to construct buildings.
- They initiated modern agriculture, so they were keen about fundamental calculations.
- They depended on rise and fall of sun to estimate time, which showed that they were keen observers of angles and geometry.
- They used cuneiforms on clay tablets.

Babylonians Contribution to Numeral System

Babylonians system of numerals based on Earlier Sumerian numeral system basically, as they were descendants of Sumerian. They relied upon a series of cuneiform marks to represent digits. This was a sexagesimal system of numbers (system with base 60). This concept is still in use today as in division of time we use 60 minutes, 60 seconds etc.

Although they carried complicated algebraic calculations and knew about the concept of nothingness but they didn’t symbolize it ever rather they used a space between digits to represent zero. It made their numbers and calculations quite ambiguous. Around Babylon, clay was abundant so they made a lot of use of clay tablets impression with cuneiforms. The cuneiforms and numerals occur together in some documents from about 3000 BC. They seem to have some convention regarding the use. Cuneiform was always used for wages due while wages paid were written in curvilinear.

Greek Contribution to Numeral System

The early Greeks, like their predecessors Egyptians and Babylonians, also repeated units to 9 and probably had “ – ” symbols for ten and “ O ” for 100. The Greek system of abbreviation in numbers called Attick numerals is present in the record of 5th century BC but probably was used much earlier. Like Babylonians and Romans, the ancient Greeks knew about nothingness but did not symbolize the concept.

Researchers had discovered that many other civilizations developed positional notation of numbers independently including the Ancient Chinese and Aztecs.

Romans Contribution to Numeral System

Romans used tally marks on tally sticks or tally bones. Like early humans they also used V to represent five and X for ten. Ancient Romans incorporated these symbols into their seven symbol system. The Roman empire was very vast and this system of numerals was used thus widely throughout Europe from early 2000 years ago to late middle ages. Like the Babylonians, the Ancient Roman Numeral system lacked to symbolize nothingness. This system was maintained for nearly 2000 years in commerce and scientific literature.

i.

Egyptian hieroglyphic numerals, 3300 B.C.

ii.

Babylonian cuneiform numerals, 2000 B.C.

iii.

Early Greek numerals, 5 B.C.

iv.

1	5	10	50	100	500	1000
I	V	X	L	C	D	M

Roman numerals, 100 A.D.

v.

Mayan numerals, 300 A.D.

vi.

0 1 2 3 4 5 6 7 8 9

Hindu-Arabic digits, present day

Discovery of Zero (the sooper Hero)

Historians believed that Mayans living in central America used the idea of zero in their calendar system but they were isolated from other world so it couldnot caoe in outer world. Some historians give the crown of symbolizing nothingness to the Indian mathematician and astronomer Brahmagupta in 628 AD. It is also narrated that a great Muslim Mathematician Abu Muhammad Musa al Khwarizmi, who was also an astronomer and a geographer, contributed much to our modern understanding of maths. He described a number system based on 10 numerals from zero to nine in 7th century in his book 'The use of the Hindu Numerals'(كتاب فی استعمال الاعداد الهندی) . He called this new digit as 'Sifr'. This useful system was soon adopted by Arabs.

Zero in the Europe

Fibonacci, the son of an Italian merchant discovered that Arab traders were using accounting system based on 0-9 numerals. He quickly understood its importance and improved book keeping and accounting in Europe. In 1202, he published a book describing this number system. He elaborated in the book about practical application i.e. how to convert one currency into other, calculation of profit and losses and other important business needs. In Italy 'Sifr' became 'zefero' which later became zero.

Discovery of zero brought a new set of numbers called set of whole numbers and it reduced the hurdles in calculation and understanding numbers.

Negative Numbers

The Chinese Mathematician Diophantus was most probably the first who used negative numbers around 200 BC in his work. He represented the amount of debt or loss. Then in 7th century the Indian Brahmagupta wrote rules for adding, subtracting, multiplying and dividing negative numbers. The discovery of negative numbers gave existence to the set of Integers.

Rational Numbers

Pythagoras the Greek mathematician used fractions for the first time in mathematics around 500BC, which was infact the discovery of rational numbers. The word rational came from ratio.

Irrational Numbers

Soon after the discovery of rational numbers by Pythagoras, one of his early follower, Hippasus of Metapontum was working to find the hypotenuse of a right isosceles triangle with 2 equal sides of length 1 unit. He came with a strange answer ($\sqrt{2}$) and concluded that the answer is not reasonable number because it cannot be written in a/b form. He called such numbers as irrational (meaning stupid, nonsense or not reasonable). It is narrated that Hippasus was drowned in the sea for his new concept of numbers as it was against their self made religion.

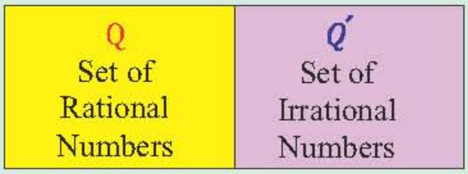
Real numbers

The set of real numbers was thus completed with the discovery of rational and irrational numbers. We know today that the set of reals contain both and there is no such real number which is neither rational nor irrational.

1.2 Introduction to Real Numbers

Earlier mathematicians particularly Richard Dedekind have precisely defined the concept of real numbers which include both rational numbers such as $\frac{2}{9}$ and irrational numbers such as $\sqrt{2}$.

The real numbers are used to solve real life problems such as finding velocity, speed or distance, the profit or loss of a business, the difference in stock market etc.

