بسم الله الرحمن الرحيم

A Textbook of

STATISTICS

For Class

12

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10 NORMAL DISTRIBUTION

10.1 NORMAL DISTRIBUTION

The normal distribution is undoubtedly the most important and frequently used of all probability laws, because

(*i*) the normal random variable does frequently occur in practical problems such as heights and weights of individuals, I.Q. scores, errors of measurements, *etc.*

(*ii*) it is the limiting form of many other probability laws and hence provides an accurate approximation to them.

(*iii*) it is also the limiting distribution on the well known *central limit theorem* (as discussed in Theorem 11.6).

Thus a great many techniques used in applied statistics are based on the normal distribution. The formal definition follows.

10.1.1 Normal Probability Density Function. A continuous random variable X is normally distributed if and only if its probability density function is

$$f(x) = n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{for } -\infty < x < \infty$$

where μ is any real number (*i. e.*, $-\infty < \mu < \infty$) and σ must be positive (*i. e.*, $\sigma > 0$), π ($= 3.141592654\cdots$) and e ($= 2.718281828\cdots$) are constants, x is the value of random variable X and $f(x)$ is the density (ordinate) at $X = x$.

A normal distribution is characterized by two parameters μ and σ, its mean and standard deviation respectively. Sometimes it is denoted by $N(\mu, \sigma^2)$. Thus

$$X \sim N(\mu, \sigma^2)$$

means that a random variable X is normally distributed with its mean μ and variance σ^2.

10.1.2 Shape of Normal Distribution. The graph of a normal probability density function is called a normal curve. As can be seen from the definition, the probability density function for a normal random variable has a unimodal, symmetrical and bell shaped distribution.

Figure 10.2 shows the graphs of some typical normal density functions for various values of the parameters μ and σ. The parameter σ controls the relative flatness of the curve.

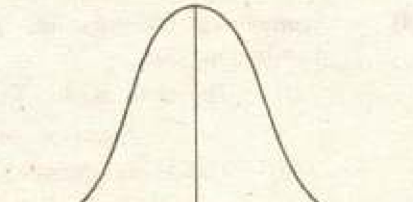
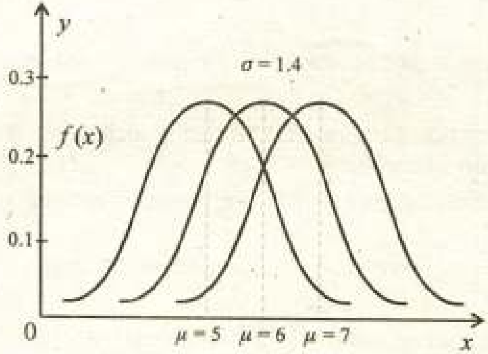


Fig 10.1 Normal distribution

(i) Keeping μ constant and decreasing σ causes the density function to become more sharply peaked, thus giving higher probabilities of X being close to μ.

(ii) Keeping μ constant and increasing σ causes the density function to flatten, thus giving lower probabilities of X being close to μ.

(iii) If σ is held constant and μ is varied, the shape of the density function remains the same with its centre moving to the location of μ.



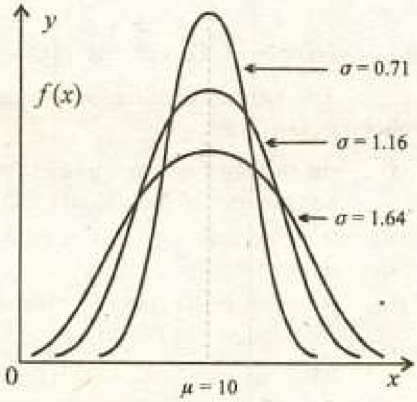


Fig. 10.2 Normal probability density functions

10.1.3 Properties of Normal Distribution. The following are the main properties of normal distribution (or curve).

(1) ***Continuous Distribution.*** The normal probability distribution is a continuous distribution that ranges from $-\infty$ to $+\infty$.

$$R_X = \{x: -\infty < x < +\infty\}$$

(2) ***Total Probability.*** The total area under the normal curve is unity. That is,

$$P(-\infty < X < +\infty) = 1$$

(3) ***Mode and Maximum Ordinate.*** The normal probability density function is unimodal (single peaked), its mode is μ and its maximum ordinate at $x = \mu$ is

$$f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}}$$

(4) ***Symmetrical Distribution.*** The normal probability distribution is a symmetrical distribution. Thus

(i) The mean, median and mode coincide at μ.

$$Mean = Median = Mode = \mu$$

(ii) The lower quartile $x_{0.25}$ or Q_1 and the upper quartile $x_{0.75}$ or Q_3 are equidistant from its mean μ.

$$x_{0.75} - \mu = \mu - x_{0.25}$$

(iii) All odd order moments about mean are zero.

$$\mu_1 = \mu_3 = \mu_5 = \cdots = 0$$

(5) ***Special areas under the curve.*** No matter what values of μ and σ are, the areas under the normal curve remain in fixed proportions within a specified number of σ on either side of μ. For example,

(*i*) $P(\mu - \sigma < X < \mu + \sigma) = 0.6827$

(*ii*) $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$

(*iii*) $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$

(6) ***Median, Quartiles and Quartile Deviation.*** In a normal probability distribution, the median, the lower and upper quartiles and the quartile deviation are

$$x_{0.5} = \mu, \qquad x_{0.25} = \mu - 0.6745\,\sigma, \qquad x_{0.75} = \mu + 0.6745\,\sigma,$$

$$Q.\,D(X) = \frac{x_{0.75} - x_{0.25}}{2} = 0.6745\,\sigma \approx \frac{2}{3}\,\sigma$$

$$\mu = \frac{x_{0.25} + x_{0.75}}{2}, \qquad \sigma = \frac{x_{0.75} - x_{0.25}}{1.349}$$

(7) ***Variance, Standard Deviation and Mean Deviation.*** In a normal probability distribution, the variance, the standard deviation and the mean deviation are

$$Var(X) = \sigma^2$$

$$S.\,D(X) = \sigma$$

$$M.\,D(X) = \sigma\sqrt{\frac{2}{\pi}} = 0.7979\,\sigma \approx \frac{4}{5}\,\sigma$$

(8) ***Moments and Moment Ratios.*** In a normal probability distribution, the first four moments about mean and the moment ratios are

$$\mu_1 = 0, \qquad \mu_2 = \sigma^2, \qquad \mu_3 = 0, \qquad \mu_4 = 3\sigma^4$$

Hence $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{0}{(\sigma^2)^3} = 0, \qquad \beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{3\sigma^4}{(\sigma^2)^2} = 3$

(9) ***Points of Inflexion.*** The points of inflexion of the normal probability density function are equidistant from mean μ, they are at $x = \mu - \sigma$ and $x = \mu + \sigma$. The normal probability distrjbution is a bell shaped distribution.

(10) ***Asymptotic Curve.*** The normal curve is *asymptotic* to the *x-axis*, that is, as $|x|$ grows without bound, the curve gets closer and closer to the *x-axis* but always stays above it. The curve has a single peak in the middle and tapers off gradually at both ends and never meets the *x-axis*.

(11) ***Reproductive Property.*** If X_1 and X_2 are two independent normal random
variables having distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively, then

their sum $X_1 + X_2$ is also a normal random variable having the distribution
$N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

10.1.4 Normal Cumulative Distribution Function. The cumulative distribution function for the normal random variable X is

$$F(x) = P(X \le x) = P(-\infty < X \le x)$$

$$= \int_{-\infty}^{x} f(u)\, du = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du$$

Unfortunately, this integration cannot be carried out in the closed form. Numerical techniques could be used to evaluate the integral for specific values of μ and σ. The various possible values of μ and σ result in a family of unlimited number of different normal distributions. It is, thus, necessary to tabulate the standard normal cumulative distribution function, that can be used to evaluate the cumulative distribution function for a normal random variable with any mean μ and any standard deviation σ.

10.2 STANDARD NORMAL RANDOM VARIABLE

A normal random variable X with mean μ and standard deviation σ can easily be transformed into a standard normal random variable Z by the transformation

$$Z = \frac{X - \mu}{\sigma}$$

which has mean 0 and variance 1.

10.2.1 Standard Normal Distribution. If the random variable X has a normal distribution with mean μ and variance σ^2, then the random variable $Z = (X - \mu)/\sigma$ has a standard normal distribution with mean 0 and variance 1.

***Theorem* 10.1** *If $X \sim N(\mu, \sigma^2)$ and $Z = (X - \mu)/\sigma$, then*

$$Z \sim N(0, 1)$$

Since the standard normal probability density function and cumulative distribution function are of such importance, we shall use ***special symbols*** for them.

10.2.2 Standard Normal Probability Density Function. The probability density function of the standard normal variable Z, denoted by $\varphi(z)$, is given as

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \text{for } -\infty < z < \infty$$

Thus $\varphi(z)$ is the value of standard normal probability density function at $Z = z$. Therefore, $\varphi(z)$ is called as the ordinate of the standard normal curve at $Z = z$. The ordinates $\varphi(z)$ have been tabulated for various values of z in Table 7.

***Theorem* 10.2** If $Z \sim N(0, 1)$, then

$$\varphi(-z) = \varphi(z)$$

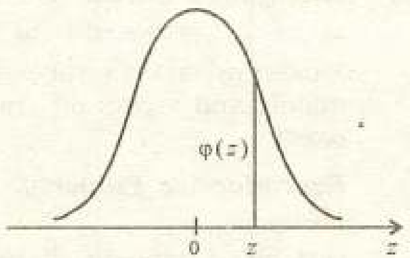


Fig. 10.3 Ordinate of standard normal probability density function at $Z = z$

10.2.3 Standard Normal Cumulative Distribution Function. The cumulative distribution function of the standard normal variable Z, denoted by $\Phi(z)$, is given as

$$\Phi(z) = P(Z \le z) = P(-\infty < Z \le z)$$

$$= \int_{-\infty}^{z} f(u)\, du = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

Thus $\Phi(z)$ is the cumulative probability up to $Z = z$ in the standard normal distribution. The cumulative probabilities $\Phi(z)$ have been tabulated for various values of z in Table 9. Note that

$$\Phi(-\infty) = 0$$

$$\Phi(+\infty) = 1$$

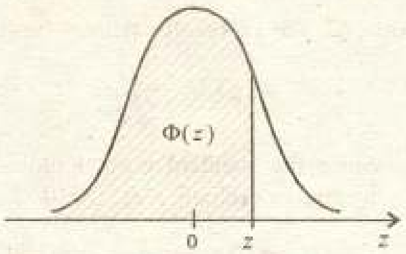


Fig. 10.4 Cumulative probability in standard normal distribution up to $Z = z$

***Theorem* 10.3** *If $Z \sim N(0, 1)$ and a, b are any real numbers, then*

(*i*) $P(Z \le a) = \Phi(a)$

(*ii*) $P(Z \ge a) = 1 - \Phi(a)$

(*iii*) $P(a \le Z \le b) = \Phi(b) - \Phi(a)$

***Theorem* 10.4** If $Z \sim N(0, 1)$, *then for any real value a*

(*i*) $\Phi(-a) = 1 - \Phi(a)$

(*ii*) $P(Z \ge a) = \Phi(-a)$

(*iii*) $P(|Z| \le a) = 2\Phi(a) - 1$

(*iv*) $P(|Z| \ge a) = 2\Phi(-a)$

10.2.4 Inverse Standard Normal Cumulative Distribution Function. The inverse standard normal cumulative distribution function determines a value z corresponding to a given value of the cumulative probability. Suppose that cumulative probability at $Z = z$ is p, then we have

$$\Phi(z) = P(Z \le z) = p$$

$$\Phi^{-1}(p) = z$$

The values of $\Phi^{-1}(p)$ have been tabulated for various values of cumulative probability p in Table 10. For example,

$$\Phi(1.96) = 0.975$$

$$\Phi^{-1}(0.975) = 1.960$$

10.2.5 Use of the Standard Normal Tables. We now show how the tables of the standard normal distribution are used illustrating their direct or inverse use.

***Example* 10.1**

(*i*) *Write down the equation of the standard normal distribution.*

(*ii*) *Find the value of maximum ordinate of the standard normal curve correct to four places of decimal.*

(*iii*) *Verify that the ordinates of the standard normal curve at* $z = 1.27$ *and* $z = -1.27$ *are equal.*

(*iv*) *Find the value* z *when the ordinate at* z *is* 0.12001.

Solution. (*i*) The probability density function of standard normal random variable Z is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

(*ii*) Since the standard normal probability density function is symmetric about zero, its maximum ordinate is at $z = 0$

$$\varphi(0) = \frac{1}{\sqrt{2\pi}} e^{-(0)^2/2} = \frac{1}{\sqrt{2\pi}} = 0.3989$$

(*iii*) Either calculating directly or using the Table 7, we have

$$\varphi(1.27) = \frac{1}{\sqrt{2\pi}} e^{-(1.27)^2/2} = 0.17810$$

$$\varphi(-1.27) = \frac{1}{\sqrt{2\pi}} e^{-(-1.27)^2/2} = 0.17810$$

Note $\varphi(-1.27) = \varphi(1.27)$

Alternately,

$$\varphi(-1.27) = \frac{1}{\sqrt{2\pi}} e^{-(-1.27)^2/2} = \frac{1}{\sqrt{2\pi}} e^{-(1.27)^2/2} = \varphi(-1.27)$$

(*iv*) By the inverse use of Table 7 and the fact that $\varphi(-z) = \varphi(z)$, we have

$$\varphi(z) = 0.12001$$

$$z = \varphi^{-1}(0.12001) = \pm 1.55$$

***Example* 10.2** *If* Z *is a standard normal random variable with mean* 0 *and variance* 1, *then find*

(*i*) $P(Z < -1.96)$ (*ii*) $P(Z > 1.26)$

(*iii*) $P(-1.96 < Z < 1.96)$ (*iv*) $P(-\infty < Z < 2.12)$

(*v*) $P(-2.72 < Z < \infty)$

Solution. From the definition of standard normal cumulative distribution function, we have

(*i*) $P(Z < -1.96) = \Phi(-1.96)$

$= 0.02500$ (From Table 9)

(*ii*) $P(Z > 1.26) = 1 - P(Z < 1.26)$

$= 1 - \Phi(1.26) = 1 - 0.89617 = 0.10383$

(*iii*) $P(-1.96 < Z < 1.96) = P(Z < 1.96) - P(Z < -1.96)$
$= \Phi(1.96) - \Phi(-1.96)$
$= 0.97500 - 0.02500 = 0.95$

(*vi*) $P(-\infty < Z < 2.12) = P(Z < 2.12) - P(Z < -\infty)$
$= \Phi(2.12) - \Phi(-\infty)$
$= 0.98300 - 0 = 0.98300$ $\{ \text{since } \Phi(-\infty) = 0 \}$

(*v*) $P(-2.72 < Z < \infty) = P(Z < \infty) - P(Z < -2.72)$
$= \Phi(\infty) - \Phi(-2.72)$
$= 1 - 0.00326 = 0.99674$ $\{ \text{since } \Phi(+\infty) = 1 \}$

***Example* 10.3** *If Z is a standard normal random variable with mean 0 and variance 1, then find*

(*i*) $P(Z < 1.282)$ (*ii*) $P(|Z| < 1.64)$

(*iii*) $P(|Z| > 2.37)$ (*iv*) $P(Z < -1.64 \text{ or } Z > 2.32)$

Solution. From the definition of standard normal cumulative distribution function, we have