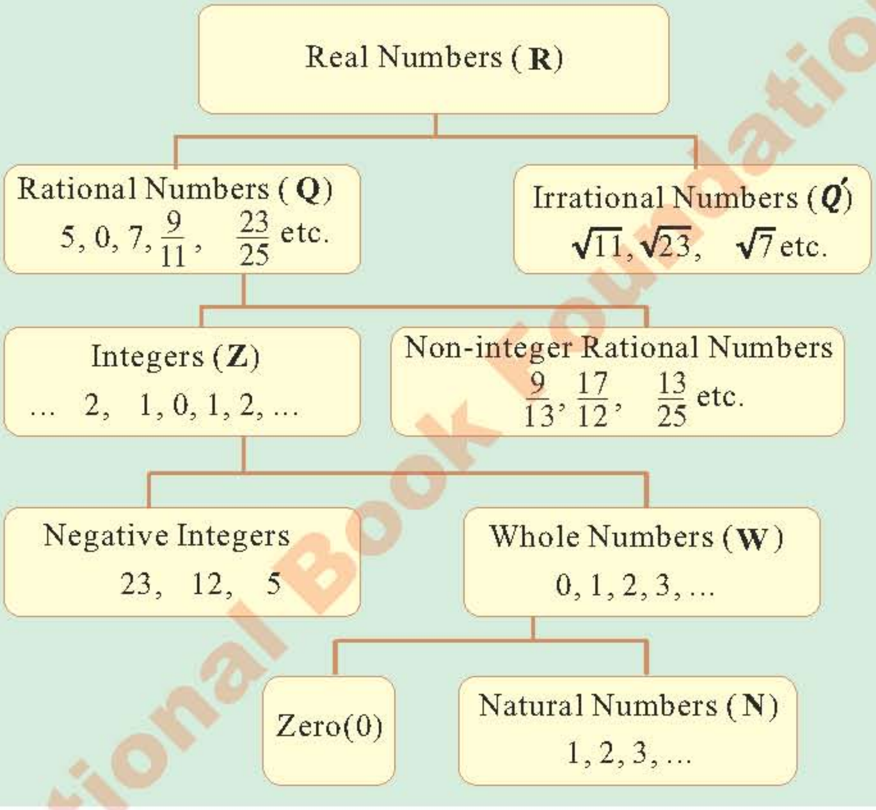


$$R = Q \cup Q'$$

$$Q \cap Q' = \phi.$$

Key Fact

Set of rational numbers (Q) and set of irrational numbers (Q') are disjoint i.e., $Q \cap Q' = \phi$. But $R = Q \cup Q'$, then Q and Q' are called exhaustive sets.



1.2.1 Number Line

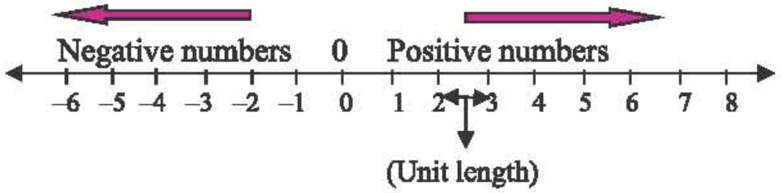
We use a number line to visualize real numbers and their relation to each other. To construct a number line, we choose a point corresponding to the number '0'. Points at equally spaced intervals are then associated with the integers. The positive integers are to the right of '0' and negative integers are to the left of 0. All other real numbers are associated with a point which is called the ***coordinate* of that point**. The point associated with zero is called the *origin*.

Enlighten Yourself

Number line is also called a real line because we express real numbers on it.

Usually few integers are shown on a number line to indicate the unit length of the line, that is, the distance between any two consecutive integers.

A number line is shown in figure below.



1.3 Properties of Real Numbers

The basic properties of real numbers are w.r.t addition and multiplication. In this section, some of the properties of these operations are reviewed. The following results are true for any real numbers a, b and c.

Name of the property	With respect to		Examples	
	+	×	+	×
Closure	$a + b \in R$	$a \bullet b \in R$	$6+4 = 10 \in R$	$6×4 = 24 \in R$
Commutative	$a + b = b + a$	$a \bullet b = b \bullet a$	$4+7 = 7+4$ $= 11$	$4×7 = 7×4$ $= 28$
Associative	$a+(b+c) = (a+b)+c$	$a \bullet (b \bullet c) = (a \bullet b) \bullet c$	$4+(6+8) = (4+6)+8$ $= 18$	$4×(6×8) = (4×6)×8$ $= 192$
Identity	$a + 0 = a = 0 + a$	$a \bullet 1 = 1 \bullet a = a$	$6+0 = 0+6$ $= 6$	$6×1 = 1×6$ $= 6$
Inverse	$a+(-a) = -a + a$ $= 0$	$a \bullet \frac{1}{a} = \frac{1}{a} \bullet a = 1$ $a \neq 0$	$14+(-14) = -14+14$ $= 0$	$14×\frac{1}{14} = \frac{1}{14}×14$ $= 1$

Key Fact

i. 0 is the additive identity and 1 is the multiplicative identity of real numbers.

ii. $-a$ is the additive inverse of a and $\frac{1}{a}$ $(a \neq 0)$ is the multiplicative inverse of a.

1.3.1 Distributive Property of Multiplication over Addition

The distributive property involves both operations i.e., addition and multiplication.
Distributive property says, for all real numbers a, b, and c:

$$\boxed{a(b + c) = ab + ac}$$

Example 1: If $a = 5$, $b = 6$ and $c = 9$, then verify that
$a(b + c) = ab + ac$.

Solution:
$$a(b + c) = 5(6 + 9) = 5(15) = 75$$
$$ab + ac = 5(6) + 5(9) = 30 + 45 = 75$$

Thus:

$$a(b + c) = ab + ac$$

Thinking Corner

- Which number has the additive inverse the number itself.
- Do all real numbers have their multiplicative inverses.
- Which number has no multiplicative inverses.

Example 2: Use the distributive property to simplify

$$32\left(\frac{1}{8}+\frac{1}{4}\right).$$

Solution: $32\left(\frac{1}{8}+\frac{1}{4}\right)=32\times\frac{1}{8}+32\times\frac{1}{4}$ $\boxed{a\,(b+c)=ab+ac}$

$$=4+8=12$$

Enlighten Yourself

Distributive property of multiplication over subtraction, $a\,(b-c)=ab-ac$ also holds.

1.3.1 Properties of Equality and Inequality of Real Numbers

Equality and Inequality Symbols

There are three symbols which can be used to show the possible relations between any two real numbers 'a' and 'b'. These are $<$, $=$ and $>$, where '=' is equality symbol and '< and >' are inequality symbols.

Following table shows the use of these symbols.

Read	Write
a is less than b	$a<b$
a is equal to b	$a=b$
a is greater than b	$a>b$

Historya Mystery

The symbol < is used for "is less than" and > is used for "is greater than". These were introduced by Thomas & Harriot around 1630.

If a and b are real numbers, then only one of the following statement is true

(i) $a<b$ (ii) $a=b$ (iii) $a>b$

This property is known as **Trichotomy Property.**

A mathematical statement with the equality sign is called an '**equality**'. A mathematical statement in which we do not use the symbol of equality but other symbols like '<' or '>' or both, is called an '**inequality**'.

Properties of Equality of Real Numbers

The following properties are true for any real numbers a, b and c.

Name of property	General statement
Reflexive	$a=a$
Symmetric	If $a=b$ then $b=a$
Transitive	If $a=b$ and $b=c$ then $a=c$
Additive	If $a=b$ then $a+c=b+c$

Multiplicative	If $a = b$ then $ac = bc$ or $ca = cb$, where $c \neq 0$
Cancellation w.r.t. addition	If $a + c = b + c$ then $a = b$
Cancellation w.r.t. multiplication	$\left.\begin{array}{l}\text{If } ac = \text{bc then } a = b \\ \text{If ca} = cb \text{ then } a = b\end{array}\right\}$ where $c \neq 0$

Properties of Inequality of Real Numbers

The following properties are true for any real numbers *a*, *b* and *c*.

Name of property	General statement	Examples:
Trichotomy	Either $a > b$ or $a = b$ or $a < b$	If $2 < 3$, then $2 \neq 3$ and $2 \not> 3$
Transitive	If $a < b$ and $b < c$ then $a < c$ If $a > b$ and $b > c$ then $a > \text{c}$	If $3 < 5$ and $5 < 7$, then $3 < 7$ If $-2 > -5$ and $-5 > -7$, then $-2 > -7$
Additive	If $a < b$, then $a + c < b + c$ If $a > b$, then $a + c > b + c$	If $3 < 5$, then $3 + 10 < 5 + 10$ If $-5 > -8$, then $-5 + 2 > -8 + 2$
Multiplicative	(a) For $c < 0$ and $a, b \in \text{R}$, (i) If $a > b$ then $ac < \text{bc}$ (ii) If $a < b$, then $ac > bc$	(a) For $-4 < 0$, (i) If $3 > 2$, then $3(-4) < 2(-4)$ $-12 < -8$ (ii) If $3 < 5$, then $3(-4) > 5(-4)$ $-12 > -20$
Multiplicative	(b) For $c > 0$ and $a, b \in \text{R}$, (i) If $a > b$, then $ac > bc$ (ii) If $a < b$, then $ac < bc$ (iii) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$	(b) For $3 > 0$ (i) If $5 > 2$, then $5(3) > 2(3)$ $15 > 6$ (ii) If $2 < 3$, then $2(5) < 3(5)$ $10 < 15$ (iii) If $2 < 3$, then $\frac{1}{2} > \frac{1}{3}$
Cancellation w.r.t addition	If $a + c < b + c$ then $a < b$ If $a + c > b + c$ then $a > b$	If $2 + 3 < 5 + 3$ then $2 < 5$ If $4 + 3 > 2 + 3$ then $4 > 2$
Cancellation w.r.t multiplication	If $ac < \text{bc}$ and $c > 0$ then $a < b$ if $ac < bc$ and $\text{c} < 0$ then $a > b$ if $ac > bc$ and $\text{c} > 0$ then $a > b$ if $ac > bc$ and $\text{c} < 0$ then $a < b$	If $2 \times 3 < 4 \times 3$ then $2 < 4$ If $-3 \times 2 > -3 \times 4$ then $2 < 4$

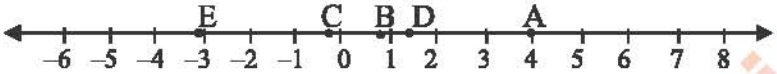
Key Fact

- The inequality sign remains unchanged if a positive number is multiplied to both the sides of an inequality.
- The inequalities are reversed if a negative number is multiplied to both the sides of an inequality.

Example 3: Show the following numbers on a number line.

(a) 4 (b) $\frac{7}{8}$ (c) $-\frac{1}{3}$ (d) $\sqrt{2}$ (e) $-\pi$

Solution:



(a) The number 4 is four units to the right of **0**, therefore, A is representing 4 on number line.

(b) $\frac{7}{8}$ = 0.875 is between 0 and 1, which is a terminating decimal. Point B in the figure is representing $\frac{7}{8}$ on the number line.

(c) $-\frac{1}{3} = -0.333\ldots$ or $-0.\bar{3}$ is between 0 and –1, which is a recurring decimal. Point C in the figure is representing $-\frac{1}{3}$ on the number line.

(d) Since $\sqrt{2} = 1.414213\ \ldots$, is between 1 and 2. Point D in the figure is the location of $\sqrt{2}$.

(e) Since $-\pi = -3.14159\ldots$ is between –3 and –4. Point E in the figure is the location of $-\pi$.