(*i*) $P(Z < 1.282) = \Phi(1.282)$ (By interpolating)

$$= \Phi(1.28) + \frac{1.282 - 1.28}{1.29 - 1.28}\{\Phi(1.29) - \Phi(1.28)\}$$

$= 0.89973 + 0.2(0.90147 - 0.89973) = 0.900078$

(*ii*) $P(|Z| < 1.64) = 2\Phi(1.64) - 1$ $\{ \text{since } P(|Z| < a) = 2\Phi(a) - 1 \}$
$= 2(0.94950) - 1 = 0.899$

(*iii*) $P(|Z| > 2.37) = 2\Phi(-2.37)$ $\{ \text{since } P(|Z| > a) = 2\Phi(-a) \}$
$= 2(0.00889) = 0.01778$

(*iv*) $P(Z < -1.64 \text{ or } Z > 2.32) = P(Z < -1.64) + P(Z > 2.32)$
$= P(Z < -1.64) + 1 - P(Z < 2.32)$
$= \Phi(-1.64) + 1 - \Phi(2.32)$
$= 0.05050 + 1 - 0.98983 = 0.06067$

***Example* 10.4** *If $Z \sim N(0, 1)$, then find the value of a such that*

(*i*) $P(Z > a) = 0.868,$ (*ii*) $P(|Z| < a) = 0.90$

(*iii*) $P(|Z| > a) = 0.238$ (*iv*) $P(Z < a) = 0.6198$

Solution. We have

(*i*) $P(Z > a) = 0.868$
$P(Z < a) = 1 - 0.868 = 0.132$
$\Phi(a) = 0.132$
$a = \Phi^{-1}(0.132) = -1.117$ {From Table 10 (*a*)}

(ii)
$$P(|Z| < a) = 0.90$$
$$2\Phi(a) - 1 = 0.90 \qquad \{\text{ since } P(|Z| < a) = 2\Phi(a) - 1\}$$
$$\Phi(a) \doteq 0.95$$
$$a = \Phi^{-1}(0.95) = 1.645$$

(iii)
$$P(|Z| > a) = 0.238$$
$$2\Phi(-a) = 0.238 \qquad \{\text{ since } P(|Z| > a) = 2\Phi(-a)\}$$
$$\Phi(-a) = 0.119$$
$$-a = \Phi^{-1}(0.119) = -1.18$$
$$a = 1.18$$

(iv)
$$P(Z < a) = 0.6198$$
$$\Phi(a) = 0.6198$$
$$a = \Phi^{-1}(0.6198) \qquad (\text{By interpolating})$$
$$= \Phi^{-1}(0.619) + \frac{0.6198 - 0.619}{0.620 - 0.619}\{\Phi^{-1}(0.620) - \Phi^{-1}(0.619)\}$$
$$= 0.3029 + 0.8(0.3055 - 0.3029) = 0.30498$$

10.2.6 Quantiles of Standard Normal Distribution. Let $0 < p < 1$, then the p-th quantile or $(100\,p)$-th percentile of the distribution of standard normal random variable Z is a value z_p such that

$$P(Z \le z_p) = p$$
$$\Phi(z_p) = p$$
$$z_p = \Phi^{-1}(p)$$

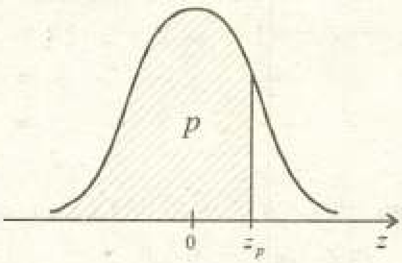


Fig. 10.5 The p-th quantile of standard normal distribution

Therefore, for the 95-th percentile $z_{0.95}$ of standard normal random variable Z, we have

$$P(Z \le z_{0.95}) = 0.95$$
$$\Phi(z_{0.95}) = 0.95$$
$$z_{0.95} = \Phi^{-1}(0.95) = 1.645 \qquad \{\text{ From Table } 10(a)\}$$

***Example* 10.5** *If Z is a standard normal random variable, then find the lower and upper quartiles, the inter quartile range, the quartile deviation and the 70-th percentile of the distribution of Z.*

Solution.

For the first quartile $z_{0.25}$, we have

$$P(Z \le z_{0.25}) = 0.25$$
$$\Phi(z_{0.25}) = 0.25$$
$$z_{0.25} = \Phi^{-1}(0.25)$$
$$z_{0.25} = -0.6745$$

{ From Table 10 (*a*) }

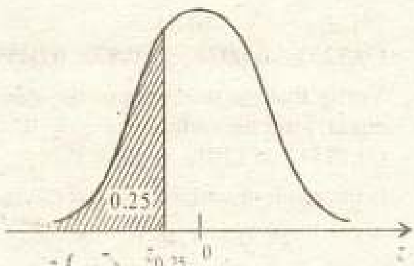


Fig 10.6 First quartile of standard normal distribution

For the third quartile $z_{0.75}$, we have

$$P(Z \le z_{0.75}) = 0.75$$
$$\Phi(z_{0.75}) = 0.75$$
$$z_{0.75} = \Phi^{-1}(0.75)$$
$$z_{0.75} = 0.6745$$

{ From Table 10 (*a*) }

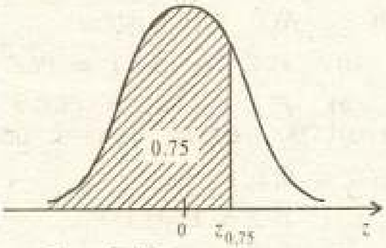


Fig 10.7 Third quartile of standard normal distribution

The inter quartile range (*I. Q. R*) and the quartile deviation (*Q. D*) are

$$I.\,Q.\,R = z_{0.75} - z_{0.25} = 0.6745 - (-0.6745) = 1.349$$

$$Q.\,D = \frac{z_{0.75} - z_{0.25}}{2} = \frac{0.6745 - (-0.6745)}{2} = 0.6745$$

For the 70-th percentile $z_{0.70}$, we have

$$P(Z \le z_{0.70}) = 0.70$$
$$\Phi(z_{0.70}) = 0.70$$
$$z_{0.70} = \Phi^{-1}(0.70)$$
$$z_{0.70} = 0.5244$$

{ From Table 10 (*a*) }

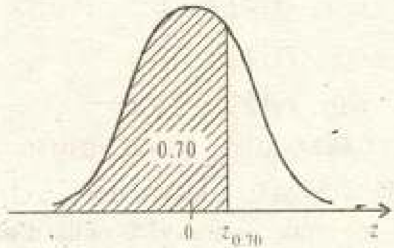


Fig 10.8 Seventieth percentile of standard normal distribution

Exercise 10.1

1. (*a*) Define the normal probability density function and the normal cumulative distribution function. Give the equation of the normal curve with mean μ and standard deviation σ.

(*b*) Define the standard normal probability density function and the standard normal cumulative distribution function. Give the equation of the normal curve with mean 0 and standard deviation 1.