1.3.2 Representation of Real Numbers on Number Line

The representation of real numbers on a number line is called graphing the real numbers or graph of the real numbers.

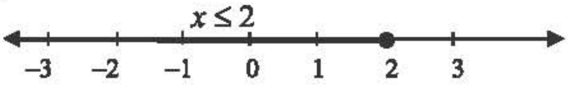
Example 4: Represent the following sets of real numbers on a number line.

(a) $x \leq 2$ (b) $-6 < x < 4$

(c) $x > -4$ (d) $-2 \leq x < 1$

Solution:

(a) The inequality $x \leq 2$ specifies all real numbers less than or equal to 2. This set is represented on a real number line as follows.



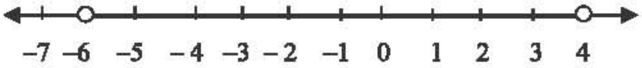
A filled circle indicate that 2 is included in the set.

Thinking zone

Try to imagine the numbers less than or equal to 2 and relate the words at least or at most which ever suitable in this case.

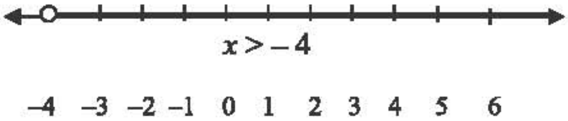
(b) The inequality $-6 < x < 4$ specifies all real numbers between – 6 and 4, as shown on the number line.

$-6 < x < 4$



We use hollow circle to indicate that both –6 and 4 are not included in the set.

(c) $x > -4$ specifies all real numbers greater than -4.



We use hollow dot to indicate that – 4 is not included in the set.

(d) $-2 \le x < 1$ is the set of all real numbers between –2 and 1 including –2 but excluding 1.

$-2 < x < 1$

EXERCISE 1.1

1. Represent each number on the number line.

(i) $\frac{3}{4}$ (ii) $\frac{-1}{3}$ (iii) $4\frac{1}{2}$ (iv) $-\sqrt{8}$

(v) $\sqrt{8}$ (vi) $-4\frac{1}{2}$ (vii) $\frac{1}{3}$ (viii) $-\frac{7}{8}$

2. Identify the property that justifies.

(i) $1 \times (y - 2) = y - 2$ (ii) $(0.2)\,5 = 1$

(iii) $(x + 2) + y = y + (x + 2)$ (iv) $-(3b) + (3b) = 0$

(v) $(x + 5) - 1 = x + (5 - 1)$ (vi) $-3\,(2 - y) = -6 + 3y$

3. Represent the following on a number line.

(i) $x < 0$ (ii) $-3 < x < 3$ (iii) $x \ge -8$

(iv) $x > 0$ (v) $x < -3$ (vi) $-4 < x \le 4$

4. Identify the properties of equality and inequality of real numbers that justifies the statement.

(i) $9x = 9x$ (ii) If $x + 2 = y$ and $y = 2x - 3$, then $x + 2 = 2x - 3$

(iii) If $2x + 3 = y$, then $y = 2x + 3$ (iv) If $3 < 4$, then $-3 > -4$

(v) If $2y + 2w = p$ and $p = 50$, then $2y + 2w = 50$

(vi) If $x + 4 > y + 4$, then $x > y$ (vii) If $2 < 5$ and $5 < 9$, then $2 < 9$

(viii) If $-18 < -16$, then $9 > 8$

1.4 Radical and Radicands

1.4.1 Square Root

A square root of a positive number ‘n’ is another number ‘m’ whose square is ‘n.’ Any positive number has two square roots, which are additive inverses of each other.

For example, 4 is a square root of 16, because $(4)^2 = 16$ and -4 is also a square root of 16, because $(-4)^2 = 16$. Therefore, the two square roots of 16 are -4 and 4, which are additive inverse of each other.

1.4.2 Principal Square Root

Positive square roots are called ‘principal square roots’. e.g. $\sqrt{25} = 5, \sqrt{81} = 9$ etc.

In expressions like $\sqrt{25}$ entire $\sqrt{25}$ is called a **square radical or radical**. The symbol $\sqrt{\ }$ is called a **radical sign** and the number ‘25’ under the radical sign is called the **radicand**.

Definition of n^th^ Root

For any real numbers ‘a’ and ‘b’ and any positive integer $n > 1$, if $a^n = b$, then $a = b^{\frac{1}{n}}$, where ‘n’ is the index of the radical.

We read $a^n = b$ as ‘b is the n^th^ power of a’ and $a = b^{\frac{1}{n}}$ as ‘a is the n^th^ root of b’.

For example, $y^3 = x$, then $y = x^{\frac{1}{3}} = \sqrt[3]{x}$

Here 3 is the index of radical and y is the cubic root of x.

Example 5:

Radical Form	Index of the Radical	Radicand
$\sqrt[3]{35}$	3	35
$\sqrt[5]{\frac{xy}{z}}$	5	$\frac{xy}{z}$
$\sqrt{-(xyz)^4}$	2	$-(xyz)^4$

Historya Mystery

The radical sign was first used in 1525 AD and was written as “$\surd$ ”.

Key Fact

(i) $\sqrt{b} = \sqrt[2]{b}$ i.e. $\sqrt{\ }$ and $\sqrt[2]{\ }$ are equivalent.

(ii) There is no real number that is a square root of a negative number.

e.g. $\sqrt{-16} \neq 4$, since $(+4)^2 \neq -16$. In Mathematics ‘imaginary numbers’ are defined to handle the square root of negative numbers.

1.4.3 Properties of Radicals

1. Product and Quotient Rules for Radicals

For any integer n > 1 and for all real numbers 'a' and 'b' for which the operations are defined

(i) $\sqrt[n]{a}.\sqrt[n]{b}=\sqrt[n]{ab}$ $\xrightarrow{\qquad\qquad}$ Product rule for radicals

e.g. $\sqrt[3]{8}\times\sqrt[3]{27}=\sqrt[3]{8\times 27}$

(ii) and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$ $\xrightarrow{\qquad\qquad}$ Quotient rule for radicals e.g. $\dfrac{\sqrt[3]{64}}{\sqrt[3]{8}}=\sqrt[3]{\dfrac{64}{8}}$

Example 6: Use the product rule for radicals to simplify the following. Assume that all variables represent positive numbers.

(a) $\sqrt{2a}.\sqrt{7b}$ (b) $\sqrt[4]{\dfrac{1}{x}}.\sqrt[4]{\dfrac{2}{y}}$ (c) $\sqrt[3]{3}.\sqrt{2}$

Solution:

(a) $\sqrt{2a}.\sqrt{7b}=\sqrt{14ab}$

(b) $\sqrt[4]{\dfrac{1}{x}}.\sqrt[4]{\dfrac{2}{y}}=\sqrt[4]{\dfrac{2}{xy}}$

(c) The product rule for radicals does not apply to $\sqrt[3]{3}.\sqrt{2}$, because the indices are not same.

Key Fact

- The product and quotient rules for radicals apply only if the indices are same.
- The plural of index is indices.

2. Reducing the Index

If the index of the radical and the exponent of the radicand have a common factor, the expression can be written with a smaller index.

We will explain it with the help of an example. Consider $\sqrt[12]{9^6}$, here the index of the radical is 12 and exponent of radicand is 6. Also 6 is the common factor of 12 and 6 . So we can write the expression with a smaller index as follows.

$$\sqrt[12]{9^6}=9^{\frac{6}{12}}=9^{\frac{1}{2}}=\sqrt{9}=3$$

Example 7: Write the given expression with a smaller index. Assume that variable t represents positive numbers.

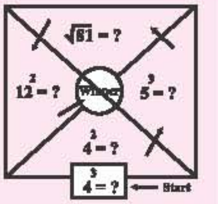
$$\sqrt[15]{t^{10}}$$

Solution : $\sqrt[15]{t^{10}}=t^{\frac{10}{15}}$

$=t^{\frac{2}{3}}=\sqrt[3]{t^2}$

Math Play Ground

1. Make a hopscotch on ground as give:
2. Ask a player to start hoping and doing sums in each box.
3. If reaches in circle without falling and doing correct calculations, he/she wins.



1.5 Laws of Exponents / Indices

1.5.1 Base and Exponents

We use exponent to indicate the repeated multiplication of the same factors.
The exponent indicates that how many times a factor, called the *base,* occurs in the multiplication form.

e.g.

$$3.3.3.3 = \underset{\text{Base}}{3}^{\overset{\text{Exponent}}{4}}$$

The expression 3^4 is an exponential form read as 'three to the fourth power', where as 3.3.3.3 is the factored form. The words 'square' and 'cube' are sometimes used for exponents '2' and '3' respectively, rather than 'to the second power' and 'to the third power'.

Historya Mystery

Rene Descartes (1596 – 1650) was the first mathematician who extensively used exponential notation as it is used today. However, for some unknown reason, he always used xx for x^2.

Key Fact

When the exponent is a natural number, the base can be any real number.
We use an exponent as a convenient way to write repeated multiplication.

1.5.2 Rational Exponents

When we perform operations with exponents, we have to define a zero exponent and a negative exponent. This may lead us to define a rational exponent.

Definition of $b^{\frac{1}{n}}$

If b is a real number and n is a positive integer, then $b^{\frac{1}{n}} = \sqrt[n]{b}$ e.g. $8^{\frac{1}{3}} = \sqrt[3]{8}$.

Definition of $b^{\frac{m}{n}}$

If m and n are positive integers with no common factor except 1 and $n \neq 0$, then $b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m = (\sqrt[n]{b})^m$ for all real numbers b. e.g. $36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3$.

We can use this definition to write expressions with rational exponents as radicals.
The number 'n' indicates the index of the radical and the number 'm' indicates the power to which the radical is to be raised.

The procedure for evaluating $b^{\frac{m}{n}}$ can be summarized as follows.

I. Determine the nth root of b.
II. Raise the result to the m power.

Example 8: In the following table:

(a) perform the operation in column-A and compare the result to the value of radical in column- B.

(b) what do you observe about the denominator of the exponent and the index of the radical?

	Column-A (exponential form)	Column-B (radical form)
(i)	$9^{\frac{1}{2}}$	$\sqrt{9}$
(ii)	$64^{\frac{1}{3}}$	$\sqrt[3]{64}$
(iii)	$81^{\frac{1}{4}}$	$\sqrt[4]{81}$
(iv)	$32^{\frac{1}{5}}$	$\sqrt[5]{32}$

Solution:

(a)

(i) $9^{\frac{1}{2}} = 3$ and $\sqrt{9} = 3$

(ii) $64^{\frac{1}{3}} = (4^3)^{\frac{1}{3}} = 4$ and $\sqrt[3]{64} = 4$

(iii) $81^{\frac{1}{4}} = 3$ and $\sqrt[4]{81} = (3^4)^{\frac{1}{4}} = 3$

(iv) $32^{\frac{1}{5}} = 2$ and $\sqrt[5]{32} = 2$

(b) In each part, the exponential and the radical expression have the same value. The denominator of the exponent is the same as the index of the radical.