2. (*a*) Find the ordinates of the standard normal curve at

(*i*) $z = 0.64$, (*ii*) $z = 2.84$,
(*iii*) $z = -0.84$ (*iv*) $z = -2.08$
(0.3251, 0.0071, 0.2897, 0.0459)

(*b*) Verify that the ordinates of the standard normal curve at $z = 1.27$ and $z = -1.27$ are equal. Find the ordinate at $z = 0$.
(0.1781, 0.1781, 0.3989)

3. (*a*) If the random variable Z has the standard normal distribution, find

(*i*) $P(Z < 1.46)$ (*ii*) $P(Z > 2.58)$
(*iii*) $P(Z < -1.48)$ (*iv*) $P(Z > -1.96)$
(*v*) $P(0.56 < Z < 1.99)$ (*vi*) $P(-1.32 < Z < 1.65)$
(0.92785, 0.00494, 0.06944, 0.97500, 0.2644, 0.8571)

(*b*) If $Z \sim N(0, 1)$, verify that

(*i*) $P(Z < -2.15) = P(Z > 2.15)$
(*ii*) $P(Z < 1.86) = P(Z > -1.86)$
(0.01578, 0.01578, 0.96856, 0.96856)

4. (*a*) If $Z \sim N(0, 1)$, find

(*i*) $P(Z > 1.645)$ (*ii*) $P(Z < -1.645)$
(*iii*) $P(Z > 1.282)$ (*iv*) $P(Z > 1.96)$
(*v*) $P(Z > 2.576)$ (*vi*) $P(Z > 2.326)$
(*vii*) $P(Z > 2.808)$ (*viii*) $P(Z < -1.96)$
(0.05, 0.05, 0.0999, 0.025, 0.005, 0.01, 0.0025, 0.025)

(*b*) If $Z \sim N(0, 1)$, find

(*i*) $P(|Z| < 1)$ (*ii*) $P(|Z| < 1.96)$
(*iii*) $P(|Z| < 3)$, (*iv*) $P(|Z| > 2)$
(*v*) $P(|Z| < 1.78)$ (*vi*) $P(|Z| < 1.645)$
(*vii*) $P(|Z| > 2.326)$ (*viii*) $P(Z < -1.97 \text{ or } Z > 2.5)$
(0.6827, 0.95, 0.9973, 0.0456, 0.925, 0.9, 0.02, 0.03063)

(*c*) If $Z \sim N(0, 1)$, show that

(*i*) the central 95% of the distribution lies between ± 1.96, *i. e.*,
$$P(-1.96 < Z < 1.96) = 0.95,$$
(*ii*) the central 99% of the distribution lies between ± 2.576, *i. e.*,
$$P(-2.576 < Z < 2.576) = 0.99.$$

5. (*a*) If $Z \sim N(0, 1)$, find a if

(*i*) $P(Z < a) = 0.325$ (*ii*) $P(Z > a) = 0.025$
(*iii*) $P(|Z| < a) = 0.9$ (*iv*) $P(|Z| > a) = 0.097$.
(-0.4538, 1.960, 1.645, 1.66)

(b) In a standard normal distribution, find

(i) a point that has 97.5% area below it, *i. e.*, $z_{0.975}$;

(ii) a point that has 97.5% area above it, *i. e.*, $z_{0.025}$;

(iii) two such points that contain central 90% area *i. e.*, $z_{0.05}$ and $z_{0.95}$.

(1.96, − 1.96, − 1.645, 1.645)

6. (a) If $Z \sim N(0,\ 1)$, find a if $P(|Z| < a)$ takes the value (i) 80% (ii) 99%.

(1.282, 2.576)

(b) If $Z \sim N(0,\ 1)$, find a if $P(|Z| > a)$ takes the value (i) 5% (ii) 2%.

(1.96, 2.326)

7. (a) Find the median, the lower and the upper quartiles, and the inter-quartile range for a standard normal random variable Z.

(b) In a standard normal distribution,

(i) what is the value of mode,

(ii) the area to the right of $z = 1$ is 0.1587, what is the area to the left of $z = 1$?

(iii) find two points on z scale such that the area between them is 80%,

(iv) find the area between -1.5 and 2.5 on z scale.

(0; 0.8413; − 1.28, 1.28; 0.9270)

10.2.7 Use of the Standard Normal Tables for Any Normal Distribution. We now show how the tables of the standard normal random variable Z can be used for any normal random variable X where $X \sim N(\mu,\ \sigma^2)$.

***Theorem* 10.5** *If $X \sim N(\mu,\ \sigma^2)$, then*

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

***Theorem* 10.6** *If $X \sim N(\mu,\ \sigma^2)$ and a, b are any real numbers, then*

(i) $P(X \le a) = \Phi\left(\dfrac{a-\mu}{\sigma}\right)$

(ii) $P(X \ge a) = 1 - \Phi\left(\dfrac{a-\mu}{\sigma}\right)$

(iii) $P(a \le X \le b) = \Phi\left(\dfrac{b-\mu}{\sigma}\right) - \Phi\left(\dfrac{a-\mu}{\sigma}\right)$

***Theorem* 10.7** *If $X \sim N(\mu,\ \sigma^2)$ and a is any real number, then*

$$f(a) = \frac{1}{\sigma}\,\varphi\left(\frac{a-\mu}{\sigma}\right)$$

***Example* 10.6** *If X is a normal random variable with $\mu = 40$ and $\sigma = 5$, write down its probability density function. Find the ordinate of its normal curve at $x = 42.5$. Also find its maximum ordinate.*

Solution. We have

$$f(a) = \frac{1}{\sigma}\varphi\left(\frac{a-\mu}{\sigma}\right)$$

$$f(42.5) = \frac{1}{5}\varphi\left(\frac{42.5-40}{5}\right)$$

$$= \frac{1}{5}\varphi(0.5) = \frac{1}{5}(0.35207) \qquad \text{(From Table 7)}$$

$$= 0.070414$$

Alternately, The probability density function of the normal random variable X with parameters $\mu = 40$ and $\sigma = 5$ is

$$f(x) = \frac{1}{5\sqrt{2\pi}}e^{-(x-40)^2/2(5)^2} = \frac{1}{5\sqrt{2\pi}}e^{-(x-40)^2/50}$$

$$f(42.5) = \frac{1}{5\sqrt{2\pi}}e^{-(42.5-40)^2/50} = 0.0704$$

The maximum ordinate of this normal curve is at $x = \mu = 40$, which is

$$f(40) = \frac{1}{5}\varphi\left(\frac{40-40}{5}\right)$$

$$= \frac{1}{5}\varphi(0) = \frac{1}{5}(0.39894) \qquad \text{(From Table 7)}$$

$$= 0.079788$$

***Example* 10.7** *The scores made by candidates in a certai: .st are normally distributed with mean 500 and standard deviation 100. What percent of the candidates received scores*

(*i*) *less than* 400,
(*ii*) *more t... n* 700,
(*iii*) *between* 400 *and* 600,
(*iv*) *which differ from mean by more than* 150,
(*v*) *if a candidate gets a score of* 680, *what percent of the candidates have higher scores than he?*

Solution. Let X be the score of a candidate, then $\mu = 500$ and $\sigma = 100$.

(*i*) $$P(X < 400) = P\left(\frac{X-\mu}{\sigma} < \frac{400-500}{100}\right)$$

$$= P(Z < -1) = \Phi(-1) = 0.15866 = 15.87\%$$

(*ii*) $$P(X > 700) = P\left(\frac{X-\mu}{\sigma} > \frac{700-500}{100}\right)$$

$$= P(Z > 2) = 1 - P(Z < 2)$$

$$= 1 - \Phi(2) = 1 - 0.97725 = 0.02275 = 2.28\%$$

(*iii*)
$$P(400 < X < 600) = P\left(\frac{400-500}{100} < \frac{X-\mu}{\sigma} < \frac{600-500}{100}\right)$$
$$= P(-1 < Z < 1) = P(Z < 1) - P(Z < -1)$$
$$= \Phi(1) - \Phi(-1) = 0.84134 - 0.15866$$
$$= 0.68266 = 68.27\%$$

(*iv*)
$$P(|X-\mu| > 150) = P\left(\left|\frac{X-\mu}{\sigma}\right| > \frac{150}{100}\right)$$
$$= P(|Z| > 1.5) = 2\,\Phi(-1.5)$$
$$= 2\,(0.06681) = 0.13362 = 13.362\%$$

(*v*)
$$P(X > 680) = P\left(\frac{X-\mu}{\sigma} > \frac{680-500}{100}\right)$$
$$= P(Z > 1.8) = 1 - P(Z < 1.8)$$
$$= 1 - \Phi(1.8) = 1 - 0.96407 = 0.03593 = 3.59\%$$

Example 10.8 *Given that the height of college boys is normally distributed with mean* $5'\text{-}2''$ *and standard deviation* $4''$ *and that the minimum height required for joining the* N.C.C. is $5'\text{-}4''$. *Find the percentage of boys who would be rejected on account of their height.*

Solution. Let X be the height of a college boy, then $\mu = 5'\text{-}2'' = 62$ inches and $\sigma = 4$ inches. The students with heights less than $5'\text{-}4'' = 64$ inches will be rejected for joining *N.C.C.* Then

$$P(X < 64) = P\left(\frac{X-\mu}{\sigma} < \frac{64-62}{4}\right)$$
$$= P(Z < 0.5) = \Phi(0.5) = 0.69146 = 69.15\%$$

Example 10.9 *In a normal distribution with mean* μ *and standard deviation* σ *find* $P(\mu - \sigma \le X \le \mu + \sigma)$.

Solution. Let X be a normal random variable with mean μ and standard deviation σ, then

$$P(\mu - \sigma \le X \le \mu + \sigma) = \left(\frac{\mu-\sigma-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{\mu+\sigma-\mu}{\sigma}\right)$$
$$= P(-1 \le Z \le 1) = P(Z \le 1) - P(Z \le -1)$$
$$= \Phi(1) - \Phi(-1) = 0.84134 - 0.15866$$
$$= 0.68268$$

Example 10.10 *If the diameters of ball bearings are normally distributed with mean* 0.6140 *inches and standard deviation* 0.0025 *inches, determine the percentage of ball bearings with diameters*

(*i*) *less than* 0.608 *inches,* (*ii*) *greater than* 0.617 *inches,*

(*iii*) *between* 0.610 *and* 0.618 *inches inclusive,* (*iv*) *equal to* 0.615 *inches.*

Solution. Let X be the diameter of a ball bearing, then $\mu = 0.6140$ inches and $\sigma = 0.0025$ inches. Considering the measurement errors, we apply continuity correction to the measurements.

(*i*) The diameter smaller than 0.608 inches is in fact the diameter less than 0.6075 inches. Then

$$P(X < 0.6075) = P\left(\frac{X-\mu}{\sigma} < \frac{0.6075 - 0.6140}{0.0025}\right)$$
$$= P(Z < -2.6) = \Phi(-2.6) = 0.00466 = 0.466\%$$

(*ii*) The diameter greater than 0.617 inches is in fact the diameter more than 0.6175 inches. Then

$$P(X > 0.6175) = P\left(\frac{X-\mu}{\sigma} > \frac{0.6175 - 0.6140}{0.0025}\right)$$
$$= P(Z > 1.4) = 1 - P(Z < 1.4) = 1 - \Phi(1.4)$$
$$= 1 - 0.91924 = 0.08076 = 8.076\%$$

(*iii*) The diameter between 0.610 and 0.618 inches inclusive is in fact the diameter between 0.6095 and 0.6185 inches. Then

$$P(0.6095 < X < 0.6185) = P\left(\frac{0.6095 - 0.6140}{0.0025} < \frac{X-\mu}{\sigma} < \frac{0.6185 - 0.6140}{0.0025}\right)$$
$$= P(-1.8 < Z < 1.8) = P(Z < 1.8) - P(Z < -1.8)$$
$$= \Phi(1.8) - \Phi(-1.8) = 0.96407 - 0.03593$$
$$= 0.92814 = 92.814\%$$

(*iv*) The diameter equal to 0.615 inches is in fact the diameter between 0.6145 and 0.6155 inches. Then

$$P(0.6145 < X < 0.6155) = P\left(\frac{0.6145 - 0.6140}{0.0025} < \frac{X-\mu}{\sigma} < \frac{0.6155 - 0.6140}{0.0025}\right)$$
$$= P(0.2 < Z < 0.6) = P(Z < 0.6) - P(Z < 0.2)$$
$$= \Phi(0.6) - \Phi(0.2) = 0.72575 - 0.57926$$
$$= 0.14649 = 14.649\%$$

10.2.8 De-standardizing. Sometimes it is required to find a value of X that corresponds to the standardized value of Z. We use the relation

$$Z = \frac{X - \mu}{\sigma} \quad \Rightarrow \quad X = \mu + \sigma Z$$

10.2.9 Quantiles of a Normal Distribution. Let $0 < p < 1$, then the p-th quantile or $(100\,p)$-th percentile of the distribution of standard normal random variable Z is a value z_p such that

$$P(Z \le z_p) = p$$
$$\Phi(z_p) = p$$
$$z_p = \Phi^{-1}(p)$$

This value z_p of the p-th quantile or $(100\,p)$-th percentile of the standard normal random variable $Z = (X - \mu)/\sigma$ can be de-standardized for determining the p-th quantile or $(100\,p)$-th percentile x_p of any normal random variable X with parameters μ and σ by the relation

$$X = \mu + \sigma Z$$

Therefore

$$x_p = \mu + \sigma\, z_p$$

***Example* 10.11** *If $X \sim N(50,\ 25)$, find the value of X which corresponds to a standardized value*

(*i*) -1.4, (*ii*) 0, (*iii*) 1.6

Solution. We have $X \sim N(50,\ 25)$, then $\mu = 50$ and $\sigma^2 = 25 \Rightarrow \sigma = 5$. Then

$$z = \frac{x-\mu}{\sigma} = \frac{x-50}{5} \quad\Rightarrow\quad x = 50 + 5\,z$$

Putting the values (*i*) $z = -1.4$, (*ii*) $z = 0$, (*iii*) $z = 1.6$, we get

(*i*) For $z = -1.4$, we get

$$x = 50 + 5\,z = 50 + 5(-1.4) = 4.3$$

(*ii*) For $z = 0$, we get

$$x = 50 + 5\,z = 50 + 5(0) = 50$$

(*iii*) For $z = 1.6$, we get

$$x = 50 + 5\,z = 50 + 5(1.6) = 58$$

***Example* 10.12** *If $X \sim N(70,\ 25)$, find*

(*i*) *a point that has 87.9% of the distribution below it,*

(*ii*) *a point that has 81.7% of the distribution above it,*

(*iii*) *two such points between which the central 70% of the distribution lies.*

Solution. We have $X \sim N(70,\ 25)$, then $\mu = 70$ and $\sigma^2 = 25 \Rightarrow \sigma = 5$

(*i*) Let a be the point that has 87.9% area below it. Then

$$P(X < a) = 87.9\% = 0.879$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{a-70}{5}\right) = 0.879$$

$$P\left(Z < \frac{a-70}{5}\right) = 0.879$$

$$\Phi\left(\frac{a-70}{5}\right) = 0.879$$

$$\frac{a-70}{5} = \Phi^{-1}(0.879) = 1.17 \qquad \{\text{From Table } 10\,(a)\}$$

$$a = 70 + 5(1.17) = 75.85$$

(ii) Let a be the point with that has 81.7% area above it. Then

$$P(X > a) = 81.7\% = 0.817$$

$$P(X < a) = 1 - 0.817 = 0.183$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{a-70}{5}\right) = 0.183$$

$$P\left(Z < \frac{a-70}{5}\right) = 0.183$$

$$\Phi\left(\frac{a-70}{5}\right) = 0.183$$

$$\frac{a-70}{5} = \Phi^{-1}(0.183) = -0.904$$

$$a = 70 + 5(-0.904) = 65.48$$

(iii) Let a, b be the two points between which 70% area lies. Then

$$P(a < X < b) = 70\% = 0.70$$

But $\quad P(X < a) + P(a < X < b) + P(X > b) = 1 \qquad$ (Total probability)

$$P(X < a) + 0.70 + P(X > b) = 1$$

$$P(X < a) + P(X > b) = 1 - 0.70 = 0.30$$

By symmetry $\quad P(X < a) = P(X > b) = 0.30/2 = 0.15$, therefore

$$P(X < a) = 0.15$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{a-70}{5}\right) = 0.15$$

$$P\left(Z < \frac{a-70}{5}\right) = 0.15$$

$$\Phi\left(\frac{a-70}{5}\right) = 0.15$$

$$\frac{a-70}{5} = \Phi^{-1}(0.15)$$

$$\frac{a-70}{5} = -1.0364$$

$$a = 70 + 5(-1.0364) = 64.818$$

$$P(X > b) = 0.15$$

$$(X < b) = 1 - 0.15 = 0.85$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{b-70}{5}\right) = 0.85$$

$$P\left(Z < \frac{b-70}{5}\right) = 0.85$$

$$\Phi\left(\frac{b-70}{5}\right) = 0.85$$

$$\frac{b-70}{5} = \Phi^{-1}(0.85)$$

$$\frac{b-70}{5} = 1.0364$$

$$b = 70 + 5(1.0364) = 75.182$$

***Example* 10.13** *If $X \sim N(24, 16)$, then find the 33-rd percentile.*

Solution. We have $X \sim N(24, 16)$, then $\mu = 24$ and $\sigma^2 = 16 \Rightarrow \sigma = 4$