Example 9: Simplify:

(a) $8^{\frac{2}{3}}$ (b) $36^{\frac{3}{2}}$

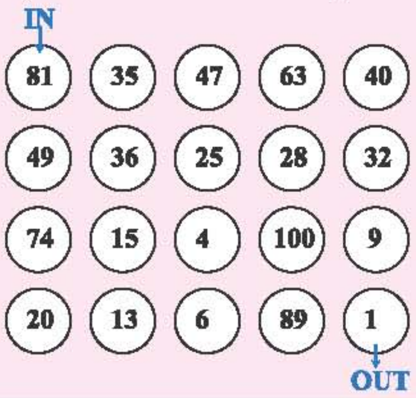
Solution:

(a)
$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left[\left(2^3\right)^{\frac{1}{3}}\right]^2 = (2)^2 = 4$$

(b)
$$36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3 = \left[\left(6^2\right)^{\frac{1}{2}}\right]^3 = (6)^3 = 216$$

Math Play Ground

Jump on the numbers which are squares of natural numbers to go out.



81	35	47	63	40
49	36	25	28	32
74	15	4	100	9
20	13	6	89	1

Negative Rational Exponents

For integral exponents, we define:

$a^{-n} = \frac{1}{a^n}$ provided $a \neq 0$. e.g. $8^{-3} = \frac{1}{8^3}$.

We can extend this definition to negative rational exponents.

If m and n are any two integers such that one of them is negative and they have no common factor other than 1 and if $b \neq 0$, then $b^{\frac{-m}{n}} = \frac{1}{b^{\frac{m}{n}}}$ for all $b \in R$, for which $b^{\frac{m}{n}}$ is defined.

Example 10: Simplify.

(a) $16^{\frac{-3}{4}}$ (b) $\left(\frac{16}{25}\right)^{\frac{-1}{2}}$

Solution:

(a)
$$16^{\frac{-3}{4}} = \frac{1}{(16)^{\frac{3}{4}}} = \frac{1}{(2^4)^{\frac{3}{4}}} = \frac{1}{2^3} = \frac{1}{8}$$

(b)
$$\left(\frac{16}{25}\right)^{\frac{-1}{2}} = \left(\frac{25}{16}\right)^{\frac{1}{2}} = \sqrt{\frac{16}{25}} = \sqrt{\frac{5^2}{4^2}} = \frac{5}{4}$$

Example 11: Write exponential expressions as an equivalent radical expression.

(a) $(-7)^{\frac{2}{3}}$ (b) $(2)^{\frac{3}{5}}$

Solution: (a)
$$(-7)^{\frac{2}{3}} = (-7)^{2\times\frac{1}{3}} = (49)^{\frac{1}{3}} = \sqrt[3]{49}$$

(b)
$$(2)^{\frac{3}{5}} = \left[(2)^3\right]^{\frac{1}{5}} = \sqrt[5]{8}$$

1.5.3 Properties of Exponents

If m and n are rational numbers, then for non zero real numbers a and b for which the expressions are defined, the following are the properties of exponents.

i) $a^m . a^n = a^{m+n} \rightarrow$ Product rule

ii) $\frac{a^m}{a^n} = a^{m-n} \rightarrow$ Quotient rule

iii) $a^{-n} = \frac{1}{a^n} \rightarrow$ Definition of negative exponent,

iv) $(a^m)^n = a^{mn} \rightarrow$ Power of a power rule

v) $(ab)^n = a^n b^n \rightarrow$ Power of a product rule

vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \rightarrow$ Power of a quotient rule

vii) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \rightarrow$ Negative power of a quotient rule

Example 12: Use the properties of exponents to evaluate each of the following.

(a) $(5^6)^{\frac{1}{2}}$ (b) $2^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$

Solution:

(a) $\left(5^6\right)^{\frac{1}{2}}$

$= 5^{6\times\frac{1}{2}} = 5^3$

$= 125$

(b) $2^{1/2} \cdot 8^{1/2}$

$= (2 \cdot 8)^{\frac{1}{2}} = (16)^{\frac{1}{2}}$

$= (4^2)^{\frac{1}{2}} = 4$

Example 13: Use properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (Write all results with positive exponents.)

(a) $a^{\frac{1}{3}}(a^{\frac{5}{3}} - a^{\frac{-2}{3}})$

(b) $\left[\dfrac{x^{\frac{1}{2}}}{y^3}\right]^{-\frac{1}{3}}$

Solution: (a) $a^{\frac{1}{3}}(a^{\frac{5}{3}} - a^{\frac{-2}{3}})$

$= a^{\frac{1}{3}}a^{\frac{5}{3}} - a^{\frac{1}{3}}a^{\frac{-2}{3}}$

$= a^{\frac{1}{3}+\frac{5}{3}} - a^{\frac{1}{3}-\frac{2}{3}}$

$= a^{\frac{6}{3}} - a^{\frac{-1}{3}}$

$= a^2 - \dfrac{1}{a^{\frac{1}{3}}} = \dfrac{a^{\frac{2}{3}} - 1}{a^{\frac{1}{3}}}$

(b) $\left[\dfrac{x^{\frac{1}{2}}}{y^3}\right]^{-\frac{1}{3}}$

$= \left[\dfrac{y^3}{x^{\frac{1}{2}}}\right]^{\frac{1}{3}} = \dfrac{\left(y^3\right)^{\frac{1}{3}}}{\left(x^{\frac{1}{2}}\right)^{\frac{1}{3}}}$

$= \dfrac{y}{x^{\frac{1}{6}}}$

EXERCISE 1.2

1. By using the property of product and quotient rule for radicals, write each expression as a single radical and simplify.

(i) $\sqrt[3]{6} \cdot \sqrt[3]{6}$ (ii) $\sqrt[5]{4} . \sqrt[5]{8}$ (iii) $\sqrt[4]{x} . \sqrt[4]{x^3}$

(iv) $\sqrt{10} \cdot \sqrt[3]{11}$ (v) $\dfrac{\sqrt[4]{x^7}}{\sqrt[4]{x^5}}$ (vi) $\dfrac{\sqrt[3]{5000}}{\sqrt[3]{5}}$

(vii) $\dfrac{\sqrt[2]{500}}{\sqrt[2]{5}}$ (viii) $\sqrt[2]{10} \cdot \sqrt[3]{7}$

2. Write each exponential expression as an equivalent radical expression and simplify if possible.

(i) $(216)^{\frac{2}{3}}$ (ii) $(29)^{\frac{1}{2}}$ (iii) $\left(\dfrac{1}{32}\right)^{\frac{1}{5}}$

(iv) $(216)^{-\frac{2}{3}}$ (v) $(1000)^{\frac{1}{3}}$ (vi) $\left(\dfrac{1}{39}\right)^{-\frac{1}{2}}$

3. Write each radical expression as an equivalent exponential expression and simplify if possible.

(i) $\left(\sqrt[3]{5}\right)^2$ (ii) $\left(\sqrt[4]{10}\right)^8$ (iii) $-\left(\sqrt[3]{6}\right)^6$

(iv) $\left(\sqrt[3]{6}\right)^6$ (v) $-\left(\sqrt[3]{5}\right)^2$ (vi) $-\left(\sqrt[4]{10}\right)^8$

4. Use the properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (write all results with positive exponents.)

(i) $\dfrac{16^{\frac{1}{5}}.16^{\frac{1}{4}}}{16^{\frac{-3}{10}}}$ (ii) $7^{\frac{-1}{3}}\left(7^{\frac{5}{3}} - 7^{\frac{4}{3}}\right)$ (iii) $\dfrac{2^{\frac{2}{3}}.2^{\frac{1}{7}}}{2^{\frac{1}{2}}}$

(iv) $\dfrac{3^{\frac{-1}{2}}.3^{\frac{1}{2}}}{3^{\frac{1}{2}}}$ (v) $\left(\dfrac{36^{\frac{1}{2}}.}{8^{\frac{1}{2}}.}\dfrac{6^{\frac{1}{2}}}{27^{\frac{1}{2}}}\right)^3$ (vi) $\left(\dfrac{2187\,a^5b^{17}}{a^{12}b^3}\right)^{\frac{1}{7}}$

(vii) $\sqrt[4]{\dfrac{a^3}{b^3}} \times \sqrt[4]{\dfrac{b^3}{c^3}} \times \sqrt[4]{\dfrac{c^3}{a^3}}$

5. Use suitable laws of exponents to show that

$$\left(\frac{x^p}{x^q}\right)^{p+q} \times \left(\frac{y^q}{y^r}\right)^{q+r} \times \left(\frac{z^r}{z^p}\right)^{r+p} \times x^{q^2} \times y^{r^2} \times z^{p^2} = x^{p^2} \times y^{q^2} \times z^{r^2}$$

1.5.4 Application of Real Numbers in Daily Life

All the numbers we use in our daily life situations are Real numbers. We cannot imagine life without numbers. For instance we use natural numbers in counting our objects in the pantry, books in the library, animals or birds at a farm, stock taking in inventory of a factory etc. Similarly, we have a vast use of integers while recording or understanding temperature, gain or loss, rise or fall etc. Rational numbers have also the vast contribution in daily life situations such as use of ratio, proportion, fractions and percentages in financial matters like income, expenditure, savings, and payment of wages to employees, rents of buildings, profit, loss sharing in business managements, risk calculations. The irrational numbers as obvious from the name are not reasonable or they don't make a sense for non-mathematicians. But for mathematicians they have really big scope of usage. Engineers, technicians, opticians while working with circles, spheres or cylinders and finding their areas, perimeters or volume, which include π are working with irrational numbers. Then we find irrational numbers like in architecture, navigation and fluid mechanics, where transcendental functions are in frequent use.

Example 14:

A cooking oil company produces four types of oils in packing of 1 litre , 5 litre & 10 litre. The inventory is shown in the table.

Name / packing size	1 litre	5 litre	10 litre
Cooking oil-I	5000	2500	1000
Cooking oil-II	5000	2500	1000
Cooking oil-III	5000	2500	1000
Cooking oil-IV	5000	2500	1000

After removal of 40% of an item, it is to be replenished. The daily removal of 1lire cooking oil-II packing is 20% and for 10 litre cooking oil-IV packing, daily removal is 5%. Find

a) Number of daily removed 1 litre cooking oil-II packing.
b) After how many days, 1 litre cooking oil-II are to be replenished?
c) Number of daily removed 10 litre cooking oil-IV packing.
d) After how many days, 10 litre cooking oil-IV packing are to be replenished?

Solution:

Total 1 litre cooking oil-II packing in inventory = 5000

a) Number of daily removed 1lire cooking oil-II packing = 20% of 5000

$$= \frac{20}{100} \times 5000$$
$$= 1000$$

b) After 40% removal, replenishment is to be made.

Here 40% of 5000 = 2000

After two days the replenishment is due.

c) Number of daily removed 10 lire cooking oil-IV packing = 5% of 1000

$$= \frac{5}{100} \times 1000$$
$$= 50$$

d) Here 40% of 1000 = 400 but packs removed per day are 50.

Therefore 400 ÷50 = 8

After 8 days the replenishment is due.

EXERCISE 1.3

1. On his last bank statement, Qasim had a balance of Rs. 1,75,000 in his checking account. He wrote one cheque for Rs. 45,790 and another for Rs. 112,921. What is his current balance?
2. Last week Wajid drove 283.4 km on 16.2 litres of petrol. He says that he averaged about 1.75 km/liter. Is his answer reasonable? Explain.
3. Salma bought 3.2 yard of fabric for a total price of Rs. 139.2. How much did the fabric cost per yard?