For the 33-rd percentile $x_{0.33}$ or P_{33}, we have

$$P(X < x_{0.33}) = 0.33$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{x_{0.33}-24}{4}\right) = 0.33$$

$$P\left(Z < \frac{x_{0.33}-24}{4}\right) = 0.33$$

$$\Phi\left(\frac{x_{0.33}-24}{4}\right) = 0.33$$

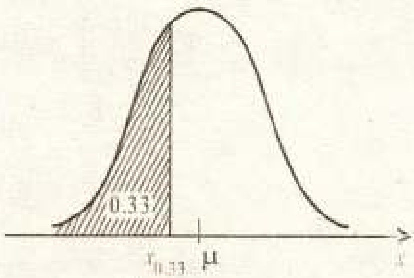


Fig 10.9 Thirty third percentile of the given normal distribution

$$\frac{x_{0.33}-24}{4} = \Phi^{-1}(0.33) = -0.4399$$

$$x_{0.33} = 24 + 4(-0.4399) = 22.24$$

10.2.10 Finding the values of μ or σ or both.

***Example* 10.14** *If $X \sim N(\mu, 25)$ and $P(X > 69.6) = 0.017$, find the value of the mean, μ.*

Solution. We have $X \sim N(\mu, 25)$, then $\sigma^2 = 25 \Rightarrow \sigma = 5$

$$P(X > 69.6) = 0.017$$

$$P(X < 69.6) = 1 - 0.017 = 0.983$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{69.6-\mu}{5}\right) = 0.983$$

$$P\left(Z < \frac{69.6-\mu}{5}\right) = 0.983$$

$$\Phi\left(\frac{69.6-\mu}{5}\right) = 0.983$$

$$\frac{69.6-\mu}{5} = \Phi^{-1}(0.983) = 2.120$$

$$\mu = 69.6 - 5(2.12) = 59$$

***Example* 10.15** *If $X \sim N(50, \sigma^2)$ and $P(X < 60.6) = 0.983$, find the value of the standard deviation, σ.*

Solution. We have $X \sim N(50, \sigma^2)$, then $\mu = 50$

$$P(X < 60.6) = 0.983$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{60.6-50}{\sigma}\right) = 0.983$$

$$P\left(Z < \frac{10.6}{\sigma}\right) = 0.983$$

$$\Phi\left(\frac{10.6}{\sigma}\right) = 0.983$$

$$\frac{10.6}{\sigma} = \Phi^{-1}(0.983) = 2.120$$

$$10.6 = 2.120\,\sigma \quad \Rightarrow \quad \sigma = 5$$

***Example* 10.16** *In a normal distribution 33% of the values are under 48 and 12.3% are over 60. Find mean and standard deviation of the distribution.*

Solution. We have

$$P(X < 48) = 33\% = 0.33$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{48-\mu}{\sigma}\right) = 0.33$$

$$P\left(Z < \frac{48-\mu}{\sigma}\right) = 0.33$$

$$\Phi\left(\frac{48-\mu}{\sigma}\right) = 0.33$$

$$\frac{48-\mu}{\sigma} = \Phi^{-1}(0.33) = -0.4399$$

$$48 - \mu = -0.4399\,\sigma \quad \text{.......................}(i)$$

$$P(X > 60) = 12.3\% = 0.123$$

$$P(X < 60) = 1 - 0.123 = 0.877$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{60-\mu}{\sigma}\right) = 0.877$$

$$P\left(Z < \frac{60-\mu}{\sigma}\right) = 0.877$$

$$\Phi\left(\frac{60-\mu}{\sigma}\right) = 0.877$$

$$\frac{60-\mu}{\sigma} = \Phi^{-1}(0.877) = 1.1601$$

$$60 - \mu = 1.1601\,\sigma \quad \text{.......................}(ii)$$

Subtracting (i) from (ii), we get

$$\begin{array}{rcl} 60 - \mu & = & 1.1601\,\sigma \\ 48 - \mu & = & -0.4399\,\sigma \\ - \quad + & & + \\ \hline 12 & = & 1.6\,\sigma \end{array} \quad \Rightarrow \quad \sigma = 7.5$$

Putting this value of σ in (ii), we have

$$60 - \mu = 1.1601(7.5) \quad \Rightarrow \quad \mu = 51.3$$

***Example* 10.17** *If X is a normal random variable with parameters $\mu = 50$ and $\sigma = 10$. Find its mean, median, mode, lower and upper quartiles, quartile deviation, mean deviation, variance, standard deviation, first four moments about mean, moment ratios and moment coefficient of skewness.*

Solution. We have $\mu = 50, \quad \sigma = 10$

The mean, median, mode, lower and upper quartiles of the distribution are

$$\text{Mean} = \mu = 50, \quad \text{Median: } x_{0.5} = \mu = 50, \quad \text{Mode} = \mu = 50$$

$$x_{0.25} = \mu - 0.6745\,\sigma = 50 - 0.6745(10) = 43.255$$

$$x_{0.75} = \mu + 0.6745\,\sigma = 50 + 0.6745(10) = 56.745$$

The quartile deviation, mean deviation, variance and standard deviation of the distribution are

$$Q.D(X) = 0.6745\,\sigma = 0.6745\,(10) = 6.745$$
$$M.D(X) = 0.7979\,\sigma = 0.7979\,(10) = 7.979$$
$$Var(X) = \sigma^2 = (10)^2 = 100$$
$$S.D(X) = \sigma = \sqrt{100} = 10$$

The first four moments about mean, moment ratios and moment coefficient of skewness of the distribution are

$$\mu_1 = 0, \qquad \mu_2 = \sigma^2 = (10)^2 = 100,$$
$$\mu_3 = 0, \qquad \mu_4 = 3\sigma^4 = 3\,(10)^4 = 30000$$
$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0)^2}{(100)^3} = 0, \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{30000}{(100)^2} = 3$$
$$Sk = \frac{\mu_3}{\sqrt{\mu_2^3}} = \frac{0}{\sqrt{(100)^3}} = 0$$

***Example* 10.18** *In a normal distribution lower and upper quartiles are* 28 *and* 55 *respectively. Find mean and standard deviation of the normal distribution.*

Solution. We have $x_{0.25} = 28$ and $x_{0.75} = 55$. Then

$$\mu = \frac{x_{0.25} + x_{0.75}}{2} = \frac{28 + 55}{2} = 41.5$$
$$\sigma = \frac{x_{0.75} - x_{0.25}}{1.349} = \frac{55 - 28}{1.349} = 20$$

10.2.11 Normal Distribution as a Limit of a Frequency Distribution of a Continuous Variable. A normal curve serves a good approximation not only for a histogram obtained from the binomial distribution but for many other histograms of observed frequency distributions of continuous random variables as well. Frequently a histogram of an observed frequency distribution with mean $\bar{x}$ and standard deviation s is well approximated for a large number of observations $n = \sum f_i$ by the normal curve whose equation is given by

$$f(x) = \frac{nh}{s\sqrt{2\pi}}\, e^{-(x-\bar{x})^2/2s^2}$$

where h is the common class interval in the grouped frequency distribution. The smaller the value of h, the better the approximation will be.

Exercise 10.2

1. (*a*) If X is a normal random variable with $\mu = 24$ and $\sigma = 4$, write down its probability density function. Find the ordinate of its normal curve at $x = 21$. Also find its maximum ordinate.
(0.075285, 0.099735)

(b) Suppose that during periods of transcendental meditation, the reduction of a person's oxygen consumption is a random variable have a normal distribution with $\mu = 37.6$ *c. c* per minute and $\sigma = 4.6$ *c. c* per minute. Find the probabilities that during a period of transcendental meditation a person's oxygen will be reduced by

(*i*) at most 35.0 *c. c* per minute,

(*ii*) at least 44.5 *c. c* per minute,

(*iii*) any where from 30.0 to 40.0 *c. c.* per minute.

(0.28604, 0.06681, 0.64900)

(c) Let $X \sim N(20, 25)$, find the area under the normal curve

(*i*) below 30,

(*ii*) above 30,

(*iii*) between 30 and 42

(0.97725, 0.02275, 0.02274)

2. (a) Suppose that it is know that IQ's for adult Pakistanis are normally distributed with $\mu = 100$ and $\sigma = 10$. If an individual with an IQ of 130 or above is classified genius, what is the probability that a random selection yields a genius?

(0.00135)

(b) The mean inside diameter of a sample of 250 washers produced by a machine is 5.05 *mm* and the standard deviation is 0.05 *mm*. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.95 *mm to* 5.10 *mm*, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine assuming the diameters are normally distributed

(18.14%)

3. (a) The length of life for a washing machine is approximately normally distributed, with a mean of 3.5 years and a standard deviation of 1.0 years. If this type of washing machine is guaranteed for 12 months, what percentage of the sales will require replacement?

(0.62%)

(b) Assume that the time X required for a runner to run a mile is a normal random variable with parameters $\mu = 4$ minutes 1 second and $\sigma = 2$ seconds. What is the probability that this athlete will run the mile

(*i*) in less than 4 minutes,

(*ii*) in more than 3 minutes 55 seconds.

(0.3085, 0.9987)

(c) Assume that the distance X that a particular athlete will be able to put a shot (on his first try) is normally distributed with parameters $\mu = 50$ feet and $\sigma = 5$ feet. Compute the probability that he tosses it not less than 55 feet and the probability that his toss travels between 50 feet and 60 feet.

(0.1587, 0.4773)

(d) If X is normally distributed with parameters μ and σ, find the area under the curve between

(*i*) $(\mu - \sigma)$ and $(\mu + \sigma)$,

(*ii*) μ and $(\mu + 2\sigma)$.

(0.68268, 0.47725)

4. (*a*) The heights of boys at a particular age follow a normal distribution with mean 150.3 *cm* and standard deviation 5.0 *cm*. Find the probability that a boy picked at random from this age group has height

(*i*) less than 153 *cm*,

(*ii*) more than 145 *cm*,

(*iii*) between 146 *cm* and 152 *cm*.

(0.67003, 0.83147, 0.5015)

(*b*) Suppose the weekly incomes X are normally distributed with mean 10.06 Rs. and variance 2.64 Rs^2., find the probability $P(8 \le X \le 12)$. Assume that the incomes are recorded to the nearest rupee.

(0.87569)

5. (*a*) Find the value of X which corresponds to a standardized value of -2.05 and 0.86 for each of the following distributions

(*i*) $X \sim N(62.3, 38)$,

(*ii*) $X \sim N(\mu, \sigma^2)$,

(*iii*) $X \sim N(a, b)$.

(49.66, 67.60; $\mu - 2.05\sigma$, $\mu + 0.86\sigma$; $a - 2.05\sqrt{b}$, $a + 0.86\sqrt{b}$)

(*b*) If $X \sim N(100, 64)$, find the value of a such that $P(X < a) = 0.95$.

(113.16)

(*c*) If $X \sim N(60, 25)$, find the value of a such that $P(X > a) = 0.837$.

(55.09)

6. (*a*) In a normal distribution $\mu = 30$ and $\sigma = 5$. Find

(*i*) a point that has 15% area below it,

(*ii*) a point that has 28% area above it,

(*iii*) two points containing middle 95% area,

(24.8; 32.9; 20.2 and 39.8)

(*b*) The time required by a nurse to inject a shot of penicillin has been observed to be normally distributed, with a mean of $\mu = 30$ seconds and a standard deviation of $\sigma = 10$ seconds. Find the

(*i*) 10-th percentile, *i. e.*, $x_{0.10}$.

(*ii*) 90-th percentile, *i. e.*, $x_{0.90}$.

(17.2 *sec*, 42.8 *sec*)

(*c*) Scores on a national education achievement test are normally distributed with $\mu = 500$ and $\sigma = 100$.

(*i*) What is the 95-th percentile of this distribution.

(*ii*) What are the lower and upper quartiles of this distribution.

(*iii*) If the university decides to accommodate the 40 percent of the students with the highest scores, what is the score that separates the successful applicants with unsuccessful?

(664.5; 432.55, 567.45; 525.33)

7. (a) The height a high jumper will clear, each time he jumps, is a normal random variable with mean 6 feet and standard deviation 2.4 inches.

(i) What is the greatest height he will jump with the probability 0.95?

(ii) What is the height he will clear only 10% of the time?

(68.06 inches, 75 inches)

(b) Suppose that the amount of vaccine required to immunize human beings against smallpox is normally distributed with $\mu = 0.250$ ounce and $\sigma = 0.040$ ounce. Increasing the dosage increase the chances of successful vaccination. What is the minimum dosage required to produce success in 99 percent of the cases.

(0.34304 *ounces*)

8. (a) If $X \sim N(70, 25)$, find the value of a such that $P(|X - 70| < a) = 0.8$. Hence find the limits within which the central 80% of the distribution lies.

(6.41, 63.59, 76.41)

(b) Bags of flour by a particular machine have masses which are normally distributed with mean 500 g and standard deviation 20 *g*. 2% of the bags are rejected for being underweight and 1% of the bags are rejected for being over weight. Between what range of values should the mass of a bag of flour lie is to accepted?

(458.92, 546.52)

9. (a) The lengths of items follow a normal distribution with mean μ *cm* and standard deviation 12 *cm*. It is known that 4.78% of the items have a length greater than 92 *cm*. Find the value of mean μ.

(72)

(b) The lengths of rods produce in a workshop follow a normal distribution with mean μ and variance 4. If 10% of the rods are less than 17.4 *cm* long. Find the probability that a rod chosen at random will be between 18 and 23 *cm* long.

(0.7725)

10. (a) Tea is soled in packages marked 750 *g*. The masses of the package are normally distributed with mean 760 *g* standard deviation σ. What is the maximum value of the σ if less than 1% of the packages are under weight?

(4.299)

(b) Suppose that the life in hours of an electric tube manufactured by a certain process is normally distributed with parameters $\mu = 160$ hours and σ hours. What is the maximum allowable value for σ, if the life X of a tube is to have probability 0.80 of being between 120 and 200 hours?

(31.21)

11. (a) Assume that we have a large number of students whose average weight is 150 *lb* and that the weights are normally distributed. If we know that 36.4% of the students have weights between 137 and 163 *lb*. What is the standard deviation of the weights?

(27.47)

(b) In a normal distribution $\mu = 40$ and $P(25 < X < 55) = 0.8662$. Find $P(20 < X < 60)$.

(0.9545)

12. (a) In a normal distribution 31% of item are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.