4. Momina walks 3.5 km/h. She took a 12 h walk. How far did she walk.
5. The hiking club went on a 7day trip. Each day they hiked between 5.5 and 7.5 miles. It is reasonable to assume that clubbing the days the club hiked.
    a. Less than 35 miles
    b. Between 35 and 55 miles
    c. Equally 55 miles
    d. More than 55 miles
6. For a class party the students council purchased 42 balloons at Rs. 1.85 each. What is the total amount the student council paid for the balloons?
7. A group of friends made 4-yard long rectangular banner. They paid Rs. 3.75 per yard for the fabric and Rs.9 for the firm to go around the banner, 10-yard perimeter. What was the width of the banner?
8. A shoe factory has an asset for Rs. 2000,000 of which $\frac{3}{5}$ is the capital and rest is the debt. Find the amount of capital and debt. (Asset = capital + debt)
9. World lowest temperature in past 100 years was recorded to be $-89.2\,{^\circ}C$ at Vostok, Antarctica on July 21, 1983. Covert this temperature into Fahrenheit and Kelvin scales.

$$(F = \frac{9}{5}C + 32\ , \qquad K = {^\circ}C + 273)$$

10. A company was penalized by the government act for low quality production. If the company has 3 share holders. Farah, Maryam and Tehreem investing in the ratios of 1 : 2 : 3 and the amount of penalty is Rs. 456,868.97. Find the amount of penalty paid by each of 3 share holders.

KEY POINTS

- Real numbers are union of rational and irrational numbers.
- Basic properties of real numbers are
    - Closure
    - Commutative
    - Associative
    - Identity
    - Inverse
    - Distributive property
- Properties of equality of real numbers:
    - Reflexive
    - Symmetric
    - Transitive
    - Additive
    - Multiplicative
    - Cancellation
- Properties of inequality of real numbers:
    - Trichotomy property
    - Transitive
    - Additive
    - Multiplicative
    - Cancellation
- Positive square roots are called principal square roots.
- For any real numbers a and b and any positive integer n >1 if $a^n = b$, then a is the nth root of b, symbolically it is represented as $a = \sqrt[n]{b}$.

- The symbol $\sqrt{\ }$ is called the radical sign, the number n is called index of the radical and b is called radicand.
- Laws of exponents

(i) $a^m . a^n = a^{m+n} \rightarrow$ Product rule (ii) $\frac{a^m}{a^n} = a^{m-n} \rightarrow$ Quotient rule

(iii) $a^{-n} = \frac{1}{a^n} \rightarrow$ Negative exponent (iv) $(a^m)^n = a^{mn} \rightarrow$ Power to a power rule

(v) $(ab)^n = a^n b^n \rightarrow$ Power of a product rule

(vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \rightarrow$ Power of a quotient rule

MISCELLANEOUS EXERCISE 1

1. Encircle the correct option in the following.

(i) a (b + c – d) equals

(a) a (b + c + d) (b) ac + ab – ad (c) ab + ac + ad (d) ab – ac – ad

(ii) $a^r . a^{-s} \div a^s$ is

(a) a^{r-s} (b) a^{r+2s} (c) $a^r . a^{2s}$ (d) $\frac{a^r}{a^{2s}}$

(iii) $\sqrt[n]{ab}$ is equal to

(a) $\sqrt{ab}$ (b) $n(ab)$ (c) $(ab)^n$ (d) $(ab)^{\frac{1}{n}}$

(iv) Which number is self-multiplicative inverse?

(a) 3 (b) –3 (c) –1 (d) 0

(v) If $a > 0$, then $\sqrt{a}$ is

(a) real (b) integer (c) irrational (d) rational

(vi) If a + b = a, what is value of b?

(a) 1 (b) –1 (c) a (d) 0

(vii) If a.b = 1, what is value of b?

(a) 1 (b) $\frac{1}{b}$ (c) $\frac{1}{a}$ (d) –1

(viii) According to reflexive property : $y^2 - 17 = ?$

(a) $y^2 + 17$ (b) $y - 17$ (c) $y^2 - 17$ (d) $-17 - y^2$

(ix) If a.b = a, what is value of b?

(a) $\frac{1}{a}$ **(b)** 1 **(c)** a **(d)** –1

(x) If a.b = 1, what is b called?

(a) multiplicative inverse of a **(b)** additive identity

(c) multiplicative identity **(d)** self-multiplicative inverse

(xi) Commutative property does not hold with respect to:

(a) addition **(b)** multiplication

(c) subtraction **(d)** both (a) and (b)

2. Represent each number on the number line.

(i) $-5\frac{1}{5}$ (ii) $\frac{17}{3}$ (iii) $-2 < x < 4$ (iv) $x \geq 6$

3. Write each exponential expression as an equivalent radical expression and simplify if possible.

(i) $(-2)^{\frac{4}{5}}$ (ii) $(-27)^{\frac{1}{3}}$ (iii) $(\sqrt{16})^4$

(iv) $(\sqrt[3]{-8})^9$ (v) $(x^{-2})^3.(x^0)^5$

4. Use the properties of exponents to simplify each of the following.

(i) $\frac{(-2)^3.(-2)^{-4}.(-2)}{(-2)^{-3}}$ (ii) $\frac{2^{\frac{1}{2}}\cdot 2^{\frac{3}{4}}}{2^{\frac{1}{2}}} \times \frac{3.3^{\frac{3}{2}}}{3^{-\frac{1}{2}}}$

5. Determine whether each statement is true or false. If false, give an example of a number that shows the statement is ture.

a. Every rational number is an integer.
b. Every real number is an irrational.
c. Every irrational number is a real number
d. Every integer is a rational number.
e. Every real number is either rational number or an irrational number.