(50, 10)

(b) If $X \sim N(\mu, \sigma^2)$ and $P(X < 35) = 0.20$ and $P(35 < X < 45) = 0.65$. Find μ and σ.
(39.5, 5.32)

13. (a) Assuming that the number of marks scored by a candidates is normally distributed, find the mean and the standard deviation, if the number of first class students (60% or more marks) is 25, the number of failed students (less than 30% marks) is 90 and the total number of candidates appearing for the examination is 450.
(40.37, 12.32)

(b) A marketing organization grades apples into three sizes, small (diameter less than 60 mm), medium (diameter between 60 and 80 mm), and large (diameter more than 80 mm). A certain grower finds that 61% of his crop falls into the small category, and 14% into the large category. Assuming that the distribution of the diameter X of the apples is described by a normal probability density function, calculate the standard deviation and mean of his crop.
(24.97, 53.03)

(c) The maximum temperature on June, 1 in a certain locality has been recorded and observed as normally distributed over year. About 15% of the time, it has exceeded 30^0 C, and about 5% of the time, it has been less than 20^0 C. What is the mean and variance of the data?
(26.13^0 C, 13.91^0 C)

14. (a) A man cuts hazel twigs to make bean poles. He says that a stick is 240 *cm* long. In fact, the length of the stick follows a normal distribution and 10% are of length 250 *cm* or more while 55% have a length over 240 *cm*. Find the probability that a stick picked at random is less than 235 *cm* long.
(0.203)

(b) The 90-th percentile of a normal distribution is 50 while the 15-th percentile is 25.

(*i*) Find μ and σ.

(*ii*) What is the value of 40-th percentile.

(36.17, 10.79, 33.44)

15. (a) The masses of articles produced in a particular workshop are normally distributed with mean μ and standard deviation σ. The 5% of the articles have a mass greater than 85 *g* and 10% have a mass less than 25 *g*. Find the value of μ and σ, and find the range symmetrical about the mean, within which 75% of the masses lie.
(51.3, 20.5, 26.72, 73.88)

(b) In a certain examination, the percentage of passes and distinctions were 80 and 10 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively, assume the distribution of marks to be normal.
(53.87, 16.48)

16. (a) What is the importance of normal distribution in statistical theory? Describe its properties.

(b) Suppose that X is normally distributed with $\mu = 25$ and $\sigma = 5$. Find

(*i*) the lower and upper quartiles,

(*ii*) the median.

(*iii*) the mean deviation.

(21.6, 28.4, 25, 4)

(c) In a normal distribution, the lower and upper quartiles are respectively 8 and 17. Find mean and standard deviation of the normal distribution.
(12.5, 6.67)

(d) The continuous random variable X is normally distributed with mean μ and standard deviation σ. Given that $P(X < 53) = 0.04$ and $P(X < 65) = 0.97$. Find the inter-quartile range of the distribution.
(4.46)

17. (a) The value of second moment about the mean in a normal distribution is 4. Find the third and the fourth moments about the mean in the distribution.
(0, 48)

(b) Find the proportion of the area under the normal curve included between the limits $\mu \pm \sigma$, $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$ where μ and σ denote the mean and the standard deviation.
(0.6827, 0.9545, 0.9973)

(c) If X is a random variable with distribution $N(12.5, 9)$ and the random variable $Y = g(X) = 2X + 5$. Find $P(-\infty \le Y \le 21)$ and $P(45 \le Y \le \infty)$.
(0.06681, 0.00621)

◆

Exercise 10.3
Objective Questions

1. Fill in the blanks.

(*i*) The normal distribution is a ——— distribution that ranges from $-\infty$ to ∞. (*continuous*)

(*ii*) The value of the parameter σ of a normal distribution is always ———. (*positive*)

(*iii*) The normal distribution is a bell shaped ——— distribution. (*symmetrical*)

(*iv*) If $X \sim N(50, 25)$, then σ = ———. (5)

(*v*) The maximum ordinate of the standard normal curve is at Z = ———. (0)

(*vi*) In a standard normal distribution, if
$P(Z < z_{0.975}) = 0.975$, then $z_{0.975}$ = ———. (1.96)

(*vii*) The maximum ordinate of a normal curve is at X = ———. (μ)

(*viii*) The total area under a normal curve is ———. (*unity*)

(*ix*) The ——— of a normal distribution corresponds to $z = 0$ in the standard normal distribution. (*mean*) (*x*) In a normal distribution, the mean, median and mode are ———— . (*equal*)

2. Fill in the blanks.

(*i*) In a normal distribution, ———— $= \mu - 0.6745\,\sigma$. (Q_1)

(*ii*) In a normal distribution, ———— $= \mu + 0.6745\,\sigma$. (Q_3)

(*iii*) In a normal distribution, $QD \cong$ ———— σ. (2/3)

(*iv*) In a normal distribution, $MD \cong$ ———— σ. (4/5)

(*v*) The limits $\mu \pm 0.6745\,\sigma$ include ———— percent area under the normal curve. (50)

(*vi*) The limits $\mu \pm \sigma$ include ———— percent area under the normal curve. (68.27)

(*vii*) The limits $\mu \pm 2\,\sigma$ include ———— percent area under the normal curve. (95.45)

(*viii*) The limits $\mu \pm 3\,\sigma$ include ———— percent area under the normal curve. (99.73)

(*ix*) In a normal distribution, all odd ordered moments about mean are ————. (*zero*)

(*x*) In a normal distribution, $\beta_1 = 0$ and $\beta_2 =$ ————. (3)

(*xi*) The normal distribution is neither platykurtic nor leptokurtic but ————. (*mesokurtic*)

(*xii*) The points of inflexion of a normal curve are ———— from mean. (equidistant)

3. Mark off the statements as true or false.

(*i*) Normal distribution has two parameters namely μ and σ^2. (*true*)

(*ii*) If X is normally distributed with mean μ and variance σ^2 then it is denoted by $X \sim N(\mu, \sigma^2)$. (*true*)

(*iii*) The standard normal distribution has mean 0 and variance 1. (*true*)

(*iv*) The maximum ordinate of a standard normal curve is at $Z = 1$. (*false*)

(*v*) The standard normal distribution is symmetrical about $Z = 0$. (*true*)

(*vi*) In a standard normal distribution, if $P(Z < z_{0.025}) = 0.025$, then $z_{0.025} = -1.96$. (*true*)

(*vii*) In a standard normal distribution, if $P(Z < z_{0.975}) = 0.975$, then $z_{0.975} = 1.96$. (*true*)

(*viii*) In a standard normal distribution, if $P(|Z| < a) = 0.95$, then $a = 1.96$. (*true*)

(*ix*) The normal curve has maximum ordinate at $X = 0$. (*false*)

4. Mark off the following statements as false or true.

(*i*) The shape of a normal distribution depends upon its parameters namely μ and σ. (*true*)

(*ii*) The parameter σ controls the relative flatness of the normal curve. (*true*)

(*iii*) The normal distribution is a bell-shaped symmetrical distribution. (*true*)

(*iv*) In a normal distribution, the mean, median and mode are equal. (*true*)

(*v*) In a normal distribution,

$$Q_1 = \mu - 0.6745\,\sigma \text{ and } Q_3 = \mu + 0.6745\,\sigma. \qquad (\textit{true})$$

(*vi*) In a normal distribution, mean and variance are always equal. (*false*)

(*vii*) The expected value of a normal distribution is μ. (*true*)

(*viii*) The standard deviation of a normal distribution is σ. (*true*)

(*ix*) The quartile deviation of a normal distribution is $0.6745\,\sigma$. (*true*)

(*x*) The mean deviation of a normal distribution is $0.7979\,\sigma$. (*true*)

(*xi*) The two points containing the middle 95.45% area under a normal curve are $\mu \pm \sigma$. (*false*)

(*xii*) In a normal distribution, all even ordered moments about mean are zero. (*false*)

(*xiii*) In a normal distribution, all odd ordered moments about mean are zero. (*true*)

(*xiv*) In a normal distribution, $\beta_1 = 0$ and $\beta_2 = 3$. (*true*)

(*xv*) The points of inflexion of the normal curve lie at $\mu \pm 2\,\sigma$. (*false*)

(*xvi*) The normal curve gets closer and closer to t[illegible]-*axis* but never touches it. (*true*)